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Vertex–regular 1–factorizations of the complete graph

Gloria Rinaldi[1]

Abstract. A 1–factorization of a complete undirected graph is said to be vertex–regular if it admits an automorphism group G acting on the vertex set in a sharply transitive manner. Which abstract groups can realize such a situation? The complete answer is still unknown, but the problem have been solved in some cases. In this survey we illustrate the state of art on this question. Most of the results were obtained via the *starter method* introduced in [7].

1. Introduction: existence and classification

A 1–*factor* in a graph is a set of pairwise disjoint edges that partition the set of vertices and a 1–*factorization* in a graph is a partition of the edge set into 1–factors. For a general graph it is not so trivial to determine whether it does admit a 1–factorization. Already the problem of determining whether a given cubic graph admits a 1–factorization is known to be computationally NP–complete, [13].

Nevertheless, it is well known that the complete undirected graph K_v admits a 1–factorization if and only if it has an even number v of vertices. In what follows we will always consider v even, if not differently specified, and we will always speak of 1–factorizations of K_v.

Such factorizations are fairly easy to construct and they probably appeared for the first time in 1847 in a paper of Kirkman, [16].

Well known is the construction given by Lucas, [18], in 1883. This construction is a particular case of a more general one which involves the notion of *starter* in a group of odd order, [12].

More precisely, let G be a group of odd order $v-1$ (written additively and with identity 0).
A *starter* in G is set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (v-2)/2\}$ that satisfies:

- $\{s_i : 1 \leq i \leq (v-2)/2\} \cup \{t_i : 1 \leq i \leq (v-2)/2\} = G \setminus \{0\}$
- $\{\pm(s_i - t_i) : 1 \leq i \leq (v-2)/2\} = G \setminus \{0\}$

[1] G. Rinaldi, Università di Modena e Reggio Emilia, Dipartimento di Scienze e Metodi dell'Ingegneria, Viale Amendola 2, 42122 Reggio Emilia, Italy; `gloria.rinaldi@unimore.it`

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This definition applies to arbitrary groups of odd order, abelian and non–abelian ones. A starter permits to construct a 1–factorization $\mathcal{F}$ of K_v. Namely, identify the vertex set of K_v with $G \cup \{\infty\}$, $\infty \notin G$, identify the pairs of distinct elements of $G \cup \{\infty\}$ with the set of edges and take the following 1–factors:
$F_0 = S \cup \{0, \infty\}$, $F_a = F_0 + a = \{\{s_i + a, t_i + a\} : \{s_i, t_i\} \in S\} \cup \{a, \infty\}$, $a \in G$, then $\mathcal{F} = \{F_a : a \in G\}$.

In any group G of odd order the set of pairs $\bar{S} = \{\{x, -x\} : x \in G \setminus \{0\}\}$ is a starter, the so called *patterned starter*. The well known 1–factorization of Lucas, [18], was obtained via the patterned starter in the cyclic group $\mathbb{Z}_{v-1}$.

As far as I am concerned, the patterned starter is mentioned in literature only for abelian groups, see [12]. Nevertheless $\bar{S}$ is a starter even if G is non–abelian. The proof is quite simple. The first condition holds: $G \setminus \{0\}$ is the disjoin union of two sets X, $-X$ such that $x \in X$ iff $-x \in -X$. For the second condition observe that when $x \neq \pm y$ then $2x \notin \{2y, -2y\}$. This is because $2x$ and x generate the same subgroup, as well as y and $2y$, if $2x \in \{2y, -2y\}$ then x and y generate the same subgroup and commute, therefore $2x = \pm 2y$ necessarily implies either $x = y$ or $x = -y$: a contradiction.

Despite the fact that 1–factorizations of a complete graph are so easy to construct, the problem of enumerating them up to isomorphism is very hard: the number of non–isomorphic ones rapidly explodes as the number of vertices increases. In particular, a technique developed in [8] permits to prove that the number of pairwise non–isomorphic 1–factorizations of K_v goes to infinity with v. So, it is clear that a classification of 1–factorizations is practically impossible. An attempt can be done requiring the 1–factorizations to satisfy additional properties. Classification results are obtained by imposing graph theoretic conditions, for example on the nature of 1–factors: think to the rich literature on *perfect, uniform, almost perfect, sequentially uniform* and *sequentially perfect* 1–factorizations which we will not consider in this survey.

An important literature goes in the direction of using symmetry criteria: 1–factorizations with non–trivial automorphism groups are considered and attempts to obtain classifications are done imposing conditions on the automorphism group and on the way this group acts on vertices, edges and 1–factors.

Recall that an *automorphism group* of the 1–factorization is a permutation group on the vertex set preserving the 1–factors. The *full* automorphism group of a 1–factorization $\mathcal{F}$ is usually denoted by $Aut(\mathcal{F})$. Each subgroup of $Aut(\mathcal{F})$ acts on the set of vertices, the set of edges and the set of 1–factors, that is $\mathcal{F}$ itself. Assumptions on one or more of these actions sometimes allow a description of the 1–factorization $\mathcal{F}$ and of the automorphism group.

As you can easily see, a 1–factorization of K_v obtained using a starter in a group G of odd order $v - 1$ (for example the patterned starter) has non trivial automorphism group: it admits G as automorphism group whose action is 1–*rotational*, i.e., G fixes one vertex and acts sharply transitively on the remaining ones. Despite the fact that these 1–factorizations exist for each v odd, 1–factorizations with non trivial symmetries seem to be rare. An automorphism–free 1–factorization is usually called *rigid*. It was proved in [19] that a rigid 1–factorization of K_v exists if and only if $v \geq 10$. Moreover, it was proved in [19] and later in [1], that the number of non–isomorphic rigid 1–factorizations of K_v goes at infinity with v. It was also proved by Cameron (unpublished) and Phelps (unpublished), that the subclass of rigid 1–factorizations asymptotically covers the class of 1–factorizations.

To confirm this fact, we see that 1–factorizations admitting an automorphism group which acts multiply transitively on the vertices are sporadic.

In fact, in [9] it is shown that a 1–factorization of K_v with an automorphism group G acting 3–*transitively* on the set of vertices is either the affine line–parallelism of $AG(d, 2)$, that is $v = 2^d$ with $d \geq 2$, or the 1–factorization of K_6 derived from the cyclic group of order 5. The full automorphism groups are respectively $AGL(d, 2)$ and $PGL(2, 5)$, [8].

1–factorizations of K_v with an automorphism group G which acts *doubly transitively* on the set of vertices are completely determined in [10]. More precisely, W. Burnside, [11, Section 3.5], showed that a doubly transitive permutation group has a transitive minimal normal subgroup which is either an elementary abelian p–group or a non–abelian simple group. In the former case the 1–factorization is the affine line–parallelism of $AG(d, 2)$, that is $v = 2^d$, while in the latter case the 1–factorization is one of the following:

(i) the unique 1–factorization of K_6;

(ii) the affine line–parallelism of $AG(3, 2)$;

(iii) the unique uniform 1–factorization of type $(6, 6)$ of K_{12}, see [8, Chapter 4];

(iv) the 1–factorization of K_{28} which is derived from $G = P\Gamma L(2, 8)$ and described in [10].

For $v = 6, 8$ and 12 (the first three cases), the automorphism groups are respectively $PGL(2, 5)$, $PSL(2, 7)$ and $PSL(2, 11)$. This last group is doubly transitive also on the 1–factors, [8].

In this paper we resume the results obtained on the problem of determining 1–factorizations of K_v which admit an automorphism group G which acts *sharply transitively* on the set of vertices and so $|G| = v$. These 1–factorizations are said to be *vertex–regular under* G, or simply *regular under* G or G*–regular*.

The class of G–regular 1–factorizations was studied mainly considering the isomorphism type of G. The first result in this direction is due to Hartman and Rosa, [14]. They investigated the existence of a *cyclic* 1*–factorization*, that is vertex–regular under the action of a cyclic group. They gave the following non–existence result.

Theorem 1.1 ([14])**.** *If* $v = 2^t$*, with* $t \geq 3$*, then no cyclic* 1*–factorization of* K_v *exists.*

In [14] they also proved the existence of a cyclic 1–factorization of K_v when v is not a power of 2.

Groups of different isomorphism type were later on considered and the main attention was deserved to the following question:

Question. *For which groups* G *of even order* v*, does there exist a* G*–regular* 1*–factorization of the complete graph* K_v*?*

When v is twice an odd number, this problem simplifies somewhat. G must be the semi–direct product of $\mathbb{Z}_2$ with its normal complement and G always realizes a 1–factorization of K_v upon which it acts sharply transitively on vertices, see [2, Remark 1].

When $v = 2n$ and n is even, the complete answer is still unknown. Nevertheless, several authors have dealt with this problem getting some interesting results.

A first answer can be found in [2]. Namely:

Theorem 1.2 ([2])**.** *For each dihedral group G of order v, there exists a G–regular 1–factorization of K_v.*

Observe also that the Question above is a restricted version of problem n.4 in the list of [24]. Namely problem n.4 asks for a 1–factorization of the complete graph K_v possessing an automorphism group with a transitive action on the vertex set. The two versions of the problem are equivalent for abelian groups since every transitive abelian permutation group is sharply transitive.

In [7] Buratti extended the result of [14] and solved problem n.4 for the abelian case. Namely he proved the following:

Theorem 1.3 ([7])**.** *For each abelian group G of even order v (except for G cyclic and $v = 2^t$, $t \geq 3$) there exists a G–regular 1–factorization of K_v.*

To prove the above Theorem, he introduced in [7] the notion of *starter* in a group of *even* order and showed how the existence of a regular 1–factorization under a group G can be entirely tested within G. The notion of starter in a group of even order is essentially different from that of a starter in a group of odd order because of the presence of the involutions. We resume the technique of [7] in the next paragraph, together with the main results obtained applying it.

2. Regular 1–factorizations via starter method

We will always consider $v = 2n$ and we denote by $V(K_v)$ and $E(K_v)$ the set of vertices and edges of K_v, respectively. Let G be a finite group of order v, in additive notation and with identity 0. We identify the vertices of K_v with the group–elements of G and we shall occasionally write K_G rather than K_v. We shall denote by $[x, y]$ the edge with vertices x and y. We always consider G in its right regular permutation representation. In other words, each group–element $g \in G$ is identified with the permutation $V(K_v) \to V(K_v)$, $x \mapsto x + g$. This action of G on $V(K_v)$ is sharply transitive and induces actions on the subsets of $V(K_v)$ and on sets of such subsets. Hence, if $g \in G$ is an arbitrary group–element and S is any subset of $V(K_v)$ then we write $S + g = \{x + g : x \in S\}$. In particular, if $S = [x, y]$ is an edge, then $[x, y]+g = [x+g, y+g]$. Furthermore, if U is a collection of subsets of $V(K_v)$, then we write $U + g = \{S + g : S \in U\}$. In particular, if U is a collection of edges of K_v then $U + g = \{[x + g, y + g] : [x, y] \in U\}$.

The G–orbit of an edge $[x, y]$ has either length $v = 2n$ or n and we speak of a *long* orbit or a *short* orbit, respectively, which corresponds to whether the orbit is a 2–factor or a 1–factor. In this case, we call $[x, y]$ a *long* edge or a *short* edge, respectively. If $[x, y]$ is a short edge, then there is a non–trivial group element g so that $[x + g, y + g] = [x, y]$. Such a g is unique ($g = -x + y$) and is an involution; we call this g the involution *associated* with the short edge $[x, y]$.

A 1–factor of K_{2n} which is fixed by G necessarily coincides with a short G–orbit of edges, see [2, Proposition 2.2].

If H is a subgroup of G then a system of distinct representatives for the left cosets of H in G will be called a *left transversal* for H in G.

If $[x, y]$ is an edge in K_G we define

$$\partial([x,y]) = \begin{cases} \{x-y, y-x\} & \text{if } [x,y] \text{ is long} \\ \{x-y\} & \text{if } [x,y] \text{ is short} \end{cases}$$

$$\phi([x,y]) = \begin{cases} \{x, y\} & \text{if } [x,y] \text{ is long} \\ \{x\} & \text{if } [x,y] \text{ is short .} \end{cases}$$

If S is a set of edges of K_G we define

$$\partial(S) = \bigcup_{e \in S} \partial(e) \qquad \phi(S) = \bigcup_{e \in S} \phi(e)$$

where, in either case, the union may contain repeated elements and so, in general, will return a multiset.

Definition 2.1 ([7, Definition 2.1]). A *starter* in a group G of even order is a set $\Sigma = \{S_1, \cdots, S_k\}$ of subsets of $E(K_G)$ together with subgroups $H_1, \cdots, H_k$ which satisfy the following conditions:

- $\partial S_1 \cup \cdots \cup \partial S_k = G \setminus \{0\}$;
- for $i = 1, \cdots, k$, the set $\phi(S_i)$ is a left transversal for H_i in G;
- for $i = 1, \cdots, k$, H_i must contain the involutions associated with any short edge in S_i.

We note that $G \setminus \{0\}$ is a set, so this definition implies that $\partial([x, y])$ are distinct for all $[x, y]$ in the multiset $S_1 \cup \cdots \cup S_k$. Hence it also follows S_i can have no edges in common with S_j for $i \neq j$.

The main Theorem of [7] proves the existence of a starter in a finite group G of order $2n$ is equivalent to the existence of a 1–factorization of the complete graph K_{2n} admitting G as an automorphism group acting sharply transitively on vertices. A starter contains the minimum amount of information which is necessary to reconstruct the 1–factorization: the first bullet in Definition 2.1 insures that every edge of K_G will occur in exactly one G–orbit of an edge from $S_1 \cup \cdots \cup S_k$. The other bullets insure the union of the H_i–orbits of edges from S_i will form a 1–factor. Namely, for each index i, we form a 1–factor as $\cup_{e \in S_i} Orb_{H_i}(e)$, whose stabilizer in G is the subgroup H_i; the G–orbit of this 1–factor, which has length $|G : H_i|$, is then included in the 1–factorization.

Suppose $g \in G$ is an element of order 2, and suppose the set $S = \{[g, 0]\}$ is an element of a starter in G. Such a set gives rise to a 1–factor which is fixed by G. Moreover, the edges of this 1–factor are short edges. Viceversa, each set of Σ which gives rise to a 1–factor which is fixed by G is necessarily of this type.
We see two very simple examples.

Example 2.2. Consider D_6, the dihedral group of order 6, in multiplicative notation with identity denoted by 1.

$$D_6 =< a, b \; : \; a^3 = b^2 = 1, \; ab = ba^2 >= \{1, a, a^2, b, ba, ba^2\}$$

A starter in D_6 is $\Sigma = \{S_1, S_2, S_3\}$ with:

$S_1 = \{[1, b], [a, a^2]\}$ $S_2 = \{[1, ba]\}$ $S_3 = \{[1, ba^2]\}$

and with associated subgroups:

$H_1 = \{1, b\}$, $H_2 = D_6$, $H_3 = D_6$.

Identify the vertex set of K_6 with the elements of D_6 and construct the 1–factors:

$F_1 = Orb_{H_1}(S_1) = \{[1, b], [a, a^2], [ba^2, ba]\}$

$F_2 = Orb_{H_2}(S_2) = \{[1, ba], [a, ba^2], [a^2, b]\}$

$F_3 = Orb_{H_3}(S_3) = \{[1, ba^2], [a, b], [a^2, ba]\}$.

The D_6–regular 1–factorization is $\mathcal{F} = \{F_1, F_1a, F_1a^2, F_2, F_3\}$ and F_2 and F_3 are fixed 1–factors.

Example 2.3. Consider Q_8, the Quaternion group of order 8, in additive notation with identity denoted by 0.

$$Q_8 =< a, b \,:\, 4a = 0,\ 2b = 2a,\ -b + a + b = -a >$$

$$Q_8 = \{0, a, 2a, 3a, b, b + a, b + 2a, b + 3a\}$$

A starter in Q_8 is $\Sigma = \{S_1, S_2, S_3, S_4\}$ with:

$S_1 = \{[0, a]\}$ $S_2 = \{[0, 2a]\}$ $S_3 = \{[0, b]\}$ $S_4 = \{[0, b + a]\}$

and with associated subgroups:

$H_1 = \{0, b, 2a, b + 2a\}$, $H_2 = Q_8$, $H_3 = H_4 = \{0, a, 2a, 3a\}$.

Identify the vertex set of K_8 with the elements of Q_8 and construct the 1–factors:

$F_1 = Orb_{H_1}(S_1) = \{[0, a], [b, b + 3a], [2a, 3a], [b + 2a, b + a]\}$

$F_2 = Orb_{H_2}(S_2) = \{[0, 2a], [a, 3a], [b, b + 2a], [b + a, b + 3a]\}$

$F_3 = Orb_{H_3}(S_3) = \{[0, b], [a, b + a], [2a, b + 2a], [3a, b + 3a]\}$

$F_4 = Orb_{H_4}(S_4) = \{[0, b + a], [a, b + 2a], [2a, b + 3a], [3a, b]\}$.

The Q_8–regular 1–factorization is $\mathcal{F} = \{F_1, F_1 + a, F_2, F_3, F_3 + b, F_4, F_4 + b + a\}$ and F_2 is the unique fixed 1–factor.

The main result of [14] states that the cyclic groups of 2–power order at least 8 never can realize a vertex–regular 1–factorization. In what follows we see how this result can be achieved via starter method.

Proposition 2.4. [7]. *A cyclic group of order 2^t, $t \geq 3$ has no starter.*

Proof. Let $G =< a >= \{0, a, \cdots, (2^t - 1)a\}$ be a cyclic group of order 2^t, $t \geq 3$ and suppose the existence of a starter $\Sigma = \{S_1, \cdots, S_r\}$ in G. Take the 1–factorization obtained via Σ. Every G–orbit of 1–factors has either even length or length 1. As the total number of 1–factors is $2^t - 1$, then at least a G–orbit of length 1 exists, i.e. the 1–factorization has at least one fixed 1–factor. A fixed 1–factor arises from a short edge and since G has a unique involution, namely $2^{t-1}a$, such a fixed 1–factor arises from the set $S_i = \{[0, 2^{t-1}a]\} \in \Sigma$. Without loss of generality we may assume $S_i = S_1$. That also means that each set S_i, with $i \geq 2$, contains only long edges, that is for each edge $[a, b] \in S_i$, $\partial[a, b] = \{a - b, b - a\}$ and $\phi(S_i) = \{a, b\}$.

We say that an edge e is of type 00 if its vertices are both even multiple of a; e is of type 11 if both its vertices are odd multiple of a and finally e is of type 01 if its vertices have distinct parity.

Given a set $S_i \in \Sigma$, with $i \geq 2$, we denote by x, y and z the number of edges in S_i of type 00, 01 and 11 respectively. Then $\phi(S_i)$ contains $2x + y$ even multiple of a and $2z + y$ odd multiple of a. Moreover, $\phi(S_i)$ is a left transversal for a subgroup H_i of G and since $H_i \neq G$, H_i does not contain odd multiple of a. This implies that in $\phi(S_i)$ the number of odd multiple of a equals the number of even multiple of a, i.e., $2z = 2x$.

As remarked above any edge of S_i is long, then ∂S_i contains $2x + 2z = 4x$ non–zero elements of G which are even multiple of a.

It follows that $|\partial\Sigma \cap < 2a > | = 4t + 1$, where t is a positive integer, that is $|\partial\Sigma \cap < 2a > | \equiv 1 \pmod 4$. But by the definition of starter, $|\partial\Sigma \cap \langle 2a \rangle| = 2^{n-1} - 1$, that is $|\partial\Sigma \cap \langle 2a \rangle| \equiv 3 \pmod 4$. That gives a contradiction and so there is no starter in G.

□

In view of the previous result, it was rather natural to extend the analysis of the existence problem for starters to arbitrary finite 2–groups and to finite groups of even order admitting a *large* cyclic subgroup, the largest possibility for "large" being namely "of index 2." As a first step in this direction, finite non–abelian 2–groups (of order ≥ 8) admitting a cyclic subgroup of index 2 were considered in [3]. These groups are known. Satz 14.9 in [15] divides them into four isomorphism types: the dihedral groups, the generalized quaternion groups (i.e., dicyclic 2–groups), the semidihedral groups and another class, respectively. The dihedral groups admit starters by the results in [2]. The other three types were examined in details in [3]. In the same paper the class of dicyclic groups of order $2n$, n even, was also studied. The following result was proved:

Theorem 2.5 ([3])**.** [3] *Let G be a finite group of order $2n$. Assume one of the following holds:*

- *$n = 2^m$, $m \geq 2$ and G is a non–cyclic group admitting a cyclic subgroup of index 2;*
- *n is even and G is a dicyclic group.*

Then G admits a starter, i.e., there exists a G–regular 1–factorization of K_{2n}.

For readers convenience we recall how these groups can be presented.
The dicyclic group of order $2n = 4s$ can be presented as follows [23, p.189]:

$$G = \langle a, b \,:\, 2sa = 0, 2b = sa, -b + a + b = -a \rangle \ .$$

We have $G = \{0, a, \cdots, (2s-1)a, b, b+a, \cdots, b + (2s-1)a\}$ and the relations $ra + b = b - ra$, $(b + ra) - (b + ta) = (t - r)a$ hold for r, $t = 0, 1, \cdots, (2s-1)$. Furthermore sa is the unique involution in G. In particular, if $s = 2^{m-1}$, then G is a generalized quaternion group of order 2^{m+1}. When $m = 2$ we have the Quaternion group Q_8 already seen in Example 2.3.
The semidihedral group of order 2^{m+1} can be presented as follows:

$$G = \langle a, b \,:\, 2^m a = 0, 2b = 0, -b + a + b = (2^{m-1} - 1)a \rangle \ .$$

The elements of G are 0, a, $2a, \cdots, (2^m-1)a$, b, $b+a$, $b+2a, \cdots, b+(2^m-1)a$ and for $r=0,1,\cdots,2^m-1$ we have $ra+b=b-ra$ if r is even and $ra+b=b+(2^{m-1}-r)a$ if r is odd, respectively. Furthermore there exist precisely $2^{m-1}+1$ involutions in G, namely all elements $b+ra$ with r even and the element $2^{m-1}a$.
The 4th isomorphism type of non–abelian 2–group of order 2^{m+1} with a cyclic subgroup of index 2 can be presented as follows ([15, p.91]:

$$G=\langle a,b\,:\,2^m a=0, 2b=0, -b+a+b=(2^{m-1}+1)a\rangle\ .$$

The elements of G are 0, a, $2a, \cdots, (2^m-1)a$, b, $b+a$, $b+2a, \cdots, b+(2^m-1)a$ and for $r=0,1,\cdots,2^{m-1}$ we have $ra+b=b+ra$ if r is even and $ra+b=b+(2^{m-1}+r)a$ if r is odd, respectively. Furthermore, there exist precisely three involutions in G, namely b, $2^{m-1}a$, $b+2^{m-1}a$.

Another result on 2–groups is the following:

Theorem 2.6 ([5])**.** *Let G be a 2–group of order 2^m, $m\geq 1$, with an elementary abelian Frattini subgroup. Then G admits a starter, i.e. there exists a G–regular 1–factorization of K_{2^m}.*

Recall that the Frattini subgroup of a group G is the intersection of all maximal subgroups of G.
In view of the previous results, one might conjecture that the cyclic groups of 2–power order at least 8 are the only 2–power order groups which do not posses starters. Indeed it is proved in [5] that the conjecture is true for the 2–groups of order ≤ 64.

In [4] a "doubling construction" for regular 1–factorizations was proposed. This construction starts from a regular 1–factorization of the complete graph K_{2n} under the action of a group H, and produces a 1–factorization of K_{4n} which is regular under the action of a group G having H as subgroup of index 2. This construction is possible under some assumptions on G and H. The main result of [4] extends the result of [2] to the entire class of generalized dihedral group. A generalized dihedral group of order $2n$ can be presented as follows, [22, p.210]: let H be an abelian group of order n possesing an element b which is not an involution, $2b\neq 0$. Let $\eta: H\longrightarrow H$ be the map defined by $\eta(a)=-a$ for every $a\in H$. It follows from $\eta(b)\neq b$ that η is an involution in $Aut(H)$. The relative holomorph $<\eta>+H$ of H is denoted by $DihH$ and called a generalized dihedral group. In particular, if H is cyclic, $DihH$ is the dihedral group D_{2n}.
The following is proved in [4]:

Theorem 2.7 ([4])**.** *Let $DihH$ be a generalized dihedral group of order $2n$. There exists a $DihH$–regular 1–factorization of K_{2n}.*

In [21] the problem of constructing starters in groups which are the direct or semidirect sum of groups having starters was considered. The aim was to enforce the conjecture that the cyclic groups of 2–power orders are the only exceptions.
The following results were obtained:

Theorem 2.8 ([21])**.** *Let G and H be finite groups of even order. Suppose that a starter exists in G as well as in H. There exists a regular 1–factorization of a complete graph under the action of $G\oplus H$ (the direct sum of G and H).*

Theorem 2.9 ([21])**.** *Let H be a group of odd order d and let G be an abelian group of even order $2n$. There exists a regular 1–factorization of K_{2nd} under the action of $G \oplus H$. (Except for $d = 1$ and $G \oplus H \simeq Z_{2^n}$, $n \geq 3$).*

Theorem 2.10 ([21])**.** *Let G be a group of even order $2n$ which is the direct sum of its Sylow 2–subgroup P with its complement. If P is either abelian or contains a cyclic subgroup of index 2, then there exists a G–regular 1–factorization of K_{2n}. (Except for $G = Z_{2^n}$, $n \geq 3$).*

Obviously many other results can be proved rearranging the previous propositions. For example, any *Hamiltonian group* (which is defined to be a non–abelian group in which every subgroup is normal) is the direct sum of a quaternion group Q_8, together with an elementary abelian 2–group A and an odd order group H (see [23, p.253]),i.e., $Q_8 \oplus A \oplus H$. If we apply Theorem 2.9 to A and H and Theorem 2.8 to Q_8 and $A \oplus H$, we can state:

Theorem 2.11 ([21])**.** *Let G be an Hamiltonian group of order $2n$. There exists a G–regular 1–factorization of K_{2n}.*

Moreover, each *nilpotent group* is the direct sum of its Sylow subgroups [23, p.144], then we can state:

Theorem 2.12 ([21])**.** *Let G be a nilpotent group of order $2n$ such that the Sylow 2–subgroup of G is either abelian or contains a cyclic subgroup of index 2. There exists a G–regular 1–factorization of K_{2n}.*

All the groups considered above are solvable. A first example of non solvable groups of even order which have a starter was given in [20]. Namely, they proved the following:

Theorem 2.13 ([20])**.** *For any prime p there exists a regular 1–factorization of $K_{(2p)!}$ under the action of the symmetric group S_{2p}.*

Up to now complete undirected graphs on a finite number of vertices were considered. Then the problem deals with finite groups. The same problem can also be addressed to complete graphs on a countable but not finite number of vertices. This was done in [6] and the following result was proved:

Theorem 2.14 ([6])**.** *For each finitely generated abelian infinite group G there exists a 1–factorization of the countable complete graph admitting G as an automorphism group acting sharply transitively on vertices.*

3. Vertex–regular 1–factorizations with an invariant 1–factor

When a regular 1–factorization of K_{2n} exists under the action of a suitable group G, it may happen that G fixes some 1–factor. We have already noticed that if this is the case, then the fixed 1–factor is the orbit under G of a short edge. Such a situation can be realized depending on the isomorphism type of the group: a certain starter type in G depends on isomorphism type of G.

We are still far from a classification of such groups, nevertheless some results were obtained.

Theorem 3.1 ([21])**.** *Let H be a group of odd order $2n + 1$ and consider the group $Z_{2^m} \oplus H$. Suppose it is either $m \geq 3$ or $m = 1$ and $|H| \equiv 3 \pmod 4$. No 1–factorization of $K_{2^m(2n+1)}$ admits $Z_{2^m} \oplus H$ as sharply vertex–transitive automorphism group fixing a 1–factor.*

To prove Theorem 3.1 the starter technique was used. A similar result was obtained in [17] without using the notion of starter. Namely:

Theorem 3.2 ([17, Theorem B]). *Let G be a nilpotent group of even order n and whose Sylow 2–subgroup is cyclic. If a 1–factorization of K_n admits G as sharply vertex–transitive automorphism group fixing a 1–factor, then it is necessarily $n \equiv 2 \pmod 8$ or $n \equiv 4 \pmod 8$.*

In [14, case 2, Theorem 3.1], a cyclic 1–factorization of K_{4d}, d odd, with a 1–factor fixed by the cyclic group was constructed. This result was extended:

Theorem 3.3 ([21]). *Let G be the direct sum of $\mathbb{Z}_4$ with a group H of odd order d. There exists a G–regular 1–factorization of K_{4d} with a 1–factor fixed by G.*

In [17, p.186-187], the non–existence of a 1–factorization of K_{2d}, $d \equiv 1 \pmod 4$, which is regular under a group G which is nilpotent and fixes a 1–factor was conjectured.

The conjecture was proved when d is a prime p, hence $G = \mathbb{Z}_2 \oplus \mathbb{Z}_p$, $p \equiv 1 \pmod 4$. Namely:

Theorem 3.4 ([21]). *Let p be a prime with $p \equiv 1 \pmod 4$. No 1–factorization of K_{2p} admits $\mathbb{Z}_2 \oplus \mathbb{Z}_p$ as a sharply vertex–transitive automorphism group fixing a 1–factor.*

Remark. The conjecture is false if we consider the complete graph on $2d$ vertices, with d not a prime and $d \equiv 1 \pmod 4$.

Here is a counterexample (see also [21]):

Example 3.5. Consider the cyclic group $\mathbb{Z}_2 \oplus \mathbb{Z}_{21}$. Let $\mathbb{Z}_2 = \langle a \rangle$ and $\mathbb{Z}_{21} = \langle b \rangle$. A starter in $\mathbb{Z}_2 \oplus \mathbb{Z}_{21}$ is $\Sigma = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ with:

$$S_1 = \{[0,a]\} \quad S_2 = \{[0,7b],[a,a+8b],[2b,a+b]\}$$

$$S_3 = \{[0,6b],[b,5b],[2b,4b],[3b,a+6b],[a,a+5b],[a+b,a+4b],[a+2b,a+3b]\}$$
$$S_4 = \{[0,9b],[a+b,a+11b],[b,a+3b],[3b,a+7b],[4b,a+9b],[5b,a+13b],\ [6b,a+12b]\}$$
$$S_5 = \{[0,a+7b]\} \quad S_6 = \{[0,a+9b]\} \quad S_7 = \{[0,a+10b]\}$$

and with associated subgroups:

$$H_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_{21} \quad , \quad H_3 = H_4 = \langle 7b \rangle \quad , \quad H_2 = \langle 3b \rangle \quad , \quad H_5 = H_6 = H_7 = \mathbb{Z}_{21} \ .$$

The fixed 1–factor is given by S_1, namely $F_1 = Orb_{\mathbb{Z}_2 \oplus \mathbb{Z}_{21}}([0,a])$.

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