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The Weierstrass representation of minimal surfaces revisited

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Abstract. We first recall the explicit solution - outlined by Weierstrass in 1866 (cf. [16]) - to the parametrization of minimal surfaces immersed in the Euclidean space $\mathbb{R}^3$. This so-called Weierstrass representation was generalized by Eisenhart in 1911 (cf. [6]) for minimal surfaces in $\mathbb{R}^4$ and then for minimal surfaces in $\mathbb{R}^n$-for any n by Beckenbach in 1933 (cf. [2]). However Beckenbach's solutions are given in terms of square roots of holomorphic forms and Taimanov in his survey [15] noticed that "there are only two cases when the Grassmannian admits a rational parametrization and only in these cases we have the Weierstrass representation of minimal surfaces".

Our first topic is to give a new presentation of the Weierstrass coordinates of minimal surfaces which will make no use of square roots of forms in dimensions bigger than 4 and which will clarify the issue raised by Taimanov.

Our second topic is the introduction of a new equivalent representation for minimal surfaces in $\mathbb{R}^{2n}$ (respectively $\mathbb{R}^{2n+1}$). This representation uses the complex structure of $\mathbb{R}^{2n}$, seen as $\mathbb{C}^n$ (respectively of $\mathbb{R}^{2n+1}$ seen as $\mathbb{C}^n \times \mathbb{R}$).

We will also compute some geometrical characteristics of the surface in terms of these representations (most are well-known for the standard Weierstrass representations (cf. for example [14])) and recall some facts on minimal surfaces and self-intersection for further applications.

1. Introduction

A minimal surface Σ in Euclidean space is a surface whose mean curvature vector is zero. Historically there are two main ways to describe the surface Σ.

A first approach is the study of minimal graphs: the surface Σ is given as a graph of a function $f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, and is minimal iff f satisfies the minimal surface equation:

$$div\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = -2H = 0\ .$$

A surface described as a graph of a function is called "non-parametric surface" (see for example [14]).

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The operator of the minimal surface equation is exactly the Laplacian of the metric induced on U by the orthogonal projection of Σ on U.

More generally a surface in Euclidean space of higher codimension m that is a graph of a map $F : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^m$ is minimal if it satisfies an elliptic system of PDEs: $\Delta_\Sigma F^\alpha = 0, \alpha = 1, \cdots, m$ where Δ_Σ is the Laplacian of the metric on U which is the metric of Σ pulled-back by F.

Explicit solutions of the minimal surface equation are scarce.

Let us give the first known nontrivial solutions: the catenoid, graph of the function $z = \operatorname{argcosh}(\sqrt{x^2 + y^2})$, the helicoid: $z = \arctan y/x$, and 50 years later Scherk's surface: $z = \log \cos x - \log \cos y$. Though explicit solutions are scarce, the minimal surface equation maybe extended for minimal surfaces in general Riemannian surfaces (cf. for example [5] & [12])

There is however another approach which doesn't extend to ambient Riemannian manifolds but which gives an explicit description- at least locally- of any minimal surface in Euclidean space: the Weierstrass representation.

If U is an open subset of a Riemann surface $\mathfrak{R}$, then the surface Σ will be described as the image of an immersion $X : \mathfrak{R} \supset U \longrightarrow \Sigma \subset \mathbb{R}^3$ parametrized by $\mathfrak{R}$. This explicit description fits in the more general theory of integrable systems. In this modern frame the "Weierstrass representation" was extended first to constant mean curvature surfaces by Kenmotsu though solutions are not explicit anymore- and then to any surface in $\mathbb{R}^3$. We will limit our topic to the study of these "parametric minimal surfaces".

From the minimal surface equation, the surface Σ is minimal iff the Euclidean coordinates functions $x^1, x^2, x^\alpha = F^\alpha, \alpha = 1, \cdots, m$ restricted to the minimal surface Σ are harmonic with respect to the metric g of $\mathbb{R}^n$ induced on Σ. Thus (Σ, g) is a Riemann surface $\mathfrak{R}$ with the associated conformal structure of g, and the induced map $X : \mathfrak{R} \longrightarrow \mathbb{R}^n$ remains harmonic for any conformal metric on $\mathfrak{R}$. Furthermore, by construction, X is also conformal.

Consequently X is minimal iff $X : \mathfrak{R} \longrightarrow \mathbb{R}^n$ is both conformal and harmonic. Let us consider briefly conditions on X that ensure conformality.

We will easily see that the conformality of X is equivalent to $[\partial X] : \mathfrak{R} \longrightarrow P(\mathbb{C}^n)$ has values in the regular complex quadric Q_{n-2} which can be identified with the Grassmannian $G_{2,n}$of two-planes in $\mathbb{R}^n$.

Applications of complex variables in geometry and in particular complex quadrics were studied extensively in the 19^{th} century. Weierstrass in particular used the regular quadric of $P(\mathbb{C}^3)$ and the Gauss map $[\partial X]$ to parametrize the surface Σ.

If furthermore X is also harmonic, we will see that $[\partial X]$ is holomorphic. Single integration with respect to z gives an explicit solution to the minimal surface parametrization. In particular, a parametrization of the quadric leads to a parametrization of a minimal surface called Weierstrass representation of a minimal surface which is an explicit solution -at least locally- of the minimal surface equation. Weierstrass representation has been extensively studied by Ossermann (see for example [4] [8]).

Part I deals more specifically with the Weierstrass form of projective quadrics. These Weierstrass forms are rational parametrizations of the regular quadrics by projective hyperplanes of the same dimension.

Part II is devoted to the geometry of minimal surfaces in terms of the Weierstrass form of the Gauss map. We will compute some geometric quantities and will introduce a rank 2 vector fibration associated to the new representation. We also add some remarks on the Gauss-Bonnet formula applied to the normal line- bundle of a minimal surface in $\mathbb{R}^4$.

1.1. **Projective quadrics in projective spaces.**

We recall first some basic notions issued from algebraic geometry on projective space. See for example [7] for more details

1.1.1. *Notations and basic facts.*

- $\mathbb{P}^{n-1}$ **is a quotient space**: let U be a vector space of dimension n; then we denote equivalently $P(U)$ or $[U]$ or $\mathbb{P}^{n-1}$ the associated projective space of complex lines of U. The correspondence is defined by the projection

$$\begin{aligned} \Pi : U &\longrightarrow [U] \\ v &\longrightarrow [v] := V \end{aligned} \tag{1}$$

  which associates to the vector $v \in U$ its class: the point V of $[U]$.
  For example, the Riemann sphere $P^1 := P(\mathbb{C}^2)$.
- $\mathbb{P}^{n-1}$ **is a complex manifold**: we cover $\mathbb{P}^n$ with the n open sets

$$U_i = \{[z_1, \cdots, z_n] \, : \, z_i \neq 0, \; i = 1, \cdots n\} \; .$$

  Consider the charts $\phi_i : U_i \longrightarrow \mathbb{C}^{n-1}$ defined by $\phi_i([z_1, \cdots, z_n]) =$ $= (z_1/z_i, \cdots, z_n/z_i)$. The transition functions are the holomorphic functions: $\phi_j \circ \phi_i^{-1} : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}$ defined by $\phi_j \circ \phi_i^{-1}(Z) = (z_i/z_j)Z$; hence $\mathbb{P}^{n-1}$ is a complex manifold.
  Notice that $\mathbb{P}^{n-1}$ is a n-1 dimensional complex manifold but also 2n-2 real manifold.
- $\mathbb{P}^{n-1}$ **is a Riemannian (Kaehler) manifold**. On $\mathbb{C}^n$, the real part (respectively imaginary part) of the canonical hermitian product defines a metric (respectively a real anti-symmetric bilinear form which is a Kaehler (1,1)- form): $h(Z, W) := h(z_1, \cdots, z_n, w_1, \cdots w_n) = \sum z_i \bar{w}_i = \langle Z, W \rangle + i\omega(Z, W)$.
  On $\mathbb{P}^{n-1}$, there is a so-called Fubini-Study metric defined by the following quadratic form on $\mathbb{C}^n$:

$$B(Z, Z) = \frac{|Z \wedge dZ|^2}{|Z|^4} \quad , \quad Z = (z_1, \cdots z_n), \; |Z|^2 = \sum |z_i|^2 \; .$$

  This form is invariant by complex multiplication and defines a metric on the quotient $\mathbb{P}^{n-1}$. In fact $P(\mathbb{C}^n) = \mathbb{S}^{2n+1} \backslash \mathbb{S}^1$ where the action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n}$ is the multiplication by complex unit numbers. The spherical distance between two classes i.e. two spherical lines is constant and induces on the quotient space $P(\mathbb{C}^n)$ the Fubini metric. Similarly there is an associated Kaehler form $\omega_{\mathbb{P}^{n-1}}$ on $\mathbb{P}^{n-1}$ defined via the map Π by

$$\Pi^*(\omega_{\mathbb{P}^{n-1}})(Z) := \frac{i}{2\pi} \partial \bar{\partial} \log |Z|^2 \; .$$

1.1.2. *Quadrics of projective spaces.* A quadric is by definition the zero locus of a homogeneous polynomial q of degree two in $\mathbb{C}^n$; the null space of q is by definition the n-1 dimensional complex cone $C(Q) = \{Z : q(Z) = B(Z,Z) = 0\} \subset \mathbb{C}^n$ where B is the polar bilinear form on associated to q on $\mathbb{C}^n$ and where $Q := \Pi(C(Q))$. If B is not degenerate, the quadric $Q_{n-2} := \{[V] \in P(\mathbb{C}^n) : q(V) = B(V,V) = 0\}$ is a complex manifold of dimension $n-2$.

Notice also that in that case, B defines a coupling on U which induces a linear isomorphism between the vector space U and its dual vector space of forms U^*. This induces in turn a duality between projective points $P = [p] \in [U]$ and projective hyperplanes $H = [v^\perp]$ of $P(\mathbb{C}^n)$ orthogonal with respect to B (cf. [7]). We will denote $P^\perp = \{[x] : B(p,x) = 0\}$.

A homogeneous quadric can be given in terms of coordinates:

$$q(Z) := q(z_1, \cdots, z_n) = \sum a_{ij} z^i z^j \ .$$

By Gauss reduction there are linear coordinates such that $q(Z) = z_1^2 + \cdots + z_k^2$ where $k := \mathrm{rk}(q)$ and $\dim Q = k-2$.

Example 1.1. In projective plane $P(\mathbb{C}^3) = \mathbb{P}^2$, a rank 3 quadric is a smooth conic (equivalent to $x^2+y^2+z^2=0$) a rank 2 quadric is a pair of lines (equivalent to $x^2+y^2=0$) and a rank 1 quadric is a double line (equivalent to $x^2=0$). In projective space $P(\mathbb{C}^4) = \mathbb{P}^3$ a rank 4 quadric is a smooth quadric (equivalent to $x^2+y^2+z^2+t^2=0$) a rank 3 quadric is a quadric cone, (equivalent to $x^2+y^2+z^2=0$), a rank 2 quadric is pair of planes (equivalent to $x^2+y^2=0$) and a rank 1 quadric is a double plane ,(equivalent to $x^2=0$).

1.1.3. *Rational parametrizations of the quadric $Q_n \subset \mathbb{P}^{n+1}$ by $\mathbb{P}^n$.* Algebraic geometry is primarily the study of properties of an algebraic variety Q over a field K that depend solely upon its "function field" $K(Q)$. In particular a"rational map" $f : Q_{n-2} \dashrightarrow P^{n-2}$ is an almost 1-1 mapping that induces a morphism between the function fields $\mathbb{C}(Q_{n-2})$ and $\mathbb{C}(P^{n-2})$. And a bi-rational map induces an isomorphism between the function fields (cf. [7]). Equivalently a rational map f admits the following description: if (U_α, ϕ_α) is a chart of the complex manifold Q_{n-2} , (respectively (V_β, ψ_β) is a chart of P^{n-2}) then $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is a rational map i.e. $\psi_\beta \circ f \circ \phi_\alpha^{-1}(z_1, \cdots z_{n-2}) = P(z_1, \cdots z_{n-2})/Q(z_1, \cdots z_{n-2})$, where $P, Q \in \mathbb{C}[z_1, \cdots z_{n-2}]$.

We will see that, in the frame of algebraic geometry, the projective quadric Q_{n-2} admits a rational parametrization by the projective space $\mathbb{P}^{n-2}$. In fact Q_{n-2} is bi-rationally equivalent to P^{n-2}.

In the frame of differential geometry, Q_{n-2} is also a complex manifold that can be described by a set of $n-1$ charts whose images are open complementary subsets, each isomorphic to $P^{n-2} \setminus \mathbb{P}^{n-3} = \mathbb{C}^{n-2}$ where $\mathbb{P}^{n-3}$ is a projective hyperplane of $\mathbb{P}^{n-2}$.

The following result of Poncelet - originally proved in the projective plane - gives a construction of the parametrization. More precisely:

Theorem 1.1 (Poncelet)**.** *Let Q_{n-2} be a nonsingular quadric in $\mathbb{P}^{n-1}$. Choose a point $A = [a] \in Q_{n-2}$ and let π be the projection of center A from Q_{n-2} into a projective hyperplane $H_{n-2} \subset \mathbb{P}^{n-1}$ that doesn't contain A. Then the projection π is a bi-rational map from Q_{n-2} to H_{n-2}. In particular Q_{n-2} is rationally parametrized by $\mathbb{P}^{n-2}$.*

Proof. Recall that $Q_{n-2} := \{[Z] \,:\, B(Z,Z) = 0\}$ where B is a nondegenerate bilinear form on $\mathbb{C}^n$. Let $A = [a] \in Q_{n-2}$ and $H_{n-2} = [V]$ a projective hyperplane not containing A, then the following projection of center A:

$$\pi : Q_{n-2} \dashrightarrow H_{n-2}$$

is constructed as follows: for any $Q = [z_1, \cdots, z_n] \in Q_{n-2}$, $\pi_{H_{n-2}}(Q)$ is defined as the intersection of the projective line (AQ) with $H_{n-2} = [V]$.
Projection is defined if $(AQ) \cap H_{n-2} \neq \emptyset$ i.e. if there are complex numbers α, β such that $\alpha a + \beta q = x \in V$ i.e. $B(x - \alpha a, x - \alpha a) = 0$ Since $B(a,a) = 0$, projection is not defined if $X = [x] \in Q'_{n-3} := Q_{n-2} \cap A^{\perp}$ Hence π is a regular map on $Q_{n-2} \setminus Q'_{n-3}$.
Reciprocally, let $X = [x] \in [V]$; since $A = [a] \in Q_{n-2}$, $B(a,a) = 0$ and the projective line $P(vect(a,x))$ cuts Q_{n-2} at point $Q = [\lambda a + \mu x]$ iff there are $\lambda, \mu \in \mathbb{C}$ such that $B(\lambda a + \mu b, \lambda a + \mu b) = 0$ i.e. either $Q = A$ or $2\lambda B(a,x) + \mu B(x,x) = 0$. Let $\lambda = (1/2)B(x,x), \mu = -B(a,x)$ then $Q = [(1/2)B(x,x)a - B(a,x)x]$.
We define thus a rational map

$$\pi^{-1} : H_{n-2} \dashrightarrow Q_{n-2}$$

with

$$\pi^{-1}[x] = [\frac{1}{2}B(x,x)a - B(a,x)x] \ .$$

Q is not defined if $B(a,x) = 0$ i.e. if X belongs to $A^{\perp}$ the projective hyperplane $\{[v] \,:\, B(v,a) = 0\}$. Hence π^{-1} is a rational map which is regular on $H_{n-2} \setminus A^{\perp} = H_{n-2} \setminus H_{n-3}$ where H_{n-3} is the projective hyperplane $A^{\perp} \cap H_{n-2}$ of H_{n-2}.
Hence π is a bi-rational map.
Notice we can choose $(n-1)$ hyperplanes H_{n-2} to define a set of complete chart of Q_{n-2}.

□

Most geometrical computations such as the Weierstrass representations of minimal surfaces require only the parametrization of the quadric by affine spaces as we shall see in last chapter, but is sometimes useful to have a description of the rational map π in terms of blow-ups.
We will consider

1.1.4. *The Weierstrass form of a quadric.* The standard Weierstrass form of a nondegenerate quadric corresponds to the choice of a given nondegenerate Q_{n-2} plus the choice of a projection π i.e. the choice of its center of projection $A \in v$ and the projective hyperplane H_{n-2} onto which Q_{n-2} is projected. Hence many equivalent Weierstrass are possible.
We will choose however the quartic Q_{n-2}, point A and hyperplane H_{n-2} such that the representation obtained coincides with the standard Weierstrass form in dimension $n = 3$ that is, the one that used in most publications.
Let then $Q_{n-2} = \{[z_1, z_2, \cdots, z_n] \,:\, z_1^2 + \cdots + z_n^2 = 0\}$ and choose point $A = [1, -i, 0, \cdots, 0] \in \mathbb{P}^{n-1}$ and projection hyperplane $H_{n-2} = \{[z_1, 0, z_3, \cdots, z_n]\}$.
Let $Q \in Q_{n-2}$. Then projection $\pi(Q)$ - when it is defined - is the intersection of line (AQ) with H_{n-2}. This defines a rational map

$$\pi : Q_{n-2} \dashrightarrow H_{n-2} \ .$$

The condition $\mathrm{vect}(a,q) \subset V$ gives

$$\pi([z_1, \cdots, z_n]) = [z_2 + iz_1, 0, \cdots, z_n]$$

π is a rational map which is not a regular map on $Q_{n-2}\cap A^{\perp}$, where $A^{\perp} = [z_1 - iz_2 = 0]$. Reciprocally, if $X \in H_{n-2}$, the line (AX) cuts Q_{n-2} at point Q iff there are complex numbers α, β such that

$$Q = \alpha \begin{pmatrix} 1 \\ -i \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \beta \begin{pmatrix} z_1 \\ 0 \\ z_3 \\ \vdots \\ z_n \end{pmatrix} \in Q_{n-2} \tag{2}$$

i.e. iff $\beta(z_1^2 + z_3^2 \cdots + z_n^2) + 2\alpha z_1 = 0$. Let $\alpha = -z_1^2 - z_3^2 - \cdots - z_n^2, \beta = 2z_1$ and plug back in equation 2 we obtain $Q = [z_1^2 - z_3^2 - \cdots - z_n^2, i(z_1^2 + z_3^2 + \cdots + z_n^2), 2z_1z_3, 2z_1z_4, \cdots, 2z_1z_n]$.

From theorem we get a bi-rational map which defines *the Weierstrass form of* Q_{n-2}:

$$\pi^{-1} : H_{n-2} \dashrightarrow Q_{n-2}$$

where

$$\pi^{-1}([z_1, 0, z_3, \cdots, z_n]) =$$
$$= [z_1^2 - z_3^2 - \cdots - z_n^2, i(z_1^2 + z_3^2 + \cdots + z_n^2), 2z_1z_3, 2z_1z_4, \cdots, 2z_1z_n] \ .$$

With the same notations as in theorem we easily see that $H_{n-3} = \{[0, 0, z_3, \cdots, z_n]\}$ and $Q'_{n-3} = \{[iz_2, z_2, z_3, \cdots, z_n] \ : \ z_3^2 + \cdots z_n^2 = 0\}$.

In particular $H_{n-2} \setminus H_{n-3} = \{[z_1, 0, z_3, \cdots, z_n] \ : \ z_1 \neq 0\}$ and dividing by z_1 we obtain a 1-1 parametrization $\phi : \mathbb{C}^{n-2} \longrightarrow H_{n-2} \setminus H_{n-3}$ defined by $\phi(w_1, w_2, \cdots, w_{n-2}) = [1, 0, z_3, \cdots, z_n]$.

We postcompose with π^{-1} and obtain the following parametrization by affine spaces which is another way to express *the Weierstrass form of* Q_{n-2}

$$\tilde{\pi}^{-1} : \mathbb{C}^{n-2} \longrightarrow Q_{n-2} \setminus Q'_{n-3}$$

where

$$\tilde{\pi}^{-1}(w_1, \cdots, w_{n-2}) =$$
$$= [1 - w_1^2 - \cdots - w_{n-2}^2, i(1 + w_1^2 + \cdots + w_{n-2}^2), 2w_1, 2w_2, \cdots, 2w_{n-2}] \ .$$

1.1.5. *The Weierstrass form(s) of the quadric* $Q_1 = \mathbb{P}^1$ *and* $Q_2 = \mathbb{P}^1 \times \mathbb{P}^1$. We apply previous formula in the particular case of $n = 3$ and $n = 4$.

- **The standard Weierstrass form of** Q_1 Apply previous formula for $n = 3$,

$$\pi^{-1} : H_1 = \mathbb{P}^1 \dashrightarrow Q_1 \subset \mathbb{P}^2 \tag{3}$$

$$[s_0, 0, s_1] \longrightarrow [s_0^2 - s_1^2, i(s_0^2 + s_1^2), 2s_0s_1] \tag{4}$$

From theorem π^{-1} is regular on $H_1 \backslash H_0$-where $H_0 = [0, 0, 1]$ - and projection π is regular on $Q_2 \setminus Q'_1$ where $Q'_1 = [i, 1, 0] = A$. But from equation (4) it is clear that π^{-1} extends regularly at $H_0 = [0, 0, 1]$ and $\pi^{-1}(H_0) = A$ hence Q_1 is diffeomorphic to $\mathbb{P}^1$.

Equivalently we cover $\mathbb{P}^1$ with the open affine subsets $U_i := \{[z_1, z_2] \ : \ z_i \neq 0\}$. The restriction of π^{-1} to U_i defines

$$\phi_1 = \tilde{\pi}^{-1} : U_1 = \mathbb{C} \longrightarrow Q_1 \setminus Q'_0$$
$$\tilde{\pi}^{-1}(w) = [1 - w^2, i(1 + w^2), 2w]$$

where $Q'_0 = [i, 1, 0] = A$. Let $z = 1/w$. We construct a new map

$$\phi_2 : U_2 = \mathbb{C} \longrightarrow Q_1 \setminus Q''_0$$

$$\tilde{\pi}^{-1}(z) = [z^2 - 1, i(z^2 + 1), 2z]$$

defined on A but not on $Q''_0 = [1, i, 0]$. We notice that U_1 and U_2 are a covering of the Riemann sphere $\mathbb{P}^1$ and that $z = 1/w$ is the transition function defining $\mathbb{P}^1$. Hence ϕ_i coincide on $\mathbb{C}^*$ and define a 1-1 map

$$\mathbb{S}^2 = \mathbb{P}^1 \longrightarrow Q_1$$

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$$\phi_2 : U_2 = \mathbb{C} \longrightarrow Q_1 \setminus Q''_0$$

$$\tilde{\pi}^{-1}(z) = [z^2 - 1, i(z^2 + 1), 2z]$$

defined on A but not on $Q''_0 = [1, i, 0]$. We notice that U_1 and U_2 are a covering of the Riemann sphere $\mathbb{P}^1$ and that $z = 1/w$ is the transition function defining $\mathbb{P}^1$. Hence ϕ_i coincide on $\mathbb{C}^*$ and define a 1-1 map

$$\mathbb{P}^1 \longrightarrow Q_1 \ .$$

In fact it is even an isometry if Q_1's metric is the induced metric of $\mathbb{P}^2$. It suffices to compare the pulled-back Kaehler form with the Kaehler form of $\mathbb{P}^1$.

- **The Weierstrass form of Q_2.** We apply theorem. π is the projection of center $A = [i, 1, 0, 0]$ on $H_2 = \{[z_1, 0, z_3, z_4]\}]$ and obtain the birational map

$$\pi^{-1} : \mathbb{P}^2 = H_2 \longrightarrow Q_2$$

$$\pi^{-1}([w_1, 0, w_2, w_3]) =$$

$$= [w_1^2 - w_2^2 - w_3^2, i(w_1^2 + w_2^2 + w_3^2), 2w_1w_2, 2w_1w_3] \ .$$

From theorem π^{-1} is regular on $H_2 \setminus H_1$-where $H_1 = \{[0, 0, z_3, z_4]\}]$. We compute that $\pi^{-1}(H_1) = A$ hence π^{-1} blows down at line H_1 to point A of Q_2. Furthermore π^{-1} blows-up at two points $P_\pm = [0, 0, 1, \pm i] \in H_1$; the blowup of each point is a line of Q_2. The union of these two lines that cut at point A is the degenerate quadric $Q'_1 = \{P_\pm = [i, 1, z, \pm iz]\}$.
Also from theorem π is regular on $Q_2 \setminus Q'_1$. This quadric is degenerate and consist of two lines intersecting at point A.
We can also use affine charts. For example in the affine plane $z_1 \neq 0$:

$$\bar{\pi}^{-1} : \mathbb{C}^2 \longrightarrow Q_2 \setminus Q'_1$$

$$\tilde{\pi}^{-1}(w_1, w_2) = [1 - w_1^2 - w_2^2, i(1 + w_1^2 + w_2^2), 2w_1, 2w_2] \ .$$

The map is a priori non regular on $H_1 := \{[0, 0, z_3, z_4]\}$ where $Q'_1 = \{[iz_2, z_2, z_3, z_4] \ : \ z_3^2 + \cdots z_4^2 = 0\}$.
But there is another form which shows directly that Q_2 isometric to the product of two spheres. Consider the Segre embedding:

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$([s_0, s_1], [t_0, t_1]) \longrightarrow [s_0t_0, -s_0t_1, s_1t_0, -s_1t_1] := [w_1, w_2, w_3, w_4] \ .$$

Note that $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \overline{Q}_2$ which is the quadric of equation $w_1w_4 - w_2w_3 = 0$. The correspondence between the two quadrics Q_2 and $\overline{Q}_2$ is given by the following coordinate change: $w_1=\xi_1+i\xi_2, w_4=\xi_1-i\xi_2, w_2=\xi_3+i\xi_4, w_3=\xi_3-i\xi_4$

$$\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2=w_1w_4-w_2w_3=0\ .$$

Let

$$g_1=\frac{w_1}{w_3}=\frac{w_2}{w_4}\quad ,\quad g_2=-\frac{w_1}{w_2}=-\frac{w_3}{w_4}\ .$$

Let us plug $h_1=g_1-g_2$ end $h_2=-i(g_1+g_2)$ in Weierstrass form: $[1-h_1^2-h_2^2, i(1+h_1^2+h_2^2), 2h_1, 2h_2]$. We find the Weierstrass form used in [11]:

$$\frac{1}{2}\left((1+g_1g_2)\,, i\,(1-g_1g_2)\,, g_1-g_2, -i\,(g_1+g_2)\right)\ .$$

We see that Q_1 and Q_2 are isometric respectively to $\mathbb{P}^1$ and $\mathbb{P}^1\times\mathbb{P}^1$. In that sense, their parametrization are simple.

1.1.6. *Other Weierstrass forms?* As we noticed earlier there are many equivalent Weierstrass forms. For example, the quadric $Q_1:\ x^2+y^2+z^2$ is equivalent to the quadric $\tilde{Q}_1=xy+z^2$ and the quadric $Q_2:\ x^2+y^2+z^2+t^2$ is equivalent to the quadric $\tilde{Q}_2=xy+zt$. More precisely, if dimension n is even $=2k+2$, we can choose as nondegenerate quadric, center of projection and projection hyperplane:

$$\tilde{Q}_{2k}:\ \sum_{i=1}^{k}x_ix_{i+k},\ \ A=[1,0,\cdots,0], H_{2k}=\{[z_1,\cdots,z_{2k}]\ :\ z_1=0\}\ .$$

In that case the projection π is simply $\pi([z_1,\cdots,z_{2k}])=[z_2,\cdots,z_{2k}]$ and Weierstrass parametrization of Q_{2k} by $\mathbb{P}^{2k}$ is given by:

$$\pi^{-1}([y_2,\cdots,y_{2k}])=[\sum_{i=2}^{k}y_iy_{k+1}, -y_{k+1}y_2,\cdots,-y_{k+1}y_{2k}]\ .$$

A similar form is obtained in odd dimensions $n=2k+3$ with the choice

$$\tilde{Q}_{2k+1}:\ \sum x_ix_{i+k}+x_{2k+1}^2\ .$$

2. PARAMETRIZATIONS OF MINIMAL SURFACES IN $\mathbb{R}^n$

An immersion

$$X:\mathfrak{R}\longrightarrow\Sigma\subset\mathbb{R}^n$$

is minimal if the image of X is a surface with zero mean curvature.
The metric of the surface Σ- or equivalently- the Euclidean metric pulled-back on $\mathfrak{R}$ defines a Laplacian Δ_Σ and a sufficiently regular immersion X satisfies the mean curvature equation $\Delta_\Sigma X=2nH$. In particular $X(\mathfrak{R})$ is minimal iff $H=0$ i.e. iff the coordinates functions of X are harmonic as functions on the Riemann $(\mathfrak{R}, g_\Sigma)$ where g_Σ is the metric induced by X. Thanks to parametrization, the minimal condition splits now into two equations. First notice that harmonicity of X is invariant by conformal change of metric on $\mathfrak{R}$. For, in a conformal change of metric, the Laplacian is divided by a the metric dilation factor and X remains a harmonic map. In particular harmonicity depends only on the Riemann surface structure of $\mathfrak{R}$. Let $U\subset\mathfrak{R}$ and let coordinate $z:=x+iy$ on U provided by the complex structure of $\mathfrak{R}$. Then X is harmonic iff $\bar\partial\partial X=X_{xx}+X_{yy}=0$.
Second, If X is conformal then , on U, the metric $g_\Sigma=\lambda^2(dx^2+dy^2)$ for some real function on U, i.e. $\lambda^2=|X_x|^2=|X_y|^2, \langle X_x, X_y\rangle=0$.
In sum, we will see that $X:U\longrightarrow\mathbb{C}$ is minimal immersion iff

1. X conformal i.e. $[\partial X] = [(X_x - iX_y)dz] \in Q_{n-2} \subset \mathbb{P}^{n-1}$
2. X harmonic: $\bar{\partial}\partial X = X_{xx} + X_{yy} = 0$

where the ambient space $\mathbb{R}^4$ has been complexified. Let us first consider first condition.

2.1. **Conformal mappings $\mathfrak{R} \longrightarrow \mathbb{R}^n \subset \mathbb{C}^n$.** The complexification of the target space gives an elegant way to characterize conformal maps.

Proposition 2.1. *The map $X : \mathfrak{R} \longrightarrow \mathbb{R}^n$ is a conformal mapping iff the image of $[\partial X] : \mathfrak{R} \longrightarrow \mathbb{P}^{n-1}$ is in Q_{n-2} i.e. $[\partial X](\mathfrak{R}) \subset Q_{n-2}$ i.e. $[\partial X]$ is a null curve.*

Proof. Choose $p \in \mathfrak{R}$ and U a open neighborhood of p in $\mathfrak{R}$. Let $z = x + iy$ ba a local complex coordinate then X is conformal iff $|X_x|^2 = |X_y|^2, \langle X_x | X_y \rangle = 0$. But consider $\partial X := X_x + iX_y \in \mathbb{C}^n$. Then the Euclidean scalar product on $\mathbb{R}^3$ extends to $\mathbb{C}^3$ and $|\partial X|^2 = 0$ iff $|X_x|^2 - |X_y|^2 + 2i\langle X_x | X_y \rangle = 0$, i.e. iff X is conformal. This is independent of the metric scale factorλ^2 as far as $\lambda \neq 0$. Hence the image in the projective space.

$$\partial X^\alpha = (X^\alpha_x - iX^\alpha_y)dz := \phi^\alpha$$

X is conformal iff

$$\sum_{\alpha=1}^{n} (\phi^\alpha)^2 = 0$$

i.e.

$$(\phi^\alpha)_\alpha \in Q := \{(z_1, \cdots, z_n) \ : \ z_1^2 + \cdots z_n^2 = 0\}$$

or $\mathrm{Im}(\phi) \subset Q$ where $\phi : U \longrightarrow \mathbb{C}^n$. We denote $X^\alpha_z = \phi^\alpha$ so that X is conformal iff $\sum (\phi^\alpha)^2 = 0$. Apply

Proposition 2.2. *The map $X : \mathfrak{R} \longrightarrow \mathbb{R}^n$ is a conformal mapping iff there exist functions $h_1, \cdots h_n : \mathfrak{R} \longrightarrow \mathbb{C}$ such that $\forall\, z \in \mathfrak{R}$*

$$[\partial X](z) = [h_1(z)^2 - h_2(z)^2 - \cdots - h_{n-1}(z)^2, i(h_1(z)^2 + h_2(z)^2 + \cdots + h_{n-1}(z)^2),$$
$$2h_1(z)h_2(z), 2h_1(z)h_3(z), \cdots, 2h_1(z)h_{n-1}(z)]dz\ .$$

We can identify $[\partial X]$ with the Gauss map of X: indeed

Lemma 2.1. *Q_{n-2} is conformal isomorphic to the Grassmannian of 2-real planes of $\mathbb{R}^n$: $G_{2,n}$.*

Proof. $P \in G_{2,n}$ choose an orthonormal direct frame (e, f) the map $(e, f) \mapsto [e + if] \in P(\mathbb{C}^n)$ is well-defined. Furthermore we can easily check that $[e + if]$ lies in the quadric $Q_{n-2} \subset P(\mathbb{C}^n)$.

□

But $G_{2,n}$ is also a metric space. We can as well induce the Fubini-study metric of $\mathbb{P}^{n-1}$ to Q_{n-2}. Equivalently we can consider the metric defined by the Euclidean space: $\bigwedge^2 \mathbb{R}^n$ and restricted to the quadric image $(e, f) \mapsto e \wedge f \in \bigwedge^2 \mathbb{R}^n$ and the quadric can be given in terms of Plucker's coordinates (see for example [1]).

Remark 2.1. The existence of local isothermic coordinates on a surface (cf. for example [3]) implies that for $C^{1,\alpha}$-immersion X, then ∂X is de facto conformal and defines a Gauss map. Then X can always be expressed locally using the Weierstrass form. This is the first step to the so-called "Weierstrass representation" for general surfaces as we shall see later.

2.2. **Conformal maps that are harmonic.** It is easy to construct harmonic maps that are necessarily not conformal; for example any map $\phi : \mathfrak{R} \longrightarrow \mathbb{C}^n$, such that $\psi(z) = (p_1(z) + \bar{q}_1(z), \cdots, p_n(z) + \bar{q}_n(z))$ where $p_i, q_i, i = 1, \cdots, n$ are holomorphic functions, is harmonic. In the section devoted to the new representation we will see that ψ is also conformal iff $\sum_{i=1}^n p_i' q_i' = 0$.
But let us consider first the standard Weierstrass representation. Suppose that X is harmonic with respect to the induced metric. Using proposition 2.2 , this is equivalent to $\bar{\partial}(\partial X) = \bar{\partial}\phi = 0$ i.e. ϕ is holomorphic. The first step to the establishment of Weierstrass representation

Proposition 2.3. *Let* $X : \mathfrak{R} \longrightarrow \mathbb{R}^n$ *be a minimal immersion; then there exists a* (*complex*) *rational map*

$$[\partial X] = \pi^{-1} \circ G : \mathfrak{R} \dashrightarrow \mathbb{P}^{n-1} \ .$$

In other terms there are affine covers A_i *of* $\mathbb{P}^{n-2}$ *and meromorphic functions* $g_j : \mathfrak{R} \longrightarrow \mathbb{C}, j = 1, \cdots, n-2$ *such that*

$$\bar{\pi}_i^{-1} \circ G : A_i \longrightarrow \mathbb{P}^{n-1} \quad , \quad i = 1, \cdots, N$$

where

$$\bar{\pi}_i^{-1} \circ G(z) = [1 - g_1^2 \cdots - g_{n-2}^2, i(1 + g_1^2 + \cdots + g_{n-2}^2), 2g_1, 2g_2, \cdots, 2g_{n-2}] \ .$$

But since $[\partial X]$ is holomorphic, when can integrate on coordinate charts. Let $z_0 \in U \subset \mathfrak{R}$ and $\phi_V : V \subset \mathbb{P}^{n-1} \longrightarrow \mathbb{C}^n$ such that there exists a function $o_{UV}(z)$ such that for any $z \in U$, $\tilde{\partial} X_{UV} := \phi_V \circ [\partial X](z) = (1 - g_1^2 - \cdots - g_{n-2}^2, i(1 + g_1^2 + \cdots + g_{n-2}^2), 2g_1, 2g_2, \cdots, 2g_{n-2}) o(z) dz$. Since charts are holomorphic, function o is also holomorphic and we integrate the form:

$$\tilde{X}_{UV} : U \subset \mathfrak{R} \longrightarrow V$$

$$\tilde{X}_{UV}(z) = \int_{z_0}^{z} (1 - g_1^2 - \cdots - g_{n-2}^2, i(1 + g_1^2 + \cdots + g_{n-2}^2), 2g_1, 2g_2, \cdots, 2g_{n-2}) o_{UV}(z)\, dz \ .$$

But $\partial X = X_x - iX_y$ hence $X(z) - X(z_0) = \Re e(\int_{z_0}^z \partial X) = \Re e(\int_{z_0}^z \phi\, dz)$.
The sections $o_{UV}(z)dz$ define a holomorphic (1,0)-form ω which is a global section $T^*(\mathfrak{R})$. To sum up $X = \int_{z_0}^z \omega\phi$.
Reciprocally, let $G : \mathfrak{R} \dashrightarrow Q_{n-2}$ be a rational map and ω a holomorphic 1-form on $\mathfrak{R}$, then locally, on say U, $G\mid_U$ is defined by meromorphic maps g_i, $i = 1, \cdots, n$ and lifts to a map $\phi : U \longrightarrow \mathbb{C}^n$ and let us define $X_U : U \longrightarrow \mathbb{R}^n$ by $X_U(z) = \Re e\left(\int_{z_0}^z \omega\right)$. Moreover, if for any closed curve $\gamma \in \pi(\mathfrak{R})$ $\Re e\left(\int_\gamma \phi\right) = 0$, then X extends to $\mathfrak{R}$ and $X(\mathfrak{R})$ is a minimal surface parametrized by the Riemann surface $\mathfrak{R}$.

2.2.1. *Examples of minimal surfaces defined by their Weierstrass coordinates.* We will give examples of surfaces i dimension 3 and four only. Roughly speaking any choice of holomorphic g_i and holomorphic ω on a surface domain $U \in \mathfrak{R}$ defines a minimal surface parametrized by U. If we allow meromorphic g_i, poles of g_i of order n should correspond to a zero of ω of order $2n$. The divisor of g_i , D_i $\sum 2D_i + D_\omega > 0$. But this parametrization maybe multivalued. To avoid this $2g+r$ period conditions should be satisfied corresponding to the number of minimal generators of $\pi_1(U)$ and the number of punctures. If an immersion is needed, he metric factor λ should never be zero.

Example 2.1. Surfaces in Euclidean space $\mathbb{R}^3$:

$\mathfrak{R} = \mathbb{C}$

$\omega = dz,\ \ g = (z-1)^k$: $k=1$ Enneper,

$$X(z) = \Re e\left(z - \frac{z^3}{3}, i\left(z + \frac{z^3}{3}\right), \frac{z^2}{2}\right)$$

$\mathfrak{R} = D \subset \mathbb{C}$ where D is the unit disk.

$\omega = z^k dz,\ \ g = z^l$: branch point of order k and index l.

$\mathfrak{R} = \mathbb{C}^*$

$\omega = \dfrac{dz}{(z)^2}$, $g = z$: Catenoid,

$$X(z) = \Re e\left(z + \frac{1}{z}, i\left(z - \frac{1}{z}\right), 2\ln z\right)$$

$\omega = \dfrac{idz}{z}$, $g = z$: Helicoid,

$$X(z) = \Re e\left(-i\left(z + \frac{1}{z}\right), -\left(z - \frac{1}{z}\right), 2i\ln z\right)$$

$\omega = \dfrac{dz}{(z-1)^2}$, $g = (z-1)^3$: $k=1$ Enneper end + catenoid end

$\mathfrak{R} = \mathbb{C} \setminus \{p_1, \cdots, p_k\}$

$\omega = \dfrac{dz}{(z^k-1)^2}$, $g = z^{k-1}$: k-noids (Jorge-Meeks)

Surfaces in Euclidean space $\mathbb{R}^4$*:*

$\mathfrak{R} = \mathbb{C}$

$\omega = dz,\ \ g_1 = z^k,\ \ g_2 = z^l$ Enneper,

$$X(z) = \Re e\left(\int_{z_0}^{z}(1 - z^{2k} - z^{2l}), i(1 + z^{2k} + z^{2l}), 2z^k, 2z^l\right)$$

2.3. **A variant of the Weierstrass representation.** The following local representation appears in [10] for surfaces in 4-dimensional spaces and used in a crucial way in [SV]. Contrary to the standard case, we start with the harmonic condition and then force the conformal condition. We distinguish odd and even dimension of the ambient space.

2.3.1. *Minimal surfaces in* $\mathbb{R}^{2n}$. Suppose Σ is an algebraic minimal surface i.e. whose meromorphic coordinates are of the first or second kind. Let U be simply-connected domain and suppose the mapping $X : U \longrightarrow \mathbb{R}^{2n}$ is harmonic. Then $\phi(z) = (\tilde{E}_1, \tilde{E}_2, \cdots, \tilde{E}_{2n})$ with $\tilde{E}_i$ real harmonic functions. Hence there exist complex holomorphic functions such that

$$X(z) = (\tilde{E}_1, \tilde{E}_2, \cdots, \tilde{E}_n) = (e_1 + \bar{f}_1, \cdots, e_n + \bar{f}_n) \in \mathbb{C}^n .$$

Indeed

$$(e_1 + \bar{f}_1, \cdots, e_n + \bar{f}_n) = \left(\frac{e_1 + \bar{e}_1}{2} + \frac{f_1 + \bar{f}_1}{2}, \frac{e_1 - \bar{e}_1}{2i} + \frac{\bar{f}_1 - f_1}{2i}, \cdots,\right.$$

$$\frac{e_n+\bar{e}_n}{2}+\frac{f_n+\bar{f}_n}{2},\frac{e_n-\bar{e}_n}{2i}+\frac{\bar{f}_n-f_n}{2i}\Big) \ .$$

There exist holomorphic function E_i such that $\tilde{E}_i = \Re e(E_i)$ and define

$$e_i = E_{2i} + \frac{i}{2}E_{2i+1} \quad , \quad f_i = E_{2i} - \frac{i}{2}E_{2i+1} \quad , \quad i = 1, \cdots, n \, .$$

X is conformal iff

$$\sum_{i=1}^{n} e'_i f'_i = 0$$

i.e. iff $[X_z] = [e'_1, \cdots, e'_n, f'_1, \cdots f'_n]$ belongs to the quadric of equation: $\sum_{i=1}^{n} z_i z_{i+n} = 0$ which is another expression of the quadric Q_{n-2}.

Example 2.2 (dimension 4)**.** Let U be an simply-connected domain there are locally holomorphic functions e, f, g, h such that $X(z) = (e+\bar{f}, g+\bar{h})$ and $e'f' + g'h' = 0$.

2.3.2. *Minimal surfaces in* $\mathbb{R}^{2n+1}$. The same method leads to the following representation in terms of holomorphic function $(e_1+\bar{f}_1, \cdots, e_n+\bar{f}_n, g+\bar{g})$ satisfying

$$\sum_{i=1}^{n} e'_i f'_i + {g'}^2 = 0 \ .$$

Let U of neighborhood $p \in \mathfrak{R}$ Since X is harmonic there exists (e, f, g) holomorphic such that

$$X(z) = (e+\bar{f}, g+\bar{g}) \in \mathbb{C} \times \mathbb{R}$$

X is conformal iff $[X_z] = [e'+f', (e'-f')/i, 2g'] \in Q_1$ iff

$$e'_1 f'_1 + {g'_1}^2 = 0 \ .$$

In that case we have chosen in Poncelet's theorem the quadric $Q_{n-2} : z_1 z_2 + z_3^2 = 0$ which, by a change of coordinates $z_1 = \phi^1 + i\phi^2, z_2 = \phi^1 - i\phi^2, z_3 = \phi_3$, gives the canonical quadric.

The complex (respectively CR) structure of the ambient space $\mathbb{R}^{2n} = \mathbb{C}^n$ (respectively $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$) yields naturally another Weierstrass representation. Since most computations concern dimension 4, we will consider only the even dimension case. Locally any harmonic map $X : U \subset \mathfrak{R} \longrightarrow \mathbb{R}^{2n} = \mathbb{C}^n$ is given by a set of holomorphic maps such that

$$X(z) = (p_1(z) + \bar{q}_1(z), \cdots, p_n(z) + \bar{q}_n(z))$$

(respectively $X : U \subset \mathfrak{R} \longrightarrow \mathbb{R}^{2n+1}$

$$X(z) = (p_1(z) + \bar{q}_1(z), \cdots, p_n(z) + \bar{q}_n(z), p_{n+1}(z) + \bar{p}_{n+1}(z))) \ .$$

Then $X_z = (p'_1(z), \cdots, p'_n(z))$ and $X_{\bar{z}} = (\bar{q}'_1(z), \cdots, \bar{q}'_n(z))$ $\langle X_z, X_{\bar{z}} \rangle = \sum p'_i q'_i = 0$ but $|X_z| \neq |X_{\bar{z}}|$ hence $\{X_z, X_{\bar{z}}\}$ is only an orthogonal frame. Hence

$$X_x = \frac{1}{2}(X_z + X_{\bar{z}}) = \frac{1}{2}\left(p'_1(z) + \bar{q}'_1(z), \cdots, p'_n(z) + \bar{q}'_n(z)\right)$$

and

$$X_y = \frac{1}{2i}(X_z - X_{\bar{z}}) = \frac{1}{2i}\left(p'_1(z) - \bar{q}'_1(z), \cdots, p'_n(z) - \bar{q}'_n(z)\right)$$

$$|X_x|^2 = \frac{1}{4}\sum |p'_i|^2 + |q'_i|^2 + 2\,\Re e(p'_i(z) q'_i) \ , \ |X_y|^2 = \frac{1}{4}\sum |p'_i|^2 + |q'_i|^2 - 2\,\Re e(p'_i(z) q'_i)$$

Hence

$$|X_z|^2 = \sum |p_i'|^2 \quad , \quad |X_{\bar z}|^2 = \sum |q_i'|^2 \, ,$$

and the metric

$$\lambda^2 dz d\bar z = (|X_z|^2 + |X_{\bar z}|^2) dz d\bar z = \left(\sum |p_i'|^2 + |q_i'|^2 \right) dz d\bar z \, .$$

2.3.3. *Kenmotsu representation.* The Weierstrass representation is reminiscent of the parametrization of convex surfaces. Such surfaces can be parametrized by the Gauss map. Similarly, we can use isothermic parameters for the induced metric of any smooth immersion $X : \mathfrak{R} \longrightarrow \mathbb{R}^n$ The map $[\partial X]$ has values in the quadric Q_{n-2} which can be parametrized using the Weierstrass form. If furthermore the mean curvature is a given, then X admits the "Weierstrass representation":

$$X = \Re e \left(\int_{z_0}^{z} f(1 - g^2, i(1 + g^2), 2g) \, dz \right)$$

with the coupling

$$(\ln f)_{\bar z} = -2 \frac{\bar g g_{\bar z}}{1 + |g|^2} \quad , \quad H = -2 \frac{\bar g_z}{f(1 + |g|^2)^2} \, .$$

Contrary to the minimal case, the solutions to these equations are however not easy to find and the two equations cannot be split (see [9] for more details).

3. GEOMETRIC QUANTITIES IN TERMS OF THE WEIERSTRASS REPRESENTATIONS

We will quickly recall the dimension 3 case using standard Weierstrass representation, since in that case commutations are well-known (see for example [14]). Since our ultimate goal is the study of isolated singularities on minimal surfaces in $\mathbb{R}^4$, most computations will be done in for geometric quantities for minimal surfaces in $\mathbb{R}^{2n}$, using the new Weierstrass representation and more particularly for $n = 4$. We will only rapidly recall the results of the computations for surfaces in $\mathbb{R}^3$ done in the standard Weierstrass representation, since these are well-known. We refer to [14] for further details and won't insist on the new representation since our applications in view are for the dimension 4 case.

3.1. **Local computations.** We define some line bundles over $\mathfrak{R}$ induced by the immersion X.

3.1.1. *Tangent bundle $T\Sigma$.* Recall that $X : U \subset \mathbb{C} \longrightarrow \mathbb{R}^n$ $X \in C(U, \mathbb{R}^n)$ local representation by the complex variable $z = x + iy$. Since $T\Sigma = \bigcup_{p \in \Sigma} T_p\Sigma$, we need to describe, at each point of $p \in \Sigma$ the 2-real plane of $\mathbb{R}^n$ tangent to Σ at point p. Let

$$X_x := \frac{\partial X}{\partial x} \quad , \quad X_y := \frac{\partial X}{\partial y} \, ,$$

$$T_pM = \operatorname{span}\{X_{,1}, X_{,2}\} \quad , \quad X_{,i} = (X_i^\alpha)_{\alpha = 1, \cdots, n}, \quad i = x, y \, .$$

X regular if $\operatorname{rg}(DX) = \operatorname{rg}(X_i^\alpha) = 2$ if not X singular. Since X is conformal $\{X_x, X_y\}$ is an orthonormal frame. Hence

$$T\Sigma := \bigcup_{p \in \Sigma} \operatorname{span}\{X_x, X_y\} \, .$$

Each two plane $T_p\Sigma$ is provided with the metric defined by the scalar product of $\mathbb{R}^4$. Using the metric, we can define on each plane a rotation J of angle $\pi/2$ that defines a complex structure on $T\Sigma$ which becomes a complex line bundle.

3.1.2. *Many line-bundles* $W\Sigma$. With respect to the natural complex structure of $\mathbb{R}^4 = \mathbb{C}^2$,

$$X_x = (e' + \bar{f}', g' + \bar{h}'), X_y = (i(e' - \bar{f}'), i(g' - \bar{h}')) \, .$$

Or, in $\mathbb{R}^4$

$$X_x = (\Re e(e' + \bar{f}'), \Im m(e' + \bar{f}'), \Re e(g' + \bar{h}'), \Im m(g' + \bar{h}')) = (e' + \bar{f}', g' + \bar{h}')$$

$$X_y = (-\Im m(e' - \bar{f}'), \Re e(e' - \bar{f}'), -\Im m(g' - \bar{h}'), \Re e(g' - \bar{h}')) = i(e' - \bar{f}', g' - \bar{h}') \, .$$

Hence

$$X_x - iX_y = \phi = (e' + f', -i(e' - f'), g' + h', -i(g' - h')) \, .$$

It is easier to derivate with respect to z and $\bar{z}$ in this representation

$$X_x = \frac{1}{2}(X_z + X_{\bar{z}}), X_y = \frac{1}{2i}(X_z - X_{\bar{z}}) \, .$$

Note that $\operatorname{span}\{X_z, X_{\bar{z}}\}$ is an orthogonal frame and generates a 2-real plane at points where the tangent plane is not complex in $\mathbb{C}^2$. This 2-plane is not tangent to Σ except when it degenerates to a line which correspond to complex planes of Σ. X_z and $X_{\bar{z}}$ are orthogonal since $\langle X_z, X_{\bar{z}} \rangle_{\mathbb{R}^4} = \Re e(e'f' + g'h') = 0$. Also $h_1^2 := \|X_z\|^2 = |e'|^2 + |f'|^2, h_2^2 := \|X_{\bar{z}}\|^2 = |g'|^2 + |h'|^2$.
Since $\Im m(e'f' + g'h') = 0$ we deduce that iX_z is orthogonal to $X_{\bar{z}}$. In particular $\{X_z, X_{\bar{z}}, iX_z, iX_{\bar{z}}\}$ spans $\mathbb{R}^4$ at points z of $\mathfrak{R}$ such that $e'\bar{h}' - f'\bar{g}' \neq 0$. We may construct many complex line fibrations

$$W_{1,0}\Sigma := \bigcup_{p \in \Sigma} \operatorname{span}\{X_z, iX_z\} \quad , W_{1,1}\Sigma := \bigcup_{p \in \Sigma} \operatorname{span}\{X_z, X_{\bar{z}}\} \, ,$$

$$W_{0,1}\Sigma := \bigcup_{p \in \Sigma} \operatorname{span}\{X_{\bar{z}}, iX_{\bar{z}}\} \, .$$

3.1.3. *Normal bundle* $N\Sigma$. Let us find now an expression for a normal basis of Σ in terms of these new coordinates.
Consider the 2-vector $p := \phi \wedge \bar{\phi} \in \bigwedge^2 \mathbb{C}^n$; then $p^{ij} := \phi_i \bar{\phi}_j - \phi_j \bar{\phi}_i$. Find "easy coordinates" i.e. ad hoc normal vectors. Let $n := n_1 + in_2$, then find solutions to

$$\langle X_z, n \rangle = 0, \langle X_z, \bar{n} \rangle = 0 \quad \text{i.e} \quad \langle X_z, n \rangle = 0, \;\; \langle \bar{X}_z, n \rangle = 0 \, .$$

This is equivalent to the system:

$$\begin{pmatrix} e' + f' & -i(e' - f') \\ \overline{e' + f'} & \overline{i(e' - f')} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = - \begin{pmatrix} g' + h' & -i(g' - h') \\ \overline{g' + h'} & \overline{i(g' - h')} \end{pmatrix} \begin{pmatrix} n_3 \\ n_4 \end{pmatrix} \, . \tag{5}$$

Solving

$$\begin{aligned} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} &= \frac{i}{2(|e'|^2 - |f'|^2)} \begin{pmatrix} i(\overline{e' - f'}) & i(e' - f') \\ -(\overline{e' + f'}) & e' + f' \end{pmatrix} \times \\ &\times \begin{pmatrix} g' + h' & -i(g' - h') \\ \overline{g' + h'} & \overline{i(g' - h')} \end{pmatrix} \begin{pmatrix} n_3 \\ n_4 \end{pmatrix} \end{aligned} \tag{6}$$

let $n_3 = 1, n_4 = 0$ then

$$\begin{aligned}\begin{pmatrix} n_1 \\ \\ n_2 \end{pmatrix} &= \frac{i}{2(|e'|^2 - |f'|^2)} \begin{pmatrix} i(\overline{e' - f'}) & i(e' - f') \\ \\ -(\overline{e' + f'}) & e' + f' \end{pmatrix} \begin{pmatrix} g' + h' \\ \\ \overline{g' + h'} \end{pmatrix} = \\ &= \frac{-1}{|e'|^2 - |f'|^2} \begin{pmatrix} \Re e((e' - f')(\overline{g' + h'})) \\ \\ \Im m((e' + f')(\overline{g' + h'})) \end{pmatrix} .\end{aligned} \tag{7}$$

Hence the first normal vector is

$$\vec{n_1} = \alpha_1(\Re e((e' - f')(\overline{g' + h'})), \Im m((e' + f')(\overline{g' + h'})), |e'|^2 - |f'|^2, 0)$$

where $\alpha_1^{-2} = (|e'|^2 + |f'|^2)|g' + h'|^2 - 2\Re e(e' f' \overline{(g' + h')}^2) + (|e'|^2 - |f'|^2)^2$.
As a vector in $\mathbb{C}^2$:

$$\vec{n_1} = \alpha_1(e'(\overline{g' + h'}) - \bar{f}'(g' + h'), |e'|^2 - |f'|^2)$$

$$\|\vec{n_1}\|^2 = (|e'|^2 + |f'|^2)|g' + h'|^2 - 2\Re e(e' f' \overline{(g' + h')^2}) + (|e'|^2 - |f'|^2)^2 \ .$$

Similarly let $n_3 = 0, n_4 = 1$ then

$$\vec{n_2} = \alpha_2(\Im m((e' - f')(\overline{g' - h'})), -\Re e((e' + f')(\overline{g' - h'})), 0, |e'|^2 - |f'|^2)$$

where $\alpha_1^{-2} = (|e'|^2 + |f'|^2)|g' - h'|^2 + 2\Re e(e' f' \overline{(g' + h')}^2) + (|e'|^2 - |f'|^2)^2$.
As a vector in $\mathbb{C}^2$

$$\vec{n_2} = i(\bar{f}'(g' - h') + e'(\overline{g' - h'}), |e'|^2 - |f'|^2)$$

$$\|\vec{n_2}\|^2 = (|e'|^2 + |f'|^2)|g' - h'|^2 + 2\Re e(e' f' \overline{(g' - h')^2}) + (|e'|^2 - |f'|^2)^2 \ .$$

Notice that if X is holomorphic

$$\vec{n_1} + i\vec{n_2} = (e'\overline{g'}, -ig'\overline{e'}, -|e'|^2, -i|f'|^2) \ .$$

As a vector in $\mathbb{C}^2$:

$$\vec{n_1} = (e'\overline{g'}, -|e'|^2), \vec{n_2} = i(\bar{e}'g', |e'|^2) \implies \vec{n_1}.\vec{n_2} = 0 \quad \text{and} \quad \|\vec{n_1}\|^2 = \|\vec{n_2}\|^2 \ .$$

Hence N is conformal as X. The non-perpendicularity measures the non-holomorphy of the surface Σ.

3.1.4. *The first fundamental form and Kaehler form.* Since the immersion X is conformal, the induced metric of the immersion X is conformal; this defines a nonnegative function λ (that is zero only at isolated points - singular points) such that:

$$ds^2 = \lambda^2(dx^2 + dy^2) \quad , \quad \lambda^2 = \sum_{i=1}^{2} \|X_i\|^2 \ .$$

In terms of the Weierstrass representation, we obtain

$$\sum_\alpha \|\phi^\alpha\|^2 = \sum_\alpha |X_x^\alpha|^2 + |X_y^\alpha|^2 = 2\lambda^2 =$$

$$= |z_1^2 - z_3^2 \cdots - z_n^2|^2 + |z_1^2 + z_3^2 + \cdots + z_n^2|^2 + 4|z_1|^2|z_3|^2 + 4|z_1|^2|z_4|^2 + \cdots + 4|z_1|^2|z_n|^2 =$$

$$= 2\left(\sum |z_1|^4 + 2\sum_{i,j\neq 1} |z_i|^2|z_j|^2 + 2|z_1|^2 \left(\sum_{i\neq 1} |z_i|^2\right)\right) = 2\left(\sum |z_i|^2\right)^2 \ .$$

In terms of the g_i:

$$\lambda^2 = 2|\omega|^2\left(1+\sum_{i=1}^{n}|g_i|^2\right)^2 .$$

We deduce from this expression the Kaehler (1,1)-form f. First recall that $\omega = i\partial\bar\partial\left(\sum_{i=1}^{n}|z_i|^2\right)$, hence, for example, in dimension 3, $\lambda^2 = |f|^2(1+|g|^2)^2$ and in dimension 4 $\lambda^2|f|^2(1+|g_1|^2+|g_2|^2)^2$

3.1.5. *Recap* $T\Sigma \subset \mathbb{R}^4$.

	Representation I	Representation II
parametrization $X : \mathfrak{R} \longrightarrow \mathbb{R}^4 = \mathbb{C}^2$	$X(z) =$ $\Re e\left(\int_{z_0}^{z}\phi(w)\,dw\right)$	$X(z) =$ $(e+\bar f, g+\bar h) \in \mathbb{C}^2$
$T\Sigma\otimes\mathbb{C}$: $\partial X = (X_x - i\,X_y)dz$ $\in \mathbb{C}^{4*}$ Segre form	$\phi = \omega\big(1-g_1^2-g_2^2,$ $i(1+g_1^2+g_2^2),$ $2g_1, 2g_2\big)$ $\omega\big((1-g_1g_2),$ $1+g_1^2-g_2^2,$ $g_1-g_2,$ $i(g_1+g_2)\big)$	$X_x = \frac{1}{2}(X_z + X_{\bar z})$ $X_y = \frac{1}{2i}(X_z - X_{\bar z})$
X is conformal iff	$\phi = [\partial X] \in Q_2$: $\sum_{\alpha=1}^{4}(\phi^\alpha)^2 = 0$	$X_z \perp X_z \,\&\, X_{z\bar z} = 0$: $e'f' + g'h' = 0$
tangent 2-vector: $X_x \wedge X_y$	$\frac{1}{i}\phi\wedge\bar\phi$	$\frac{1}{4}(X_z\wedge JX_z +$ $+X_{\bar z}\wedge JX_{\bar z})$
normal 2-vector: $\mathrm{span}(n_1, n_2)$	$\langle\phi, n_1+in_2\rangle =$ $= \langle\bar\phi, n_1+in_2\rangle =$ $= 0$	see 3.1.3
1st fundamental form $I = \lambda^2\,\lvert dz\rvert^2$	$\lambda^2 =$ $= 2\lvert\omega\rvert^2\left(1+\sum_{i=1}^{n}\lvert g_i\rvert^2\right)^2$	$\lambda^2 = \lvert e'\rvert^2+\lvert f'\rvert^2+$ $+\lvert g'\rvert^2+\lvert h'\rvert^2$
K : Curvature of $T\mathfrak{R}$	$K = -4\,\frac{\lvert\phi\wedge\phi'\rvert^2}{\lvert\phi\rvert^6}$	see 3.2.2

3.2. **Global computations.** We will study almost exclusively minimal surfaces in $\mathbb{R}^4$. We will consider various complex line bundles-or real vector bundles of rank 2- over Σ namely the tangent bundle $T\Sigma$, the normal bundle $N\Sigma$ and other fibrations $W_{i,j}\Sigma$ which maybe singular at some points of $\mathfrak{R}$.
These bundles are all pencils of planes of $\mathbb{R}^4$ parametrized by Σ. These planes are all provided with a scalar product- induced by the scalar product of $\mathbb{R}^4$. Then on each fiber a rotation J of angle $+\pi/2$ is well-defined and defines a complex structure on the bundle. Let us give a general description of the "Gauss-Bonnet" formula for all these line bundles.

3.2.1. *Curvature of a line bundle and Gauss-Bonnet formula.*

Lemma 3.1. *Let Σ be a surface with boundary and let $F : L \longrightarrow \Sigma$ be a line-bundle. We denote by $\langle\, ,\, \rangle$ the scalar product on L and by J the complex structure on L. We consider a section s of L which vanishes nowhere on the boundary of Σ. We let ∇ be a connection on L and we define a connection 1-form ω by*

$$\omega(u) = \frac{\langle \nabla_u s, Js\rangle}{\|s\|^2} = \langle \nabla_u(\frac{s}{\|s\|}), J(\frac{s}{\|s\|})\rangle \ .$$

The curvature K^L of L is defined by

$$\Omega := \nabla\omega = d\omega = K^L dA_\Sigma$$

where dA_Σ is the area element on Σ.
We have

$$\int_\Sigma K^L\, dA_\Sigma = \int_{\partial\Sigma} \omega + \sum_{i=1}^m \text{index}(z_i)$$

where the z_i's $i = 1, \cdots, m$ are the zeroes of s inside Σ.

Proof. Then the connection and connection form satisfy: $\nabla_X s = \omega_s(X)$. By definition the curvature $\Omega = D\omega = d\omega + \omega \wedge \omega = d\omega$ since fibers are lines.

Suppose section s is zero at points $z_i, i = 1, \cdots n$. For a small real number ϵ, we consider the balls $B(z_i, \epsilon)$ centered at z_i $i = 1, \cdots, m$ and of radius ϵ. We apply Stokes' formula on $\Sigma \setminus \cup B(z_i, \epsilon)$ to the form ω.

□

3.2.2. *The curvature of the tangent and normal bundles.* We denote by ∇^T (respectively ∇^N) the connection on $T\Sigma$ (respectively $N\Sigma$) induced by the projection of the Levi-Civita connection on M. The goal of this paragraph is to compute its curvature Ω^T (respectively Ω^N).
We choose e_1, e_2 (respectively e_3, e_4) a positive orthonormal basis of $T\Sigma$ (respectively $N\Sigma$) and we let ω^T (respectively ω^N) be the connections 1-forms for ∇^T (respectively ∇^N). We write

$$\omega^T(X) = \langle \nabla_X e_2, e_1\rangle \quad , \quad \omega^N(X) = \langle \nabla_X e_4, e_3\rangle \ ,$$

where X is a vector tangent to Σ; so the curvature forms are $\Omega^T = d\omega^T$ and $\Omega^N = d\omega^N$.
We recall Gauss'equation

Proposition 3.1.

$$\Omega^T(e_1, e_2) = -\|B(e_1, e_2)\|^2 + \langle B(e_1, e_1), B(e_2, e_2)\rangle + \\ + \langle R^M(e_1, e_2)e_1, e_2\rangle \ . \tag{8}$$

If Σ *is a minimal surface, then the righthand side of* (8) *is*

$$-\|B(e_1,e_2)\|^2 - \|B(e_1,e_1)\|^2 + \langle R^M(e_1,e_2)e_1, e_2\rangle \ .$$

We now turn to Ω^N and compute

$$\Omega^N(e_1,e_2) = e_1\omega(e_2) - e_2\omega(e_1) - \omega([e_1,e_2]) =$$
$$= e_1\langle \nabla_{e_2}e_4, e_3\rangle - e_2\langle \nabla_{e_1}e_4, e_3\rangle - \langle \nabla_{[e_1,e_2]}e_4, e_3\rangle = E_1 + E_2$$

where

$$E_1 = \langle \nabla_{e_1}\nabla_{e_2}e_4 - \nabla_{e_2}\nabla_{e_1}e_3 - \nabla_{[e_1,e_2]}e_4, e_3\rangle \ , \tag{9}$$

$$E_2 = \langle \nabla_{e_2}e_4, \nabla_{e_1}e_3\rangle - \langle \nabla_{e_1}e_4, \nabla_{e_2}e_3\rangle \ . \tag{10}$$

E_1 is equal to $\langle R^M(e_1,e_2)e_3, e_4\rangle$ where R^M is the curvature of the ambient manifold M. To estimate E_2, we notice that only the components of ∇e_3 and ∇e_4 along the tangent vectors e_1, e_2 will contribute to $\langle \nabla e_3, \nabla e_4\rangle$. So

$$\begin{aligned} E_2 &= \langle \nabla_{e_2}e_4, e_1\rangle\langle \nabla_{e_1}e_3, e_1\rangle + \langle \nabla_{e_2}e_4, e_2\rangle\langle \nabla_{e_1}e_3, e_2\rangle - \\ &-\langle \nabla_{e_1}e_4, e_1\rangle\langle \nabla_{e_2}e_3, e_1\rangle - \langle \nabla_{e_1}e_4, e_2\rangle\langle \nabla_{e_2}e_3, e_2\rangle = \\ &= \langle e_4, \nabla_{e_2}e_1\rangle\langle e_3, \nabla_{e_1}e_1\rangle + \langle e_4, \nabla_{e_2}e_2\rangle\langle e_3, \nabla_{e_1}e_2\rangle - \\ &-\langle e_4, \nabla_{e_1}e_1\rangle\langle \nabla_{e_2}e_1, e_3\rangle - \langle e_4, \nabla_{e_1}e_2\rangle\langle e_3, \nabla_{e_2}e_2\rangle \ . \end{aligned}$$

We identify the elements of $\Lambda^2(N\Sigma)$ to real numbers; hence we write

Proposition 3.2.

$$\Omega^N(e_1,e_2) = (B(e_1,e_1) - B(e_2,e_2)) \wedge B(e_1,e_2) + \langle R^M(e_1,e_2)e_3, e_4\rangle \ .$$

The Gauss map is given by a couple of holomorphic functions (g_1, g_2), $Q_2 = \mathbb{S}^2\times\mathbb{S}^2$. *The conformal metric factor of the surface metric is* $\lambda^2 = (1/4)|f|^2(1+|g_1|^2)^2(1+|g_2|^2)^2$ *for a holomorphic* f. *Furthermore, the Jacobian of the Gauss map* J *decomposes into* J_1 *and* J_2 *and the curvature of the tangent curvature and normal curvature respectively are* $K_T = J_1 + J_2$, $K_N = J_1 - J_2$. *The respective total curvatures - when defined - are given in terms of the degrees* d_i *of the map* g_i, $i = 1, 2$*:*

$$\int K_T = 2\pi(d_1 + d_2); \int K_N = 2\pi(d_1 - d_2) \ .$$

If Σ *is minimal, then*

$$\Omega^N(e_1,e_2) = 2B(e_1,e_1) \wedge B(e_1,e_2) + \langle R^M(e_1,e_2)e_3, e_4\rangle \ .$$

Corollary 3.1. *If* Σ *is minimal and* M *is flat, then* $|\Omega^N(e_1,e_2)| \leq -\Omega^T(e_1,e_2)$.

Remark 3.1. If L is the tangent bundle $T\Sigma$, and s restricts to the unit positive tangent vector field T on $\partial\Sigma$, then $\omega(T)$ is the geodesic curvature and the Lemma is the Gauss-Bonnet formula.

We can also compute curvature as follows using the Christoffel symbols of the connection defined as

$$X_{,ij} = \Gamma^k_{ij}X_k + L^\alpha_{ij}n_\alpha \quad , \quad i,j = x,y \ .$$

We can also use isothermal coordinates. Then (see for instance [14]).

$$K = -\frac{\Delta_{Eucl}\ln\lambda}{\lambda^2} \ .$$

Equivalently, if h is a holomorphic section of L then $\Omega = i\partial\bar{\partial}\ln(|h|^2)$.

3.2.3. *Gauss-Bonnet for the normal bundle $N\Sigma$ and self-intersection.* We now want to write the Lemma for the normal bundle of a surface Σ embedded in a 4-manifold M and we want to give a topological interpretation to the number of zeroes of s. If Σ has no boundary, we know that

$$\int_\Sigma K^{N\Sigma}\, dA_\Sigma = [\Sigma].[\Sigma]$$

where $[\Sigma].[\Sigma]$ is the self-intersection number of Σ in M. If Σ is a surface with boundary, it does not have a well-defined self-intersection number. Rather, this number depends on a *framing* of $\partial\Sigma$ which we now define.
To simplify matters, we assume from now on that Σ is in $\mathbb{B}^4$ and its boundary lies in $\mathbb{S}^3$. We have

Definition 3.1. A *framing* of a knot K in $\mathbb{S}^3$ is a vector field Y along K on $\mathbb{S}^3$ which is never tangent to K.

Since the normal bundle to K in $\mathbb{S}^3$ is trivial, the homotopy classes of framings are parametrized by $\pi_1(\mathbb{R}^2 \setminus \{0\})$, i.e. $\mathbb{Z}$. Now assume that K bounds the smooth surface Σ embedded in $\mathbb{B}^4$ We denote by $\tilde{K}$ the knot obtained by pushing K in the direction of Y and we let $\tilde{\Sigma}$ be a smooth surface *embedded* in $\mathbb{B}^4$ bounded by $\tilde{K}$. The number of intersection points of Σ and $\tilde{\Sigma}$ (counted with respect to the sign) is equal to the linking number of K and $\tilde{K}$. It is also called the *self-intersection number of Σ with respect to the framing Y.*
Using the notations from the previous paragraph, we suppose that Y is orthogonal to the restriction to K of the tangent bundle to Σ and that it extends to a section (which we also denote Y) of the normal bundle $N\Sigma$ above Σ. We denote by $N(Y)$ the number of zeroes of Y on $N\Sigma$; we have

Proposition 3.3.

1) *If Σ is embedded*

$$N(Y) = lk(K, \tilde{K}) \ .$$

2) *If Σ is not embedded but immersed with only transverse double points, and Y extends to a section of $N\Sigma$ which does not vanish at any double point,*

$$N(Y) = lk(K, \tilde{K}) - 2D_\Sigma$$

where D_Σ denotes the number of double points of Σ counted with respect to sign.

3) *If Σ is immersed with interior branch points (to simplify, we assume that there is only one (branched) disk going through each branch points) $p_1, \cdots, p_k$,*

$$N(Y) = lk(K, \tilde{K}) - \sum_{i=1}^{k} e(K_i)$$

where $e(K_i)$ is the writhe number of the braid defined by the branch point p_i.

The proof of this proposition is analogous to the proof of the corresponding facts for embedded/immersed/branched immersed surfaces without boundary.

3.2.4. *If the boundary $\partial\Sigma$ is a closed braid.* We now consider the case when the boundary $\partial\Sigma$ is a closed braid K with writhe number $e(K)$ and the framing Y is parallel to the axis of the braid. Then Bennequin has shown in his thesis that

Proposition 3.4. $lk(K,\tilde{K}) = e(K)$.

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