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A survey on the asymptotic Dirichlet problem for some elliptic PDE's on Cartan-Hadamard manifolds

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Abstract. This is an expanded version of a talk given by the author in "Università degli Studi della Basilicata", Potenza, Italy, in February of 2013, on the solvability of the asymptotic Dirichlet problem for some quasi-linear elliptic PDE's in a Hadamard manifold.

1. General overview

In these notes we survey partially what has been done, mainly more recently, on the asymptotic Dirichlet problem on some quasilinear elliptic PDE's in a Cartan-Hadamard manifold.

Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian n-manifold, $n \geq 2$, of non-positive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o : T_oM \to M$ is a diffeomorphism for every point $o \in M$. Consequently, M is diffeomorphic to $\mathbb{R}^n$.

A Cartan-Hadamard manifold M can be compactified by adding a *sphere at infinity*. Denoted by $M(\infty)$, the sphere at infinity is defined as the set of all equivalence classes of unit speed geodesic rays in M; two such rays γ_1 and γ_2 are equivalent if $\sup_{t\geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$. The compactification $\bar{M}$ of M is then $\bar{M} := M \cup M(\infty)$, with the following topology.

For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y} : \mathbb{R} \to M$ such that $\gamma_0^{x,y} = x$ and $\gamma_t^{x,y} = y$ for some $t \in (0, \infty]$. If $v \in T_xM \setminus \{0\}$, $\alpha > 0$, and $r > 0$, we define a cone

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\} \,:\, \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \alpha\}$$

and a truncated cone

$$T(v, \alpha, r) = C(v, \alpha) \setminus \bar{B}(x, r) \ ,$$

where $\sphericalangle(v, \dot{\gamma}_0^{x,y})$ is the angle between vectors v and $\dot{\gamma}_0^{x,y}$ in T_xM. All cones and open balls in M form a basis for the so called *cone topology* on $\bar{M}$. The space $\bar{M}$, equipped with the cone topology is homeomorphic to a closed Euclidean ball. For more details see [15].

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We consider on M operators of the form

$$\mathcal{Q}[u] := \operatorname{div} \mathcal{A}(|\nabla u|^2)\nabla u \tag{1}$$

with $\mathcal{A} : (0, \infty) \to [0, \infty)$ a smooth function such that

$$\mathcal{A}(t) \leq A_0 t^{(p-2)/2} \tag{2}$$

for all $t > 0$, with some constants $A_0 > 0$ and $p \geq 1$, and that $\mathcal{B} := \mathcal{A}'/\mathcal{A}$ satisfies

$$-\frac{1}{2t} < \mathcal{B}(t) \leq \frac{B_0}{t} \tag{3}$$

for all $t > 0$ with some constant $B_0 > -1/2$. Furthermore, we assume that $t\mathcal{A}(t^2) \to 0$ as $t \to 0^+$ and therefore we set $\mathcal{A}(|X|^2)X = 0$ whenever X is a zero vector. As a consequence of (3), the function $t \mapsto t\mathcal{A}(t^2)$ is strictly increasing.

An example of equation that satisfy the conditions above is the *minimal graph equation*

$$\mathcal{Q}[u] = \mathcal{M}[u] := \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \ , \tag{4}$$

in which case

$$\mathcal{A}(t) = \frac{1}{\sqrt{1+t}} \quad \text{and} \quad \mathcal{B}(t) = -\frac{1}{2(1+t)} \ ,$$

and therefore (2) and (3) hold with constants $A_0 = 1$ and $B_0 = 0$, respectively. We note that u satisfies (4) if and only if $G := \{(x, u(x)) \, : \, x \in \Omega\}$ is a minimal hypersurface in $M \times \mathbb{R}$.

The class of equations considered above also includes other classical PDE's as the p-Laplace

$$\mathcal{Q}[u] = \Delta_p[u] := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \quad , \quad 1 < p < \infty \ ,$$

in which case

$$\mathcal{A}(t) = t^{(p-2)/2} \quad \text{and} \quad \mathcal{B}(t) = \frac{p-2}{2t} \ ,$$

and so $A_0 = 1$ and $B_0 = (p-2)/2$. In particular, when $p = 2$, one obtains the usual, extensively studied, Laplace-Beltrami equation $\Delta u = 0$, with $\mathcal{A}(t) \equiv 1$ and $\mathcal{B}(t) \equiv 0$.

We are interested here in the *asymptotic Dirichlet problem* on M for the operator $\mathcal{Q}$, namely: given a continuous function h on $M(\infty)$ does there exist a (unique) function $u \in C(\bar{M})$ such that $\mathcal{Q}[u] = 0$ on M and $u|_{M(\infty)} = h$?

We recall that a function u is a (weak) solution to the equation $\mathcal{Q}[u] = 0$ in M if it belongs to the local Sobolev space $W^{1,p}_{\text{loc}}(M)$ and

$$\int_M \left\langle \mathcal{A}\left(|\nabla u|^2\right) \nabla u, \nabla \varphi \right\rangle \, dm = 0 \tag{5}$$

for every $\varphi \in C_0^\infty(M)$. Such function u will be called a $\mathcal{Q}$-*solution* in M.

For future references in the text, it is convenient to state the asymptotic Dirichlet problem for $\mathcal{Q}$ in the following short form

$$\begin{cases} \mathcal{Q}[u] = 0 \quad \text{on} \ M \\ u|_{M(\infty)} = h \end{cases} \quad , \quad u \in W^{1,p}_{\text{loc}}(M) \cap C(\bar{M}) \ . \tag{6}$$

The study of (6) in a Cartan-Hadamard manifold began with the particular case of the Laplace-Beltrami equation, gaining strength with the seminal work of R. E. Greene and H. Wu [19]. Motivated by their search for a higher dimensional counterpart for the uniformization theorem of Riemann surfaces they conjectured in [19] that if the sectional curvature on a Cartan-Hadamard manifold M satisfies $K_M \leq -C/\rho^2$ outside a compact set, where $C > 0$ is a positive constant, then there exists a non-constant bounded harmonic function on M. This is the well-known *Green-Wu conjecture.*

Here and throughout these notes, $\rho(x)$ stands for the distance between $x \in M$ and a fixed point $o \in M$. Also, when one states a condition on K_M we mean that this condition is satisfied at all points outside a compact subset of M and on the sectional curvatures on all planes through the origin of the tangent spaces at these points.

Despite numerous partial results, the conjecture is still open in its general form in dimensions three and above. In fact, in the present state of art, it is not even known if this conjecture is true under the stronger hypothesis $K_M \leq c < 0$.

A natural framework to discuss the existence of globally defined bounded harmonic functions is the Dirichlet problem at infinity.

The asymptotic Dirichlet problem for the Laplace-Beltrami operator was solved affirmatively by Choi [10] under assumptions that sectional curvatures satisfy $K_M \leq -a^2 < 0$ and any two points in $M(\infty)$ can be separated by convex neighborhoods. Such appropriate convex sets were constructed by Anderson [5] for manifolds of pinched sectional curvature $-b^2 \leq K_M \leq -a^2 < 0$. Independently, Sullivan [35] solved the Dirichlet problem at infinity under the same pinched curvature assumption by using probabilistic arguments. In [6], Anderson and Schoen presented a simple and direct solution to the Dirichlet problem again in the case of pinched negative curvature. By modifying Anderson's argument, Borbély [7] was able to construct appropriate convex sets under a weaker curvature lower bound $K_M \geq -g(\rho(x))$, where $g(t) \approx e^{\lambda t}$, with $\lambda < 1/3$.

Major contributions to the Dirichlet problem were given by Ancona in a series of papers [1], [2], [3], and [4]. In particular, he was able to replace the curvature lower bound with a bounded geometry assumption that each ball up to a fixed radius is L-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$ for some fixed $L \geq 1$; see [1]. On the other hand, in [4] Ancona constructed a $3-$dimensional Cartan-Hadamard manifold with sectional curvatures bounded from above by -1 where the asymptotic Dirichlet problem is not solvable. Another example of a (3-dimensional) Cartan-Hadamard manifold, with sectional curvatures ≤ -1, on which the asymptotic Dirichlet problem is not solvable was constructed by Borbély [8].

To the best of our knowledge, the most general curvature bounds under which the asymptotic Dirichlet problem for the Laplace-Beltrami equation is solvable are given in the following theorems by Hsu (see also Theorems 3 and 4 below).

Theorem 1. *Let M be a Cartan-Hadamard manifold. Suppose that there exist a positive constant a and a positive and non-increasing function h with $\int_0^\infty th(t)dt < \infty$ such that*

$$-he^{\rho^{2a}} \leq \mathrm{Ric}_M \quad \textit{and} \quad K_M \leq -a^2 \, .$$

Then the Dirichlet problem (6) *is uniquely solvable for* $\mathcal{Q} = \Delta$ *for any* $h \in C(\bar{M})$.

Theorem 2. *Let* M *be a Cartan-Hadamard manifold. Suppose that there exist positive constants* r_0, $\alpha > 2$ *and* $\beta < \alpha - 2$ *such that*

$$-\rho^{2\beta} \leq \mathrm{Ric}_M \quad \textit{and} \quad K_M \leq -\frac{\alpha(\alpha-1)}{\rho^2}$$

with $\rho \geq r_0$. *Then* (6) *is uniquely solvable for* $\mathcal{Q} = \Delta$ *for any* $h \in C(\bar{M})$.

Different approaches to the asymptotic Dirichlet for the Laplace-Beltrami equation can also be found in ([33], Ch. II, Section 5) and in ([27], Section 13).

The asymptotic Dirichlet problem has been studied also in a more general context of p-harmonic and $\mathcal{A}$-harmonic functions as well as for operators $\mathcal{Q}$. For the p-Laplace equation the asymptotic Dirichlet problem was solved by Ilkka Holopainen [22] on Cartan-Hadamard manifolds of pinched negative sectional curvature by modifying the direct approach of Anderson and Schoen [6]. In ([25], Theorem 3.21) Holopainen and Vähäkangas studied the asymptotic Dirichlet problem for the p-Laplace equation on a Cartan-Hadamard manifold M under a curvature assumption

$$-b(\rho(x))^2 \leq K_M \leq -a(\rho(x))^2 \tag{7}$$

on M, where $a, b : [0, \infty) \to [0, \infty)$, $b \geq a$, are smooth functions satisfying certain explicit conditions (see [25], Section 3). The following two special cases of functions a and b are of particular interest:

Theorem 3 ([25], Corollary 3.22)**.** *Let* M *be a Cartan-Hadamard manifold. Suppose that*

$$-\rho^{2\phi-4-\varepsilon} \leq K_M \leq -\frac{\phi(\phi-1)}{\rho^2}$$

for some $\phi > 1$ *and* $\varepsilon > 0$. *Then problem* (6) *is uniquely solvable for* $\mathcal{Q} = \Delta_p$ *for every* $p \in (1, 1 + (n-1)\phi)$ *and for any* $h \in C(\bar{M})$.

Theorem 4 ([25], Corollary 3.23)**.** *Let* M *be a Cartan-Hadamard manifold. Suppose that*

$$-\rho^{-2-\varepsilon} e^{2k\rho} \leq K_M \leq -k^2$$

for some $k > 0$ *and* $\varepsilon > 0$. *Then problem* (6) *is uniquely solvable for* $\mathcal{Q} = \Delta_p$ *for every* $p \in (1, \infty)$ *and for any* $h \in C(\bar{M})$.

The case of the usual Laplacian ($p = 2$) is covered by Theorem 3 for every $\phi > 1$ since then $1 + (n-1)\phi > 2$. Thus the assumptions in Theorem 3 are slightly weaker than those in Theorem 2. On the other hand, Theorem 4 and Theorem 1 are closely related in the case $p = 2$ but, nevertheless, slightly different and neither one implies the other directly.

Note that using the Ricci curvature instead of the sectional makes no essential difference since all sectional curvatures are nonpositive.

In [37] Vähäkangas generalized the method and results due to Cheng [9] and showed the solvability of (6) for $\mathcal{Q} = \Delta_p$ if M satisfies a pointwise curvature pinching condition

$$|K_x(P)| \leq C\,|K_x(P')|$$

for some constant C and

$$K_M \leq -\frac{\phi(\phi-1)}{\rho^2(x)}$$

for some constant $\phi > 1$ with $1 < p < 1 + \phi(n-1)$. Above P and P' are any 2-dimensional subspaces of T_xM containing the (radial) vector $\nabla\rho(x)$. It is worth observing that no curvature lower bounds are needed here.

In fact, Vähäkangas considered even a more general case of $\mathcal{A}$-harmonic functions (of type $p \in (1, \infty)$), i.e. continuous weak solutions to the equation

$$-\text{div } \mathcal{A}(\nabla u) = 0 ,$$

where $\mathcal{A}$ is subject to certain conditions; for instance $\langle \mathcal{A}(V), V\rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. This class of equations is different from the ones satisfied by $\mathcal{Q}$ in this survey, although both include the p-Laplace equation. Recently, Vähäkangas generalized Theorems 3 and 4 to cover the case of $\mathcal{A}$-harmonic functions as well; see ([38], Corollary 3.7, Corollary 3.8 and Remark 3.9).

We turn now our attention to the minimal graph PDE (4). In [12] Collin and Rosenberg constructed harmonic diffeomorphisms from the complex plane $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$ disproving a conjecture of Schoen and Yau [33]. A bit later Gálvez and Rosenberg [18] extended the result to any Hadamard surface M whose curvature is bounded from above by a negative constant by proving the existence of harmonic diffeomorphisms from $\mathbb{C}$ onto M. The proofs in both papers are based on the construction of an entire minimal surface $\Sigma = (x, u(x)) \subset \mathbb{H}^2 \times \mathbb{R}$ ($\Sigma \subset M \times \mathbb{R}$, resp.) of conformal type $\mathbb{C}$, and thus on the construction of an entire solution u to the minimal graph equation that is unbounded both from above and from below. Harmonic diffeomorphisms $\mathbb{C} \to \mathbb{H}^2$ ($\mathbb{C} \to M$, resp.) are then obtained by composing conformal mappings (diffeomorphisms) $\mathbb{C} \to \Sigma$ with harmonic vertical projections $\Sigma \to \mathbb{H}^2$ ($\Sigma \to M$, resp.). A crucial method in the construction of an entire unbounded solution u to the minimal graph equation is to solve the Dirichlet problem on unbounded ideal polygons with boundary values $\pm\infty$ on the sides of the ideal polygons.

The unexpected result of Collin and Rosenberg has raised interest in (entire) minimal graphs in the product space $M \times \mathbb{R}$, where M is a Cartan-Hadamard manifold. Also, recent research in this field (see for example, [13], [16], [17], [28], [29], [30], [31], [32], [34]) motivated the author of the present survey, in collaboration with J-b. Casteras and I. Holopainen, the investigation of possible extensions of the results for the Laplacian and for the p-Laplacian to the minimal graph PDE. And, very recently, in ([11], Theorem 1.6), they proved the solvability of (6) for operators $\mathcal{Q}$ that satisfy (2), (3), under the curvature assumption (7), with a and b satisfying the same conditions of [25], Section 3, already mentioned above. As a main consequence they obtained:

Theorem 5. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that*

$$-\rho^{2(\phi-2)-\varepsilon} \leq K_M \leq -\frac{\phi(\phi-1)}{\rho^2} \tag{8}$$

for some constants $\phi > 1$ and $\varepsilon > 0$. Then the asymptotic Dirichlet problem (6) *for $\mathcal{Q} = \mathcal{M}$ is uniquely solvable for any $h \in C(\bar{M})$.*

Until [11], the solvability of the asymptotic Dirichlet problem for the minimal graph equation had been established only under hypothesis which included the condition $K_M \leq c < 0$ (see [18], [30]). In [30] Ripoll and Telichevesky introduced the following *strict convexity condition* (SC condition) that applies to equations (1).

A Cartan-Hadamard manifold M satisfies the strict convexity condition (SC condition) if, for every $x \in M(\infty)$ and relatively open subset $W \subset M(\infty)$ containing x, there exists a C^2 open subset $\Omega \subset M$ such that $x \in \Omega(\infty) \subset W$ and $M \setminus \Omega$ is convex. They proved that the asymptotic Dirichlet problem for (1) on M is solvable for any $h \in C(\bar{M})$ if $K_M \leq -k^2 < 0$ and M satisfies the SC condition; see [30, Theorem 7]. Furthermore, they showed by modifying Anderson's and Borbély's arguments that M satisfies the SC condition provided there exist constants $k > 0$, $\varepsilon > 0$ such that

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq K_M \leq -k^2 \ .$$

Another special case of Theorem 1.6 of [11], where the curvature is bounded from above by a negative constant $-k^2$, generalizes Theorem 4 and gives another proof for the above mentioned result of Ripoll and Telichevesky ([30], Theorem 14). It reads:

Theorem 6. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that*

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq K_M \leq -k^2 \tag{9}$$

for some constants $k > 0$ and $\varepsilon > 0$ and for all $x \in M \setminus B(0, R_0)$. Then the asymptotic Dirichlet problem (6) *for $\mathcal{Q} = \mathcal{M}$ is uniquely solvable for any $h \in C(\bar{M})$.*

We close this section with comments on the necessity of curvature bounds. It is worth of pointing out that the curvature bounds mentioned so far are essentially the most general ones under which the asymptotic Dirichlet problem is known to be solvable, for instance, for the usual Laplace equation ([26]), for the p-Laplace equation or the $\mathcal{A}$-harmonic equation ([25], [38]), or for the minimal graph equation ([30] and [11]). On the other hand, Ancona's and Borbély's examples ([4], [8]) show that a (strictly) negative curvature upper bound alone is not sufficient for the solvability of the asymptotic Dirichlet problem for the Laplace equation. In [23], Holopainen generalized Borbély's result to the p-Laplace equations, and very recently, Holopainen and Ripoll [24] extended these nonsolvability results to equations (1), in particular, to the minimal graph equation.

2. Overview of one the main techniques (PDE approach)

We say that a relatively compact open set $\Omega \Subset M$ is $\mathcal{Q}$-*solvable* if for any continuous boundary data $h \in C(\partial\Omega)$ there exists a unique $u \in C(\bar{\Omega})$ which is a $\mathcal{Q}$-solution in Ω and $u|_{\partial\Omega} = h$.

In addition to the growth conditions on $\mathcal{A}$, conditions (2) and (3), the solvability of (6) depends heavily on the following two facts:

(A) existence of an exhaustion of M by an increasing sequence of $\mathcal{Q}$-solvable domains Ω_k,
(B) compactness on compact subsets of M of locally uniformly bounded sequences of continuous $\mathcal{Q}$-solutions.

For the minimal graph equation, condition (A) follows from ([14], Theorem 2) where Ω_k may be chosen as a geodesic ball with radius k centered at a fixed point of M, and condition (B) follows from ([34],Theorem 1.1); see also([14], Theorem 1).

It is well-known that the properties (A) and (B) above hold for the p-Laplace equation and that (weak) solutions of the p-Laplace equation have Hölder-continuous representatives, usually called p-harmonic functions; see [21].

Once conditions (A) and (B) are satisfied the solvability of (6), for a given $h \in C(\bar{M})$, depends on the continuous extension to $M(\infty)$ of the bounded solution u (call prospective solution) obtained, using (A) and (B), as the limit of sequences of solutions on the exhaustion Ω_k, which boundary values on Ω_k coincide with $h|_{\partial\Omega_k}$, and that $u|_{M(\infty)} = h|_{M(\infty)}$.

In the case of bounded domains, the continuous extension to the boundary of the domain of a prospective solution of an elliptic PDE (for example, the one obtained by Perron's method is typical; see [20]) depends on the *regularity* of the domain with respect to the PDE (see also [20]). To deal with the asymptotic Dirichlet problem in M we extend this notion to the asymptotic boundary $M(\infty)$ of M as follows.

We recall that a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a $\mathcal{Q}$-*subsolution* in a domain Ω of M if $\mathcal{Q}[u] \geq 0$ weakly in Ω, that is

$$\int_\Omega \left\langle \mathcal{A}\left(|\nabla u|^2\right) \nabla u, \nabla\varphi \right\rangle \, dm \leq 0 \tag{10}$$

for every non-negative $\varphi \in C^\infty_0(\Omega)$. Similarly, a function $v \in W^{1,p}_{\text{loc}}(\Omega)$ is called a $\mathcal{Q}$-*supersolution* in Ω if $-v$ is a $\mathcal{Q}$-subsolution in Ω. Note that $u + c$ is a $\mathcal{Q}$-solution (respectively, $\mathcal{Q}$-subsolution, $\mathcal{Q}$-supersolution) for every constant c if u is a $\mathcal{Q}$-solution (respectively, $\mathcal{Q}$-subsolution, $\mathcal{Q}$-supersolution). It follows from the growth condition (2) that test functions φ in (5) and (10) can be taken from the class $W^{1,p}_0(\Omega)$ if $|\nabla u| \in L^p(\Omega)$.

We say that M is *regular at infinity* with respect to $\mathcal{Q}$ if, given $C > 0$, $x \in M(\infty)$, and a relatively open subset $W \subset M(\infty)$ containing x, there are an open subset $\Omega \subset M$ such that $x \in \text{Int}(\Omega(\infty)) \subset W$,sub and supersolutions $\sigma, \Sigma \in C(M)$ of $\mathcal{Q} = 0$ in M (called *barriers* at x) such that $\sigma \leq 0 \leq \Sigma$, $\lim_{p\to x} \sigma(p) = \lim_{p\to x} \Sigma(p) = 0$ and $\sigma|_{M\setminus\Omega} \leq -C$ and $\Sigma|_{M\setminus\Omega} \geq C$. We then have:

Theorem 7. *Let M be a Hadamard manifold and let*

$$\mathcal{Q}[u] = \operatorname{div} \mathcal{A}(|\nabla u|^2 \nabla u)$$

be a differential operator in M where $\mathcal{A} : (0, \infty) \to [0, \infty)$ satisfies the growth conditions (2) *and* (3) *and conditions* (*A*) *and* (*B*) *as well. Assume that M is regular at infinity with respect to $\mathcal{Q}$. Then the asymptotic Dirichlet problem of $\mathcal{Q}$ is solvable for any continuous boundary data $\varphi \in C(M(\infty))$.*

Proof. Let $h \in C^0(\overline{M})$ be an extension of φ. Condition (A) allows us to solve the Dirichlet problem

$$\begin{cases} \mathcal{Q}[u] = 0 \quad \text{in } \Omega_k \quad , \quad u \in W^{1,p}_{\text{loc}}(\Omega_k) \cap C^0(\overline{\Omega}_k) \\ u|_{\partial\Omega_k} = h \ , \end{cases}$$

and find a solution $u_k \in C^0(\overline{\Omega}_k)$.

Condition (B) together with the diagonal method show that there exists a subsequence of (u_k) (which we suppose to be (u_k)) converging uniformly on compacts subsets of M to a global solution u of $\mathcal{Q} = 0$. We show that u extends continuously to $M(\infty)$ and satisfies $u|_{M(\infty)} = \varphi$. Choose $x \in M(\infty)$ and let $\varepsilon > 0$ be given.

Since φ is continuous, there exists an open neighborhood $W \subset M(\infty)$ of x such that $\varphi(y) < \varphi(x)+\varepsilon/2$ for all $y \in W$. Furthermore, the regularity of M with respect to $\mathcal{Q}$ implies the existence of an open subset $\Omega \subset M$ such that $x \in \mathrm{Int}(\Omega(\infty)) \subset W$ and and upper barrier $\Sigma : M \to \mathbb{R}$ with respect to x and Ω with height $C := \max_{\overline{M}} |h|$.

Defining

$$v(q) := \Sigma(q) + \varphi(x) + \varepsilon \ ,$$

we claim that $u \leq v$ in Ω.

Since ϕ is continuous, $\exists k_0 >> 0$ such that $\phi(q) < \varphi(x) + \varepsilon/2$ for all $q \in \partial\Omega_k \cap \Omega$, $k \geq k_0$; we may chose k_0 such that $\Omega_{k_0} \cap \Omega \neq \emptyset$.

Setting $V_k = \Omega \cap \Omega_k$, $k \geq k_0$, we claim that $u_k \leq v$ in V_k. First we observe that the inequality holds on

$$\partial V_k = \overline{(\partial\Omega_k \cap \Omega)} \cup \overline{(\partial\Omega \cap \Omega_k)} \ .$$

In fact: on $\partial\Omega_k \cap \Omega$, it is true due to the choice of k_0; on $\partial\Omega \cap \Omega_k$, it holds because $\Sigma \geq \max |h|$ on $\partial\Omega$, which implies that $\Sigma \geq u_k$, by the Maximum Principle.

Also the Maximum Principle implies that $u_k \leq v$ in V_k; since it holds for all $k \geq k_0$, we have $u \leq v$ on Ω.

It is also possible to define $v_- : M \to \mathbb{R}$ by $v_-(q) := \varphi(x) - \varepsilon - \Sigma(q)$ in order to obtain $u \geq v_-$ in Ω. It then holds that

$$|u(q) - \varphi(x)| < \varepsilon + \Sigma(q) \quad , \quad \forall\, q \in \Omega \ ,$$

and hence

$$\limsup_{p \to x} |u(p) - \varphi(x)| \leq \varepsilon \ .$$

The proof is complete, since $\varepsilon > 0$ is arbitrary.

□

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