Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 8(2009), pp. 1–41.

An invitation to the study of evolution equations by means of positive linear operators

Francesco Altomare and Vita Leonessa

Abstract[1]. This paper provides a detailed survey on a series of methods and results which are centered around the problem of the constructive approximation of the positive semigroups generated by degenerate second order differential operators by means of iterates of positive linear operators.

This problem, first stated in [7], has been tackled in the last fifteen years from different angles and in different settings such as spaces of bounded and unbounded continuous functions on real intervals as well as on multidimensional convex compact subsets.

For the sake of simplicity the presentation is restricted in the framework of compact intervals. The exposition includes several applications as well as several references for further possible deepening and developments.

Introduction

This survey paper intends to provide a detailed and rigorous introduction into a field of researches which is concerned with the study of some degenerate second order elliptic-parabolic differential problems by means of semigroup theory and the problem of the constructive approximation of the relevant solutions in terms of iterates of positive linear operators.

Among the major motivations which justify an analysis of these problems we mention that they are deeply involved while describing models of diffusion in different branches of applied sciences like, for instance, in population genetics and mathematical finance.

The results discussed in the paper fall within an investigation area which involves methods from Real Analysis, Operator Theory and Approximation Theory and which has been developed during the last fifteen years by several mathematicians, mainly from Italy, Rumania and Germany, after the appearance of the pioneering paper [7] where these problems were stated for the first time.

For the sake of simplicity we restrict our exposition in the framework of compact intervals.

---

[1]Authors' address: F. Altomare, Università degli Studi di Bari, Dipartimento di Matematica, Campus Universitario, Via E. Orabona 4, 70125 Bari, Italy; e-mail: `altomare@dm.uniba.it` .

V. Leonessa, Università degli Studi della Basilicata, Dipartimento di Matematica e Informatica, Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy; e-mail: `vita.leonessa@unibas.it` .

Keywords. Degenerate differential operator, degenerate evolution equation, positive C_0- semigroup, approximation of semigroups, Borel measure, positive approximation process, Bernstein-Schnabl operator, modified Bernstein-Schnabl operator, asymptotic formula.

AMS Subject Classification. 47D06, 35A35, 41A36.

We shall mainly focus upon the study of a class of degenerate elliptic differential operators of the form

$$Lu = \alpha u'' + \beta u' + \gamma u$$

defined on a suitable domain $D(L)$ of the Banach lattice $C([a,b])$, the space of all real-valued continuous functions on a real interval $[a,b]$. The coefficients α, β and γ are continuous functions on $[a,b]$ satisfying suitable assumptions.

It is well-known that, if the operator $(L, D(L))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$, then the Cauchy problems associated with the evolution equation generated by L, i.e.

$$\frac{\partial u}{\partial t}(x,t) = \alpha(x)\,\frac{\partial^2 u}{\partial x^2}(x,t) + \beta(x)\,\frac{\partial u}{\partial x}(x,t) + \gamma(x)u(x)$$

($0 < x < 1$, $t \geq 0$), have a unique solution which depends continuously on the initial data and can be expressed by means of the semigroup itself.

The main target is to construct suitable sequences of positive linear operators in order to approximate the semigroup $(T(t))_{t\geq 0}$ by means of their iterates.

After that, an additional problem comes into play which consists in performing, by means of these approximating operators, a quantitative as well as a qualitative analysis of both the semigroup and, hence, of the solutions of the associated evolution equations.

Among other things it is possible to investigate the asymptotic behaviour of the semigroup and the propagation of regularity properties of the initial data to similar spatial regularity properties of the relevant solutions.

These targets appear tightly connected both to the study of sequences of positive linear operators and to different classical themes of the approximation theory.

To facilitate access to the main results, in Section 2 we collect several preliminaries. In particular we review some definitions and results concerning C_0-semigroups of bounded linear operators on Banach spaces and we discuss the main properties of Bernstein-Schnabl operators which play a fundamental role in the approximation of the semigroups and in the study of their qualitative properties.

In Section 3 we enter in the hearth of the matter and we discuss some generation results for the operator L by imposing either maximal boundary conditions or Ventcel boundary conditions.

We also determine a core for these domains by opening the door to obtain the desired approximation in terms of iterates of Bernstein-Schnabl operators in the case $\beta = \gamma = 0$ and of a suitable modification of them in the general case.

As a consequence we discuss several applications of the approximation formula by showing, for instance, that the semigroups preserve continuous increasing functions, Hölder continuous functions, continuous convex functions, continuously differentiable functions having Lipschitz continuous first derivatives and continuous functions having bounded continuous second derivatives.

In the final part of the paper we list several references for further possible deepening and developments.

1. Statement of the problem

Below we state the main problem we shall deal with in this paper. This problem was first stated in [7] in a more general setting but, for the sake of simplicity, we restate it in the framework of compact intervals.

In the sequel we shall denote by $C([0,1])$ (resp. $C^2([0,1])$) the space of all continuous (resp. twice continuously differentiable) functions on $[0,1]$.

The space $C([0,1])$, endowed with the sup-norm $\|\cdot\|_\infty$, i.e.

$$\|f\|_\infty := \sup_{0\le x\le 1} |f(x)| \qquad (f \in C([0,1]))\ ,$$

and the natural (pointwise) ordering, i.e.

$$f \le g \quad \text{if and only if} \quad f(x) \le g(x) \quad \text{for every } \ 0 \le x \le 1\ ,$$

$(f, g \in C([0,1]))$, is a Banach lattice.

Consider a degenerate second order differential operator $L : D(L) \to C([0,1])$ such that

$$Lu(x) = \alpha(x)u''(x) + \beta(x)u'(x) + \gamma(x)u(x) \tag{1.1}$$

for every $u \in D(L)$ and $x \in]0,1[$.

Here $\alpha, \beta, \gamma \in C([0,1])$, $\alpha(0) = \alpha(1) = 0$ and $\alpha(x) > 0$ for $0 < x < 1$, and $D(L)$ is a suitable linear subspace of $\{u \in C([0,1]) \ : \ u_{|]0,1[} \in C^2(]0,1[)\}$.

If the operator $(L, D(L))$ is the generator of a strongly continuous semigroup $(T(t))_{t\ge 0}$ in $C([0,1])$ then, for every $u_0 \in D(L)$, the initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) + \beta(x)\dfrac{\partial u}{\partial x}(x,t) + \gamma(x)u(x) & 0<x<1,\ t\ge 0\ , \\ u(x,0) = u_0(x) & 0 \le x \le 1\ , \\ u(\cdot,t) \in D(L) & t \ge 0\ , \end{cases} \tag{1.2}$$

has a unique solution given by

$$u(x,t) = T(t)(u_0)(x) \qquad (0 \le x \le 1, t \ge 0) \tag{1.3}$$

and the solution continuously depends on the initial datum u_0 (for more details on semigroups of operators see, e.g., [42], [55]).

Usually $D(L)$ incorporates some boundary conditions imposed to 0 and 1.

By using Feller's theory we already know several conditions under which the operator $(L, D(L))$ generates a strongly continuous (positive) semigroup on $C([0,1])$ (see, e.g., [42, Section VI.4]).

Our general aim is to investigate the possibility of constructing a suitable approximation process $(T_n)_{n\ge 1}$ of linear operators on $C([0,1])$ (i.e., $\lim_{n\to\infty} T_n(f) = f$ uniformly on $[0,1]$ for every $f \in C([0,1])$) such that, for every $t \ge 0$ and $f \in C([0,1])$,

$$T(t)f = \lim_{n\to\infty} T_n^{k(n)} f \quad \text{uniformly on} \quad [0,1] \tag{1.4}$$

for a suitable sequence $(k(n))_{n\ge 1}$ of positive integers (independent of f), where each $T_n^{k(n)}$ denotes the power of T_n of order $k(n)$.

Having formula (1.4) at our disposal and taking (1.3) into account, it is clear that we can try to derive both numerical and qualitative information about the solutions of the Cauchy problems (1.2) from the study of the operators T_n.

For instance, if each T_n is positive, then $T(t)$ is positive ($t \ge 0$) and so to a positive initial datum u_0 there corresponds a positive solution. In particular, in this case, the rich theory of positive semigroups is also available (see, e.g., [52]).

Moreover, it $u_0 \in D(L)$ and $u_0 \le T_n(u_0)$ for every $n \ge 1$, then by iteration we obtain $u_0 \le T(t)(u_0)$, i.e., $u_0(x) \le u(x,t)$ for every $x \in [0,1]$ and $t \ge 0$.

If each T_n leaves invariant special closed subspaces of $C([0,1])$ (for instance, Lipschitz continuous functions, Hölder continuous functions and so on), the same holds true for the semigroup $(T(t))_{t\geq 0}$ and hence the solutions belong to this subspace provided the initial data belong to it (regularity results).

Of course the saturation properties and the Favard class of the semigroup could be investigated by studying the corresponding properties of the sequence $(T_n)_{n\geq 1}$.

Another aspect which justifies a further possible interest in formula (1.4) lies on the fact that, when $(T(t))_{t\geq 0}$ is a Markov semigroup (i.e., $T(t)$ is positive and $T(t)\mathbf{1} = \mathbf{1}$ for each $t \geq 0$), then there exists a right-continuous normal Markov process with state space $[0,1]$ whose transition function is expressed in terms of it (see, e.g., [64, Chapter 9], for more details about these probabilistic aspects). Hence, by means of the operators T_n, we may also investigate some qualitative properties of these transition functions which are one of the main objects of interest in the theory of Markov processes (see, e.g., [14, pp. 461-466]).

Clearly the first (difficult) problem to deal with is to find a suitable sequence of operators $(T_n)_{n\geq 1}$ (which is not unique in general) which may be expected to satisfy the representation formula (1.4).

Anyhow, if one has at his disposal an approximation process $(T_n)_{n\geq 1}$ of linear operators on $C([0,1])$, in order to obtain (1.4) it is sufficient that the following properties are satisfied:

There exist a dense (in $C([0,1])$) subspace D_0 of $D(L)$, a decreasing sequence $(\rho_n)_{n\geq 1}$ of positive real numbers tending to 0 as $n \to \infty$ and real numbers $M \geq 0$ and $\omega \in \mathbb{R}$ such that

(1.a) (Stability condition) $\|T_n^k\| \leq M \exp(\omega \rho_n k)$ for every $k, n \geq 1$;

(1.b) (Asymptotic formula)

$$\lim_{n\to\infty} \frac{T_n u - u}{\rho_n} = Lu \qquad \text{for every } \ u \in D_0 \ ;$$

(1.c) (Range condition) the image $(\lambda I - L)(D_0)$ is dense in $C([0,1])$ for some $\lambda > \omega$ (here I denotes the identity operator).

This is the content of a theorem of Trotter [65] (see also [55, Chapter 3, Theorem 6.7]) (cf. Theorem 2.23 of this paper).

Moreover, according to a result of Schnabl [63], if $\|T_n\| \leq 1$ for every $n \geq 1$, then condition (1.c) can be replaced by

(1.d) (Local invariance property) there exists a family $(F_i)_{i\in I}$ of finite dimensional subspaces of D_0 which are invariant under each T_n (i.e., $T_n(F_i) \subset F_i$ for every $n \geq 1$ and $i \in I$) and whose union is dense in $C([0,1])$.

In both cases, as a sequence $(k(n))_{n\geq 1}$ in (1.4) one can choose an arbitrary sequence $(k(n))_{n\geq 1}$ satisfying $\lim_{n\to\infty} k(n)\rho_n = t$ (in particular $k(n) = [t/\rho_n]$, where $[x]$ denotes the integer part of a real number x).

Moreover, the following estimate holds true:

$$\|T(t)\| \leq M \exp(\omega t) \qquad (t \geq 0) \ .$$

In concrete applications condition (1.c) represents the main problem to verify. However this problem can be overcome by showing that D_0 is a core for $(L, D(L))$ (see Section 2.4).

Having this scheme as a guide in Sections 3 and 4 we shall illustrate some of the main results which have been obtained in this direction (see also [10], [19]).

For additional results concerning, in particular, unbounded intervals or multidimensional settings we refer, e.g., to [14, Chapter 6], [16], [17], [15] and the references therein.

2. Preliminaries

In this section we recall some notation which will be useful throughout this paper.

As usual the space of all real valued functions on $[0,1]$ will be denoted by $F([0,1])$ and $C([0,1])$ (resp. $C^2([0,1])$) will stand for the space of all continuous (resp. twice continuously differentiable) functions on $[0,1]$.

Moreover we shall denote by $C_b(]0,1[)$ the space of all continuous bounded functions defined on $]0,1[$ and by $C_0(]0,1[)$ the space of all continuous functions on $]0,1[$ which vanish at 0 and 1. Observe that $C_0(]0,1[)$ is a closed subspace of $C_b(]0,1[)$.

We recall that, given a $f \in F([0,1])$, its *support* is defined as

$$\text{Supp}(f) := \overline{\{x \in [0,1] \ : \ f(x) \neq 0\}} \ .$$

The symbol $\mathbf{1}$ will stand for the constant function 1; for every $n \geq 1$, e_n will denote the function $e_n(t) := t^n$, $t \in [0,1]$, and, for every $0 \leq x \leq 1$, ψ_x will be the function

$$\psi_x(t) := t - x \qquad (0 \leq t \leq 1) \ . \tag{2.1}$$

Moreover, $\mathbf{1}_B$ will denote the *characteristic function* of a subset B of $[0,1]$, i.e.,

$$\mathbf{1}_B(t) = \begin{cases} 1 & \text{if } t \in B, \\ 0 & \text{if } t \notin B \ . \end{cases}$$

Now we proceed to introduce some further definitions and results useful for the understanding of the next sections. For the reader convenience, we share out them into the following four subsections.

2.1. **Borel measures on** $[0,1]$**.** Let $\mathcal{B}([0,1])$ be the σ-algebra of all Borel subsets of $[0,1]$. A finite measure $\mu : \mathcal{B}([0,1]) \to \mathbb{R}_+$ is also called a *Borel measure* on $[0,1]$. In particular, if $\mu([0,1]) = 1$ we say that μ is a *probability Borel measure* on $[0,1]$. We denote by $M^+([0,1])$ (resp. $M_1^+([0,1])$) the cone of all (regular) (resp. probability) Borel measures on $[0,1]$.

Observe that, if $\mu \in M^+([0,1])$, then every $f \in C([0,1])$ is μ-integrable. Therefore we may consider the following linear positive form $\nu : C([0,1]) \to \mathbb{R}$ defined by setting, for every $f \in C([0,1])$,

$$\nu(f) := \int_0^1 f \, d\mu \ . \tag{2.2}$$

Clearly ν is a *Radon measure* on $[0,1]$, that is, ν is continuous with respect to the sup-norm, since $|\nu(f)| \leq \|f\|_\infty \, \mu([0,1])$, and it is positive (i.e., $\nu(f) \geq 0$ for every $f \in C([0,1])$, $f \geq 0$). In such a way every Borel measure leads to a positive Radon measure. Actually the converse is also true thanks to the Riesz representation theorem. However throughout this paper we shall prefer to handle Borel measures instead of Radon measures.

We shall say that a measure μ is *concentrated* on a Borel subset B of $[0,1]$ if $\mu([0,1] \setminus B) = 0$. Moreover, for every $x \in [0,1]$, we shall denote by ε_x the *point-mass* *measure concentrated at* x, i.e., for every $B \in \mathcal{B}([0,1])$,

$$\varepsilon_x(B) := \begin{cases} 1 & x \in B, \\ 0 & x \notin B\,. \end{cases}$$

We observe that, for every $x \in [0,1]$ the corresponding Radon measure of ε_x, denoted by δ_x, is the *Dirac measure at* x defined by

$$\delta_x(f) := f(x) \qquad (f \in C([0,1]))\ ;$$

hence, taking (2.2) into account, for every $f \in C([0,1])$ and $x \in [0,1]$,

$$\int_0^1 f(t)\, d\varepsilon_x(t) = f(x)\ . \tag{2.3}$$

For a given $\mu \in M^+([0,1])$ the *support* of μ is the subset $\mathrm{Supp}(\mu)$ of $[0,1]$ which is defined as the complement of the largest open subset of $[0,1]$ of measure zero with respect to μ.

Observe that a given element $x_0 \in [0,1]$ belongs to the $\mathrm{Supp}(\mu)$ if and only if $\mu(V) \neq 0$ for any open neighborhood V of x_0.

Moreover, the following properties of $\mathrm{Supp}(\mu)$ hold true.

Proposition 2.1.

(a) *Let* B *be a closed subset of* $[0,1]$. *Then the following propositions are equivalent:*
  (i) $\mathrm{Supp}(\mu) \subset B$*;*
  (ii) $\int_0^1 f\, d\mu = 0$ *for every* $f \in C([0,1])$ *with* $\mathrm{Supp}(f) \cap B = \emptyset$*;*
  (iii) μ *is concentrated on* B.

(b) *If* $f \in C([0,1])$, $f \geq 0$ *and* $\int_0^1 f\, d\mu = 0$ *then* $\mathrm{Supp}(\mu) \subset \{x \in [0,1] : f(x) = 0\}$.

(c) *If* $\mu := \sum_{i=1}^n \lambda_i \varepsilon_{x_i}$ *with* $\lambda_i > 0$ *and* $x_i \in [0,1]$ *for every* $i = 1, \ldots, n$*, then* $\mathrm{Supp}(\mu) = \{x_1, \cdots, x_n\}$. *Conversely, if* $\mu \in M^+([0,1])$ *and* $\mathrm{Supp}(\mu) = \{x_1, \cdots, x_n\}$*, then there exist* $\lambda_1, \cdots, \lambda_n \in \mathbb{R}^+$ *such that* $\mu = \sum_{i=1}^n \lambda_i \varepsilon_{x_i}$.

(For the details see, for instance, [40, Section 11]).

We end his subsection by recalling two important definitions for the sequel.

Fix two Borel measures μ and ν on $[0,1]$. Then there exists a unique Borel measure λ on $[0,1] \times [0,1]$ such that, for every $A, B \in \mathcal{B}([0,1])$, $\lambda(A \times B) = \mu(A)\nu(B)$. Such a measure is called the *tensor product measure* of μ and ν and it will be denoted by the symbol $\mu \otimes \nu$.

By Fubini theorem, for every $f \in C([0,1] \times [0,1])$,

$$\begin{aligned}\int_{[0,1]\times[0,1]} f\, d\mu \otimes \nu &= \int_{[0,1]} \left(\int_{[0,1]} f(x,y)\, d\nu(y) \right) d\mu(x) = \\ &= \int_{[0,1]} \left(\int_{[0,1]} f(x,y)\, d\mu(x) \right) d\nu(y)\ .\end{aligned}$$

Analogously one can define the tensor product of a finite family of measures $\mu_1, \ldots, \mu_n$, $n \geq 3$, on $[0,1]$.

If $\mu \in M^+([0,1])$ and $\varphi : [0,1] \to [0,1]$ is continuous, we define the *image measure of* μ *under the mapping* φ as the measure ν defined by setting

$$\nu(B) := \varphi(\mu)(B) := \mu(\varphi^{-1}(B))$$

for every $B \in \mathcal{B}([0,1])$. If $f \in C([0,1])$, then

$$\int_0^1 f\, d\varphi(\mu) = \int_0^1 f \circ \varphi\, d\mu\,.$$

For more details and deepening we refer, e.g., to [28].

2.2. **Positive linear operators and approximation processes on $C([0,1])$.** Consider a linear mapping $T : C([0,1]) \to Y$ ($Y = C([0,1])$ or $Y = \mathbb{R}$). The mapping T is said to be *positive* if $T(f) \geq 0$ for every $f \in C([0,1])$, $f \geq 0$.

For such positive mappings we have the following important properties and consequences (see, e.g. [4]).

Proposition 2.2. *Let $T : C([0,1]) \to Y$ be a positive linear mapping. Then the following statements hold true.*

(*1*) *For every $f, g \in C([0,1])$, if $f \leq g$, then $T(f) \leq T(g)$.*
(*2*) *For every $f \in C([0,1])$, $|T(f)| \leq T(|f|)$.*
(*3*) *If $Y = \mathbb{R}$ then, for every $f, g \in C([0,1])$,*

$$T(|fg|) \leq \sqrt{T(f^2)T(g^2)}\,.$$

Proposition 2.3. *Let $T : C([0,1]) \to Y$ be a positive linear mapping. Then T is continuous and*

$$\|T\| = \begin{cases} \|T(\mathbf{1})\|_\infty & \text{if } Y = C([0,1])\,, \\ T(\mathbf{1}) & \text{if } Y = \mathbb{R}\,. \end{cases}$$

We proceed now to recall the concept of approximation process on the space $C([0,1])$.

A sequence $(L_n)_{n\geq 1}$ of linear operators from $C([0,1])$ into $C([0,1])$ is said to be an (uniform) approximation process if

$$\lim_{n\to\infty} L_n(f) = f \quad \text{uniformly on} \quad [0,1]$$

for every $f \in C([0,1])$.

A simple and useful tool which allows to decide whether a given sequence of positive linear operators on $C([0,1])$ is an approximation process is due to Korovkin [47] (see also [14, Theorem 4.2.7]).

Theorem 2.4. *Given a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C([0,1])$ into itself, if*

$$\lim_{n\to\infty} L_n(h) = h \quad \textit{uniformly on} \quad [0,1] \tag{2.4}$$

for every $h \in H := \{\mathbf{1}, e_1, e_2\}$, then

$$\lim_{n\to\infty} L_n(f) = f \quad \textit{uniformly on} \quad [0,1]$$

for every $f \in C([0,1])$.

Given an approximation process $(L_n)_{n\geq 1}$ on $C([0,1])$, from a theoretical as well numerical point of view it is important to estimate the order of convergence of $\|L_n f - f\|$ toward 0 as $n \to \infty$. Often it happens that it is also possible to evaluate

$$\lim_{n\to\infty} n(L_n(f) - f) = A(f)$$

for suitable "smooth" functions $f \in C([0,1])$.

Such a formula is referred to as an "asymptotic formula". The first formula of this type, due to Voronovskaja [66], states that, for the sequence of classical Bernstein operators on $[0,1]$ (for their definition see (2.10)), if $f \in C^2([0,1])$, then

$$\lim_{n\to\infty} n[\mathcal{B}_n(f)(x) - f(x)] = \frac{x(1-x)}{2}\, f''(x)$$

uniformly with respect to $x \in [0,1]$. Such a result shows that the convergence of $\mathcal{B}_n(f)$ to f cannot be too much fast, even if the approximating functions are very smooth.

In order to obtain asymptotic formulae for general sequences of operators we shall refer us to the following generalization of the Voronovskaja's result due to Mamedov [48].

Theorem 2.5. *Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from $C([0,1])$ into itself and consider $\alpha, \beta, \gamma \in F([0,1])$. For every $x \in [0,1]$ consider the function ψ_x defined by* (2.1) *and assume that*

(*i*) $\lim_{n\to\infty} n[L_n(\mathbf{1}) - \mathbf{1}] = \gamma$ *uniformly on* $[0,1]$,

(*ii*) $\lim_{n\to\infty} nL_n(\psi_x)(x) = \beta(x)$ *uniformly with respect to* $x \in [0,1]$,

(*iii*) $\lim_{n\to\infty} nL_n(\psi_x^2)(x) = 2\alpha(x)$ *uniformly with respect to* $x \in [0,1]$,

(*iv*) *there exists an even integer* $q \geq 4$ *such that* $\lim_{n\to\infty} nL_n(\psi_x^q)(x) = 0$ *uniformly with respect to* $x \in [0,1]$.

Then, for every $f \in C^2([0,1])$,

$$\lim_{n\to\infty} n[L_n(f) - f] = \alpha f'' + \beta f' + \gamma f \quad \textit{uniformly on} \quad [0,1] \,.$$

For a proof see [9, Theorem 1] where the previous result is proved for any arbitrary (not necessarily compact) real interval.

2.3. **Bernstein-Schnabl operators on $C([0,1])$.** In this section we shall consider a noteworthy sequence $(B_n)_{n\geq 1}$ of positive linear operators acting on the space $C([0,1])$ and we shall investigate their approximation properties as well as their shape preserving properties.

By means of them we shall represent the semigroup quoted in Section 1 as in (1.4), and hence the solutions of the problems (1.2).

All the results presented below are taken from [19].

In order to define the operators B_n $(n \geq 1)$ we fix a *continuous selection* of probability Borel measures on $[0,1]$, that is a family $(\mu_x)_{0\leq x\leq 1}$ in $M_1^+([0,1])$ such that, for every $f \in C([0,1])$, the function $T(f)$, defined as

$$T(f)(x) := \int_0^1 f\, d\mu_x \qquad (0 \leq x \leq 1)\,, \tag{2.5}$$

is continuous on $[0,1]$. Moreover we assume that $(\mu_x)_{0\leq x\leq 1}$ satisfies the additional condition

$$T(e_1) = e_1, \tag{2.6}$$

i.e. $\int_0^1 e_1\, d\mu_x = x$ for every $x \in [0,1]$. Observe that, thanks to Proposition 2.3, the operator $T : C([0,1]) \to C([0,1])$ is continuous and $\|T\| = 1$, since $T(\mathbf{1}) = \mathbf{1}$.

Moreover we point out that, by virtue of the Riesz representation theorem, every positive linear operator $T : C([0,1]) \to C([0,1])$ such that $T(\mathbf{1}) = \mathbf{1}$ generates a continuous selection of probability Borel measures on $[0,1]$ satisfying (2.5).

By Jensen's inequality (see [27, Theorem 3.9]) and by the monotonicity of positive operators (see Proposition 2.2, (1)), $x^2 \le T(e_2)(x) \le T(e_1)(x) = x$ for every $0 \le x \le 1$, then

$$0 \le T(e_2)(x) - x^2 \le x - x^2 = x(1-x) \quad \text{for every} \quad 0 \le x \le 1 \,. \tag{2.7}$$

For every $n \ge 1$, the *n-th Bernstein-Schnabl operator* associated with the above family $(\mu_x)_{0\le x\le 1}$ is the linear operator $B_n : C([0,1]) \to C([0,1])$ defined by setting, for every $f \in C([0,1])$ and $x \in [0,1]$,

$$\begin{aligned} B_n(f)(x) : &= \int_0^1 f \, d\mu_{x,n} = \\ &= \int_{[0,1]^n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x^n(x_1, \cdots, x_n) = \\ &= \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x(x_1) \cdots d\mu_x(x_n) \,, \end{aligned} \tag{2.8}$$

where μ_x^n and $\mu_{x,n}$ denote, respectively, the product measure on $[0,1]^n$ of μ_x with itself n-times and the image measure of μ_x^n under the mapping $\pi_n : [0,1]^n \to [0,1]$ defined by $\pi_n(x_1, \cdots, x_n) := (x_1 + \cdots + x_n)/n$ $(0 \le x_1, \cdots, x_n \le 1)$.

Because of the continuity of the function $T(f)$, each positive linear operator B_n maps $C([0,1])$ into itself. Moreover, for any $n \ge 1$, B_n is positive and

$$B_n(\mathbf{1}) = \mathbf{1} \,, \tag{2.9}$$

then it is continuous with respect to the sup-norm and $\|B_n\| = 1$.

Bernstein-Schnabl operators were first introduced by Schnabl (see [61, 62]) in the context of sets of probability Radon measures on compact Hausdorff spaces. Subsequently Grossman in [46] proposed a general method of constructing Bernstein-Schnabl operators on an arbitrary convex compact subset of a locally convex space and he showed that they are an approximation process for continuous functions. A particular class of these operators has been also studied by the first author [5] (see also [6]) and, subsequently, by several other authors (see [14, Chapter 6] and the relevant Notes and References). Their construction essentially involves positive projections [14, Section 6.1] and they satisfy many additional properties useful for the study of evolution problems [14, Section 6.2].

Below we discuss some examples.

Example 2.6.

1. For every $x \in [0,1]$ consider the measure $\mu_x = x\varepsilon_1 + (1-x)\varepsilon_0$. Then the sequence $(\mu_x)_{0\le x\le 1}$ is a continuous selection of probability Borel measures on $[0,1]$ satisfying (2.6). Moreover, taking (2.5) and (2.3) into account, we have

$$T(f)(x) = \int_0^1 f(t) \, d\mu_x(t) = xf(1) + (1-x)f(0)$$

for every $f \in C([0,1])$ and $x \in [0,1]$. By induction it is easy to show that the Bernstein-Schnabl operators associated with $(\mu_x)_{0\le x\le 1}$ and defined by (2.8) turn into the classical Bernstein operators defined by

$$\mathcal{B}_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{2.10}$$

for every $n \ge 1$, $f \in C([0,1])$ and $x \in [0,1]$.

2. In this second example we consider a linear combination of three point-mass measures, that is, for every $x \in [0,1]$, we define $\mu_x := \alpha(x)\varepsilon_0 + \beta(x)\varepsilon_{1/2} + \gamma(x)\varepsilon_1$ where $\alpha, \beta, \gamma \in C([0,1])$, $0 \leq \alpha, \beta, \gamma \leq 1$, $\alpha + \beta + \gamma = \mathbf{1}$ and $\beta + 2\gamma = 2\mathbf{1}$. Then $(\mu_x)_{0\leq x\leq 1}$ is a continuous selection of probability Borel measures on $[0,1]$ satisfying (2.6) and the Bernstein-Schnabl operators associated with $(\mu_x)_{0\leq x\leq 1}$ can be written as

$$B_n(f) := \sum_{h=0}^{n} \sum_{k=0}^{n-h} \binom{n}{h} \binom{n-h}{k} \alpha(x)^{n-h-k} \beta(x)^h \gamma(x)^k f\left(\frac{h+2k}{2n}\right)$$

for every $n \geq 1$, $f \in C([0,1])$ and $x \in [0,1]$.

3. Let $\lambda \in C_b(]0,1[)$ a function satisfying $0 \leq \lambda \leq \mathbf{1}$ and, for every $x \in [0,1]$, let

$$\mu_x := \begin{cases} \lambda(x)[x\varepsilon_1 + (1-x)\varepsilon_0] + (1-\lambda(x))\varepsilon_x & \text{if } 0 < x < 1 , \\ \varepsilon_x & \text{if } x = 0, 1 . \end{cases}$$

Then $(\mu_x)_{0\leq x\leq 1}$ is a continuous selection satisfying (2.6). Moreover, the operators B_n turn into the so-called Lototsky-Schnabl operators defined, for every $n \geq 1$, $f \in C([0,1])$ and $x \in [0,1]$, as follows

$$L_{n,\lambda}(f)(x) = \tag{2.11}$$

$$= \begin{cases} \displaystyle\sum_{h=0}^{n} \sum_{k=0}^{n-h} \binom{n}{h} \binom{n-h}{k} x^k (1-x)^{n-h-k} \lambda(x)^{n-h} (1-\lambda(x))^h \times \\ \times f\left(\frac{k}{n} + \frac{h}{n}x\right) & \text{if } 0 < x < 1 , \\ f(x) & \text{if } x = 0, 1 , \end{cases}$$

(see [14, Subsection 6.1, p. 399]).

In order to study the convergence of the B_n's we shall apply Theorem 2.4, so we need to evaluate the operators B_n on the test functions $\mathbf{1}$, e_1 and e_2.

Fix $n \geq 1$. We have already got (2.9), hence we go on to compute $B_n(e_1)$. For every $x \in [0,1]$, taking (2.6) into account, we have

$$\begin{aligned} B_n(e_1)(x) &= \int_0^1 \cdots \int_0^1 e_1\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x(x_1) \cdots d\mu_x(x_n) = \\ &= \int_0^1 \cdots \int_0^1 \frac{x_1 + \cdots + x_n}{n} \, d\mu_x(x_1) \cdots d\mu_x(x_n) = \\ &= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 x_i \, d\mu_x(x_i) = \frac{1}{n} \sum_{i=1}^{n} \int_0^1 e_1(x_i) \, d\mu_x(x_i) = \\ &= \frac{1}{n} \sum_{i=1}^{n} T(e_1)(x) = T(e_1)(x) = e_1(x) , \end{aligned}$$

hence

$$B_n(e_1) = e_1 . \tag{2.12}$$

Finally, for every $x \in [0,1]$,

$$B_n(e_2)(x) = \int_0^1 \cdots \int_0^1 e_2\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x(x_1)\cdots d\mu_x(x_n) =$$

$$= \int_0^1 \cdots \int_0^1 \left(\frac{x_1 + \cdots + x_n}{n}\right)^2 d\mu_x(x_1)\cdots d\mu_x(x_n) =$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^n \int_0^1 x_i^2\, d\mu_x(x_i) + 2\sum_{1\le i<j\le n}\int_0^1\int_0^1 x_i x_j\, d\mu_x(x_i)d\mu_x(x_j)\right) =$$

$$= \frac{1}{n^2}\, n\, T(e_2)(x) + \frac{2}{n^2}\sum_{1\le i<j\le n}(T(e_1)(x))^2 =$$

$$= \frac{1}{n}\, T(e_2)(x) + \frac{2}{n^2}\,\frac{n(n-1)}{2}\, e_1^2(x)\ .$$

So

$$B_n(e_2) = \frac{1}{n}\, T(e_2) + \frac{n-1}{n}\, e_2\ . \tag{2.13}$$

Theorem 2.7. *For every $f \in C([0,1])$,*

$$\lim_{n\to\infty} B_n(f) = f \quad \textit{uniformly on} \quad [0,1]\ .$$

Proof. Taking (2.9), (2.12) and (2.13) into account, clearly conditions (2.4) are fulfilled and hence the claim follows from Theorem 2.4.

□

We recall here some further formulae, which will be useful in the sequel (for the details see [19, p. 357]; see also [14, Section 6.1]). For every $n \ge 1$,

$$B_n(e_3) = \frac{1}{n^3}\, T(e_3) + \frac{3(n-1)}{n^2}\, e_1\, T(e_2) + \frac{(n-1)(n-2)}{n^2}\, e_3 \tag{2.14}$$

and

$$\begin{aligned} B_n(e_4) \quad &= \frac{1}{n^3}\, T(e_4) + \frac{4(n-1)}{n^3}\, e_1\, T(e_3) + \frac{3(n-1)}{n^3}\, T(e_2)^2 + \\ &+ \frac{6(n-1)(n-2)}{n^3}\, e_2\, T(e_2) + \frac{(n-1)(n-2)(n-3)}{n^3}\, e_4\ . \end{aligned} \tag{2.15}$$

Now we shall show several estimates, both pointwise and uniform, of the convergence described in Theorem 2.7 by means of the usual first and second moduli of continuity of $f \in C([0,1])$ with respect to δ defined, resp. by

$$\omega(f,\delta) := \sup\{|f(x) - f(y)|\ :\ |x-y| \le \delta,\ x,y \in [0,1]\}$$

and

$$\omega_2(f,\delta) := \sup\left\{\left|f(x) - f\left(\frac{x+y}{2}\right) + f(y)\right|\ :\ |x-y| \le 2\delta,\ x,y\in[0,1]\right\}\ .$$

We shall also use the so-called second order Ditzian-Totik modulus $\omega_2^{\varphi}(f,\delta)$ $(\delta > 0)$ defined by means of the weight function $\varphi(x) := \sqrt{x(1-x)}$ $(0 \le x \le 1)$ as

$$\omega_2^{\varphi}(f,\delta) := \sup\left\{\left|f(x) - f\left(\frac{x+y}{2}\right) + f(y)\right| : |x-y| \le 2\delta\varphi\left(\frac{x+y}{2}\right), x,y\in[0,1]\right\}$$

(see, for instance, [41] for more details about moduli of smoothness).

Taking formulae (2.12), (2.13) and definition (2.1) into account, for every $n \geq 1$ and $x \in [0,1]$, the moments of the first and second order of the operator B_n are

$$B_n(\psi_x)(x) = 0\ , \tag{2.16}$$

$$B_n(\psi_x^2)(x) = \frac{1}{n}(T(e_2)(x) - x^2)\ . \tag{2.17}$$

Moreover, because of (2.7) we get

$$B_n(\psi_x^2)(x) \leq \frac{M}{n} \leq \frac{1}{4n}\ ,$$

where

$$M := \max\{T(e_2)(x) - x^2\ :\ 0 \leq x \leq 1\}\ .$$

Theorem 2.8. *Fix $n \geq 1$, $f \in C([0,1])$ and $0 \leq x \leq 1$ and consider M as before. Then*

$$|B_n(f)(x) - f(x)| \leq (1 + T(e_2)(x) - x^2)\,\omega\left(f, \frac{1}{\sqrt{n}}\right) \tag{2.18}$$

and hence,

$$\|B_n(f) - f\|_\infty \leq (1+M)\,\omega\left(f, \frac{1}{\sqrt{n}}\right)\ . \tag{2.19}$$

Moreover,

$$|B_n(f)(x) - f(x)| \leq \left(1 + \frac{T(e_2)(x) - x^2}{2}\right)\,\omega_2\left(f, \frac{1}{\sqrt{n}}\right) \tag{2.20}$$

and thus hence

$$\|B_n(f) - f\|_\infty \leq \left(1 + \frac{M}{2}\right)\,\omega_2\left(f, \frac{1}{\sqrt{n}}\right)\ . \tag{2.21}$$

Finally,

$$|B_n(f)(x) - f(x)| \leq \frac{3}{2}\left(1 + \frac{T(e_2)(x) - x^2}{x(1-x)}\right)\,\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right)\ , \tag{2.22}$$

so that

$$\|B_n(f) - f\|_\infty \leq 3\,\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right)\ . \tag{2.23}$$

Proof. From [14, Proposition 5.1.5] and (2.13) it follows that, given $\delta > 0$,

$$|B_n(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta^2}(B_n(e_2)(x) - e_2(x))\right)\omega(f,\delta) \leq$$

$$\leq \left(1 + \frac{1}{\delta^2}\ \frac{T(e_2)(x) - x^2}{n}\right)\omega(f,\delta)\ .$$

Setting $\delta = 1/\sqrt{n}$ we get the pointwise estimate (2.18) and hence (2.19).

By [54] we get

$$|B_n(f)(x) - f(x)| \leq |B_n(\mathbf{1})(x) - 1||f(x)| + \frac{1}{\delta}\,|B_n(\psi_x)(x)|\omega(f,\delta)+$$

$$+\left(B_n(\mathbf{1})(x) + \frac{1}{2\delta^2}\,B_n(\psi_x^2)(x)\right)\omega_2(f,\delta) \leq$$

$$\leq \left(1 + \frac{1}{2\delta^2}\,\frac{T(e_2)(x) - x^2}{x}\right)\omega_2(f,\delta)\ ,$$

the last inequality being true thanks to (2.9), (2.16) and (2.17). Setting $\delta = 1/\sqrt{n}$ we get (2.20) and hence (2.21). Finally, by using a general estimate for positive linear operators reproducing linear functions contained in [44, Theorem 15], we obtain

$$|B_n(f)(x) - f(x)| \leq \frac{3}{2}\left(1 + \frac{1}{\delta^2}\,\frac{B_n(\psi_x^2)(x)}{x(1-x)}\right)\omega_2^{\varphi}(f,\delta)\ ;$$

on account of (2.17), setting $\delta = 1/\sqrt{n}$, we get (2.22) and hence (2.23).

□

Now we proceed to illustrate some shape preserving properties of Bernstein-Schnabl operators.

In particular we shall see that, under suitable hypotheses, the B_n's preserve the class of increasing continuous functions as well as the one of Hölder continuous functions. We shall also study their behavior on convex functions.

All these shape preserving properties will reflect similar "spatial regularity" properties of the solutions of problems (1.2).

Suppose $n \geq 2$ and consider $f \in C([0,1])$ and $x \in [0,1]$. We consider the following auxiliary functions introduced in [14, Theorem 6.1.21].

For every $x_1, \cdots, x_{n-1} \in [0,1]$ let $f^x_{x_1,\cdots,x_{n-1}} : [0,1] \to \mathbb{R}$ be the function defined by

$$f^x_{x_1,\cdots,x_{n-1}}(t) := f\left(\frac{x_1 + \cdots + x_{n-1} + t}{n}\right) \qquad (0 \leq t \leq 1)\ . \tag{2.24}$$

Moreover, for every $k = 2, \cdots, n-1$, consider the functions on $[0,1]$ defined by the following recursive formula, which involves the operator T defined in (2.5),

$$f^x_{x_1,\cdots,x_{n-k}}(t) := T(f^x_{x_1,\cdots,x_{n-k},t})(x) \qquad (0 \leq t \leq 1)\ . \tag{2.25}$$

Finally, set

$$f^x(t) := T(f^x_t)(x) \qquad (0 \leq t \leq 1)\ . \tag{2.26}$$

Theorem 2.9. *Assume that the operator T, defined by* (2.5)*, maps increasing functions into increasing functions. Then, for every $n \geq 1$, B_n maps increasing functions into increasing functions too.*

Proof. If $n = 1$ the statement is true since $B_1 = T$. Assuming $n \geq 2$, let f be an increasing continuous function on $[0,1]$, $x \in [0,1]$ and consider the auxiliary functions defined in (2.24)-(2.26). By finite induction it is easy to prove that $f^x_{x_1,\cdots,x_h}$ is increasing for any $h = 1, \cdots, n-1$. Moreover, observe that f^x is increasing as well. Now let $y \in [0,1]$, $x < y$ and note that

$$T(f^x_{x_1,\cdots,x_h})(x) \leq T(f^x_{x_1,\cdots,x_h})(y) \quad , \quad \text{for every} \quad h = 1, \cdots, n-1\ , \tag{2.27}$$

$$T(f^x)(x) \le T(f^x)(y) \ . \tag{2.28}$$

Since $f^x_{x_1,\cdots,x_{n-1}} = f^y_{x_1,\cdots,x_{n-1}}$, by (2.5) and (2.8) we get

$$\begin{aligned} B_n(f)(x) = & \int_{[0,1]^{n-1}} \int_0^1 f^x_{x_1,\cdots,x_{n-1}}(x_n)\, d\mu_x(x_n)\, d\mu_x^{n-1}(x_1,\cdots,x_{n-1}) = \\ = & \int_{[0,1]^{n-1}} T(f^x_{x_1,\cdots,x_{n-1}})(x)\, d\mu_x^{n-1}(x_1,\cdots,x_{n-1}) = \\ = & \int_{[0,1]^{n-1}} T(f^y_{x_1,\cdots,x_{n-1}})(x)\, d\mu_x^{n-1}(x_1,\cdots,x_{n-1}) \le \\ \le & \int_{[0,1]^{n-1}} T(f^y_{x_1,\cdots,x_{n-1}})(y)\, d\mu_x^{n-1}(x_1,\cdots,x_{n-1}) \ , \end{aligned}$$

the last inequality being true thanks to (2.27).

Now, since $T(f^y_{x_1,\cdots,x_{n-1}})(y) = f^y_{x_1,\cdots,x_{n-2}}(x_{n-1})$, we obtain

$$\begin{aligned} B_n(f)(x) \le & \int_{[0,1]^{n-2}} \int_0^1 f^y_{x_1,\cdots,x_{n-2}}(x_{n-1})\, d\mu_x(x_{n-1})\, d\mu_x^{n-2}(x_1,\cdots,x_{n-2}) = \\ = & \int_{[0,1]^{n-2}} T(f^y_{x_1,\cdots,x_{n-2}})(x)\, d\mu_x^{n-2}(x_1,\cdots,x_{n-2}) \le \\ \le & \int_{[0,1]^{n-2}} T(f^y_{x_1,\cdots,x_{n-2}})(y)\, d\mu_x^{n-2}(x_1,\cdots,x_{n-2}) \end{aligned}$$

and so on. Finally

$$\begin{aligned} B_n(f)(x) \le & \int_0^1 T(f^y_{x_1})(x)\, d\mu_x(x_1) \le \int_0^1 T(f^y_{x_1})(y)\, d\mu_x(x_1) = \\ = & \int_0^1 f^y(x_1)\, d\mu_x(x_1) = T(f^y)(x) \le T(f^y)(y) \ , \end{aligned}$$

because of (2.28). On the other hand $B_n(f)(y) = T(f^y)(y)$ because

$$\begin{aligned} B_n(f)(y) = & \int_{[0,1]^{n-1}} \int_0^1 f^y_{x_1,\cdots,x_{n-1}}(x_n)\, d\mu_y(x_n)\, d\mu_y^{n-1}(x_1,\cdots,x_{n-1}) = \\ = & \int_{[0,1]^{n-1}} T(f^y_{x_1,\cdots,x_{n-1}})(y)\, d\mu_y^{n-1}(x_1,\cdots,x_{n-1}) = \\ = & \int_{[0,1]^{n-1}} f^y_{x_1,\cdots,x_{n-2}}(x_{n-1})\, d\mu_y(x_{n-1})\, d\mu_y^{n-1}(x_1,\cdots,x_{n-1}) = \\ = & \int_{[0,1]^{n-2}} T(f^y_{x_1,\cdots,x_{n-2}})(y)\, d\mu_y^{n-2}(x_1,\cdots,x_{n-2}) = \\ = & \cdots = \int_0^1 T(f^y_{x_1})\, d\mu_y(x_1) = T(f^y)(y) \end{aligned}$$

and hence the proof is complete.

□

Remark 2.10. In the particular case of Examples 2.6, 2, with λ constant, the above result has been proved in [1, Theorem 1].

As usual for given $M > 0$ and $0 \le \alpha \le 1$ we shall denote by $Lip\,(\alpha, M)$ the set of all $f \in C([0,1])$ such that

$$|f(x) - f(y)| \le M|x-y|^\alpha \qquad (0 \le x, y \le 1) \ .$$

We have the following result.

Theorem 2.11. *Assume that there exists $c \geq 1$ such that*

$$T(f) \in Lip\,(1,c) \quad \textit{for every} \quad f \in Lip(1,1)\ , \tag{2.29}$$

where T is given in (2.5)*. Then, for every $n \geq 1$, $B_n(f) \in Lip\,(1, cM)$ provided that $f \in Lip\,(1,M)$.*

Proof. If $n = 1$ the result is obvious because $B_1 = T$ and, by (2.29), $T(f) \in Lip\,(1, cM)$ for every $f \in Lip\,(1,M)$. Suppose $n \geq 2$ and consider $f \in Lip\,(1,M)$ and $x \in [0,1]$. Consider again the auxiliary functions (2.24)-(2.26); then by finite induction we obtain that, for every $h = 1, \cdots, n-1$, $f^x_{x_1,\cdots,x_h} \in Lip\,(1, M/n)$; moreover, $f^x \in Lip\,(1, M/n)$ too.

Now let $y \in [0,1]$ and observe that, for every $h = 1, \cdots, n-1$,

$$T(f^y_{x_1,\cdots,x_h})(x) \leq T(f^y_{x_1,\cdots,x_h})(y) + \frac{cM}{n}\ |x-y|\ ,$$

$$T(f^y)(x) \leq T(f^y)(y) + \frac{cM}{n}\ |x-y|\ .$$

Arguing as in the proof of Theorem 2.9 we obtain the claim. □

Since $\|B_n\| = 1$, according to the above theorem together with Corollary 6.1.20 in [14], we immediately have the further result.

Corollary 2.12. *If $T(Lip(1,1)) \subset Lip(1,c)$ for some $c \geq 1$ then, for every $n \geq 1$, $f \in C([0,1])$, $\delta > 0$, $M > 0$ and $0 < \alpha \leq 1$,*

$$\omega(B_n(f),\delta) \leq (1+c)\omega(f,\delta) \quad \textit{and} \quad B_n(Lip(\alpha,M)) \subset Lip(\alpha, c^\alpha M)\ .$$

In particular, if $T(Lip(1,1)) \subset Lip(1,1)$, then

$$\omega(B_n(f),\delta) \leq 2\omega(f,\delta) \quad \textit{and} \quad B_n(Lip(\alpha,M)) \subset Lip(\alpha,M)\ .$$

As regards the behaviour of the B_n's on convex functions it is already known from general results for convex compact sets that, if $f \in C([0,1])$ is convex, then

$$f \leq B_{n+1}(f) \leq B_n(f) \leq T(f)\ . \tag{2.30}$$

Moreover the following statements are equivalent (see [14, 56, 57]):

(i) $B_{n+1}(f) = B_n(f)$ for any $n \geq 1$;
(ii) $B_n(f) = f$ for every $n \geq 1$;
(iii) $T(f) = f$.

Now we investigate under which conditions our operators preserve the convexity. This is certainly true for the classical Bernstein operators and the Lototsky-Schnabl operators with λ constant (see [1, Theorem 1]; see also [14, Theorem 6.1.21]) .

For the general case we shall require the following additional hypotheses about T:

(c_1) The operator T, given in (2.5), maps continuous convex functions into (continuous) convex functions;
(c_2) For every convex function $f \in C([0,1])$ and for every $x, y \in [0,1]$,

$$2\int_{[0,1]^2} \varphi_f\, d(\mu_x \otimes \mu_y) \leq \int_{[0,1]^2} \varphi_f\, d(\mu_x \otimes \mu_x + \mu_y \otimes \mu_y)\ , \tag{2.31}$$

where $\varphi_f(s,t) := f\,((s+t)/2)$, $0 \leq s,t \leq 1$.

Given $n \geq 1$ and $f \in C([0,1])$, define

$$F_n(f; x_1, \cdots, x_n) := \int_0^1 \cdots \int_0^1 f\left(\frac{t_1 + \cdots + t_n}{n}\right) d\mu_{x_1}(t_1) \cdots d\mu_{x_n}(t_n) ,$$

where $x_i \in [0,1]$ for any $i = 1, \cdots, n$. In particular

$$F_n(f; x, \cdots, x) = B_n(f)(x) \qquad (0 \leq x \leq 1) .$$

First of all we observe that $F_n(f; \cdots)$ is invariant with respect to any permutation of the indices $1, \cdots, n$ and that it is convex with respect to each x_i (for $i = 1, \cdots, n$). Moreover, from (2.31) it follows that

$$\begin{aligned} &2F_n(f; x_1, \cdots, x_i, x_{i+1}, \cdots, x_n) \leq \\ &\leq F_n(f; x_1, \cdots, x_i, x_i, \cdots, x_n) + F_n(f; x_1, \cdots, x_{i+1}, x_{i+1}, \cdots, x_n) . \end{aligned} \tag{2.32}$$

Consider the following quantity defined by setting, for every $p, q \geq 0$, $p + q = n$ and for every $x, y \in [0,1]$,

$$S_{n,p,q}(f; x, y) := F_n(f; \underbrace{x, \cdots, x}_{p}, \underbrace{y, \cdots, y}_{q}) + F_n(f; \underbrace{x, \cdots, x}_{q}, \underbrace{y, \cdots, y}_{p}) . \tag{2.33}$$

We get that

$$S_{n,p,q}(f; x, y) = S_{n,q,p}(f; x, y) \tag{2.34}$$

and that the following lemma holds true.

Lemma 2.13. *Under hypotheses* (c_1) *and* (c_2)*, for every convex function* $f \in C([0,1])$*,* $k \geq 1$ *and* $x, y \in [0,1]$*, we obtain*

$$S_{k,k-1,1}(f; x, y) \leq S_{k,k,0}(f; x, y) .$$

Proof. The claim can be proved by induction on $k \geq 1$ taking (2.32)-(2.34) into account (for the details see [19, Lemma 2.5]).

□

Theorem 2.14. *Assume that conditions* (c_1) *and* (c_2) *are satisfied. Then, for every* $n \geq 1$ *and for every convex function* $f \in C([0,1])$*,* $B_n(f)$ *is convex on* $[0,1]$*.*

Proof. Fix a convex function $f \in C([0,1])$. Since $B_1 = T$ the result is true if $n = 1$. Suppose that $n \geq 1$; because of the continuity of $B_n(f)$, in order to prove the claim it is sufficient to show that, for any $x, y \in [0,1]$,

$$B_n(f)\left(\frac{x+y}{2}\right) \leq \frac{1}{2} B_n(f)(x) + \frac{1}{2} B_n(f)(y) . \tag{2.35}$$

Consider the function define by (2.24). Observe that condition (c_1) implies that the function $T(f^x_{x_1,\cdots,x_{n-1}})$ is convex on $[0,1]$, since $f^x_{x_1,\cdots,x_{n-1}}$ is convex. Moreover,

$$T(f^x_{x_1,\cdots,x_{n-1}})\left(\frac{x+y}{2}\right) \leq \frac{1}{2} T(f^x_{x_1,\cdots,x_{n-1}})(x) + \frac{1}{2} T(f^y_{x_1,\cdots,x_{n-1}})(y) , \tag{2.36}$$

for every $x, y \in [0,1]$, since $f^x_{x_1,\cdots,x_{n-1}} = f^y_{x_1,\cdots,x_{n-1}}$. By reasoning by induction on n, suppose that (2.35) is true for $n - 1$ and consider the function $\delta_n(f, x_n)$ defined by setting, for every $t \in [0,1]$,

$$\delta_n(f, x_n) := f\left(\frac{n-1}{n} t + \frac{1}{n} x_n\right) . \tag{2.37}$$

Then, on the light of (2.8),

$$B_n(f)\left(\frac{x+y}{2}\right) = \int_{[0,1]^n} \delta_n(f,x_n)\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right) d\mu^n_{(x+y)/2}(x_1,\cdots,x_n) = $$
$$= \int_0^1 B_{n-1}(\delta_n(f,x_n))\left(\frac{x+y}{2}\right) d\mu_{(x+y)/2}(x_n)\ .$$

Since (2.35) holds for $n-1$, we get

$$B_n(f)\left(\frac{x+y}{2}\right) \le \frac{1}{2}\int_0^1 B_{n-1}(\delta_n(f,x_n))(x) d\mu_{(x+y)/2}(x_n) + $$
$$+ \frac{1}{2}\int_0^1 B_{n-1}(\delta_n(f,x_n))(y) d\mu_{(x+y)/2}(x_n)$$

and, by definition of B_{n-1} and (2.37), we obtain

$$B_n(f)\left(\frac{x+y}{2}\right) = \frac{1}{2}\int_{[0,1]^{n-1}} T\left(f^{(x+y)/2}_{x_1,\cdots,x_{n-1}}\right)\left(\frac{x+y}{2}\right) d\mu^{n-1}_x(x_1,\cdots,x_n) + $$
$$+ \frac{1}{2}\int_{[0,1]^{n-1}} T\left(f^{(x+y)/2}_{x_1,\cdots,x_{n-1}}\right)\left(\frac{x+y}{2}\right) d\mu^{n-1}_y(x_1,\cdots,x_n)\ .$$

Inequality (2.36) then implies

$$B_n(f)\left(\frac{x+y}{2}\right) \le \frac{1}{4}\, B_n(f)(x) + \frac{1}{2}\, S_{n,n-1,1}(f;x,y) + \frac{1}{4}\, B_n(f)(y)$$

and, by Lemma 2.13,

$$S_{n,n-1,1}(f;x,y) \le S_{n,n,0}(f;x,y)\ .$$

Accordingly we get the result since

$$\frac{1}{2}\, S_{n,n-1,1}(f;x,y) \le \frac{1}{4}\, B_n(f)(x) + \frac{1}{4}\, B_n(f)(y)\ .$$

□

Example 2.15. Examples of measures satisfying the hypotheses (c_1) and (c_2) of the previous theorem are illustrated below:

1. For every $x \in [0,1]$ set $\mu_x := (1-x)\varepsilon_0 + x\varepsilon_1$.
2. For a given $\lambda \in [0,1]$, set $\mu_x := (1-\lambda)\varepsilon_x + \lambda(1-x)\varepsilon_0 + \lambda x\varepsilon_1$ $(0 \le x \le 1)$.
3. Consider a concave function $\beta \in \mathcal{C}([0,1])$ such that $0 \le \beta(x) \le \min\{x, 1-x\}$ and set $\mu_x := (1-x-\beta(x))\varepsilon_0 + 2\beta(x)\varepsilon_{1/2} + (x-\beta(x))\varepsilon_1$.

2.4. **C_0-semigroups on Banach spaces.** In this section we shall recall some of the main definitions and results concerning the theory of (strongly continuous) semigroups of operators on Banach spaces. For the sake of simplicity we have compiled only some basic facts and we refer the reader, e.g., to [42], [55] for more details.

Actually this theory was developed mainly to study *abstract Cauchy problems* defined as follows. Fix a Banach space X on $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), a linear subspace $D(A) \subset X$, a linear operator $A : D(A) \to X$ and consider the following abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\, u(t) = Au(t) & t \ge 0\ , \\ u(0) = u_0 & u_0 \in D(A)\ . \end{cases} \tag{2.38}$$

A *solution* of (2.38) is a continuously differentiable function $u : [0, +\infty[\to X$ such that $u(0) = u_0$, $u(t) \in D(A)$ and $(d/dt)u(t) = Au(t)$ for every $t \geq 0$.

Of course, when the space X is a concrete function space, the problem (2.38) may come from a *concrete Cauchy problem* associated with a parabolic partial differential equation.

Several examples of concrete problems associated with abstract problems of type (2.38) can be found, e.g., in [42, Chapter 6, Sections V and VI].

In order to show the close connection between the solvability of (2.38) and the semigroup theory we have to introduce some definitions.

A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is said to be a *semigroup of operators* or, briefly, a *semigroup*, if

(s_1) $T(0) = I_X$, where I_X is the identity operator on X (i.e., $I_X(f) = f$ for any $f \in X$);

(s_2) $T(s + t) = T(s)T(t)$ for every $s, t \geq 0$.

A semigroup $(T(t))_{t\geq 0}$ on X is named a *strongly continuous semigroup* or C_0-*semigroup* if, for every $t_0 \geq 0$ and $f \in X$,

(s_3) $\lim_{t\to t_0} T(t)f = T(t_0)f$.

Strongly continuous semigroups may be characterized in the following way.

Theorem 2.16. *Let $(T(t))_{t\geq 0}$ be a semigroup on X. Then the following propositions are equivalent.*

(*a*) *$(T(t))_{t\geq 0}$ is strongly continuous.*

(*b*) *$\lim_{t\to 0} T(t)f = f$ for every $f \in X$.*

(*c*) *There exist a dense subspace D of X, a constant $M \geq 1$ and $t_0 \geq 0$ such that*

  (*i*) *$\lim_{t\to t_0} T(t)u = u$ for every $f \in D$,*

  (*ii*) *$\sup_{0\leq t\leq t_0} \|T(t)\| \leq M$.*

Moreover, if one of the above statements is true, then

(*1*) *For every $t_0 \geq 0$, $\sup_{0\leq t\leq t_0} \|T(t)\| \leq +\infty$;*

(*2*) *There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$.*

Examples of C_0-semigroups can be found, e.g., in [42, Chapters I and II].

A *contraction C_0-semigroup* $(T(t))_{t\geq 0}$ is a C_0-semigroup of linear contractions (i.e., $\|T(t)\| \leq 1$ for any $t \geq 0$).

If X is a space of real valued functions (or, more in general, if X is a Banach lattice), then a C_0-semigroup $(T(t))_{t\geq 0}$ is called a *positive C_0-semigroup* if, for every $t \geq 0$, the operator $T(t)$ is positive.

A semigroup $(T(t))_{t\geq 0}$ is said to be a *Feller semigroup* on X if it is a positive contraction C_0-semigroup. In particular, if $X = C(K)$, K compact, and if $T(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$, then it will be called *Markov semigroups* on X.

We recall that, given an arbitrary linear operator $A : D(A) \to X$ acting on a linear subspace $D(A)$ of $(X, \|\cdot\|)$, a linear subspace D of $D(A)$ is called a *core* for $(A, D(A))$ if it is dense in $D(A)$ for the graph norm $\|\cdot\|_A$ defined by

$$\|u\|_A := \|u\| + \|Au\| \qquad (u \in D(A)) ,$$

i.e., for every $u \in D(A)$ there exists a sequence $(u_n)_{n\geq 1}$ in D such that $u_n \to u$ and $Au_n \to Au$ in X.

In general the space $D(A)$, endowed with the graph-norm, is not a Banach space, but if it is so then the operator $(A, D(A))$ is said to be *closed*. This also means that, for every sequence $(f_n)_{n\geq 1}$ of elements in $D(A)$ such that $f_n \to f \in X$ and $Af_n \to g \in X$, then $f \in D(A)$ and $Af = g$.

For instance, given a real interval $[a,b]$, the operator $Au := u''$ defined on $C^2([a,b])$ is closed.

We point out that every bounded operator on X is closed and that, if $(A, D(A))$ is closed and B is bounded on X, then $(A+B, D(A))$ is closed too, the sum operator $A+B$ being naturally defined as $(A+B)f := Af + Bf$ for any $f \in D(A)$.

Given a linear operator $A : D(A) \to X$ we say that a linear operator $B : D(B) \subset X \to X$ is an *extension of* A if $D(A) \subset D(B)$ and $Af = Bf$ for every $f \in D(A)$, in symbols $B_{|D(A)} = A$.

An operator A will be said *closable* if it admits a closed extension. It is easy to verify that an operator $(A, D(A))$ is closable if and only if, for every sequence $(u_n)_{n\geq 1}$ in $D(A)$ such that $u_n \to 0$ in X and $Au_n \to v \in X$, then $v = 0$.

If $(A, D(A))$ is closable, the smallest closed extension $\overline{A} : D(\overline{A}) \to X$ of A is called the *closure of* A and it is defined as follows

$$D(\overline{A}) := \{f \in X \ : \ \text{there exists} \ \ (f_n)_{n\geq 1} \ \text{ in } \ D(A) \ \text{ such that } \ f_n \to f \\ \text{and} \ \ (Af_n)_{n\geq 1} \ \text{ is convergent}\}$$

and

$$\overline{A}f := \lim_{n\to\infty} Af_n \quad \text{for every} \quad f \in D(\overline{A}) \ ,$$

$(f_n)_{n\geq 1}$ being an arbitrary sequence in $D(A)$ such that $f_n \to f$ and $(Af_n)_{n\geq 1}$ is convergent.

If $(A, D(A))$ is closed, then a linear subspace D of $D(A)$ is a core for $(A, D(A))$ if and only if the restriction $A_{|D}$ of A to D is closable and its closure $\overline{A_{|D}}$ coincides with A.

Cores are strictly connected with the range condition (1.c) discussed in Section 1. By using the closed graph theorem it is easy to show that if $(A, D(A))$ is closed and if, in addition, the resolvent set of A

$$\rho(A) := \{\lambda \in \mathbb{K} \ : \ \text{the resolvent operator} \ \ \lambda I_{D(A)} - A : D(A) \to X \ \text{ is invertible}\}$$

is non empty, then D is a core for $(A, D(A))$ if and only if $(\lambda I_{D(A)} - A)D$ is dense in E for one/all $\lambda \in \rho(A)$ ($I_{D(A)}$ stands for the identity operator on $D(A)$).

Given a C_0-semigroup $(T(t))_{t\geq 0}$ we may consider the following linear subspace of X

$$D(A) := \left\{ u \in X \ : \ \text{there exsists} \ \lim_{t\to 0} \frac{T(t)u - u}{t} \in X \right\}$$

and the linear operator $A : D(A) \to X$ defined by setting, for every $u \in D(A)$,

$$Au := \lim_{t\to 0} \frac{T(t)u - u}{t} \ .$$

The operator $A : D(A) \to X$ is called the *generator* of the semigroup $(T(t))_{t\geq 0}$ and it verifies interesting properties.

Proposition 2.17. *Let $A : D(A) \to X$ be the generator of a C_0- semigroup $(T(t))_{t\geq 0}$ on X. Then the following properties hold true.*

(*1*) *$D(A)$ is dense in X, $(A, D(A))$ is closed and the semigroup is uniquely determined by $(A, D(A))$.*

(*2*) *$T(t)(D(A)) \subset D(A)$ for any $t \geq 0$.*

(3) *Given $f \in X$, the mapping $T(\cdot)f$ on $[0,+\infty[$ is differentiable if and only if $f \in D(A)$. In this case $(T(t)f)' = T(t)Af = AT(t)f$ for every $t \geq 0$.*

(4) *For every $t \geq 0$ and $f \in X$,*

$$\int_0^t T(s)f\,ds \in D(A) \quad \text{and} \quad A\left(\int_0^t T(s)f\,ds\right) = T(t)f - f\ .$$

(5) *For each $t \geq 0$ and $f \in D(A)$,*

$$\int_0^t T(s)Af\,ds = T(t)f - f\ .$$

(6) *If $X = C(K)$, K compact, then $T(t)\mathbf{1} = \mathbf{1}$ for any $t \geq 0$ if and only if $\mathbf{1} \in D(A)$ and $A\mathbf{1} = 0$.*

For the details see [42, Chapter II, Section 1].

The connection between semigroups and abstract Cauchy problems of type (2.38) is indicated below.

Theorem 2.18. *Let $A : D(A) \to X$ be a linear operator. The following statements are equivalent.*

(a) *The operator $(A, D(A))$ is closable, $\rho(A)$ is not empty and the relevant abstract Cauchy problem* (2.38) *admits a unique solution.*

(b) *The operator $(A, D(A))$ is the generator of a C_0-semigroup on X.*

Moreover, if (a) *or* (b) *is true and if $(T(t))_{t\geq 0}$ is the semigroup generated by $(A, D(A))$ then, for every $u_0 \in D(A)$, the unique solution of* (2.38) *is given by*

$$u(t) := T(t)u_0 \quad \text{for every} \quad t \geq 0\ .$$

In particular, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|u(t)\| \leq Me^{\omega t}\|u_0\| \quad \text{for every} \quad t \geq 0\ .$$

For a proof see [42, Chapter II, Theorem 6.7].

Therefore, in order to solve problem (2.38), one may investigate the generation properties of the operator $(A, D(A))$.

Several generation results can be found in [42, Chapters II and III]. Here we limit ourselves to collect some of them which will be useful in the next sections.

Theorem 2.19 ([43])**.** *Let K be a compact space and let $(A, D(A))$ be a linear operator on $C(K)$. Then the following statements are equivalent:*

(a) *$(A, D(A))$ generates a Markov semigroup on $C(K)$;*

(b) (i) *$D(A)$ is dense in $C(K)$;*

(ii) *$(A, D(A))$ verifies the maximum principle, i.e., if $u \in D(A)$ and $x_0 \in K$ verify $u(x_0) = \sup_{x\in K} u(x) > 0$, then $Au(x_0) \leq 0$;*

(iii) *There exists $\lambda > 0$ such that $(\lambda I - A)(D(A)) = C(K)$;*

(iv) *$\mathbf{1} \in D(A)$ and $A(\mathbf{1}) = 0$.*

Corollary 2.20 ([22], Corollary 2.2)**.** *Let $(A, D(A))$ be the generator of a Feller semigroup on $C(K)$, K compact. Given $\gamma \in C(K)$, then the operator $(A+\gamma I, D(A))$ is the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on $C(K)$ satisfying*

$$\|T(t)\| \leq e^{\gamma_\infty t} \qquad (t \geq 0)\ ,$$

where $\gamma_\infty := \sup_{x\in K} \gamma(x)$.

Theorem 2.21. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ and let B a bounded linear operator on X. Then the operator $(A+B, D(A))$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X such that $\|S(t)\| \leq M\exp((\omega + M\|B\|)t)$ for every $t \geq 0$.*

For a proof see [42, Chapter III] where the reader can find other perturbation results like the previous one (see also [12]).

From Corollary 2.7 of [12] we obtain the next useful result.

Corollary 2.22. *Let $(A, D(A))$ be the generator of a positive C_0-semigroup on $C_0(]0,1[)$ satisfying $\|T(t)\| \leq e^{\omega t}$ for some $\omega \in \mathbb{R}$ and for every $t \geq 0$. Let $m \in C_b(]0,1[)$ and assume that $m(x) > 0$ for every $x \in]0,1[$. Then $(mA, D(A))$ is closable and its closure generates a positive C_0-semigroup $(S(t))_{t\geq 0}$ on $C([0,1])$ satisfying $\|S(t)\| \leq e^{\omega_1 t}$ for every $t \geq 0$, where $\omega_1 := \omega^+ \max\{\|m\|_\infty, 1\}$.*

Now we state a generation result due to Trotter [65] which involves C_0-semigroups and sequences of bounded linear operators.

To this end we introduce the following notation. Given a linear operator L on a Banach space X and $m \geq 1$, the symbol L^m denotes the *m-th iterate of L* or the *power of L of order m*, i.e.,

$$L^m = \begin{cases} L & \text{if } m = 1 \,, \\ L^{m-1} \circ L & \text{if } m \geq 2 \,. \end{cases}$$

Theorem 2.23 (Trotter's theorem)**.** *Let $(L_n)_{n\geq 1}$ be a sequence of bounded linear operators on X and let $(\rho_n)_{n\geq 1}$ be a decreasing null sequence of positive real numbers. Assume that there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|L_n^k\| \leq Me^{\omega\rho_n k}$ for every $k, n \geq 1$. Set*

$$D(A) := \left\{ u \in X \ : \ \text{there exists } \lim_{n\to\infty} \frac{L_n u - u}{\rho_n} \in X \right\}$$

and

$$Au := \lim_{n\to\infty} \frac{L_n u - u}{\rho_n} \quad \text{for any} \quad u \in D(A) \,.$$

If $D(A)$ is dense in X and if there exists $\lambda > \omega$ such that the range $(\lambda I_{D(A)} - A)(D(A))$ is dense in X, then $(A, D(A))$ is closable and its closure $(\overline{A}, D(\overline{A}))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ such that

(*1*) $\|T(t)\| \leq Me^{\omega t}$ *for every* $t \geq 0$

and

(*2*) *for every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $k(n)\rho_n \to t$ as $n \to \infty$, then*

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)} f \qquad (f \in X) \,. \tag{2.39}$$

For the details see [65, Theorem 5.2](see also [55, Chapter 3, Theorem 6.7]).

We point out that formula (2.39) has a certain theoretical interest since it allows to investigate the qualitative properties of the semigroup, and hence of the solution to the abstract Cauchy problem associated to its generator, by similar ones satisfied by the L_n's.

There are further results which go in the same direction of Trotter's theorem ([63]; see also [53]).

The next result concerns a kind of converse problem, that is, given a generator of a suitable semigroup $(T(t))_{t\geq 0}$, how recognizing when a sequence $(L_n)_{n\geq 1}$ of operators leads to a representation formula of type (2.39)?

An answer to such a problem is due by Theorem 2.24 which, among other things, also shows the usefulness of cores for the approximation of semigroups. It will be our main tool in the sequel.

Although it is a simple consequence of Trotter's theorem, we present here a proof for the reader convenience.

Theorem 2.24. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and suppose that there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$. Moreover, let D be a core for $(A, D(A))$ and $(L_n)_{n\geq 1}$ a sequence of bounded linear operators on X such that*

(i) $||L_n^k|| \leq Me^{\omega\rho_n k}$ *for every* $n, k \geq 1$

and

(ii) $\lim_{n\to\infty}(L_n u - u)/\rho_n = Au$ *for every* $u \in D$,

where $(\rho_n)_{n\geq 1}$ is a null sequence of positive real numbers. Then, for every $f \in X$ and $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)\rho_n \to t$,

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)} f\ .$$

Proof. Let $B : D(B) \subset X \to X$ be the linear operator defined by

$$Bu := \lim_{n\to\infty} \frac{L_n u - u}{\rho_n}\ ,$$

for every $u \in D(B) := \{u \in X\ :\ \text{there exists}\ \lim_{n\to\infty}(L_n u - u)/\rho_n \in X\}$.

By (ii), $D \subset D(B)$ and D is dense in X, because $D(A)$ is dense in X. Therefore $D(B)$ is dense in X too. Moreover, since $(A, D(A))$ is a generator, there exists $\lambda > \omega$ such that $\lambda I - A$ is invertible, so that $(\lambda I - B)(D) = (\lambda I - A)(D)$ is dense in X. Then, from Theorem 2.23 it follows that the operator $(B, D(B))$ is closable and its closure $(\overline{B}, D(\overline{B}))$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X such that $\|S(t)\| \leq Me^{\omega t}$ for every $t \geq 0$ and $S(t)f = \lim_{n\to\infty} L_n^{k(n)} f$, for every $f \in X$, $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)\rho_n \to t$ as $n \to \infty$.

The result will be proved once we show that $S(t) = T(t)$ $(t \geq 0)$. To this end it suffices to prove that $(\overline{B}, D(\overline{B})) = (A, D(A))$.

First observe that, for $\lambda > \omega$, the operator $\lambda I - \overline{B}$ is invertible as well and $(\lambda I - \overline{B})(D) = (\lambda I - B)(D) = (\lambda I - A)(D)$ is dense in X; hence D is a core for $(\overline{B}, D(\overline{B}))$. Therefore $A = \overline{A_{|D}} = \overline{B_{|D}} = \overline{B}$.

□

Corollary 2.25. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and suppose that $\|T(t)\| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ and for all $t \geq 0$. Let D be a core for $(A, D(A))$ and consider a C_0-semigroup $(S(t))_{t\geq 0}$ on X satisfying*

(i) $\|S(\rho_n)\| \leq Me^{\omega\rho_n}$ $(n \geq 1)$

and

(ii) $\lim_{n\to\infty}(S(\rho_n)(u) - u)/\rho_n = Au$ *for every* $u \in D$,

where $(\rho_n)_{n\geq 1}$ is a null sequence of positive real numbers. Then $S(t) = T(t)$ for every $t \geq 0$.

Proof. It suffices to apply Theorem 2.24 to the sequence $(S(\rho_n))_{n\geq 1}$.

□

3. Degenerate evolution problems

This second part of the paper is concerned with the study of initial boundary problems (1.2).

Our approach is based on the theory of semigroup of operators which will guarantee existence, uniqueness and continuous dependence on initial conditions of the relevant solutions.

We shall be mainly interested in the approximation of the solutions in terms of iterates of suitable (constructively defined) positive linear operators as in Theorem 2.24.

From such approximation formula we shall infer some information about the long-time behaviour of the solutions as well as some spatial regularity properties of them.

According to (1.2) we are interested in studying the differential operator defined as

$$Lu(x) := \begin{cases} \alpha(x)u''(x) + \beta(x)u'(x) + \gamma(x)u(x) & \text{if } 0 < x < 1\,, \\ \lim\limits_{t\to x} \alpha(t)u''(t) + \beta(t)u'(t) + \gamma(t)u(t) & \text{if } x = 0, 1\,, \end{cases} \tag{3.1}$$

on the following two domains

$$\begin{aligned} D_M(L) := \Big\{ u \in C([0,1]) \cap C^2(]0,1[) \ \mid \ & \lim_{x\to 0^+} \alpha(x)u''(x) + \beta(x)u'(x) \in \mathbb{R} \\ & \text{and } \lim_{x\to 1^-} \alpha(x)u''(x) + \beta(x)u'(x) \in \mathbb{R} \Big\} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} D_V(L) := \Big\{ u \in C([0,1]) \cap C^2(]0,1[) \ \mid \ & \lim_{x\to 0^+} \alpha(x)u''(x) + \beta(x)u'(x) = \\ & = \lim_{x\to 1^-} \alpha(x)u''(x) + \beta(x)u'(x) = 0 \Big\}\,, \end{aligned} \tag{3.3}$$

where

$$\alpha, \beta, \gamma \in C([0,1])\,, \tag{3.4}$$

$$\alpha(0) = \alpha(1) = 0 \quad , \quad 0 < \alpha(x) \leq \frac{x(1-x)}{2} \quad \text{for every} \quad 0 < x < 1\,. \tag{3.5}$$

Note that $Lu \in C([0,1])$ for every $u \in D_M(L)$ and hence for every $u \in D_V(L) \subset D_M(L)$. Moreover the boundary (or lateral) conditions incorporated in the domains $D_M(L)$ and $D_V(L)$ are usually referred to as maximal as well as Ventecel boundary conditions, respectively.

Finally we point out that the operator L has been widely studied often coupled with other boundary conditions (see, e.g., [42, Section VI.4]). However, for the sake of simplicity, we shall restrict to consider only the domains (3.2) and (3.3).

We shall develop our analysis by considering separately the case $\beta = \gamma = 0$ and then the general case.

3.1. **The differential operator** $Au = \alpha u''$**.** Consider $\alpha \in C([0,1])$ such that $\alpha(0) = \alpha(1) = 0$ and $0 < \alpha \leq (x(1-x))/2$ for every $0 < x < 1$. If we set

$$\lambda(x) := \frac{2\alpha(x)}{x(1-x)} \qquad (0 < x < 1)\ , \tag{3.6}$$

the function λ is continuous in $]0,1[$ and, for any $0 < x < 1$, $0 < \lambda(x) \leq 1$ and

$$\alpha(x) = \frac{x(1-x)}{2}\lambda(x)\ .$$

We shall focus our analysis on the following two differential operators defined by

$$Au(x) := \begin{cases} \alpha(x)u''(x) & \text{if } 0 < x < 1\ , \\ 0 & \text{if } x = 0, 1\ , \end{cases} \tag{3.7}$$

for every $u \in D_V(A)$ and $x \in [0,1]$, and

$$Bu(x) := \begin{cases} \dfrac{x(1-x)}{2}u''(x) & \text{if } 0 < x < 1\ , \\ 0 & \text{if } x = 0, 1\ , \end{cases} \tag{3.8}$$

for every $u \in D_V(B)$ and $x \in [0,1]$, where

$$D_V(A) := \{u \in C([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0^+} \alpha(x)u''(x) = \lim_{x\to 1^-} \alpha(x)u''(x) = 0\}$$

and

$$D_V(B) = \Big\{u \in C([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0^+} x(1-x)u''(x) = \\ = \lim_{x\to 1^-} x(1-x)u''(x) = 0\Big\}\ .$$

Clearly $D_V(B) \subset D_V(A)$.

We point out that for the above defined differential operator A it is superfluous to consider the maximal domain

$$D_M(A) := \Big\{u \in C([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0^+} \alpha(x)u''(x) \in \mathbb{R}\ , \\ \lim_{x\to 1^-} \alpha(x)u''(x) \in \mathbb{R}\Big\}\ .$$

Indeed, according to the Feller theory ([43], [42, Chapter VI, Theorem 4.15]), $(A, D_M(A))$ is the generator of a Feller semigroup if and only if 0 and 1 are both either entrance points or natural points. From (3.5) it follows that $2/(x(1-x)) \leq 1/\alpha(x)$ $(0 < x < 1)$ and hence $1/\alpha(x) \notin L^1(0,1/2)$ and $1/\alpha(x) \notin L^1(1/2,1)$. Therefore 0 and 1 cannot be entrance points and hence, if $(A, D_M(A))$ generates a Feller semigroup, then 0 and 1 must be natural points but, if this is the case, then $D_M(A) = D_V(A)$ ([43, Corollary after Theorem 13.1], [42, Exercises 4.19, (4), p. 400]).

Theorem 3.1 ([39], see also [43])**.** *Both operators $(B, D_V(B))$ and $(A, D_V(A))$ are generators of Markov semigroups on $C([0,1])$.*

Proof. For the sake of simplicity we shall prove the result only for $(B, D_V(B))$. For the result concerning $(A, D_V(A))$ we refer to the original article [39] or to [42, Chapter VI, Theorem 4.18].

To show the result we shall apply Theorem 2.19. Since $C^2([0,1])$ is contained in $D_V(B)$, $D_V(B)$ is dense in $C([0,1])$. As regards the maximum principle (ii), consider $u \in D_V(B)$ and $x_0 \in [0,1]$ such that $u(x_0) = \sup_{0\le x\le 1} u(x) > 0$. If $x_0 = 0,1$, then $Bu(x_0) = 0$. If $x_0 \in]0,1[$, then $u''(x_0) \le 0$ and hence $Bu(x_0) \le 0$ as well.

Property (iv) of part (b) of Theorem 2.19 being obvious, it remains to show property (iii).

Consider the operator $B_0 := B_{|D(B_0)} : D(B_0) \to C_0(]0,1[)$ defined on $D(B_0) := D_V(B) \cap C_0(]0,1[)$.

We preliminarily show that B_0 is closed. Given, indeed, a sequence $(u_n)_{n\ge 1}$ in $D(B_0)$ converging to some $u \in C_0(]0,1[)$ and such that $(B_0 u_n)_{n\ge1}$ converges to some $v \in C_0(]0,1[)$, for every compact subintervals $[a,b]$ of $]0,1[$, $u''(x) \to v(x)/(x(1-x))$ uniformly with respect to $x \in [a,b]$. Therefore, u is two times continuously differentiable on $[a,b]$ and $u''(x) = v(x)/(x(1-x))$ for every $x \in [a,b]$. Since the interval $[a,b]$ was arbitrarily chosen, we obtain that $u \in C^2(]0,1[)$ and $x(1-x)u''(x) = v(x)$ for every $x \in]0,1[$. Therefore, since $v \in C_0(]0,1[)$, we conclude that $u \in D(B_0)$ and $B_0 u = v$.

As a second step we proceed to show that B_0 is invertible. Consider, indeed, $u \in D(B_0)$ such that $B_0 u = 0$, i.e., $u''(x) = 0$ for $x \in]0,1[$. Therefore there exist $a,b \in \mathbb{R}$ such that $u(x) = ax + b$ for every $x \in]0,1[$ and hence, by continuity, for every $x \in [0,1]$. Accordingly, since $u \in C_0(]0,1[)$, $a = b = 0$ and so $u = 0$.

The previous reasoning shows that B_0 is injective. In order to prove the surjectivity of B_0, given $f \in C_0(]0,1[)$ define

$$u(x) := \int_0^x \frac{x-1}{1-y} f(y)\,dy - \int_x^1 \frac{x}{y} f(y)\,dy \qquad (0 < x < 1)\,.$$

Then $u \in D(B_0)$ and $B_0 u = f$.

Having proved that $B_0 : D(B_0) \to C_0(]0,1[)$ is closed and invertible, by the open mapping theorem the inverse $B_0^{-1} : C_0(]0,1[) \to D(B_0) \subset C_0(]0,1[)$ is continuous with respect to the graph norm $\|\cdot\|_{B_0}$ and hence with respect to the ordinary norm on $C_0(]0,1[)$ (because $\|\cdot\|_\infty \le \|\cdot\|_{B_0}$ on $D(B_0)$).

Therefore, fixing $0 < \lambda < 1/\|B_0^{-1}\|$, $I - \lambda B_0^{-1}$ is invertible on $C_0(]0,1[)$ and hence

$$\lambda I - B_0 = -B_0(I - \lambda B_0^{-1})$$

is invertible on $C_0(]0,1[)$.

After these preliminaries we can now proceed to show property (iii) of Theorem 2.19.

Fix $0 < \lambda < 1/\|B_0^{-1}\|$ as above and consider $f \in C([0,1])$. Set $g(x) := xf(0) + (1-x)f(1)$ $(0 \le x \le 1)$, we get $f - g \in C_0(]0,1[)$ and then we may choose $v \in D(B_0)$ such that $\lambda v - B_0 v = f - g$. Therefore the function $u := v + (g/\lambda) \in D_V(B)$ and $\lambda u - Bu = \lambda v + g - B_0 v = f$.

□

Our next aim is to determine a core for the operators $(B, D_V(B))$ and $(A, D_V(A))$. We begin by considering $(B, D_V(B))$.

Theorem 3.2. *The subspace $C^2([0,1])$ is a core for $(B, D_V(B))$.*

Proof. Given $u \in D_V(B)$, we first observe that

$$\lim_{x\to 0^+} xu'(x) = 0 \quad , \quad \lim_{x\to 0^+} xu''(x) = 0 \tag{1}$$

and

$$\lim_{x\to 1^-}(1-x)u'(x)=0 \quad , \quad \lim_{x\to 1^-}(1-x)u''(x)=0\,. \tag{2}$$

Actually, setting $M := \sup_{0<x<1} x(1-x)|u''(x)|$, for every $0 < x \leq 1/2$ we get

$$\left|\int_x^{1/2} u''(t)\,dt\right| \leq \int_x^{1/2} \frac{M}{t(1-t)}\,dt \leq 2M \int_x^{1/2} \frac{1}{t}\,dt = 2M\left(\log\left(\frac{1}{2}\right) - \log x\right)\,,$$

i.e.,

$$\left|u'\left(\frac{1}{2}\right) - u'(x)\right| \leq 2M\left(\log\left(\frac{1}{2}\right) - \log x\right)\,.$$

Therefore

$$x|u'(x)| \leq x\left(\left|u'\left(\frac{1}{2}\right)\right| + 2M\log\left(\frac{1}{2}\right)\right) - 2Mx\log x$$

and hence the first formula of (1) follows. The second one is obvious on account of the Ventcel condition imposed on 0.

In a similar manner formula (2) can be proved.

For every $n \geq 1$ and $0 \leq x \leq 1$ consider

$$u_n(x) := \begin{cases} u\left(\frac{1}{n}\right) + u'\left(\frac{1}{n}\right)\left(x-\frac{1}{n}\right) + \frac{1}{2}\,u''\left(\frac{1}{n}\right)\left(x-\frac{1}{n}\right)^2 & 0 \leq x \leq \frac{1}{n}\,, \\ u(x) & \frac{1}{n} \leq x \leq 1-\frac{1}{n}\,, \\ u\left(1-\frac{1}{n}\right) u'\left(1-\frac{1}{n}\right)\left(x-1+\frac{1}{n}\right) + & \\ +\frac{1}{2}u''\left(1-\frac{1}{n}\right)\left(x-1+\frac{1}{n}\right)^2 & 1-\frac{1}{n} \leq x \leq 1\,. \end{cases}$$

Then $u_n \in C^2([0,1])$. We claim that $u_n \to u$ and $Bu_n \to Bu$ uniformly on $[0,1]$.

Indeed, given $\varepsilon > 0$, since both u and Bu are uniformly continuous, there exists $\delta > 0$ such that for every $x, y \in [0,1]$, $|x-y| \leq \delta$,

$$|u(x)-u(y)| \leq \frac{\varepsilon}{3} \quad \text{and} \quad |Bu(x)-Bu(y)| \leq \frac{\varepsilon}{3}\,.$$

Moreover, formulae (1) and (2) imply that there exists $\nu \in \mathbb{N}$, $\nu \geq 1/\delta$, such that, for every $n \geq \nu$,

$$\frac{1}{n}\left|u'\left(\frac{1}{n}\right)\right| \leq \frac{\varepsilon}{3} \quad , \quad \frac{1}{n}\left|u'\left(1-\frac{1}{n}\right)\right| \leq \frac{\varepsilon}{3}\,,$$

$$\frac{1}{n}\left|u''\left(\frac{1}{n}\right)\right| \leq \frac{\varepsilon}{3} \quad , \quad \frac{1}{n}\left|u''\left(1-\frac{1}{n}\right)\right| \leq \frac{\varepsilon}{3}\,.$$

Accordingly, it is not difficult to show that

$$|u_n(x)-u(x)| \leq \varepsilon \quad \text{and} \quad |Bu_n(x)-Bu(x)| \leq \varepsilon$$

for every $n \geq \nu$ and $x \in [0,1]$ and hence the result follows.

□

In order to show that $C^2([0,1])$ is a core also for $(A, D_V(A))$ we need some more results.

We recall that a Feller semigroup on $C_0(]0,1[)$ is a C_0-semigroup of positive linear contractions on $C_0(]0,1[)$.

Proposition 3.3. *The operators*

$$(3.9) \qquad D(A_0) := D_V(A) \cap C_0(]0,1[) \quad , \quad A_0 := A_{|D(A_0)} : D(A_0) \to C_0(]0,1[)$$

and

$$D(B_0) := D_V(B) \cap C_0(]0,1[) \quad , \quad B_0 := B_{|D(B_0)} : D(B_0) \to C_0(]0,1[)$$

are the generators of some Feller semigroups on $C_0(]0,1[)$. *Moreover* $C([0,1]) \cap C^2(]0,1[)$ *is a core for* $(B_0, D(B_0))$.

Proof. The first part of the statement is a direct consequence of Proposition 2.2 in [11], hence we proceed to show the last part. Fix an element $u \in D(B_0)$; then there exists a sequence $(u_n)_{n\geq 1}$ in $C^2([0,1])$ such that $u_n \to u$ and $Bu_n \to Bu$ uniformly on $[0,1]$ because $D(B_0) \subset D_V(B)$ and $C^2([0,1])$ is a core for$(B, D_V(B))$ (the functions Bu_n and Bu are extended to 0 at the boundary points 0 and 1). In particular $u_n(0) \to 0$ and $u_n(1) \to 0$. For any $n \geq 1$ set $v_n(x) := u_n(x) - xu_n(1) - (1-x)u_n(0)$; then $v_n \in C^2([0,1]) \cap C_0(]0,1[)$, $v_n \to u$ uniformly on $[0,1]$ and finally, taking (3.8) into account, $Bv_n = Bu_n \to Bu$ uniformly on $[0,1]$.

□

Theorem 3.4. *Consider the following subset of* $D(A_0)$

$$D_*(A_0) := \{u \in D(A_0) \ : \ \textit{there exists } (u_n)_{n\geq 1} \textit{ in } D(B_0) \textit{ such that } u_n \to u \textit{ and } A_0 u_n \to A_0 u \textit{ uniformly on } [0,1]\}$$

which contains $D(B_0)$. *Then* $(A_0, D_*(A_0))$ *generates a Feller semigroup on* $C_0(]0,1[)$ *and* $C([0,1]) \cap C^2(]0,1[)$ *is a core for* $(A_0, D_*(A_0))$.

Proof. Among other things, from Proposition 3.3 it follows that $(B_0, D(B_0))$ generates a Feller semigroup on $C_0(]0,1[)$. Thanks to Theorem 2.22, the operator $(\lambda B_0, D(B_0))$ is closable and its closure $(C, D(C))$ is the generator of a Feller semigroup on $C_0(]0,1[)$.

Moreover, since $D(B_0) \subset D(A_0)$ and $A_{0|D(B_0)} = \lambda B_0$, then the operator $(A_0, D(A_0))$ is an extension of $(\lambda B_0, D(B_0))$ and $D(C) = D_*(A_0)$; on the other hand $(A_0, D(A_0))$ is closed, therefore $C = A_0$ for any $u \in D_*(A_0)$, so the first part of the statement is proved.

In order to prove that $C([0,1]) \cap C^2(]0,1[)$ is a core for $(A_0, D_*(A_0))$, let $u \in D_*(A_0)$. For a fixed $\varepsilon > 0$ there exists $v \in D(B_0)$ such that $\|u - w\| \leq \varepsilon/2$ and $\|A_0 u - A_0 v\| \leq \varepsilon/2$. On the other hand, by Proposition 3.3, there exists $w \in C([0,1]) \cap C^2(]0,1[)$ such that $\|v - w\| \leq \varepsilon/2$ and $\|B_0 v - B_0 w\| \leq \varepsilon/2$. Then $\|u-w\| \leq \|u-v\| + \|v-w\| \leq \varepsilon$ and $\|A_0 u - A_0 w\| \leq \|A_0 u - A_0 v\| + |\lambda B_0 v - \lambda B_0 w\| \leq \varepsilon$.

□

At this point we may show the desired result on the existence of a core for $(A, D_V(A))$. To this end, set

$$(3.10) \qquad D_V^*(A) := \{u \in D_V(A) \ : \ \text{there exists } (u_n)_{n\geq 1} \text{ in } D_V(B) \text{ such that } u_n \to u \text{ and } Au_n \to Au \text{ uniformly on } [0,1]\} \ .$$

It is clear that $D_V(B) \subset D_V^*(A)$. Moreover,

$$(3.11) \qquad D_V^*(A) \cap C_0(]0,1[) = D_*(A_0) \ .$$

Indeed, being the inclusion $D_*(A_0) \subseteq D_V^*(A) \cap C_0(]0,1[)$ obvious, we proceed to show its converse. Fix $u \in D_V^*(A) \cap C_0(]0,1[)$; then $u \in D_V(A) \cap C_0(]0,1[) = D(A_0)$ and there exists a sequence $(u_n)_{n\geq 1}$ in $D_V(B)$ such that $u_n \to u$ and $Au_n \to Au$ uniformly on $[0,1]$, in particular $u_n(0) \to 0$ and $u_n(1) \to 0$. For every $n \geq 1$ and $0 \leq x \leq 1$ set $v_n(x) := u_n(x) - xu_n(1) - (1-x)u_n(0)$. Then $v_n \in D(B_0)$, $v_n \to u$ uniformly on $[0,1]$ and $A_o v_n = Av_n \to Au = A_0 u$ uniformly on $[0,1]$; hence $u \in D_*(A)$.

Theorem 3.5. *$C^2([0,1])$ is a core for $(A, D_V(A))$.*

Proof. First we shall prove that the operator $(A, D_V^*(A))$ is the generator of a Markov semigroup $(T(t))_{t\geq 0}$ on $C([0,1])$ by applying a general result due to Bony, Courrège and Priouret [30]. Indeed, $D_V^*(A)$ is dense in $C([0,1])$, since $D_V(B)$ is dense in $C([0,1])$. Moreover the operator $(A, D_V^*(A))$ satisfies the positive maximum principle (see [30]), since $(A, D_V(A))$ satisfies it. Finally, we have to show that the range $R(\lambda I - A) = C([0,1])$ for $\lambda > 0$ fixed. Let $f \in C([0,1])$ and define the function $g \in C_0(]0,1[)$ by setting $g(x) := f(x) - xf(1) - (1-x)f(0)$ for every $x \in]0,1[$. From Theorem 3.4, $(\lambda I - A_0)(D_*(A)) = C_0(]0,1[)$ and so there exists $u \in D_*(A_0)$ such that $\lambda u - A_0 u = g$. Now set $v(x) := u(x) + 1/\lambda(xf(1) + (1-x)f(0))$ $(0 \leq x \leq 1)$ (we continuously extend u at 0 and 1). Since $u \in D(A_0) \subset D_V(A)$, $v \in D_V(A)$ and $Av = A_0 u$ on $[0,1]$. Moreover $\lambda v - Av = f$.

Now we proceed to show that $C^2([0,1])$ is a core for $(A, D_V^*(A))$. Let $u \in D_V^*(A)$ and $\varepsilon > 0$ and consider the function $\lambda \in C_b(]0,1[)$ defined by (3.6), so that $\alpha(x) = (x(1-x)/2)\,\lambda(x)$ $(0 \leq x \leq 1)$. Let v and element in $D_V(B)$ such that $\|u - v\| \leq \varepsilon/2$ and $\|Au - Av\| \leq \varepsilon/2$ and take $w \in C^2([0,1])$ such that $\|v - w\| \leq \varepsilon/2$ and $\|Bv - Bw\| \leq \varepsilon/2$. Then $\|u - w\| \leq \varepsilon$. Moreover if $x = 0, 1$, then $|Au(x) - Aw(x)| = 0 \leq \varepsilon$ and, if $0 < x < 1$,

$$|Au(x) - Aw(x)| \leq |Au(x) - Av(x)| + |Av(x) - Aw(x)| \leq$$

$$\leq \frac{\varepsilon}{2} + |\lambda(x)Bv(x) - \lambda Bw(x)| \leq \frac{\varepsilon}{2} + |Bv(x) - Bw(x)| \leq \varepsilon\ .$$

Accordingly, $\|Au - Aw\| \leq \varepsilon$.

At this point we can able to show the claim. Choosing $\lambda > 0$, $(\lambda I - A)(C^2([0,1])$ is dense in $C([0,1])$ because $C^2([0,1])$ is a core for $(A, D_V^*(A))$. Since $(A, D_V(A))$ generates a Feller semigroup too, the above density relation implies in turn that $C^2([0,1])$ is a core for $(A, D_V(A))$.

□

We continue our program of investigations by trying to represent the Markov semigroups involved in Theorem 3.1 in terms of iterates of positive linear operators as in (1.4).

We begin by considering the seeming particular case when the coefficient α is associated with a continuous selection of Borel measures (see (3.14) below). In fact we shall see that all functions α satisfying (3.5) satisfy this assumption.

Consider, therefore, a continuous selection $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ satisfying (2.6). We have already observed in (2.7) that

$$0 \leq T(e_2)(x) - x^2 \leq x(1-x) \qquad (0 \leq x \leq 1)\ . \tag{3.12}$$

Proposition 3.6. *The following propositions are true.*

1. $\mu_0 = \varepsilon_0$.
2. $\mu_1 = \varepsilon_1$.

3. If $x \in]0,1[$, then $\mu_x = \varepsilon_x$ if and only if $T(e_2)(x) = x^2$.

Proof. From (3.12) it follows that $T(e_2)(0) = 0$ and so $\mathrm{Supp}(mu_0) = \{0\}$. Then $\mu_0 = \varepsilon_0$. Again from (3.12), $T(e_2)(1) = 1$, so

$$\int_0^1 (e_1 - e_2)\, d\mu_1 = 0$$

and $\mathrm{Supp}(\mu_1) \subset \{0,1\}$. Hence, on the one hand we can write $\mu_1 = \alpha\varepsilon_0 + \beta\varepsilon_1$ with $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$, and on the other hand we have

$$1 = \int_0^1 e_1\, d\mu_1 = \alpha \cdot 0 + \beta \cdot 1 = \beta\ ,$$

hence $\alpha = 0$ and $\mu_1 = \varepsilon_1$.

Finally, first suppose that $T(e_2)(x) = x^2$ and consider the function ψ_x given by (2.1). We have $\int_0^1 \psi_x^2\, d\mu_x = 0$, so $\mathrm{Supp}(\mu_x) = \{x\}$ and $\mu_x = \varepsilon_x$. The converse is obvious.

□

From now on we shall assume that the selection $(\mu_x)_{0 \le x \le 1}$ satisfies the following requirement

$$\mu_x \neq \varepsilon_x \quad \text{for every} \quad x \in]0,1[\ . \tag{3.13}$$

For every $0 \le x \le 1$ set

$$\alpha(x) := \frac{1}{2}(T(e_2)(x) - x^2) = \frac{1}{2}\left(\int_0^1 e_2\, d\mu_x - x^2\right)\ . \tag{3.14}$$

Then $\alpha \in C([0,1])$, $\alpha(0) = \alpha(1) = 0$ and $0 < \alpha(x) \le x(1-x)/2$ for every $0 < x < 1$.

Let $(B_n)_{n \ge 1}$ be the sequence of Bernstein-Schnabl associated with the continuous selection $(\mu_x)_{0 \le x \le 1}$. From definition (2.8) and Proposition 3.6 we have

$$B_n(f)(x) = f(x) \quad \text{for} \quad x = 0, 1\ ; \tag{3.15}$$

in particular the B_n's map the space $C_0(]0,1[)$ into itself.

Theorem 3.7. *For every $u \in C^2([0,1])$,*

$$\lim_{n\to\infty} n[B_n(u) - u] = \alpha u'' \quad \textit{uniformly on} \quad [0,1]\ , \tag{3.16}$$

where α is given in (3.14).

Proof. From (2.9), (2.16) and (2.17) it follows that

$$\lim_{n\to\infty} n[B_n(\mathbf{1}) - \mathbf{1}] = 0 = \lim_{n\to\infty} nB_n(\psi_x)(x) \quad \text{uniformly with respect to} \quad x \in [0,1]$$

and

$$\lim_{n\to\infty} nB_n(\psi_x^2)(x) = T(e_2)(x) - x^2 \quad \text{uniformly with respect to} \quad x \in [0,1]\ .$$

On the other hand, since $\psi_x^4 = e_4 - 4xe_3 + 6x^2e_2 - 4x^3e_1 + x^4\mathbf{1}$ $(0 \le x \le 1)$, by using (2.14) and (2.15), we obtain

$$B_n(\psi_x^4)(x) = \frac{1}{n^3}\left[T(e_4)(x) - 4xT(e_3)(x) + 3(n-1)T(e_2)^2(x) - \right.$$
$$\left. -6(n-2)x^2T(e_2)(x) + 3(n-2)x^4\right]\ ,$$

therefore $\lim_{n\to\infty} nB_n(\psi_x^4)(x) = 0$ uniformly with respect to $x \in [0,1]$. Accordingly, the result follows from Theorem 2.5.

□

Remark 3.8. In [8, Theorem 13] it is shown that for a given $u \in C([0,1])$ there exists $\lim_{n\to\infty} n(B_n(u) - u)$ uniformly on $[0,1]$ if and only if $u \in C^2(]0,1[)$ and u'' is bounded on $]0,1[$ (see Theorem 3.15 of this section).

We have now prepared all the necessary tools to easily get the approximation result for our semigroups.

Theorem 3.9. *Let α be the function defined by* (3.14) *and consider the operators $(A_0, D_*(A_0))$, given in* (3.9), *and $(A, D_V(A))$, defined by* (3.7) *and* (3.10). *Denote by $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ the relevant Feller semigroups on $C_0(]0,1[)$ and $C([0,1])$ respectively. Then, for any $t \geq 0$, $S(t) = T(t)_{|C_0(]0,1[)}$ and, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ and for every $f \in C([0,1])$,*

$$T(t)f = \lim_{n\to\infty} B_n^{k(n)} f \quad \text{uniformly on} \quad [0,1] \,. \tag{3.17}$$

Proof. By Theorem 3.7, for every $u \in C^2([0,1]) \subset D_V(A)$,

$$\lim_{n\to\infty} n(B_n(u) - u) = Au \quad \text{uniformly on} \quad [0,1]$$

and $C^2([0,1])$ is a core for $(A, D_V(A))$. Finally, for every $n \geq 1$ and $p \geq 1$, $\|B_n^p\| = 1$, since $B_n^p(\mathbf{1}) = \mathbf{1}$. Therefore, formula (3.17) easily follows from Theorem 2.24.

Finally, take $u_0 \in D_*(A_0) \subset D_V(A)$ and consider the function $u(t) := T(t)u_0$ $(t \geq 0)$. Then, taking Theorem 2.18 into account, $u(t) \in D_V(A)$ for any $t \geq 0$, u is strongly differentiable in $[0,+\infty[$ and $(d/dt)u(t) = Au(t)$ for every $t \geq 0$. From (3.17), (3.16) and (3.11), it follows that $T(t)u_0 \in D_V(A) \cap C_0(]0,1[) = D_*(A)$ and so, by the uniqueness of the solution of the Cauchy problem associated with $(A_0, D_*(A_0))$, we get $u(t) = S(t)u_0$ for every $t \geq 0$. Therefore $T(t) = S(t)$ on $D_*(A_0)$ and hence on $C_0(]0,1[)$, because $D_*(A_0)$ is dense in $C_0(]0,1[)$.

□

Next we shall see that formula (3.17) holds true for the Markov semigroup generated by the operator $(A, D_V(A))$ associated with an arbitrary function $\alpha \in C([0,1])$ satisfying (3.5), because under these assumptions it is always possible to construct a continuous selection of Borel measures in terms of which α can be expressed as in (3.14).

Theorem 3.10. *Consider $\alpha \in C([0,1])$ satisfying* (3.5) *and, according to Theorem 3.1, denote by $(T(t))_{t\geq 0}$ the Markov semigroup generated by the operator $(A, D_V(A))$. Set $\lambda(x) := 2\alpha(x)/(x(1-x))$ $(0 < x < 1)$ and, for every $x \in [0,1]$, consider the measure*

$$\mu_x := \begin{cases} \lambda(x)(x\varepsilon_1 + (1-x)\varepsilon_0) + (1-\lambda(x))\varepsilon_x & 0 < x < 1 \,, \\ \varepsilon_x & x = 0,1 \,. \end{cases} \tag{3.18}$$

Let $(B_n)_{n\geq 1}$ be the sequence of Bernstein-Schnabl operators associated with the selection $(\mu_x)_{0\leq x\leq 1}$ and defined by (2.11). *Then, for every $t \geq 0$, for each sequence $(k(n))_{n\geq 1}$ of natural integers such that $k(n)/n \to t$ as $n \to \infty$ and for every $f \in C([0,1])$,*

$$T(t)f = \lim_{n\to\infty} B_n^{k(n)}(f) \quad \text{uniformly on} \quad [0,1] \,. \tag{3.19}$$

Proof. First we note that $\lambda \in C_b(]0,1[)$, $0 < \lambda \leq 1$, that $\alpha(x) = (x(1-x)/2)\,\lambda(x)$ for any $x \in [0,1]$ and that the family $(\mu_x)_{0\leq x\leq 1}$ is a continuous selection satisfied (2.6) and (3.13). Moreover $\int_0^1 e_2\, d\mu_x = x^2 + x(1-x)\lambda(x) = x^2 + 2\alpha(x)$. According to Theorem 3.9 the semigroup $(T(t))_{t\geq 0}$ can be represented as the limit of powers of the B_n's corresponding to the selection of measures (3.18) and this finishes the proof.

□

Remark 3.11. When $\alpha(x) := x(1-x)/2$, then the approximating Bernstein-Schnabl operators are, indeed, the classical Bernstein operators (2.10).

The main significance of Theorem 3.10 lies in the possibility to numerically approximate and to infer qualitative properties of the semigroup $(T(t))_{t\geq 0}$ and hence of the solutions of the initial boundary problems (1.2) because of formula (1.3).

In this case problem (1.2) turns into

$$
(3.20)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) & 0 < x < 1\,,\ \ t \geq 0\,, \\[2ex] u(x,0) = u_0(x) & 0 \leq x \leq 1,\ u_0 \in D_V(A)\,, \\[2ex] \lim\limits_{\substack{x\to 0^+ \\ x \to 1^-}} \alpha(x)\,\dfrac{\partial^2 u}{\partial x^2}(x,t) = 0 & t \geq 0\,. \end{cases}
$$

Such a problem is strictly related to a stochastic model from genetics which is involved in the study of the fluctuations of gene frequency under the influence of mutation and selection (see [14, pp. 465-466] for more details).

We also point out that the semigroups involved in Theorem 3.1 are the transition semigroups of some normal Markov processes, having $[0,1]$ as state space, with absorbing barriers at 0 and 1 and with mean instantaneous velocity 0 and variance instantaneous velocity $\alpha(x)$ at position $x \in [0,1]$ (see [64, Section I.4]).

From formula (3.19) it is then possible to infer several useful information on such Markov processes (see [14, pp. 461-465]).

For the sake of brevity we also omit to discuss some results concerning numerical estimates of

$$\|T(t)f - B_n^{k(n)} f\| \qquad (n \to \infty)$$

($f \in C([0,1])$, $t \geq 0$, $k(n)/n \to t$) and we refer the interested reader to [45], [2], [51], [37], [38] for more deepening.

Below we restrict ourselves to discuss some qualitative properties of the semigroup, which have their counterparts in terms of the solutions of problem (3.20) because of formula (1.3).

Corollary 3.12. *Under the same hypotheses of Theorem 3.9 the following statements hold true.*

(*1*) $T(t)f(x) = f(x)$ *for every* $f \in C([0,1])$ *and* $x = 0, 1$.

(*2*) *If the operator* T *given in* (2.5) *maps continuous increasing functions into* (*continuous*) *increasing functions, then each* $T(t)$ *maps continuous increasing functions into increasing functions.*

(*3*) *If* $T(Lip(1,1)) \subset Lip(1,1)$ *then, for every* $M > 0$, $0 < \alpha \leq 1$ *and* $t \geq 0$, $T(t)(Lip(\alpha, M)) \subset Lip(\alpha, M)$.

(4) *If $f \in C([0,1])$ the following propositions are equivalent:*
(i) *f is convex;*
(ii) *$f \le T(t)f$ for any $t \ge 0$.*

(5) *Under assumptions (c_1) and (c_2) stated at p. 15, if $f \in C([0,1])$ is convex, then each $T(t)f$ is convex ($t \ge 0$) and $(T(t)f)_{t\ge0}$ is increasing.*

(6) *For every $f \in C([0,1])$*

$$\lim_{t\to+\infty} T(t)f = T(f) \quad \text{uniformly on} \quad [0,1]\ .$$

Therefore $\lim_{t\to+\infty} T(t)f = 0$ uniformly on $[0,1]$ if and only if $f(0) = f(1) = 0$. Moreover, if $T(Lip(1,1)) \subset Lip(1,1)$ and conditions (c_1) and (c_2) are satisfied, then for every $f \in C([0,1])$, $x \in [0,1]$ and $t \ge 0$

$$|T(t)f(x) - T(f)(x)| \le M\omega_2(f, \sqrt{\lambda_t(x)})$$

where $\lambda_t(x) := |T(t)e_2(x) - x|$ and M is an absolute constant.

Proof. Statement (1) follows from (3.15). Statements (2) and (3) follows from Theorem 2.9 and Corollary 2.12. The implication $(i) \Rightarrow (ii)$ follows from (2.30). Conversely, assume that $f \le T(t)f$ for every $t \ge 0$ and set $u(r) := 1/r \int_0^r T(s)f\,ds \in D_V(A)$ for every $r > 0$. Then $Au(r) = 1/r(T(r)f - f) \ge 0$, so that $u(r)$ is convex for every $r > 0$. Accordingly, $f = \lim_{r\to0^+} u(r)$ is convex too.

The first part of (4) is a consequence of Theorem 2.14. As regard to the second part, first take $u \in D_V(A)$, u convex. Then for every $t \ge 0$, $T(t)u \in D_V(A)$ and $T(t)u$ is convex too. Therefore $AT(t)u \ge 0$ and hence $(d/dt)T(t)u = AT(t)u \ge 0$ ($t \ge 0$), so that $(T(t)u)_{t\ge0}$ is increasing. Define $u(r) := 1/r \int_0^r T(s)f\,ds \in D_V(A)$ for every $r > 0$. Then $\lim_{r\to0^+} u(r) = f$ uniformly on $[0,1]$ and each $u(r)$ is convex. So, for $0 \le s < t$, we get $T(s)u(r) \le T(t)u(r)$ and hence $T(s)f \le T(t)f$ because of the continuity of the operators $T(s)$ and $T(t)$ and this completes the proof of (4).

For the proof of (5) we refer to [19, Theorem 3.9] or [60, Section 3].

□

The approximation formula (3.19) can be fruitfully used to determine the saturation class of Bernstein-Schnabl operators and the Favard class of the semigroup $(T(t))_{t\ge0}$ which are, respectively, the linear subspaces of all functions $f \in C([0,1])$ such that $\sup_{n\ge1} \|B_n f - f\|_\infty < +\infty$, resp. $\sup_{t>0} \|T(t)f - f\|_\infty/t < +\infty$ (see, e.g., [29, Section 2.1]). Among other things, the characterization of the Favard class reveals some further "spatial regularity" properties preserved under the evolution governed by the semigroup $(T(t))_{t\ge0}$.

For the complete proof we refer to [8, Theorem 7].

Theorem 3.13. *Given $f \in C([0,1])$, the following statements are equivalent:*

(i) *There exists $M_1 \ge 0$ such that for every $x \in [0,1]$ and $n \ge 1$*

$$|B_n(f)(x) - f(x)| \le \frac{M_1\alpha(x)}{n}\ .$$

(ii) *There exists $M_2 \ge 0$ such that for every $n \ge 1$*

$$\|B_n f - f\|_\infty \le \frac{M_2}{n}\ .$$

(iii) *There exists $M_3 \ge 0$ such that for every $n \ge 1$*

$$\|T(t)f - f\|_\infty \le M_3 t\ .$$

(*iv*) $f \in C^1([0,1])$ *and* f' *is Lipschitz continuous with some Lipschitz constant* $M > 0$.

If, in addition, α *is concave, then statements* (i)-(iv) *are also equivalent to*

(*v*) *there exists* $M_4 \geq 0$ *such that for every* $t \geq 0$ *and* $x \in [0,1]$

$$|T(t)f(x) - f(x)| \leq M_4 t\alpha(x) \ .$$

Moreover $M_2 = M_3 = M_4\|\alpha\|_\infty = 4\|\alpha\|_\infty M = M_1\|\alpha\|_\infty$.

As a consequence of the previous result we have some further "spatial regularity" properties which are preserved by the semigroup $(T(t))_{t\geq 0}$.

Corollary 3.14. *Let* $f \in C^1([0,1])$ *with* f' *Lipschitz continuous. Then, for every* $s \geq 0$, $T(s)f$ *is continuously differentiable on* $[0,1]$ *and its first derivative is Lipschitz continuous.*

Proof. For every $t \geq 0$ we get

$$\|T(t)T(s)f - T(s)f\|_\infty \leq \|T(s)\|\|T(t)f - f\|_\infty \leq \|T(t)f - f\|_\infty$$

and hence the result follows from Theorem 3.13 both to f and $T(s)f$.

□

We end this section by pointing out the following result obtained in [8, Theorem 13]. Below we denote by $C_b^2(]0,1[)$ the subspace of all $u \in C([0,1]) \cap C^2(]0,1[)$ such that u'' is bounded in $]0,1[$.

Theorem 3.15. *Given* $u \in C([0,1])$*, the following statements are equivalent:*

(*i*) *There exists* $\lim_{n\to\infty} n(B_n u - u)$ *uniformly on* $[0,1]$*;*

(*ii*) $u \in C_b^2(]0,1[)$*;*

(*iii*) *There exists* $w \in C([0,1])$ *such that* $\sup_{0<x<1} w(x)/\alpha(x) < +\infty$ *and* $\lim_{t\to 0^+} (T(t)u - u)/t = w$ *uniformly on* $[0,1]$.

By using the above theorem we then obtain another regularity result.

Corollary 3.16. *Assume that* α *is concave. Then* $T(s)(C_b^2(]0,1[)) \subset C_b^2(]0,1[)$ *for every* $s \geq 0$.

Proof. Since α is concave, by Corollary 3.12, (3), $T(s)\alpha \leq \alpha$. If $u \in C_b^2(]0,1[)$, by Theorem 3.15 there exists $w \in C([0,1])$ such that $\lim_{t\to 0^+} (T(t)u - u)/t = w$ and $w \leq M\alpha$ on $]0,1[$ for some $M \geq 0$. By continuity $w \leq M\alpha$ on $[0,1]$ and hence $T(s)w \leq MT(s)\alpha \leq M\alpha$. Moreover

$$\lim_{t\to 0^+} \frac{T(t)T(s)u - T(s)u}{t} = T(s)w \ .$$

Therefore, by Theorem 3.15, $T(s)u \in C_b^2(]0,1[)$.

□

3.2. **The complete operator** $Lu = \alpha u'' + \beta u' + \gamma u$**.** In the same spirit of the previous section we shall now study the complete operator L defined by (1.1) on the domains (3.2) and (3.3). The coefficients α, β, γ are assumed to verify (3.4) and (3.5).

As regards the generation property of $(L, D_M(L))$ and $(L, D_V(L))$, according to Corollary 2.20 we may restrict the analysis to the differential operator $A^* := L - \gamma I$

defined on $D_M(A^*) := D_M(L)$ and $D_V(A^*) := D_V(L)$. Thus

$$A^*u(x) := \begin{cases} \alpha(x)u''(x) + \beta(x)u'(x) & \text{if } 0 < x < 1 , \\ \lim\limits_{t\to x} \alpha(t)u''(t) + \beta(t)u'(t) & \text{if } x = 0, 1 \end{cases}$$

for every $u \in D_M(A^*)$ or $u \in D_V(A^*)$ and $0 \le x \le 1$.

The generation property of both $(A^*, D_M(A^*))$ and $(A^*, D_V(A^*))$ can be obtained by using the results of Feller ([43]; see also [39, Theorem 2], [42, Chapter VI, Section 4]).

Fix $x_0 \in]0, 1[$ and consider the Wronskian

$$W(x) := \exp\left(-\int_{x_0}^{x} \frac{\beta(s)}{\alpha(s)}\, ds\right) \qquad (0 < x < 1)$$

and the auxiliary functions

$$Q(x) := \frac{1}{\alpha(x)W(x)} \int_{x_0}^{x} W(t)\, dt \qquad (0 < x < 1)$$

and

$$R(x) := W(x) \int_{x_0}^{x} \frac{1}{\alpha(t)W(t)}\, dt \qquad (0 < x < 1) .$$

Theorem 3.17. *Assume that $R \notin L^1(0, x_0)$ and $R \notin L^1(x_0, 1)$. Then $(A^*, D_M(A^*))$ is the generator of a Markov semigroup on $C([0, 1])$ and hence $(L, D_M(L))$ is the generator of a positive C_0-semigroup $(S(t))_{t\ge 0}$ on $C([0, 1])$ satisfying $\|S(t)\| \le e^{\gamma_\infty t}$ $(t \ge 0)$ where $\gamma_\infty := \sup_{0\le x\le 1} \gamma(x)$.*

Theorem 3.18. *Assume that*

(*i*) *$Q \notin L^1(x_0, 1)$ or $R \in L^1(x_0, 1)$ or both,*
(*ii*) *$Q \notin L^1(0, x_0)$ or $\in L^1(0, x_0)$ or both.*

Then $(A^*, D_V(A^*))$ is the generator of a Markov semigroup on $C([0, 1])$ and hence $(L, D_V(L))$ is the generator of a positive C_0-semigroup $(S(t))_{t\ge 0}$ on $C([0, 1])$ satisfying $\|S(t)\| \le e^{\gamma_\infty t}$ $(t \ge 0)$ where γ_∞ is defined as above.

In order to achieve the representation of the semigroup in terms of iterates of suitable positive linear operators we have to impose some further assumptions on α and β.

From now on we shall further assume that

$$\alpha \text{ is differentiable at } 0 \text{ and } 1 \text{ and } \alpha'(x) \ne 0 \ne \alpha'(1) .$$

Therefore we can express α as

$$\alpha(x) = \frac{x(1-x)}{2}\lambda(x) ,$$

where

$$\lambda(x) := \begin{cases} 2\alpha'(0) & x = 0 , \\ \dfrac{2\alpha(x)}{x(1-x)} & 0 < x < 1 , \\ -2\alpha'(1) & x = 1 \end{cases}$$

and $\lambda \in C([0, 1])$, $0 < \lambda \le 1$ for every $0 \le x \le 1$.

Theorem 3.19. *Assume that β/λ is Hölder continuous at* 0 *and* 1*. The following statements hold true:*

(*1*) *If $\beta(0) \geq \alpha'(0)$ and $\beta(1) \leq \alpha'(1)$, then $(L, D_M(L))$ is the generator of a positive C_0-semigroup as in Theorem 3.17 and $C^2([0,1])$ is a core for both $(A^*, D_M(A^*))$ and $(L, D_M(L))$.*

(*2*) *If $\beta(0) = \beta(1) = 0$, $\beta(x) = O(x)$ as $x \to 0^+$ and $\beta(x) = O(1-x)$ as $x \to 1^-$, then $(L, D_V(L))$ is the generator of a positive C_0-semigroup as in Theorem 3.18 and $C^2([0,1])$ is a core for both $(A^*, D_V(A^*))$ and $(L, D_V(L))$.*

For a proof we refer to [34, Proposition 4.3] and to [11, Theorem 3.1 and Lemma 3.2] (see also [26, pp. 120-121 and Theorem 2.3]).

We also point out that in [11, Theorem 3.1] it was also proved that $(L, D_V(L))$ generates a positive C_0-semigroup provided that $0 < \beta(0) < \alpha'(0)$ and $\alpha'(1) < \beta(1) < 0$.

As a final step we now proceed to construct a sequence of approximating positive operators for both the semigroups considered in Theorem 3.19. Actually these new operators will be constructed by modifying the Bernstein-Schnabl operators considered in the previous section, according to an idea first developed in [10] and [11].

From now on we shall assume that there exists $n_0 \geq 1$ such that, for every $n \geq n_0$ and $x \in [0,1]$,

$$0 \leq x + \frac{\beta(x)}{n} \leq 1 \qquad \text{and} \qquad 1 + \frac{\gamma(x)}{n} \geq 0 \,. \tag{3.21}$$

For instance conditions (3.21) are satisfied by choosing a natural number $n_0 \geq \max\{\beta_0, \beta_1, \|\gamma\|_\infty\}$ provided that

$$-\beta_0 := \inf_{0<x\leq 1} \frac{\beta(x)}{x} > -\infty \qquad \text{and} \qquad \beta_1 := \sup_{0\leq x<1} \frac{\beta(x)}{x} < +\infty \,.$$

Consider a continuous selection $(\mu_x)_{0\leq x\leq 1}$ of Borel measures on $[0,1]$ in terms of which α can be expressed as in (3.14) (for instance, the selection defined by (3.18)) and denote by B_n ($n \geq 1$) the relevant Bernstein-Schnabl operators.

For every $n \geq n_0$, define $M_n : C([0,1]) \to C([0,1])$ by setting

$$M_n(f) := B_n\left(\left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right)\right) \qquad (f \in C([0,1])) \,. \tag{3.22}$$

In the particular case where $\alpha(x) = x(1-x)/2$, the B_n's are the classical Bernstein operators (defined in (2.10)) and the operators M_n turn into

$$M_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left(1 + \frac{\gamma(k/n)}{n}\right) f\left(\frac{k}{n} + \frac{\beta(k/n)}{n}\right)$$

($f \in C([0,1]$, $0 \leq x \leq 1$).

Proposition 3.20. *The sequence $(M_n)_{n\geq 1}$ is equibounded and, for every $f \in C([0,1])$,*

$$\lim_{n\to\infty} M_n(f) = f \quad \textit{uniformly on} \quad [0,1] \,.$$

Proof. For every $n \geq n_0$ and $f \in C([0,1])$ we have

$$\|M_n(f)\|_\infty \leq \|B_n\| \left\| \left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right) \right\|_\infty \leq \left(1 + \frac{\gamma_\infty}{n}\right) \|f\|_\infty \,,$$

where $\gamma_\infty := \sup_{0\le x\le 1} \gamma(x)$; therefore $\|M_n\| \le 1 + (\gamma_\infty/n)$. Moreover

$$\|M_n(f) - f\|_\infty \le \left\| B_n\left(f\circ\left(e_1+\frac{\beta}{n}\right)\right) - B_n(f)\right\|_\infty + \\ +\frac{1}{n}\left\| B_n\left(\gamma f\circ\left(e_1+\frac{\beta}{n}\right)\right)\right\|_\infty + \|B_n(f)-f\|_\infty \le \\ \le \left\| f\circ\left(e_1+\frac{\beta}{n}\right) - f\right\|_\infty + \frac{1}{n}\|\gamma\|_\infty\|f\|_\infty + \|B_n(f)-f\|_\infty \,.$$

At this point we can observe that, for every $f \in C([0,1])$,

$$\lim_{n\to\infty} f\circ\left(e_1+\frac{\beta}{n}\right) = f \quad \text{uniformly on} \quad [0,1]\,. \tag{3.23}$$

Indeed, given a function $f \in C^1([0,1])$, from Lagrange theorem we get

$$\left| f\left(x+\frac{\beta(x)}{n}\right) - f(x)\right| \le \|f'\|_\infty \frac{|\beta(x)|}{n}\,,$$

for every $x \in [0,1]$. So

$$\|f\circ\left(e_1+\frac{\beta}{n}\right) - f\|_\infty \le \|f'\|_\infty\|\beta\|_\infty \frac{1}{n} \to 0$$

and (3.23) follows easily because $C^1([0,1])$ is dense in $C([0,1])$. Moreover $\|B_n(f) - f\|_\infty \to 0$ thanks to Theorem 2.7. Then $\|M_n(f)-f\|_\infty \to 0$ as well.

□

The construction of the M_n's by means of the B_n's allows to obtain estimates of the rate of the above convergence from the corresponding ones which we know for the operators B_n as stated in Theorem 2.8.

Analogously, we can obtain some shape-preserving properties for the M_n's by the similar ones studied for the B_n's (see Theorems 2.9-2.14). We collect them in the following proposition.

Proposition 3.21. *The following statements hold true.*

(1) *Under the same hypotheses of Theorem 2.9, if in addition $e_1 + \beta/n$ is increasing and if, for every $f \in C([0,1])$, $(1+\gamma/n)f$ is increasing too, then each M_n ($n \ge n_0$) maps increasing functions into increasing functions.*

(2) *Assume that $T(Lip(1,1)) \subset Lip(1,1)$, where T given in (2.5), and suppose that $\beta \in Lip(1,B)$ and $\gamma \in Lip(1,C)$ (resp. γ is constant) for some $B, C > 0$. Then, for every $f \in Lip(\alpha,\widetilde{M})$, $\widetilde{M} > 0$, $0 < \alpha \le 1$ and $n \ge n_0$,*

$$M_n(f) \in Lip\left(\alpha, \widetilde{M}\left(1+\frac{\|\gamma\|_\infty}{n}\right)\left(1+\frac{B}{n}\right)^\alpha + \|f\|_\infty \frac{C}{n}\right)$$

(*resp.*

$$M_n(f) \in Lip\left(\alpha, \widetilde{M}\left(1+\frac{|\gamma|}{n}\right)\left(1+\frac{B}{n}\right)^\alpha\right)\,.$$

(3) *If $T(Lip(1,1)) \subset Lip(1,c)$ (T as in (2.5)) for some $c \ge 1$ and $\beta \in Lip(1,B)$ for some $B > 0$. Then, for every $f \in C([0,1])$, $\delta > 0$ and $n \ge n_0$,*

$$\omega(M_n(f),\delta) \le (1+c)\left(1+\frac{\|\gamma\|_\infty}{n}\right)\omega\left(f,\left(1+\frac{B}{n}\right)\delta\right) + (1+c)\frac{\|f\|_\infty}{n}\,\omega(\gamma,\delta)\,.$$

If γ is constant then, for every $f \in C([0,1])$, $\delta > 0$ and $n \geq n_0$,

$$\omega(M_n(f), \delta) \leq (1+c)\left(1 + \frac{|\gamma|}{n}\right) \omega\left(f, \left(1+\frac{B}{n}\right)\delta\right) .$$

(*4*) *Assume that β is affine (resp. convex) on $[0,1]$ and that, for every convex function $f \in C([0,1])$, $(1+\gamma/n)f$ is convex too. Then each M_n ($n \geq n_0$) maps convex (resp. increasing convex) functions into convex functions provided that conditions (c_1) and (c_2) stated at p. 15 are satisfied.*

Proof. Statement (1) is a direct consequence of the assumptions and Theorem 2.9. As regard (2), taking Theorem 2.12 into account with

$$M := \widetilde{M}\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\left(1 + \frac{B}{n}\right)^\alpha + \|f\|_\infty \frac{C}{n} ,$$

it is sufficient to show that, if $f \in Lip(\alpha, \widetilde{M})$, then

$$\left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right) \in Lip\left(\alpha, \widetilde{M}\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\left(1+\frac{B}{n}\right)^\alpha + \|f\|_\infty \frac{C}{n}\right) .$$

Indeed, for every $x, y \in [0,1]$,

$$\begin{aligned}
&\left|\left(1 + \frac{\gamma(x)}{n}\right) f\left(x + \frac{\beta(x)}{n}\right) - \left(1 + \frac{\gamma(y)}{n}\right) f\left(y + \frac{\beta(y)}{n}\right)\right| \leq \\
&\leq \left| f\left(x + \frac{\beta(x)}{n}\right) - f\left(y + \frac{\beta(y)}{n}\right)\right| + \\
&+ \frac{|\gamma(x) - \gamma(y)|}{n} \left| f\left(x + \frac{\beta(x)}{n}\right)\right| + \frac{|\gamma(y)|}{n}\left| f\left(x + \frac{\beta(x)}{n}\right) - f\left(y + \frac{\beta(y)}{n}\right)\right| \leq \\
&\leq \widetilde{M}\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\left| x - y + \frac{\beta(x) - \beta(y)}{n}\right|^\alpha + \frac{C}{n}|x-y|^\alpha \|f\|_\infty \leq \\
&\leq \left[\widetilde{M}\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\left(1 + \frac{B}{n}\right)^\alpha + \|f\|_\infty \frac{C}{n}\right]|x-y|^\alpha .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\omega\left(\left(1 + \frac{\gamma}{n}\right) f \circ \left(e_1 + \frac{\beta}{n}\right), \delta\right) \leq \\
&\leq \left(1 + \frac{\|\gamma\|_\infty}{n}\right) \omega\left(f, \left(1 + \frac{B}{n}\right)\delta\right) + \frac{\|f\|_\infty}{n}\omega(f,\delta) .
\end{aligned}$$

which gives statement (3). Finally, statement (4) follows from the hypotheses and Theorem 2.14.

□

From Theorem 3.7 and the remarks in the relevant proof, we get the following asymptotic formula for the sequence $(M_n)_{n\geq 1}$.

Theorem 3.22. *For every $f \in C^2([0,1])$*

$$\lim_{n\to\infty} n[M_n(f) - f] = \alpha f'' + \beta f' + \gamma f \quad \textit{uniformly on} \quad [0,1] .$$

Proof. The proof can be achieved by means of Theorem 2.5 and formula (3.16). We left the details to the reader.

□

Collecting together Theorems 3.19 and 3.22 we may use Theorem 2.24 to obtain our final result. First observe that, as it was shown in the proof of Proposition 3.20,

$$\|M_n\| \leq 1 + \frac{\gamma_\infty}{n} \qquad (n \geq n_0)$$

and hence

$$\|M_n^p\| \leq \left(1 + \frac{\gamma_\infty}{n}\right)^p \leq e^{\gamma_\infty p/n} \qquad (n \geq n_0, p \geq 1) .$$

Theorem 3.23 ([11], Theorem 3.3)**.** *Under assumptions* (3.4)*,* (3.5) *and* (3.1) *and under the ones of Theorem 3.19, denoted by* $(S(t))_{t\geq 0}$ *the positive* C_0*-semigroup generated by* $(L, D_M(L))$ *or by* $(L, D_V(L))$*, then for every* $f \in C([0,1])$ *and* $t \geq 0$

$$S(t)f = \lim_{n\to\infty} M_n^{k(n)} f \quad \textit{uniformly on} \quad [0,1]$$

where the operators M_n *are defined by* (3.22) *and* $(k(n))_{n\geq 1}$ *is an arbitrary sequence of positive integers such that* $k(n)/n \to t$ *as* $n \to \infty$*.*

By means of Theorem 3.23 and Proposition 3.21 we obtain the following qualitative properties of the semigroup $(S(t))_{t\geq 0}$ or, equivalently, of the solutions to the initial boundary value problems (1.2) where $D(L) = D_M(L)$ or $D(L) = D_V(L)$.

Proposition 3.24. *The following statements hold true.*

(*1*) *Under the same hypotheses of Theorem 2.9, if in addition* $e_1 + \beta/n$ *is increasing and if, for every* $f \in C([0,1])$*,* $(1 + \gamma/n) f$ *is increasing too, then each* $S(t)$ $(t \geq 0)$ *maps increasing functions into increasing functions.*

(*2*) *Assume that* $T(Lip(1,1)) \subset Lip(1,1)$*, where* T *given in* (2.5)*, and suppose that* $\beta \in Lip(1,B)$*, for some* $B > 0$*, and* γ *is constant. Then, for every* $f \in Lip(\alpha, \widetilde{M})$*,* $\widetilde{M} > 0$*,* $0 < \alpha \leq 1$ *and* $n \geq n_0$*,*

$$S(t)f \in Lip\left(\alpha, \widetilde{M}\exp((\alpha B + |\gamma| t))\right) .$$

(*3*) *Assume that* β *is affine* (*resp. convex*) *on* $[0,1]$ *and that, for every convex function* $f \in C([0,1])$*,* $(1 + \gamma/n) f$ *is convex too. Then each* $S(t)$ $(t \geq 0)$ *maps convex* (*resp. increasing convex*) *functions into convex functions provided that conditions* (c_1) *and* (c_2) *states at p. 15 are satisfied.*

Proof. Because of Theorem 2.9 and 2.14, we have only to prove part (2). Fix $t \geq 0$ and choose a sequence $(k(n))_{n\geq 1}$ such that $k(n)/n \to t$. Without loss of generality, we can assume that $\widetilde{M} = 1$. From Proposition 3.21, (2), it follows that, if $f \in Lip(\alpha, 1)$, then for any $n \geq n_0$,

$$M_n^{k(n)} f \in Lip\left(\alpha, \left(1 + \frac{|\gamma|}{n}\right)^{k(n)} \left(1 + \frac{B}{n}\right)^{k(n)}\right) .$$

Since $(1+|\gamma|/n)^{k(n)}(1+B/n)^{\alpha k(n)} \to \exp(\alpha B+|\gamma| t)$ the result follows from Theorem 3.23.

□

Remark 3.25. We finally refer to [18], [23], [25], [32, 33], [34], [58, 59] where other particular cases of the differential operators (3.1) have been studied in the same spirit of this paper and where other approximating sequences of positive linear operators have been introduced.

In particular, in [23], [58] and [59] the limit behaviour of the semigroup $(S(t))_{t\geq 0}$ as $t \to +\infty$ has been determined in some very special cases.

It would to be interesting and important to get some further results to this respect in the general contest of Theorems 3.19 and 3.23.

3.3. **Final remarks.** The problem of the constructive approximation of semigroups by means of iterates of positive linear operators and the relevant qualitative analysis have been developed in the last fifteen years also in the framework of continuous function spaces on general real intervals and on convex compact subsets.

Without no claim of completeness we mention, e.g., for one-dimensional intervals [11], [16], [17], [20], [21], [24], [34], [36], [49], [50] and the references therein, and for the convex compact sets [3], [14, Chapter 6], [15], [31], [35] and the references therein.

Despite such a high number of papers, the results are far from being conclusive and there seem to be still not devoid of interest a further deepening and a more comprehensive development of these topics together with some significant applications.

References

[1] J.A. Adell, J. de la Cal & I. Raşa, *Lototsky-Schnabl operators on the unit interval*, Rend. Circ. Mat. Palermo (2), 48(3)(1999), 517–536.

[2] A. Albanese, M. Campiti & E. Mangino, *Approximation formulae for C_0-semigroups and their risolvent operators*, J. Appl. Funct. Anal., 1(3)(2006), 343–358.

[3] A. Albanese, M. Campiti & E. Mangino, *Regularity properties of semigroups generated by some Fleming-Viot type operators*, J. Math. Anal. Appl., 335(2)(2007), 1259–1273.

[4] C.D. Aliprantis & O. Burkinshaw, *Positive operators*, Academic Press, New York, 1985.

[5] F. Altomare, *Limit semigroups of Bernstein-Schnabl operators associated with positive projections*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. (4), 16(2)(1989), 259–279.

[6] F. Altomare, *Positive projections, approximation processes and degenerate diffusion equations*, in Recent developments in mathematical analysis and its applications, Bari, Italy, 1990, Conf. Sem. Mat. Univ. Bari, 241(1991), 43–68.

[7] F. Altomare, *Approximation theory methods for the study of diffusion equations*, Approximation Theory, Proc. IDOMAT, 75 M.W. Müller, M. Felten, D.H. Mache Eds., Math. Res., 86, Akademie Verlag, Berlin, 1995, 9–26.

[8] F. Altomare, *Asymptotic formulae for Bernstein-Schnabl operators and smoothness*, Boll. U.M.I. (9), II(1)(2009), 135–150.

[9] F. Altomare & R. Amiar, *Asymptotic formulae for positive linear operators*, Math. Balkanica (NS), 16(2002), 283–304.

[10] F. Altomare & R. Amiar, *Approximation by positive operators of the C_0-semigroups associated with one-dimensional diffusion equations: Part I*, Numer. Funct. Anal. Optim., 26(1)(2005), 1–15.

[11] F. Altomare & R. Amiar, *Approximation by positive operators of the C_0-semigroups associated with one-dimensional diffusion equations: Part II*, Numer. Funct. Anal. Optim., 26(1)(2005), 17–33.

[12] F. Altomare & A. Attalienti, *Degenerate evolution equations in weighted continuous function spaces, Feller process and the Black-Scholes equation, Part I*, Result. Math., 42(2002), 193–211.

[13] F. Altomare & A. Attalienti, *Degenerate evolution equations in weighted continuous function spaces, Feller process and the Black-Scholes equation, Part II*, Result. Math., 42(2002), 212–228.

[14] F. Altomare & M. Campiti, *Korovkin-type approximation theory and its applications*, de Gruyter Studies in Mathematics 17, Walter de Gruyter & Co., Berlin, 1994.

[15] F. Altomare, M. Cappelletti Montano & S. Diomede, *Degenerate elliptic operators, Feller semigroups and modified Bernstein-Schnabl operators*, to appear in Math. Nach., 2010.

[16] F. Altomare, V. Leonessa & S. Milella, *Cores for second-order differential operators on real intervals*, Comm. in Appl. Anal., 13(4)(2009), 447–496.

[17] F. Altomare, V. Leonessa & S. Milella, *Bernstein-Schnabl operators on non compact real intervals*, Jaen J. Approx., 1(2)(2009), 223–256.
[18] F. Altomare & V. Leonessa, *Continuous selections of Borel measures, positive operators and degenerate evolution problems*, Rev. Anal. Numér. Théor. Approx., 36(1)(2007), 9–23.
[19] F. Altomare, V. Leonessa & I. Raşa, *On Bernstein-Schnabl operators on the unit interval*, Z. Anal. Anwend., 27(2008), 353–379.
[20] F. Altomare & S. Milella, *On the C_0-semigroups generated by second-order differential operators on the real line*, Taiwanese J. Math., 13(1)(2009), 25–46
[21] F. Altomare & S. Milella, *Degenerate differential equations and modified Szász-Mirakjan operators*, Preprint, 2009.
[22] F. Altomare & G. Musceo, *Degenerate second order differential operators with generalized reflecting barriers boundary conditions*, to appear in Math. Reports, 2010.
[23] F. Altomare & I. Raşa, *On some classes of diffusion equations and related approximation problems*, in Trends and Applications in Constructive Approximation, M.G. de Bruin, D.H. Mache, J. Szabados Eds., Internat. Series of Numer. Math., 151, Birkhäuser-Verlag, Basel, 2005, 13–26.
[24] F. Altomare & I. Raşa, *On a class of exponential-type operators their limit semigroups*, J. Approx. Theory, 135(2)(2005), 258–275.
[25] A. Attalienti, *Generalized Bernstein-Durrmeyer operators and the associated limit semigroup*, J. Approx. Theory, 99(1999), 289–309.
[26] A. Attalienti & M. Campiti, *Degerate evolution problems and beta-type operators*, Studia Math., 140(2000), 117–139.
[27] H. Bauer, *Probability Theory*, de Gruyter Studies in Mathematics, 23, Walter de Gruyter & Co., Berlin, 1996.
[28] H. Bauer, *Measure and Interation Theory*, de Gruyter Studies in Mathematics, 26, W. de Gruyter & Co., Berlin, 2001.
[29] P.L. Butzer & H. Berens, *Semi-groups of Operators and Approximation*, Die Grundlehren der Mathematischen Wissenschaften, 145, Springer-Verlag, Berlin, 1967.
[30] J.-M. Bony, P. Courège & P. Priouret, *Semigroupes of Feller sur une variété à bord compacte et problèmes aux limites intégro différentielsdu second ordre donnant lieu au principe de maximum*, Ann. Inst. Fourier (Grenoble), 18(1968), 369–521.
[31] M. Campiti, *Recursive Bernstein operators and degenerate diffusion processes*, Acta Sci. Math. (Szeged), 68(2002), 179–201.
[32] M. Campiti & G. Metafune, *Evolution equations associated with recursively defined Bernstein-type operators*, J. Approx. Theory, 87(3)(1996), 270–290.
[33] M. Campiti & G. Metafune, *Approximation of solutions of some degenerate parabolic problems*, Numer. Funct. Anal. Optim., 17(1-2)(1996), 23–35.
[34] M. Campiti, G. Metafune & D. Pallara, *General Voronovskaja formula and solutions of second-order degenerate differential equations*, Rev. Roumaine Math. Pures Appl., 44(5-6)(1999), 755–766.
[35] M. Campiti & I. Raşa, *Qualitative properties of a class of Fleming-Viot operators*, Acta Math. Hungar., 103(1-2)(2004), 55–69.
[36] M. Campiti, I. Raşa & C. Tacelli, *Steklov operators and semigroups in weighted spaces of continuous real functions*, Acta Math. Hungar., 120(1-2)(2008), 103–125.
[37] M. Campiti & C. Tacelli, *Rate of convergence in Trotter's approximation theorem*, Constr. Approx., 28(2008), 333–341.
[38] M. Campiti & C. Tacelli, *Trotter's approximation of semigroups and order of convergence in $C^{2,\alpha}$-spaces*, Preprint, 2009.
[39] Ph. Clément & C.A. Timmermans, *On C_0-semigroups generated by differential operators satisfying Ventcel's boundary conditions*, Indag. Math., 89(1986), 379–387.
[40] G. Choquet, *Lectures on Analysis*, Vol. I, A. Benjamin Inc., New York-Amsterdam, 1969.
[41] Z. Ditzian & V. Totik, *Moduli of smoothness*, Springer Ser. Comput. Math., 9, New York, Springer, 1987.
[42] K.-J. Engel & R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Text in Mathematics, 194, Springer, New York-Berlin, 2000.
[43] W. Feller, *The parabolic differential equations and the associated semigroups of transformations*, Ann. of Math., 55(1952), 468–519.

[44] I. Gavrea, H. Gonska, R. Păltănea & G. Tachev, *General estimates for the Ditzian-Totik modulus*, East J. Approx., 9(2)(2003), 175–194.

[45] H. Gonska & I. Raşa, *The limiting semigroup of the Bernstein iterates: degree of convergence*, Acta Math. Hungar., 111(2006), 111–122.

[46] M.W. Grossman, *Note on a generalized Bohman-Korovkin theorem*, J. Math. Anal. Appl., 45(1974), 43–46.

[47] P.P. Korovkin, *On convergence of linear positive operators in the spaces of continuous functions* (russian), Doklady Akad. Nauk. SSSR (N.S.), 90(1953), 961–964.

[48] R.G. Mamedov, *On the order of the approximation of differentiable functions by linear positive operators* (russian), Doklady SSSR, 146(1962), 1013–1016.

[49] E.M. Mangino, *Differential operators with second-order degeneracy and positive approximation processes*, Constr. Approx., 18(3)(2002), 443–466.

[50] E.M. Mangino, *A positive approximtion sequence related to Black and Scholes equation*, Proceedings of the Fourth International Conference on Functional Analysis and Approximation Theory, Vol. I, (Potenza 2000), Rend. Circ. Mat. Palermo (2) Suppl., 68(2002), part I, 359–372.

[51] E.M. Mangino & I. Raşa, *A quantitative version of Trotter's approximation theorem*, J. Approx. Theory, 146(2007), 149–156.

[52] R. Nagel (Ed.), *One-parameter semigroups of positive operators*, Lecture Notes in Math., 1184, Springer-Verlag, Berlin, 1986.

[53] T. Nishishiraho, *Saturation of bounded linear operators*, Tôhoku Math. J., 30(1978), 69–81.

[54] R. Păltănea, *Approximation theory using positive operators*, Boston (MA), Birkhäuser, 2004.

[55] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Science, 44, Springer Verlag, New York, 1983.

[56] I. Raşa, *Generalized Bernstein operators and convex functions*, Studia Univ. Babeş-Bolyai Math., 33(2)(1988), 36–39.

[57] I. Raşa, *On the monotonicity of sequences of Bernstein-Schnabl operators*, Anal. Numér. Théor. Approx., 17(2)(1988), 185–187.

[58] I. Raşa, *Semigroups associated to Mache operators*, in *Advanced problems in constructive approximation*, M.D. Buhmann & D.H. Mache Eds., ISNM, 142, Birkhäuser-Verlag, Basel, 2002.

[59] I. Raşa, *One-dimensional diffusion and approximation*, Mediterr. J. Math., 2(2005), 153–169.

[60] I. Raşa, *Asymptotic behaviour of iterates of positive linear operators*, Jaen J. Approx., 1(2)(2009), 195–204.

[61] Schnabl, R., *Eine Verallgemeinerung der Bernsteinpolynome*, Math. Ann., 179(1968), 74–82.

[62] Schnabl, R., *Zur Approximation durch Bernstein Polynome auf gewissen Räumen von Wahrscheinlichkeitsmaßen*, Math. Ann., 180(1969), 326–330.

[63] R. Schnabl, *Über gleichmäßige Approximation durch positive lineare Operatoren*, in *Constructive theory of functions*, Proc. Internat. Conf., Varna, 1970, 287–296, Izdat. Bolgar. Akad. Nauk, Sofia, 1972.

[64] K. Taira, *Diffusion processes and partial differential equations*, Academic Press, Boston-San Diego-London-Tokyo, 1988.

[65] H. Trotter, *Approximation of semigroups of operators*, Pacific J. Math., 8(1958), 887–919.

[66] E.V. Voronovskaya, *The asymptotic properties of the approximation of functions with Bernstein polynomials*, Dokl. Akad. Nauk SSSR, A(1932), 79–85.