Lecture Notes of
*Seminario Interdisciplinare di Matematica*
Vol. 8(2009), pp. 43–68.

# Concentration and compactness arguments in coupled nonlinear Schrödinger - Maxwell equations

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**Abstract**[1]. In this survey we give an overview on the Schrödinger-Maxwell system and show how concentration and compactness arguments can be used to prove the existence of solutions with special properties.

## 1. An overview on the problem

In the recent years, the following electrostatic nonlinear Schrödinger-Maxwell equations, also known as nonlinear Schrödinger-Poisson system,

$$(\mathcal{SM}) \qquad \begin{cases} -\Delta u + q\phi u = g(x,u) & \text{in } \mathbb{R}^3 , \\ -\Delta \phi = qu^2 & \text{in } \mathbb{R}^3 , \end{cases}$$

have been object of interest for many authors. Indeed a similar system arises in many mathematical physics contexts, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, but also in semiconductor theory, in nonlinear optics and in plasma physics. The variational structure of the problem allows us to deal with it looking for critical points of the associated functional. Even if the functional presents a strongly indefinite behaviour, a simple application of the reduction method (see [8]) permits to remove this difficulty and approach the problem by the classical technique of critical points theory. In particular, the problem of finding solutions for $(\mathcal{SM})$ can be reduced to that of looking for critical points of a functional depending on the variable $u$ in a suitable functional space. The greatest part of the literature has focused the attention on the study of the previous system for the very special nonlinearity $g(x,u) = -V(x)u + |u|^{p-1}u$. In [7], the autonomous case has been considered, and the linear version of the problem (i.e. $g(x,u) = -\omega u$) has been studied as an eigenvalue problem for a bounded domain. The linear Schrödinger-Maxwell equations have been treated also in [12, 14], where the potential $V$ has been supposed radial.

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The author is supported by M.I.U.R. - P.R.I.N. "Metodi variazionali e topologici nello studio di fenomeni non lineari"

*Keywords.* Schrödinger-Maxwell system, concentration and compactness, general nonlinearity.

*AMS Subject Classification.* 35J60.

The nonlinear case has been considered in many papers, where existence and multiplicity results have been stated when $V$ is a positive constant. In [13] the exponent of the nonlinearity has been taken in the interval $]3,5[$ and the equivariant version of the mountain pass theorem has been used to get a multiplicity result on the set of radial functions. The result has been improved in [16], where a radial solution has been found by means of the mountain pass theorem also for $p=3$. The same authors in [17] proved that no nontrivial solution exists for $p\leq 1$ or $p\geq 5$ (actually they proved it for more general nonlinearities including these). By means of the Pohozaev's fibering method, a multiplicity result has been proved in [27] also in the non-homogeneous case, namely when $g(x)=-u+|u|^{p-1}u+r(u)$ with $r\in L^2(\mathbb{R}^3)$ (see also [10]). In [26], the nonlinearity has assumed the form $g(x,u)=-u+|u|^{p-1}u$ and the case $p\in\,]1,3[$ has been treated. This case is very delicate from a technical point of view, because of the difficulty in proving the boundedness of the so called *Palais-Smale sequences*. By solving a suitable minimizing problem related with the functional constrained on a suitable manifold, it has been proved the existence of a radial solution for any $q>0$ if $p\in]2,3[$. When $p\in]1,2]$ the solution has been obtained for sufficiently small $q$ whereas a nonexistence result was stated for $q$ larger than a suitable value. The same results have been obtained independently in [21], where the author used in an original way the monotonicity trick introduced in [28] and formalized in [19] to prove the boundedness of a special Palais-Smale sequence of the associated functional. In [1], the results obtained in [21, 26] have been improved and multiple radial solutions have been founded for $p\in]1,3]$. In [29] it has been treated the system where a non pure power nonlinearity appears. In fact, the authors considered $g(x,u)=-V(x)u+f(x,u)$ where $f$ was supposed asymptotically linear at infinity with respect to $u$. Existence and non existence results, depending on $q$, have been proved also in this case. Recently a new interest for the nonlinear elliptic equations with a general nonlinearity has motivated the study of this system in presence of a *Berestycki-Lions* type nonlinearity. Using exactly the same assumptions on $g$ introduced in [9], in [3] an existence result for small $q$ has been proved by means of a method combining the monotonicity trick and a penalization argument due to Jeanjean and Le Coz [20].

Apart from [29], which however does not include nonlinearities as the pure powers, in all these papers the set of the radial functions has been introduced as a nice functional setting to overcome the difficulty of the lack of compactness due to the unbondedness of the domain $\mathbb{R}^3$. However, it could happen that such a restriction is either not allowed or not suitable to our aim. For example, consider these three situations:

- $g(\cdot,s)$ is not invariant under the action of the group of rotations (for example in presence of a breaking-symmetry potential),
- we are looking for a *ground state solution*,
- we are looking for non-radial solutions of the problem.

The present survey is aimed in showing how the concentration and compactness method can be successfully used in these three situations to recover the compactness.

The paper is so divided: in section 2 we briefly show the physical interpretation of the system. In section 3 we give some preliminary results useful to introduce the variational problem and the difficulties related with the application of the critical points theory. In section 4 we spend some words on the concentration and compactness principle introduced by Lions [23] and explain the fundamental role that it plays in solving $(\mathcal{SM})$ in the three situations previously mentioned. In section 5 we present a first application of this method: we show how it has been used in [4] to prove the existence of a ground state solution to $(\mathcal{SM})$ for $g(x,u) = -u + |u|^{p-1}u$. In section 6, following [4], we use a concentration and compactness argument to study the existence of a ground state to $(\mathcal{SM})$ for $g(x,u) = -V(x)u+|u|^{p-1}u$ when $V$ is a potential *à la* Rabinowitz. At the end, in section 7 we use the concentration and compactness to show that, if $g$ is a Berestycki-Lions type nonlinearity, there exists $q$ for which the system possesses a nonradial solution. In this last section we will follow [2].

## 2. The physical origin

The nonlinear Schrödinger equation

$$(\mathcal{NLS}) \qquad -i\hbar\,\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\,\Delta\psi + F'(x,\psi)$$

where $\psi = \psi(x,t)\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ is a field function, $\hbar$ denotes the Planch constant, $F : \mathbb{R}^3 \times \mathbb{C} \to \mathbb{R}$ and $m$ the mass, is well-known in quantistic mechanics and arises as a model equation from several areas of physics.

It is the Euler-Lagrange equation with respect to the action

$$\mathcal{S} = \iint \mathcal{L}\, dx\, dt\ ,$$

being $\mathcal{L}$ the Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\left[\hbar\left\langle i\frac{\partial\psi}{\partial t},\psi\right\rangle - \frac{\hbar^2}{2m}\,|\nabla\psi|^2\right] + F(x,\psi)$$

with

$$\langle z_1, z_2\rangle = x_1x_2 + y_1y_2 \quad , \quad |z| = \sqrt{\langle z, z\rangle}\ ,$$

$z_j = x_j + iy_j \in \mathbb{C}$, $x_j$, $y_j \in \mathbb{R}$. If we assign the electromagnetic field $(\mathbf{E},\mathbf{B})$, which is described by the gauge potential

$$\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \quad , \quad \mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$$

by means of the following equations

$$\mathbf{E} = -\left(\nabla\phi + \frac{\partial\mathbf{A}}{\partial t}\right) \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}\ ,$$

the interaction between the electromagnetic field $(\mathbf{E},\mathbf{B})$ and the field $\psi$ is described by the rule of "minimal coupling": formally in the lagrangian density $\mathcal{L}$ the ordinary derivatives $\partial/\partial t, \nabla$ are substituted by Weyl covariant derivatives, namely

$$\frac{\partial}{\partial t} \leftrightarrow \frac{\partial}{\partial t} + \frac{iq}{\hbar}\phi \quad , \quad \nabla \leftrightarrow \nabla - \frac{iq}{\hbar}\,\mathbf{A}\ ,$$

$q$ being the electrical charge. Therefore $\mathcal{L}$ becomes

$$\mathcal{L}_0 = \frac{1}{2}\left[\hbar\left\langle i\frac{\partial\psi}{\partial t},\psi\right\rangle - q\phi|\psi|^2 - \frac{\hbar^2}{2m}\,|\nabla\psi - \frac{iq}{\hbar}\,\mathbf{A}\psi|^2\right] + F(x,\psi)\ .$$

whose Euler Lagrange equation describes the dynamic of a charged particle immersed in the assigned electromagnetic field $(\mathbf{E},\mathbf{B})$.

If we suppose that the electromagnetic field $(\mathbf{E},\mathbf{B})$ is not assigned, then we need to add the lagrangian density of the electromagnetic field

$$\mathcal{L}_1 = \frac{1}{8\pi}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) = \frac{1}{8\pi}\left(\left|\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right|^2 - |\nabla\times\mathbf{A}|^2\right) .$$

Therefore the total action is:

$$\mathcal{S}_{tot}(\psi,\phi,\mathbf{A}) = \iint (\mathcal{L}_0 + \mathcal{L}_1)\, dx\, dt .$$

If we set

$$\psi(x,t) = u(x,t)e^{iS(x,t)/\hbar} ,$$

with $u, S \in \mathbb{R}$, and suppose that $F(\cdot, e^{i\theta}\psi) = F(\cdot,\psi)$, the Euler-Lagrange equations of the functional $\mathcal{S}_{tot} = \mathcal{S}_{tot}(u,S,\phi,\mathbf{A})$ with respect to $u, S, \phi$ and $\mathbf{A}$ are:

$$-\frac{\hbar^2}{2m}\,\Delta u + \left(\frac{\partial S}{\partial t} + q\phi + \frac{1}{2m}\,|\nabla S - q\mathbf{A}|^2\right) u - F'(x,u) = 0 , \tag{1}$$

$$\frac{\partial u^2}{\partial t} + \frac{1}{m}\,\mathrm{div}\left((\nabla S - q\mathbf{A})u^2\right) = 0 , \tag{2}$$

$$qu^2 = -\frac{1}{2\pi}\,\mathrm{div}\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right) , \tag{3}$$

$$\frac{q}{2m}\left(\nabla S - q\mathbf{A}\right) u^2 = \frac{1}{4\pi}\left(\frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right) + \nabla\times(\nabla\times\mathbf{A})\right) . \tag{4}$$

Particular solutions of the system can be found as standing waves in the electrostatic case, namely

$$u = u(x) \quad , \quad S = \omega t \quad , \quad \phi = \phi(x) \quad , \quad \mathbf{A} = 0 ,$$

where $\omega$ is a positive constant.

Then (2) and (4) are identically satisfied, while (1) and (3) become

$$-\frac{\hbar^2}{2m}\,\Delta u + \omega u + q\phi u - F'(x,u) = 0 , \tag{5}$$

$$-\Delta\phi = 2\pi q u^2 . \tag{6}$$

Equations (5) and (6) are called *nonlinear Schrödinger-Maxwell equations* and describes the dynamics of a nonrelativistic charged particle interacting with the electromagnetic field. The previous system is formally equivalent to $(\mathcal{SM})$, setting $-\omega u + F'(x,u) = g(x,u)$. It is also known with the name *Schrödinger-Poisson* since the second equation corresponds with the Poisson's.

## 3. Some preliminaries

We denote by $H^1(\mathbb{R}^3)$, $\mathcal{D}^{1,2}(\mathbb{R}^3)$, $L^p(\mathbb{R}^3)$ the usual Sobolev and Lebesgue spaces with the respective norms:

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\right)^{1/2} \;,\; \|u\|_{D^{1,2}}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{1/2} \;,\; \|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p\right)^{1/p} .$$

By $\|\cdot\|$ we could mean also a norm equivalent to that defined. Under suitable growth conditions on the nonlinearity, the functional

$$\mathcal{E}_q(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{q}{2}\int_{\mathbb{R}^3} u^2\phi - \int_{\mathbb{R}^3} G(x,u)\ , \tag{7}$$

where $u \in H^1(\mathbb{R}^3)$, $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, and $(dG/ds)(x,s) = g(x,s)$ for all $x \in \mathbb{R}^3$ and $s \in \mathbb{R}$, is finite and $C^1$, and $(\mathcal{SM})$ can be solved looking for its critical points. The functional $\mathcal{E}_q$ exhibits a strong indefinite nature, in the sense that it is unbounded below and above in infinite dimensional spaces. This fact represents a difficulty in approaching the problem using the techniques of the critical points theory. To overcome this difficulty, we use the reduction method. We recall the following well-known fact (see, for instance [7, 16]).

**Lemma 3.1.** *For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = qu^2 \quad , \quad \text{in } \mathbb{R}^3\ .$$

*Moreover such a solution has the following explicit form*

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\, dy \tag{8}$$

*Proof.* Take $u \in H^1(\mathbb{R}^3)$. Then by well known embedding theorems, certainly $u \in L^{12/5}(\mathbb{R}^3)$. As a consequence, $u \in L^{6/5}(\mathbb{R}^3)$ and then $u \in (\mathcal{D}^{1,2}(\mathbb{R}^3))'$, the dual space of $\mathcal{D}^{1,2}(\mathbb{R}^3)$. So, by inverting the Laplacian operator, we find just one function in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ solving the second equation. The remaining part of the theorem is a consequence of the fact that $1/|x|$ is the fundamental solution of the Poisson equation.

□

It is an obvious consequence the following

**Corollary 3.2.** *For every $u \in L^{12/5}(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = qu^2 \quad , \quad \text{in } \mathbb{R}^3\ .$$

By Corollary 3.2, it is well defined the map $\Phi : L^{12/5}(\mathbb{R}^3) \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that for every $u \in L^{12/5}(\mathbb{R}^3)$ we set $\Phi(u) = \phi_u$. Applying the implicit functions theorem to $\partial\mathcal{A}_q/\partial\phi$ where

$$\mathcal{A}_q(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 - q\int_{\mathbb{R}^3} u^2\phi\ ,$$

we deduce that $\Phi$ is $C^1$. Then it is well defined and of class $C^1$ the functional

$$I_q(u) = \mathcal{E}_q(u,\Phi(u)) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 + \frac{q}{2}\int_{\mathbb{R}^3} u^2\phi_u - \int_{\mathbb{R}^3} G(x,u)\ .$$

**Lemma 3.3.** *For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = qu^2 \quad , \quad \text{in } \mathbb{R}^3\ .$$

*Moreover*

- *i)* $\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = q\int_{\mathbb{R}^3} \phi_u u^2$;
- *ii)* $\phi_u \geq 0$;
- *iii)* *for any $t > 0$ and $\theta > 0$: $\phi_{tu_\theta}(x) = t^2\theta^2\phi_u(x/\theta)$, where $u_\theta(x) = u(x/\theta)$;*

*iv) there exist $C, C' > 0$ independent of $u \in H^1(\mathbb{R}^3)$ such that*

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leq Cq\|u\|^2_{12/5} \,,$$

*and*

$$\int_{\mathbb{R}^3} \phi_u u^2 \leq C' q\|u\|^4_{12/5} \,;$$

*Proof.* The proofs can be found in [16, 25]. We report it here. Property $i$ follows from the second equation, multiplying by $\phi$ and integrating. To prove $ii$ it is enough to observe that actually $\phi_u$ is the minimizer of the functional $\mathcal{A}(u, \cdot)$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Since $\mathcal{A}(u, |\phi_u|) \leq \mathcal{A}(u, \phi_u)$, by uniqueness we deduce that $\phi_u = |\phi_u|$. Property $iii$ is a consequence of a simple computation. Finally, $iv$ derives from $i$, Hölder inequality and Sobolev embeddings. In fact

$$\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = q \int_{\mathbb{R}^3} \phi_u u^2 \leq q \left( \int_{\mathbb{R}^3} \phi_u^6 \right)^{1/6} \left( \int_{\mathbb{R}^3} u^{12/5} \right)^{5/6} \leq$$

$$\leq C\, q \left( \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \right)^{1/2} \left( \int_{\mathbb{R}^3} u^{12/5} \right)^{5/6} = C\, q\, \|\phi_u\|_{D^{1,2}} \|u\|^2_{12/5} \,.$$

□

By i) of the previous Lemma, we can write the functional $I_q$ in the following equivalent way

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} u^2 \phi_u - \int_{\mathbb{R}^3} G(x, u) \,.$$

**Theorem 3.4.** *The couple $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of $(\mathcal{SM})$ (critical point of functional $\mathcal{E}_q$) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I_q \colon H^1(\mathbb{R}^3) \to \mathbb{R}$ and $\phi = \phi_u$.*

*Proof.* Indeed let $u \in H^1(\mathbb{R}^3)$ be a critical point of $I_q$ and $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. Then

$$\frac{\partial \mathcal{E}_q}{\partial u}(u, \phi) + \frac{\partial \mathcal{E}_q}{\partial \phi}(u, \phi)\Phi'(u) = 0 \quad \text{and} \quad \frac{\partial \mathcal{E}_q}{\partial \phi}(u, \phi) = 0 \,,$$

so we deduce that also

$$\frac{\partial \mathcal{E}_q}{\partial u}(u, \phi) = 0 \,.$$

The "only if" part is deduced inverting the arguments.

□

$I_q$ is called *the reduced functional* and it does not exhibit the strong indefiniteness presented by $\mathcal{E}_q$.

## 4. The concentration and compactness principle

In this section we present the Lions' concentration and compactness principle, and explain its fundamental role inside the critical points theory. It is well known that the lack of compactness in unbounded domain constitutes one of the main difficulty to deal with when we are looking for the critical points of a $C^1$ functional $I$. Indeed, usually we find a critical point by detecting a Palais-Smale (briefly (P.S.)) level, namely a value $c \in \mathbb{R}$ such that there exists a sequence $u_n$ (called (P.S.) sequence) for which $I'(u_n) \to 0$ and $I(u_n) \to c$, and then extracting a converging subsequence whose limit is the critical point we are looking for. When the domain is unbounded, the compact immersions of the Sobolev spaces do not hold anymore, and one has to study deeply the behaviour of the (P.S.) sequences. The concentration and compactness principle is a useful tool in this sense. Now we are going to state precisely what it consists in.

**Theorem 4.1.** *Let $(\sigma_n)_n$ be a sequence in $L^1(\mathbb{R}^N)$, and $\lambda > 0$ such that*

$$\sigma_n \geq 0 \quad , \quad \int_{\mathbb{R}^N} \sigma_n \, dx \to \lambda \, .$$

*Then, up to a subsequence, one of the following possibilities holds:*

1. (*vanishing*) $\lim_n \sup_{y \in \mathbb{R}^N} \int_{y+B_R} \sigma_n(x)\, dx = 0$ , *for all* $R < \infty$
2. (*dichotomy*) *there exists* $\alpha \in ]0, \lambda[$ *such that for all* $\varepsilon > 0$, *there exist* $n_0 \geq 1$ *and* $\sigma_n^1$, $\sigma_n^2 \in L^1(\mathbb{R}^N)$, $\sigma_n^1 \geq 0$, $\sigma_n^2 \geq 0$, *satisfying for any* $n \geq n_0$

$$\|\sigma_n - \sigma_n^1 - \sigma_n^2\|_{L^1} \leq \varepsilon \, , \quad \left| \int_{\mathbb{R}^N} \sigma_n^1 \, dx - \alpha \right| \leq \varepsilon \, , \quad \left| \int_{\mathbb{R}^N} \sigma_n^2 \, dx - (\lambda - \alpha) \right| \leq \varepsilon \, ,$$

$$\mathrm{dist}(\mathrm{supp}\, \sigma_n^1, \mathrm{supp}\, \sigma_n^2) \to +\infty$$

3. (*concentration*) *there exists* $y_n \in \mathbb{R}^N$ *such that* $\sigma_n(\cdot + y_n)$ *concentrates in a bounded set, that is*

$$\forall \varepsilon > 0 \, , \ \exists \, R \in ]0, +\infty[ \, , \ \textit{such that} \int_{y_n + B_R} \sigma_n(x) \, dx \geq \lambda - \varepsilon \, , \ \forall n \, .$$

The proof of this celebrated result is in [23]. Here we just want to make some comments on it, to give an idea about its applications. Roughly speaking, one can interpret the integral $\int_{\mathbb{R}^N} \sigma_n(x)\, dx$ as the total *mass* associated to the function $\sigma_n$ and the integral $\int_\Omega \sigma_n(x)\, dx$ as the part of the mass which is inside the set $\Omega \subset \mathbb{R}^N$. The three possibilities above describe what happens to the total mass as $n$ goes to infinity. Vanishing means that the total mass spreads on the whole space. Dichotomy means that the total mass "breaks" in two separated parts, each one travelling away from the other. Concentration means that the total mass tends to accumulate inside a travelling bounded region of $\mathbb{R}^N$. Then, if concentration holds, we have that almost the total mass lies, uniformly with respect to $n$, inside a bounded region (up to translations). This allows us to use compact embedding theorems in the bounded region where $\sigma_n$ concentrates, and obtain strong convergence. Actually, the most general version of the concentration and compactness principle involves the positive measures. We have presented the previous one since it is more purposeful in our contest.

## 5. Ground state solution (autonomous case)

In this section we will assume that

$$g(x,u) = -u + |u|^{p-1}u \quad , \quad \text{with } 2 < p < 6 \, . \tag{9}$$

Since $p$ is subcritical, by standard arguments we deduce that $I_q \in C^1(H^1(\mathbb{R}^3))$. The value of the parameter $q$ is not important in what follows, so we do not lose generality if we set $q = 1$ and we use the symbol $I$ instead of $I_q$ to denote the functional. So

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 + \frac{q}{4} \int_{\mathbb{R}^3} u^2 \phi_u - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, .$$

It can be proved (see [17, 26]) that if $(u,\phi)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of $(\mathcal{SM})$, then it satisfies the following Pohozaev type identity

$$\int_{\mathbb{R}^3}\frac{1}{2}\,|\nabla u|^2+\frac{3}{2}\,u^2+\frac{5}{4}\,\phi u^2-\frac{3}{p+1}\,|u|^{p+1}=0\,. \tag{10}$$

As in [26], we introduce the following manifold

$$\mathcal{M}:=\left\{u\in H^1(\mathbb{R}^3)\setminus\{0\}\;\mid\;P(u)=0\right\}\,,$$

where

$$P(u):=\int_{\mathbb{R}^3}\frac{3}{2}\,|\nabla u|^2+\frac{1}{2}\,u^2+\frac{3}{4}\,\phi_u u^2-\frac{2p-1}{p+1}\,|u|^{p+1}\,.$$

**Remark 5.1.** Observe that if $u\in H^1(\mathbb{R}^3)$ is a nontrivial critical point of $I$, then $u\in\mathcal{M}$. Indeed $G(u)=0$ can be obtained by a linear combination of $\langle I'(u),u\rangle=0$ and (10), with $\phi=\phi_u$. As a consequence, if $(u,\phi)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of $(\mathcal{SM})$, then $u\in\mathcal{M}$.

The next lemma describes some properties of the manifold $\mathcal{M}$:

**Lemma 5.2.**

1. *For any $u\in H^1(\mathbb{R}^3)$, $u\neq 0$, there exists a unique number $\overline{\theta}>0$ such that $u_{\overline{\theta}}\in\mathcal{M}$, where $u_\theta=\theta^2u(\theta\cdot)$. Moreover*
$$I(u_{\overline{\theta}})=\max_{\theta\geq 0}I(u_\theta)\;;$$
2. *there exists a positive constant $C$, such that for all $u\in\mathcal{M}$, $\|u\|_{p+1}\geq C$;*
3. *$\mathcal{M}$ is a natural constraint of $I$, namely every critical point of $I|_{\mathcal{M}}$ is a critical point for $I$.*

*Proof.* The proofs are in [26]. We report only those concerning 1 and 2. By *iii)* of the Lemma 3.3 and a change of variable, we have that for any $u\in H^1(\mathbb{R}^3)\setminus\{0\}$ and $\theta>0$

$$P(u_\theta)=\theta^3\int_{\mathbb{R}^3}\left(\frac{3}{2}\,|\nabla u|^2+\frac{3}{4}\,\phi_u u^2\right)+\theta\int_{\mathbb{R}^3}\frac{1}{2}\,u^2-\theta^{2p-1}\int_{\mathbb{R}^3}\frac{(2p-1)}{p+1}\,|u|^{p+1}\,.$$

So, since $p>2$, we deduce that there exists a unique $\overline{\theta}>0$ such that $P(u_{\overline{\theta}})=0$. On the other hand,

$$f(\theta):=I(u_\theta)=\frac{\theta^3}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{\theta}{2}\int_{\mathbb{R}^3}u^2+\frac{\theta^3}{4}\int_{\mathbb{R}^3}\phi_u u^2-\frac{\theta^{2p-1}}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,,$$

so for any critical point $\overline{\theta}$ of $f$ there is a corresponding a solution $u_{\overline{\theta}}$ of the equation $P(v)=0$. By the geometry of the function $f$, the unique critical point is its maximum. We have so proved 1.

Now we pass to prove 2. Suppose $u\in\mathcal{M}$, then for a suitable $C>0$

$$\frac{1}{2}\,\|u\|^2\leq\int_{\mathbb{R}^3}\frac{3}{2}\,|\nabla u|^2+\frac{1}{2}\,u^2+\frac{3}{4}\,\phi_u u^2=\frac{2p-1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\leq C\,\frac{2p-1}{p+1}\,\|u\|^{p+1}\,,$$

so we deduce that, for a suitable $C>0$ independent from $u$, $C<\|u\|$. Since $u\in\mathcal{M}$, we infer also that for another constant $C'>0$ independent from $u$, $C'<\|u\|_{p+1}$.

□

5.1. **Proof of the existence of the ground state.** A *ground state* is a solution which minimizes the functional of the action among all the solutions. The problem to look for such a type of solution is very classical, and dates back to a work of Coleman, Glaser and Martin [15]. Taking into account the results of the previous section, we deduce the following fact

**Remark 5.3.** By point 3 of Lemma 5.2 and Remark 5.1, we argue that if $u \in \mathcal{M}$ is such that $I(u) = c$, then $(u, \phi_u)$ is a ground state solution of $(\mathcal{SM})$.

Here we prove the existence of a ground state solution for $(\mathcal{SM})$ assuming (9). According to what we have established in remark (5.3), we want to solve the following minimizing problem:

$$\text{to find } \overline{u} \in \mathcal{M} \text{ such that } I(\overline{u}) = \min_{u \in \mathcal{M}} I(u) .$$

To do this, we have to investigate the compactness properties of a minimizing sequence.

Let $(u_n)_n \subset \mathcal{M}$ be such that

$$\lim_n I(u_n) = c . \tag{11}$$

We define the functional $J \colon H^1(\mathbb{R}^3) \to \mathbb{R}$ as:

$$J(u) = \int_{\mathbb{R}^3} \frac{p-2}{2p-1} |\nabla u|^2 + \frac{p-1}{2p-1} u^2 + \frac{p-2}{2(2p-1)} \phi_u u^2 .$$

Observe that for any $u \in \mathcal{M}$, $I(u) = J(u)$ and, by *ii* of Lemma 3.3 we have $J(u) \geq 0$. By (11), we deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\overline{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \rightharpoonup \overline{u} \quad \text{weakly in } H^1(\mathbb{R}^3) , \tag{12}$$

$$u_n \to \overline{u} \quad \text{in } L^s(B) , \text{ with } B \subset \mathbb{R}^3 , \text{ bounded, and } 1 \leq s < 6 .$$

We use a concentration-compactness argument. In particular we are going to rule out vanishing and dichotomy for the sequence of $L^1$ functions,

$$\sigma_n = \frac{p-2}{2p-1} |\nabla u_n|^2 + \frac{p-1}{2p-1} u_n^2 + \frac{p-2}{2(2p-1)} \phi_{u_n} u_n^2 . \tag{13}$$

Arguing as in [30], we prove the following

**Lemma 5.4.** *Compactness holds for the sequence of measures $(\nu_n)_n$, defined in* (13).

*Proof.* Vanishing does not occur

Suppose by contradiction, that for all $R > 0$

$$\lim_n \sup_{y \in \mathbb{R}^N} \int_{y + B_R} \sigma_n(x)\, dx = 0 .$$

In particular, we deduce that there exists $\overline{r} > 0$ such that

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_{\overline{r}}(\xi)} u_n^2 = 0 .$$

By [24, Lemma I.1], we have that $u_n \to 0$ in $L^s(\mathbb{R}^3)$, for $2 < s < 6$. As a consequence, since $(u_n)_n \subset \mathcal{M}$ and by Lemma 3.3, we get

$$\begin{aligned} 0 \le I(u_n) &\le \int_{\mathbb{R}^3} \frac{3}{2}\,|\nabla u_n|^2 + \frac{1}{2}\,u_n^2 + \frac{1}{4}\,\phi_{u_n} u_n^2 - \frac{1}{p+1}\,|u_n|^{p+1} = \\ &= -\frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{2p-2}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \to 0 \end{aligned}$$

which contradicts (11).

Dichotomy does not occur

Suppose by contradiction that there exists $\tilde{c} \in ]0, c[$ such that for all $\varepsilon > 0$, there exist $n_0 \ge 1$ and $\sigma_n^1$, $\sigma_n^2 \in L^1(\mathbb{R}^N)$, $\sigma_n^1 \ge 0$, $\sigma_n^2 \ge 0$, satisfying for any $n \ge n_0$

$$\|\sigma_n - \sigma_n^1 - \sigma_n^2\|_{L^1} \le \varepsilon \quad , \quad \left|\int_{\mathbb{R}^N} \sigma_n^1\,dx - \tilde{c}\right| \le \varepsilon \quad , \quad \left|\int_{\mathbb{R}^N} \sigma_n^2\,dx - (c - \tilde{c})\right| \le \varepsilon\,,$$

$$\operatorname{dist}(\operatorname{supp}\sigma_n^1, \operatorname{supp}\sigma_n^2) \to +\infty\,.$$

So there exist two sequences $(\xi_n)_n \subset \mathbb{R}^3$ and $(r_n)_n$, with $r_n \to +\infty$, and two sequences of $L^1$ functions $\sigma_n^1$ and $\sigma_n^2$ such that

$$0 \le \int_{\mathbb{R}^3} \sigma_n^1 + \sigma_n^2 \le \int_{\mathbb{R}^3} \sigma_n \quad , \quad \int_{\mathbb{R}^3} \sigma_n^1 \to \tilde{c} \quad , \quad \int_{\mathbb{R}^3} \sigma_n^2 \to c - \tilde{c}\,,$$

$$\operatorname{supp}(\sigma_n^1) \subset B_{r_n}(\xi_n) \quad , \quad \operatorname{supp}(\sigma_n^2) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n)\,.$$

Let $\rho_n \in C^1(\mathbb{R}^3)$ be such that $\rho_n \equiv 1$ in $B_{r_n}(\xi_n)$, $\rho_n \equiv 0$ in $\mathbb{R}^3 \setminus B_{2r_n}(\xi_n)$, $0 \le \rho_n \le 1$ and $|\nabla \rho_n| \le 2/r_n$. We set

$$v_n := \rho_n u_n \quad , \quad w_n := (1 - \rho_n) u_n\,.$$

It is easy to see that

$$\liminf_n J(v_n) \ge \tilde{c} \quad , \quad \liminf_n J(w_n) \ge c - \tilde{c}\,.$$

Moreover, denoting $\Omega_n := B_{2r_n}(\xi_n) \setminus B_{r_n}(\xi_n)$, we have

$$\int_{\Omega_n} \sigma_n \to 0 \quad , \quad \text{as } n \to \infty\,,$$

namely

$$\int_{\Omega_n} |\nabla u_n|^2 + u_n^2 \to 0 \quad , \quad \int_{\Omega_n} \phi_{u_n} u_n^2 \to 0 \quad , \quad \text{as } n \to \infty\,. \tag{14}$$

By simple computations, we infer also

$$\int_{\Omega_n} |\nabla v_n|^2 + v_n^2 \to 0 \quad , \quad \int_{\Omega_n} |\nabla w_n|^2 + w_n^2 \to 0 \quad , \quad \text{as } n \to \infty\,.$$

Hence, we deduce that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2 = \int_{\mathbb{R}^3} |\nabla v_n|^2 + v_n^2 \quad + \int_{\mathbb{R}^3} |\nabla w_n|^2 + w_n^2 + o_n(1)\,, \tag{15}$$

$$\int_{\mathbb{R}^3} |u_n|^{p+1} = \int_{\mathbb{R}^3} |v_n|^{p+1} + \int_{\mathbb{R}^3} |w_n|^{p+1} + o_n(1)\,. \tag{16}$$

Moreover, by point $v$ of Lemma 3.3 and (14), we have

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + 2 \int_{B_{r_n}} \int_{B^c_{2r_n}} \frac{u_n^2(x) u_n^2(y)}{|x-y|}\, dx\, dy + o_n(1) \geq$$

$$\geq \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + o_n(1)\ . \tag{17}$$

Hence, by (15) and (17), we get

$$J(u_n) \geq J(v_n) + J(w_n) + o_n(1)\ .$$

Then

$$c = \lim_n J(u_n) \geq \liminf_n J(v_n) + \liminf_n J(w_n) \geq \tilde{c} + (c - \tilde{c}) = c\ ,$$

hence

$$\lim_n J(v_n) = \tilde{c} \quad , \quad \lim_n J(w_n) = c - \tilde{c}\ . \tag{18}$$

We recall the definition of the functional $G \colon H^1(\mathbb{R}^3) \to \mathbb{R}$

$$G(u) = \int_{\mathbb{R}^3} \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3}{4} \phi_u u^2 - \frac{2p-1}{p+1} |u|^{p+1}$$

and that if $u \in \mathcal{M}$, then $G(u) = 0$. By (15), (16) and (17), we have

$$0 = G(u_n) \geq G(v_n) + G(w_n) + o_n(1)\ . \tag{19}$$

By Lemma 5.2, for any $n \geq 1$, there exists $\theta_n > 0$ such that $(v_n)_{\theta_n} \in \mathcal{M}$, and then

$$\int_{\mathbb{R}^3} \frac{3}{2} \theta_n^2 |\nabla v_n|^2 + \frac{1}{2} v_n^2 + \frac{3}{4} \theta_n^2 \phi_{v_n} v_n^2 = \int_{\mathbb{R}^3} \frac{2p-1}{p+1} \theta_n^{2p-2} |v_n|^{p+1}\ . \tag{20}$$

We have to distinguish three cases.

Case 1: up to a subsequence, $G(v_n) \leq 0$.
By (20) we have

$$\int_{\mathbb{R}^3} \frac{3}{2} (\theta_n^{2p-2} - \theta_n^2) |\nabla v_n|^2 + \frac{1}{2} (\theta_n^{2p-2} - 1) v_n^2 + \frac{3}{4} (\theta_n^{2p-2} - \theta_n^2) \phi_{v_n} v_n^2 \leq 0\ ,$$

which implies that $\theta_n \leq 1$. Therefore, for all $n \geq 1$

$$c \leq I\left((v_n)_{\theta_n}\right) = J\left((v_n)_{\theta_n}\right) \leq J(v_n) \to \tilde{c} < c\ ,$$

which is a contradiction.

Case 2: up to a subsequence, $G(w_n) \leq 0$. We can argue as in the previous case.

Case 3: up to a subsequence, $G(v_n) > 0$ and $G(w_n) > 0$.
By (19), we infer that $G(v_n) = o_n(1)$ and $G(w_n) = o_n(1)$. If $\theta_n \leq 1 + o_n(1)$, we can repeat the arguments of Case 1. Suppose that

$$\lim_n \theta_n = \theta_0 > 1\ .$$

We have

$$o_n(1) = G(v_n) = \int_{\mathbb{R}^3} \frac{3}{2} |\nabla v_n|^2 + \frac{1}{2} v_n^2 + \frac{3}{4} \phi_{v_n} v_n^2 - \frac{2p-1}{p+1} |v_n|^{p+1} =$$

$$= \int_{\mathbb{R}^3} \frac{3}{2} \left(1 - \frac{1}{\theta_n^{2p-4}}\right) |\nabla v_n|^2 + \frac{1}{2} \left(1 - \frac{1}{\theta_n^{2p-2}}\right) v_n^2 + \int_{\mathbb{R}^3} \frac{3}{4} \left(1 - \frac{1}{\theta_n^{2p-4}}\right) \phi_{v_n} v_n^2$$

and so $v_n \to 0$ in $H^1(\mathbb{R}^3)$, but we get a contradiction with (18). Hence we conclude that dichotomy can not occur.

□

Now we are able to yield the following

**Theorem 5.5.** *There exists a ground state.*

*Proof.* Let $(u_n)_n$ be a sequence in $\mathcal{M}$ such that (11) holds. We define $(\sigma_n)_n$ as in (13); by Lemma 5.4 there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^N$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_r^c(\xi_n)} \frac{p-2}{2p-1}\,|\nabla u_n|^2 + \frac{p-1}{2p-1}\,u_n^2 + \frac{p-2}{2(2p-1)}\,\phi_{u_n}u_n^2 < \delta\,. \tag{21}$$

We define the new sequence of functions $v_n := u_n(\cdot - \xi_n) \in H^1(\mathbb{R}^3)$. It is easy to see that $\phi_{v_n} = \phi_{u_n}(\cdot - \xi_n)$, and hence $v_n \in \mathcal{M}$. Moreover, by (21), we have that for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\|v_n\|_{H^1(B_r^c)} < \delta \quad \text{uniformly for} \quad n \geq 1\,. \tag{22}$$

Since, by (12), $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$. certainly there exist a subsequence (likewise labelled) and $\overline{v} \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup \overline{v} \quad \text{weakly in} \quad H^1(\mathbb{R}^3)\,, \tag{23}$$

$$v_n \to \overline{v} \quad \text{in} \ \ L^s(B)\,, \ \ \text{with} \ \ B \subset \mathbb{R}^3\,, \ \ \text{bounded, and} \ \ 1 \leq s < 6\,. \tag{24}$$

By (22), (23) and (24), we have that, taken $s \in [2,6[$, for any $\delta > 0$ there exists $r > 0$ such that, for any $n \geq 1$ large enough

$$\begin{aligned}\|v_n - \overline{v}\|_{L^s(\mathbb{R}^3)} &\leq \|v_n - \overline{v}\|_{L^s(B_r)} + \|v_n - \overline{v}\|_{L^s(B_r^c)} \leq \\ &\leq \delta + C\left(\|v_n\|_{H^1(B_r^c)} + \|\overline{v}\|_{H^1(B_r^c)}\right) \leq (1+2C)\delta\,,\end{aligned}$$

where $C > 0$ is the constant of the embedding $H^1(B_r^c) \hookrightarrow L^s(B_r^c)$. We deduce that

$$v_n \to \overline{v} \quad \text{in} \ \ L^s(\mathbb{R}^3) \quad , \quad \text{for any} \ \ s \in [2,6[\,. \tag{25}$$

Since $\phi$ is continuous from $L^{12/5}(\mathbb{R}^3)$ to $\mathcal{D}^{1,2}(\mathbb{R}^3)$, from (25) we deduce that

$$\phi_{v_n} \to \phi_{\overline{v}} \quad \text{in} \ \ \mathcal{D}^{1,2}(\mathbb{R}^3) \quad , \quad \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \to \int_{\mathbb{R}^3} \phi_{\overline{v}}\overline{v}^2 \quad , \quad \text{as} \ \ n \to \infty\,. \tag{26}$$

Since $(v_n)_n$ is in $\mathcal{M}$, by 2 of Lemma 5.2 $(\|v_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (25) implies that $\overline{v} \neq 0$. Proceeding as in [26, Theorem 3.2, Step 4], by (25) and (26) we can show that $v_n \to \overline{v}$ in $H^1(\mathbb{R}^3)$ so that $\overline{v} \in \mathcal{M}$ and $I(\overline{v}) = c$. By Remark 5.3, we have that $(\overline{v}, \phi_{\overline{v}})$ is a ground state solution of $(\mathcal{SM})$. □

## 6. Ground state solution (nonautonomous case)

In this section the value of $q$ is not important, so we again assume it as 1. We suppose that $g(x,u) = -V(x)u + |u|^{p-1}u$ with $p \in ]3,5[$ and the potential $V$ satisfying

- **(V1)** $V\colon \mathbb{R}^3 \to \mathbb{R}$ is a measurable function;
- **(V2)** $V_\infty := \lim_{|y|\to\infty} V(y) \geq V(x)$, for almost every $x \in \mathbb{R}^3$, and the inequality is strict in a non-zero measure domain;

**(V3)** there exists $\overline{C} > 0$ such that, for any $u \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 \geq \overline{C}\|u\|^2 \ .$$

A potential of this type has been considered in [22], where the Schödinger equation has been studied.

In order to get our result, we will use a very standard device: we will look for a minimizer of the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \frac{q}{4} \int_{\mathbb{R}^3} u^2 \phi_u - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}$$

restricted to the Nehari manifold

$$\mathcal{N} = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \widetilde{G}(u) = 0 \right\} \ ,$$

where

$$\widetilde{G}(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{p+1} \ .$$

The following lemma describes some properties of the Nehari manifold $\mathcal{N}$:

**Lemma 6.1.**

1. *For any $u \neq 0$ there exists a unique number $\bar{t} > 0$ such that $\bar{t}u \in \mathcal{N}$ and*
$$I(\bar{t}u) = \max_{t \geq 0} I(tu) \ ;$$
2. *there exists a positive constant $C$, such that for all $u \in \mathcal{N}$, $\|u\|_{p+1} \geq C$;*
3. *$\mathcal{N}$ is a $C^1$ manifold.*

*Proof.* Points 1 and 2 can be proved using standard arguments which follows those in the proof of Lemma 5.2.
3. Observe that for any $u \in H^1(\mathbb{R}^3)$ we have

$$\widetilde{G}(u) = 4I(u) - \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) - \frac{p-3}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \ ,$$

and then, by point 2, for any $u \in \mathcal{N}$ we have

$$\langle \widetilde{G}'(u), u \rangle = -2 \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) - (p-3) \int_{\mathbb{R}^3} |u|^{p+1} \leq -C < 0 \ .$$

□

The functional $I$ restricted to the Nehari manifold is positive. We assume the following definition

$$c_V := \inf_{u \in \mathcal{N}} I(u) \ .$$

We recall some preliminary lemmas which can be obtained by using the same arguments as in [25] (see also [4]).

As a consequence of the Lemma 6.1, we are allowed to define the map $t : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}_+$ such that for any $u \in H^1(\mathbb{R}^3)$, $u \neq 0$ :

$$I\left(t(u)u\right) = \max_{t \geq 0} I(tu) \ .$$

**Lemma 6.2.** *Let $u_n \in H^1(\mathbb{R}^3)$, $n \geq 1$, such that $\|u_n\| \geq C > 0$ and*

$$\max_{t\geq 0} I(tu_n) \leq c_V + \delta_n \ ,$$

*with $\delta_n \to 0^+$. Then, there exist a sequence $(y_n)_n \subset \mathbb{R}^N$ and two positive numbers $R$, $\mu > 0$ such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 \, dx > \mu \ .$$

*Proof.* We have

$$C \geq \int_{\mathbb{R}^3} |\nabla u_n|^2 + V(x)u_n^2 = t_n^2 \left( t_n^{p-3} \int_{\mathbb{R}^3} |u_n|^{p+1} - \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \right) \ .$$

The conclusion follows from $i$ of Lemma 3.3 and Lemma 6.2.

□

**Lemma 6.3.** *Suppose that $V$, $V_n \in L^\infty$, for all $n \geq 1$. If $V_n \to V$ in $L^\infty(\mathbb{R}^N)$ then $c_{V_n} \to c_V$.*

Now define

$$\begin{aligned} I_\infty(u) \ &:= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \ , \\ c_\infty \ &:= c_{V_\infty} \ . \end{aligned}$$

As in [25], we have

**Lemma 6.4.** *If $V$ satisfies (**V1-3**), we get $c_V < c_\infty$.*

*Proof.* As proved in the previous section, there exists $(w, \phi_w) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ a ground state solution of the problem

$$\begin{cases} -\Delta u + V_\infty u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3 \ , \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \ . \end{cases}$$

Let $t(w) > 0$ be such that $t(w)w \in \mathcal{N}$. By (**V2**), we have

$$c_\infty = I_\infty(w) \geq I_\infty(t(w)w) = I(t(w)w) + \int_{\mathbb{R}^N} (V_\infty - V(x))|t(w)w|^2 > c_V \ ,$$

and then we conclude.

□

6.0.1. *Proof of the existence of the ground state.* The Nehari manifold $\mathcal{N}$ is a natural constraint for the functional $I$, therefore we are allowed to look for critical points of $I$ restricted to $\mathcal{N}$.

In view of this, our goal is to find $\overline{u} \in \mathcal{N}$ such that $I(\overline{u}) = c_V$, from which we would deduce that $(\overline{u}, \phi_{\overline{u}})$ is a ground state solution of $(\mathcal{SM})$.

Let $(u_n)_n \subset \mathcal{N}$ such that

$$\lim_n I(u_n) = c_V \ . \tag{27}$$

We define the functional $J \colon H^1(\mathbb{R}^3) \to \mathbb{R}$ as:

$$J(u) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \left( \frac{1}{4} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} \phi_u u^2 \ .$$

Observe that for any $u \in \mathcal{N}$, we have $I(u) = J(u)$. By (**V3**) and (27), we deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\overline{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup \overline{u} \quad \text{weakly in} \quad H^1(\mathbb{R}^3)\ , \\ &u_n \to \overline{u} \quad \text{in } L^s(B)\ , \ \text{ with } B \subset \mathbb{R}^3\ , \ \text{ bounded, and } 1 \le s < 6\ . \end{aligned} \tag{28}$$

We need some compactness on the sequence $(u_n)_n$. We denote by $\sigma_n$ the $L^1$ functions

$$\sigma_n = \left(\frac{1}{2} - \frac{1}{p+1}\right) |\nabla u_n|^2 + V(x)u_n^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) \phi_{u_n} u_n^2\ .$$

Observe that, since there is no lower boundedness condition on the potential $V$, the functions $\sigma_n$ may be not positive, and then we are not allowed to use the Lions' concentration arguments [23, 24] on them. However, using a variant presented in [11], in the following theorem we are able to show that the functions $u_k$ concentrate in the $H^1(\mathbb{R}^3)$-norms.

**Theorem 6.5.** *For any $\delta > 0$ there exists $\tilde{R} > 0$ such that for any $n \ge \tilde{R}$*

$$\int_{|x|>\tilde{R}} (|\nabla u_n|^2 + |u_n|^2) < \delta\ .$$

*Proof.* By contradiction, suppose that there exist $\delta_0 > 0$ and a subsequence $(u_k)_k$ such that for any $k \ge 1$

$$\int_{|x|>k} (|\nabla u_k|^2 + |u_k|^2) \ge \delta_0\ . \tag{29}$$

We define

$$\rho_k(\Omega) = \int_\Omega |\nabla u_k|^2 + |u_k|^2 + \int_\Omega \phi_{u_k} u_k^2\ .$$

and, for any $r > 0$, we set $A_r := \{x \in \mathbb{R}^3 \mid r \le |x| \le r+1\}$. We claim that

$$\text{for any } \ \mu > 0 \ \text{ and } \ R > 0, \ \text{ there exists } \ r > R \ \text{ such that } \ \rho_k(A_r) < \mu \tag{30}$$

for infinitely many $k$. If not, then there should exist $\hat{\mu} > 0$ and $\widehat{R} \in \mathbb{N}$ such that, for any $m \ge \widehat{R}$, there exists $p(m)$ such that, for any $k \ge p(m)$,

$$\rho_k(A_m) \ge \hat{\mu}\ .$$

We are allowed to take $(p(m))_m$ not decreasing, so that for every $m \ge \widehat{R}$ we could get $u_k$ such that, using $i$ of Lemma 3.3,

$$C\|u_k\|^2(1 + \|u_k\|^2) \ge \|u_k\|^2 + \int_{\mathbb{R}^3} \phi_{u_k} u_k^2 \ge \rho_k(B_m \setminus B_{\widehat{R}}) \ge \left(m - \widehat{R}\right) \hat{\mu}$$

contradicting the boundedness in $H^1(\mathbb{R}^3)$ of the sequence $(u_n)_n$ So, we assume that (30) holds. Taking into account Lemma 6.3 and Lemma 6.4, consider $\mu > 0$ such that

$$c < c(V_\infty - \mu) < c(V_\infty)\ .$$

Using (**V2**), there exists $R_\mu \in \mathbb{N}$ such that for almost every $|x| \ge R_\mu$

$$V(x) \ge V_\infty - \mu > 0\ ; \tag{31}$$

we take $r > R_\mu$ such that, up to a subsequence,

$$\rho_k(A_r) < \mu \quad , \quad \text{for all } \ k \ge 1\ . \tag{32}$$

In particular, (31) and (32) imply

$$(33)\qquad \int_{A_r} |\nabla u_k|^2 + V(x)u_k^2 = O(\mu) \quad , \quad \int_{A_r} \phi_{u_k} u_k^2 = O(\mu) \quad , \quad \text{for all } \; k \geq 1 \; .$$

Let $\chi \in C^\infty$, such that $\chi = 1$ in $B_r$ and $\chi = 0$ in $(B_{r+1})^c$, $0 \leq \chi \leq 1$ and $|\nabla\chi| \leq 2$. Set $v_k = \chi u_k$ and $w_k = (1-\chi)u_k$. By simple computations, by (31) and (33) we infer

$$\int_{A_r} |\nabla v_k|^2 + V(x)v_k^2 = O(\mu) \quad , \quad \int_{A_r} |v_k|^{p+1} = O(\mu) \; ,$$
$$\int_{A_r} |\nabla w_k|^2 + V(x)w_k^2 = O(\mu) \quad , \quad \int_{A_r} |w_k|^{p+1} = O(\mu) \; .$$

Hence, we deduce that

$$(34)\quad \int_{\mathbb{R}^3} |\nabla u_k|^2 + V(x)u_k^2 = \int_{\mathbb{R}^3} |\nabla v_k|^2 + V(x)v_k^2 + \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 + O(\mu) \; ,$$

$$(35)\qquad \int_{\mathbb{R}^3} |u_k|^{p+1} = \int_{\mathbb{R}^3} |v_k|^{p+1} + \int_{\mathbb{R}^3} |w_k|^{p+1} + O(\mu) \; ;$$

for large $k \geq 1$, by (29) and (31), we also deduce that there exists $\delta' > 0$ such that

$$(36)\qquad \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)|w_k|^2 \geq \delta' + O(\mu) \; .$$

Moreover, arguing as in (17), we have

$$(37)\qquad \int_{\mathbb{R}^3} \phi_{u_k} u_k^2 \geq \int_{\mathbb{R}^3} \phi_{v_k} v_k^2 + \int_{\mathbb{R}^3} \phi_{w_k} w_k^2 + O(\mu) \; .$$

Hence, by (34) and (37), we get

$$J(u_k) \geq J(v_k) + J(w_k) + O(\mu) \; ,$$

and then, using (36) and (**V3**), we deduce

$$(38)\qquad J(u_k) - C\delta' \geq J(v_k) + O(\mu) \; ,$$

$$(39)\qquad J(u_k) \geq J(w_k) + O(\mu) \; .$$

We recall the definition of the functional $\widetilde{G}\colon H^1(\mathbb{R}^3) \to \mathbb{R}$

$$\widetilde{G}(u) = \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{p+1}$$

and that if $u \in \mathcal{N}$, then $\widetilde{G}(u) = 0$. By (34), (35) and (37), we have

$$(40)\qquad 0 = \widetilde{G}(u_k) \geq \widetilde{G}(v_k) + \widetilde{G}(w_k) + O(\mu) \; .$$

We have to distinguish three cases.

Case 1: up to a subsequence, $\widetilde{G}(v_k) \leq 0$.
By Lemma 6.1, for any $k \geq 1$, there exists $\theta_k > 0$ such that $\theta_k v_k \in \mathcal{N}$, and then

$$(41)\qquad \int_{\mathbb{R}^3} |\nabla v_k|^2 + V(x)v_k^2 + \theta_k^2 \phi_{v_k} v_k^2 = \int_{\mathbb{R}^3} \theta_k^{p-1} |v_k|^{p+1} \; .$$

By (41) we have

$$(\theta_k^{p-1} - 1)\int_{\mathbb{R}^3} |\nabla v_k|^2 + V(x)v_k^2 + (\theta_k^{p-1} - \theta_k^2)\int_{\mathbb{R}^3} \phi_{v_k} v_k^2 \leq 0 \; ,$$

and, by (**V3**), we deduce that $\theta_k \leq 1$. Therefore, for all $k \geq 1$, by (**V3**) and (38),

$$c_V \leq I(\theta_k v_k) = J(\theta_k v_k) \leq J(v_k) \leq J(u_k) - C\delta' + O(\mu) =$$

$$= c_V - C\delta' + o_k(1) + O(\mu) \ ,$$

which is a contradiction.

CASE 2: up to a subsequence, $\widetilde{G}(w_k) \leq 0$.
Let $(\eta_k)_k$ be such that, for any $k \geq 1$, $\eta_k w_k \in \mathcal{N}$. Arguing as in the previous case, we deduce that $\eta_k \leq 1$. Define $\tilde{w}_k = \eta_k w_k$. Let $(t_k)_k$ be such that, for any $k \geq 1$, $t_k \tilde{w}_k \in \mathcal{N}_{V_\infty - \mu}$. By (31),

$$\int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + (V_\infty - \mu)\tilde{w}_k^2 + \phi_{\tilde{w}_k}\tilde{w}_k^2 \leq \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + V(x)\tilde{w}_k^2 + \phi_{\tilde{w}_k}\tilde{w}_k^2 = \int_{\mathbb{R}^3} |\tilde{w}_k|^{p+1} \ ,$$

and then $t_k \leq 1$. By (39) and (**V3**), we conclude that

$$c(V_\infty - \mu) \leq \frac{t_k^2}{2} \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + (V_\infty - \mu)\tilde{w}_k^2 + \frac{t_k^4}{4} \int_{\mathbb{R}^3} \phi_{\tilde{w}_k}\tilde{w}_k^2 - \frac{t_k^{p+1}}{p+1} \int_{\mathbb{R}^3} |\tilde{w}_k|^{p+1} \leq$$

$$\leq \frac{t_k^2}{2} \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + V(x)\tilde{w}_k^2 + \frac{t_k^4}{4} \int_{\mathbb{R}^3} \phi_{\tilde{w}_k}\tilde{w}_k^2 - \frac{t_k^{p+1}}{p+1} \int_{\mathbb{R}^3} |\tilde{w}_k|^{p+1} =$$

$$= \left(\frac{t_k^2}{2} - \frac{t_k^{p+1}}{p+1}\right) \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + V(x)\tilde{w}_k^2 + \left(\frac{t_k^4}{4} - \frac{t_k^{p+1}}{p+1}\right) \int_{\mathbb{R}^3} \phi_{\tilde{w}_k}\tilde{w}_k^2 \leq$$

$$\leq J(\tilde{w}_k) = J(\eta_k w_k) \leq J(w_k) \leq J(u_k) + O(\mu) = c_V + o_k(1) + O(\mu) \ ,$$

but, letting $\mu$ go to zero and $k$ go to $\infty$, by Lemma 6.3, this yields a contradiction with Lemma 6.4.

CASE 3: up to a subsequence, $\widetilde{G}(v_k) > 0$ and $\widetilde{G}(w_k) > 0$. By (40), we infer that $\widetilde{G}(v_k) = O(\mu)$ and $\widetilde{G}(w_k) = O(\mu)$. Let $(\eta_k)_k$ be such that $\eta_k w_k \in \mathcal{N}$. If $\eta_k \leq 1 + O(\mu)$, we can repeat the arguments of Case 2. Suppose that

$$\lim_k \eta_k = \eta_0 > 1 \ .$$

We have

$$O(\mu) = \widetilde{G}(w_k) = \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 + \phi_{w_k}w_k^2 - |w_k|^{p+1} =$$

$$= \left(1 - \frac{1}{\eta_k^{p-1}}\right) \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 + \left(1 - \frac{1}{\eta_k^{p-3}}\right) \int_{\mathbb{R}^3} \phi_{w_k}w_k^2$$

and so

$$\int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 = O(\mu) \ ,$$

which contradicts (36).

□

**Theorem 6.6.** *There exists a ground state.*

*Proof.* By Theorem 6.5, for any $\delta > 0$ there exists $r > 0$ such that

$$\|u_n\|_{H^1(B_r^c)} < \delta \quad , \quad \text{uniformly for} \quad n \geq 1 \; .$$

Hence, arguing as in the constant potential case, we deduce that

$$u_n \to \overline{u} \quad \text{in} \;\; L^s(\mathbb{R}^3) \quad , \quad \text{for any} \;\; s \in [2, 6[\; . \tag{42}$$

Moreover

$$\phi_{u_n} \to \phi_{\overline{u}} \quad \text{in} \;\; \mathcal{D}^{1,2}(\mathbb{R}^3) \quad , \quad \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_{\overline{u}} \overline{u}^2 \quad , \quad \text{as} \;\; n \to \infty \; , \tag{43}$$

and for any $\psi \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \to \int_{\mathbb{R}^3} \phi_{\overline{u}} \overline{u} \psi \; . \tag{44}$$

By (27), we can suppose (see [31]) that $(u_n)_n$ is a Palais-Smale sequence for $I|_{\mathcal{N}}$ and, as a consequence, it is easy to see that $(u_n)_n$ is a Palais-Smale sequence for $I$. By (28), (42) and (44), we conclude that $I'(\overline{u}) = 0$. Since $(u_n)_n$ is in $\mathcal{N}$, by 3 of Lemma 6.1 $(\|u_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (42) implies that $\overline{u} \neq 0$ and so $\overline{u} \in \mathcal{N}$. Finally, by (27), (28), (42) and (43) and by (**V2-3**) we get

$$c_V \leq I(\overline{u}) \leq \liminf I(u_n) = c_V \; ,$$

so we can conclude that $(\overline{u}, \phi_{\overline{u}})$ is a ground state solution of $(\mathcal{SM})$.

□

**Remark 6.7.** The result proved in this section has been recently improved in [32], by means of an adaptation of the well known splitting lemma of Benci and Cerami [6].

## 7. Nonradial solution

In this section we will look for a nonradial solution of the problem $(\mathcal{SM})$ assuming on $g$ the following Berestycki-Lions type hypotheses

- (**g1**) $g \in C(\mathbb{R}, \mathbb{R})$, $g$ odd;
- (**g2**) $-\infty < \liminf_{s \to 0^+} g(s)/s \leq \limsup_{s \to 0^+} g(s)/s = -\omega < 0$;
- (**g3**) $-\infty \leq \limsup_{s \to +\infty} g(s)/s^p \leq 0$, $1 < p < 5$;
- (**g4**) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)\, ds > 0$.

The get such a type of solution, following [18] we will introduce a suitable natural constraint where no radial function is contained. However first we need to make some preliminary considerations on the nonlinearity in order to use on the problem a variational approach.

Following [9], define $s_0 := \min\{s \in [\zeta, +\infty[ \mid g(s) = 0\}$ ($s_0 = +\infty$ if $g(s) \neq 0$ for any $s \geq \zeta$) and set $\tilde{g} : \mathbb{R} \to \mathbb{R}$ the function such that

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on} \;\; [0, s_0] \; ; \\ 0 & \text{on} \;\; \mathbb{R}_+ \setminus [0, s_0] \; ; \\ -\tilde{g}(-s) & \text{on} \;\; \mathbb{R}_- \; . \end{cases} \tag{45}$$

By the strong maximum principle and by ii) of Lemma 3.3, a solution of $(\mathcal{SM})$ with $\tilde{g}$ in the place of $g$ is a solution of $(\mathcal{SM})$. So we can suppose that $g$ is defined as in (45), so that (**g1**), (**g2**) and (**g4**) hold, and we have also the following limit

$$\lim_{s\to\infty} \frac{|g(s)|}{|s|^p} = 0 \; .$$

Moreover, we set for any $s \geq 0$,

$$g_1(s) := (g(s) + \omega s)^+ \quad , \quad g_2(s) := g_1(s) - g(s) \; ,$$

and we extend them as odd functions. Since

$$\lim_{s\to 0} \frac{g_1(s)}{s} = 0 \quad , \quad \lim_{s\to\infty} \frac{g_1(s)}{|s|^p} = 0 \; , \tag{46}$$

and

$$g_2(s) \geq \omega s \quad , \quad \forall s \geq 0 \; , \tag{47}$$

by some computations, we have that for any $\varepsilon > 0$ there exist $C_\varepsilon$, $C'_\varepsilon > 0$ such that

$$g_1(s) \leq C_\varepsilon s^p + \varepsilon s \quad , \quad \forall s \geq 0 \tag{48}$$

$$g_1(s) \leq C'_\varepsilon s^5 + \varepsilon s \quad , \quad \forall s \geq 0 \tag{49}$$

$$g_1(s) \leq C_\varepsilon s^p + \varepsilon g_2(s) \quad , \quad \forall s \geq 0 \tag{50}$$

$$g_1(s) \leq C'_\varepsilon s^5 + \varepsilon g_2(s) \quad , \quad \forall s \geq 0 \tag{51}$$

If we set

$$G_i(t) := \int_0^t g_i(s)\, ds \quad , \quad i = 1, 2 \; ,$$

then, by (47), we have

$$G_2(s) \geq \frac{\omega}{2}\, s^2 \quad , \quad \forall s \in \mathbb{R} \tag{52}$$

and by (48), (49), (50) and (51), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ and $C'_\varepsilon > 0$ such that

$$G_1(s) \leq \frac{C_\varepsilon}{6}\, |s|^6 + \varepsilon s^2 \quad , \quad \forall s \in \mathbb{R}$$

$$G_1(s) \leq \frac{C'_\varepsilon}{p+1}\, |s|^{p+1} + \varepsilon s^2 \quad , \quad \forall s \in \mathbb{R}$$

$$G_1(s) \leq \frac{C_\varepsilon}{6}\, |s|^6 + \varepsilon G_2(s) \quad , \quad \forall s \in \mathbb{R} \tag{53}$$

$$G_1(s) \leq \frac{C'_\varepsilon}{p+1}\, |s|^{p+1} + \varepsilon G_2(s) \quad , \quad \forall s \in \mathbb{R} \; . \tag{54}$$

Now, let $\mathcal{O}(2)$ denote the orthogonal group of the rotation matrices in $\mathbb{R}^2$, that is

$$\mathcal{O}(2) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \;\middle|\; \alpha \in [0, 2\pi) \right\} \; .$$

For any $g \in \mathcal{O}(2)$ define the following action $\mathcal{T}_g$ on $H^1(\mathbb{R}^3)$:

$$\mathcal{T}_g u(x) = -u(\tilde{g}x) \in H^1(\mathbb{R}^3) \quad , \quad \tilde{g} = \begin{pmatrix} g & 0 \\ 0 & -1 \end{pmatrix} \; .$$

Now we set

$$H^1_{cyl,o}(\mathbb{R}^3) = \{u \in \mathcal{D}^1(\mathbb{R}^3, \mathbb{R}^3) \mid \mathcal{T}_g u = u \ \ \forall g \in \mathcal{O}(2)\} \ .$$

It is easy to see that $H^1_{cyl,o}(\mathbb{R}^3)$ is the setting of the functions cylindrically symmetric with respect to $(x_1, x_2)$ and odd with respect to $x_3$.

Since $g$ is odd (and consequently $G$ is even) and since we have that for any $u \in H^1(\mathbb{R}^3)$ and $g \in \mathcal{O}(2)$

$$-\mathcal{T}_g \phi_u = \phi_{\mathcal{T}_g u} \tag{55}$$

by the Palais' symmetrical criticality principle we can prove that $H^1_{cyl,o}(\mathbb{R}^3)$ is a natural constraint for the action functional $J_q$ (see [18] for details). We point out that, since $u \in H^1_{cyl,o}(\mathbb{R}^3)$, we have that $\phi_u \in \mathcal{D}^{1,2}_{cyl,e}(\mathbb{R}^3)$, the set of the functions in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ that are cylindrically symmetric with respect to the first two variables, and even with respect to the third. To improve the notations, we will often use $r$ in the place of $\sqrt{x_1^2 + x_2^2}$.

We will proceed as follows: we consider the manifold

$$\mathcal{M} = \{u \in H^1_{cyl,o}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} G(u) = 1\} \ .$$

As proved in [5] (see also [9]), $\mathcal{M}$ is nonempty. Consider indeed a family of functions $\rho_R(r, x_3) = \xi \alpha_R(r) \beta_R(x_3)$, for $R > 1$, with

$$\alpha_R(t) := \begin{cases} 1 & \text{if } \ |t| < R \ , \\ R + 1 - |t| & \text{if } \ R \le |t| < R + 1 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

and

$$\beta_R(t) := \begin{cases} 0 & \text{if } \ 0 < t \le 1 \\ t - 1 & \text{if } \ 1 < t \le 2 \ , \\ 1 & \text{if } \ 2 < t \le R \ , \\ R + 1 - t & \text{if } \ R < t \le R + 1 \ , \\ -\beta_R(-t) & \text{if } \ t \le 0 \ . \end{cases}$$

We have $\rho_R \in H^1_{cyl,o}(\mathbb{R}^3)$, and for large $\overline{R}$

$$\int_{\mathbb{R}^3} G(\rho_{\overline{R}}) > 0 \ .$$

So, if $\sigma$ is a suitable rescaling parameter, the function

$$\rho_{\overline{R},\sigma} : (r, x_3) \mapsto \rho_{\overline{R}}(\sigma r, \sigma x_3)$$

belongs to $\mathcal{M}$. Then, we consider the functional

$$J_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} \phi_u u^2$$

restricted on $\mathcal{M}$, and we look for a minimizer $\overline{u}$. Solving the minimizing problem, we find a Lagrange multiplier $\lambda \in \mathbb{R}$ such that the tern $(\overline{u}, \phi_{\overline{u}}, \lambda)$ solves the system

$$\begin{cases} -\Delta u + q\phi u = \mu g(u) & \text{in } \ \mathbb{R}^3 \ , \\ -\Delta \phi = q u^2 & \text{in } \ \mathbb{R}^3 \ . \end{cases}$$

Then we apply the following

**Theorem 7.1.** *Let $\overline{u} \in \mathcal{M}$ a minimizer for $J_q|_{\mathcal{M}}$, and let $\lambda$ be the Lagrange multiplier. Then $\lambda$ is positive, and the couple $(\tilde{u}, \tilde{\phi}) \in H^1_{cyl,o}(\mathbb{R}^3) \times \mathcal{D}^{1,2}_{cyl,e}(\mathbb{R}^3)$ defined rescaling as follows*

$$\tilde{u} = \overline{u}(\cdot/\sqrt{\lambda}) \quad \tilde{\phi} = \phi_{\overline{u}}(\cdot/\sqrt{\lambda}) \tag{56}$$

*solves the system*

$$\begin{cases} -\Delta u + q'\phi u = g(u) & \text{in } \mathbb{R}^3 \ , \\ -\Delta \phi = q' u^2 & \text{in } \mathbb{R}^3 \ . \end{cases} \tag{57}$$

*with $q' = q/\lambda$.*

7.1. **Compactness.** In this section we present the main tool to get our result. We first need to introduce some notations and definitions. Set $m_q = \inf_{u \in \mathcal{M}} J_q(u)$, and denote by $(u_n)_n := (u_n^q)_n$ a sequence such that

$$u_n \in \mathcal{M} \quad \text{and} \quad J_q(u_n) \to m_q \tag{58}$$

and by $\phi_n = \phi_{u_n}$.

As in [3, 20, 21] we introduce the cut-off function $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \chi(s) = 1 \ , & \text{for } s \in [0,1] \ , \\ 0 \le \chi(s) \le 1 \ , & \text{for } s \in ]1,2[ \ , \\ \chi(s) = 0 \ , & \text{for } s \in [2, +\infty[ \ , \\ \|\chi'\|_\infty \le 2 \ , & \end{cases}$$

and, for every $T > 0$, we denote

$$k_T(u) = \chi\left(\frac{\|u\|^2}{T^2}\right) \ .$$

Moreover, assume the following definitions

$$\begin{aligned} J_q^T(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4}\, k_T(u) \int_{\mathbb{R}^3} \phi_u u^2 \\ \sigma_n^{T,q} &= \frac{1}{2}\, |\nabla u_n|^2 + G_2(u_n) + \frac{q}{4}\, k_T(u_n)\phi_n u_n^2 \ , \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$. Set also $m_q^T = \inf_{u \in \mathcal{M}} J_q^T(u)$, and denote by $(u_n^{T,q})_n$ a minimizing sequence of $J_q^T|_{\mathcal{M}}$. It is trivial to see that $m_q^T \le m_q \le m_{\overline{q}}$ for any $T > 0$ and any $q \le \overline{q}$.

**Lemma 7.2.** *For any $T, q > 0$ the $L^1$ functions $\sigma_n^{T,q}$ are positive and bounded in the $L^1$-norm. Moreover $\sigma_n^{T,q}$ is bounded $T$-uniformly.*

*Proof.* The positiveness is a trivial consequence of the definition of the functions. As to boundedness, by the very definition of $u_n$ we have only to check if $(\int_{\mathbb{R}^3} G_2(u_n))_n$ is bounded. But by (53) we have

$$1 + \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u_n) \le \int_{\mathbb{R}^3} \varepsilon G_2(u_n) + C \int_{\mathbb{R}^3} |u_n|^6$$

and then

$$1 + (1-\varepsilon)\int_{\mathbb{R}^3} G_2(u_n) \le C'\left(\int_\Omega |\nabla u_n|^2\right)^3 \tag{59}$$

for $0 < \varepsilon < 1$ and $C$, $C'$ suitable positive constants. The $T$-uniform boundedness is a consequence of the fact that for any $n \ge 1$ and for any $T > 0$ $k_T(u_n) \le 1$.

□

Let $c = c_q^T$ be the limit (up to a subsequence) of $\int_{\mathbb{R}^3} \sigma_n^{T,q}$. Of course $c > 0$ because, otherwise, we would contradict (59).

**Lemma 7.3.** *For any $\overline{q}$ there exists $\overline{T}$ such that*

$$\limsup_n \|u_n^q\| \le T \quad , \quad \limsup_n \|u_n^{T,q}\| \le T \tag{60}$$

*for all $q \le \overline{q}$ and $T \ge \overline{T}$. As a consequence, every minimizing sequence for $J_q|_{\mathcal{M}}$, is a minimizing sequence also for $J_q^T|_{\mathcal{M}}$.*

*Proof.* Fix $\overline{q} > 0$ and $q \le \overline{q}$ and consider a minimizing sequence $u_n = u_n^q$ as in (58). Consider also $\overline{T} > 0$ whose precise estimate will be given later, $T \ge \overline{T}$ and $(u_n^{T,q})_n$ a minimizing sequence of $J_q^T|_{\mathcal{M}}$. Certainly we have that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \le 2m_q + o_n(1) \le 2m_{\overline{q}} + o_n(1)\ . \tag{61}$$

By (52) and (59) we have also

$$\int_{\mathbb{R}^3} |u_n|^2 \le \frac{\omega}{2}\int_{\mathbb{R}^3} G_2(u_n) \le C\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^3 \le$$

$$\le C'(2m_q + o_n(1))^3 = 8C'm_q^3 + o_n(1) \le 8C'm_{\overline{q}}^3 + o_n(1)\ . \tag{62}$$

Since $m_q^T \le m_q$, the same estimates can be proved also for $(u_n^{T,q})_n$. By (61) and (62) we conclude the first part of the proof taking $\overline{T} > \max(2m_{\overline{q}}, 8C'm_{\overline{q}}^3)$.

To prove the final part of the theorem, it is sufficient to show that $m_q^T = m_q$. But for a sufficiently large $\nu \ge 1$ and any $n \ge \nu$, by (60) we have that $k_T(u_n^{T,q}) = 1$ and $J_q^T(u_n^{T,q}) = J_q(u_n^{T,q}) \ge m_q$. We deduce that $m_q^T \ge m_q$ and then $m_q^T = m_q$.

□

Now we are studying the behaviour of the sequence $(\sigma_n^{T,q})_n$

**Theorem 7.4.** *Vanishing does not occur.*

*Proof.* Suppose by contradiction, that for all $R > 0$

$$\lim_n \sup_{\xi\in\mathbb{R}^3} \int_{B_R(\xi)} \sigma_n^{T,q} = 0\ .$$

In particular, we deduce that there exists $\overline{R} > 0$ such that

$$\lim_n \sup_{\xi\in\mathbb{R}^3} \int_{B_{\overline{R}}(\xi)} u_n^2 = 0\ .$$

By this and Lemma 7.3, we have that $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$ (see [24, Lemma I.1]). As a consequence, since $(u_n)_n \subset \mathcal{M}$ and by (54), we get for $0 < \varepsilon < 1$ and $C'_\varepsilon > 0$

$$1 + \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u_n) \le \int_{\mathbb{R}^3} \varepsilon\, G_2(u_n) + C'_\varepsilon \int_{\mathbb{R}^3} |u_n|^{p+1}$$

and then

$$1 + (1 - \varepsilon)\int_{\mathbb{R}^3} G_2(u_n) \le C'_\varepsilon \int_{\mathbb{R}^3} |u_n|^{p+1} \to 0 \, .$$

□

From now on, if the notation of a ball does not present explicitly expressed the center, than we assume it is the origin

**Theorem 7.5.** *For any $\overline{q} > 0$, there exist $\overline{T} > 0$ such that for any $T \ge \overline{T}$ and a suitable $0 < q(T) \le \overline{q}$, either $\mu_n^{T,q(T)}$ concentrates in a ball $B_R$ (namely compactness holds for $\xi_n = (0,0,0)$, $n \ge 1$) or it exhibits the following dichotomic behaviour: there exist $R > 0$ and a divergent sequence $\xi_n = (0,0,x_3^n)_n$ in $\mathbb{R}^3$ such that*

$$\int_{B_R(\xi_n)} \sigma_n^{T,q} \to \frac{c}{2} \quad , \quad \int_{B_R(-\xi_n)} \sigma_n^{T,q} \to \frac{c}{2} \, .$$

Since the proof is quite involved, we refer to [2] for the details. Here we just want to point out the meaning of the previous result. Because of the particular symmetry of the functions in our constraint, we are not able to avoid a priori any dichotomic behaviour of the minimizing sequences. In fact, we have certainly no dispersion of mass along the $x_1$ and $x_2$ directions, thanks to the cylindrical symmetry. However, on the $x_3$ direction the functions are odd, so if we have a part of the total mass concentrating on a bounded region which travels on the *positive semispace* $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$, then we necessarily have an equal quantity of mass travelling symmetrically in $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 < 0\}$. However, this dichotomy is not a *bad one*. In fact, as showed in the next subsection, up to a suitable rearrangement of the minimizing sequence we can always reduce to concentration on a bounded region.

7.2. **Existence of a nonradial solution.** We are going to prove the following

**Theorem 7.6.** *Assume* **(g1)**,$\cdots$,**(g4)**. *Then there exists $q > 0$ such that the system $(\mathcal{SM})$ possesses a solution $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ with the following features*

1. *$u$ and $\phi$ are respectively odd and even with respect to the third variable,*
2. *$u$ and $\phi$ are cylindrically symmetric with respect to the first two variables,*
3. *$u$ is positive on the half space $x_3 > 0$ (and, consequently, negative in the half space $x_3 < 0$), $\phi$ is positive everywhere.*

From now on, all the sequences considered have their $\limsup$ in the norm of $H^1(\mathbb{R}^3)$ less than $\overline{T}$, being $\overline{T}$ the same as in Lemma 7.3. Therefore there is no difference between $J_q$ and $J_q^T$ evaluated on them.

**Theorem 7.7.** *Let $q$ be as in Theorem 7.5, then the infimum $m_q$ is achieved.*

*Proof.* Suppose that the dichotomy situation described in Theorem 7.5 holds. Since $x_3^n \to +\infty$, we can suppose that for any $n \ge 1$ we have $x_3^n > 3R$. Then, consider a sequence of $\xi_n$-radially symmetric cut-off functions $\rho_n \in C^1(\{x \in \mathbb{R}^3 \mid x_3 > 0\})$ such that $\rho_n \equiv 1$ in $B_R(\xi_n)$, $\rho_n \equiv 0$ in $\{x \in \mathbb{R}^3 \mid x_3 > 0\} \setminus B_{2R}(\xi_n)$, $0 \le \rho_n \le 1$ and $|\nabla \rho_n| \le 2/R$, and define $\sigma_n \in C^1(\mathbb{R}^3)$ by evenness with respect to the third variable. Set $v_n = \sigma_n u_n \in H^1_{cyl,o}(\mathbb{R}^3)$ and for any $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ define

$$\tilde{v}_n(x) = \begin{cases} v_n(x_1,x_2,x_3+\xi_n-3R) & \text{if } x_3 > 0 \\ v_n(x_1,x_2,x_3-\xi_n+3R) & \text{if } x_3 < 0 \, . \end{cases} \tag{63}$$

We would have that, for $R' = 4R$,

$$\frac{1}{2}\int_{B_{R'}} |\nabla \tilde{v}_n|^2 + \int_{B_{R'}} G_2(\tilde{v}_n) + \frac{q}{4}\int_{B_{R'}} \phi_{\tilde{v}_n}\tilde{v}_n^2 \to c$$

and it is easy to verify also that a sequence so defined is such that

$$\int_{\mathbb{R}^3} G(\tilde{v}_n) \to 1 \quad \text{and} \quad J_q(\tilde{v}_n) \to m_q \ .$$

So, in any case, by Theorem 7.5 we are able to obtain a minimizing sequence that we label $(u_n)_n$ for the functional restricted to $\mathcal{M}$, which concentrates on a ball centered at the origin and with a sufficiently large radius. By boundedness of the sequence, we can extract a subsequence weakly convergent in $H^1$-norm to a function $u$. As a consequence of the weak convergence, the Fatou lemma and the weak lower semicontinuity of $\|\nabla \cdot\|_2$, we have

$$J_q(u) \le \liminf_n J_q(u_n) = m_q \ . \tag{64}$$

Since we also have

$$\begin{aligned} &u_n \to u \quad \text{pointwise} \\ &u_n \to u \quad \text{in } L^q(B)\ , \ \text{ for any bounded set } B \text{ and any } q \in [1,6[\ , \end{aligned} \tag{65}$$

we deduce that $u \in H^1_{cyl,o}(\mathbb{R}^3) \setminus \{0\}$ and $G_1(u_n(x)) \to G_1(u(x))$ for any $x \in \mathbb{R}^3$. Since

$$G_1(s) = o_n(s^2 + |s|^{p+1}) \ \text{ for } \ s \to 0 \ \text{ and } \ s \to \infty\ ,$$

and by concentration we have

$$\int_{\mathbb{R}^3 \setminus B_R} u_n^2 + |u_n|^{p+1} \to 0\ ,$$

by standard compactness arguments (see for instance the proof of Theorem A.I. in the Appendix in [9]) we deduce that

$$\int_{\mathbb{R}^3} G_1(u_n) \to \int_{\mathbb{R}^3} G_1(u)\ .$$

On the other hand, we also have that

$$1 + \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u_n) \to \int_{\mathbb{R}^3} G_1(u)$$

and then, by (65)

$$\int_{\mathbb{R}^3} G_2(u) \le \liminf_n \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_1(u) - 1\ .$$

that is

$$\int_{\mathbb{R}^3} G(u) \ge 1\ .$$

We deduce that

$$\int_{\mathbb{R}^3} G(u) = 1\ ,$$

otherwise we set $\overline{u} = u(K\,\cdot) \in \mathcal{M}$ with

$$K = \sqrt[3]{\int_{\mathbb{R}^3} G(u)} > 1$$

and by (64) we have,

$$m_q \le J_q(\overline{u}) = \frac{1}{2\sqrt[3]{K}} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4\sqrt[3]{K^5}} \int_{\mathbb{R}^3} \phi_u u^2 < J_q(u) \le m_q$$

which is a contradiction. So

$$\int_{\mathbb{R}^3} G(u) = 1 \, ,$$

and by (64) $J_q(u) = m_q$.

□

*Proof of Theorem 7.1.* Let $\overline{u} \in \mathcal{M}$ be such that $J_q(\overline{u}) = m_q$ and let $\lambda \in \mathbb{R}$ be the Lagrange multiplier. To show that $\lambda > 0$, we can proceed as in [9, p. 327]. Now define $\tilde{u}$ and $\tilde{\phi}$ as in (56). We prove that $(\tilde{u}, \tilde{\phi})$ satisfies the second equation of the system (57)

$$-\Delta\tilde{\phi} = -\frac{1}{\lambda}\, \Delta\phi_{\overline{u}}(\cdot/\sqrt{\lambda}) = \frac{1}{\lambda}\, q\overline{u}^2(\cdot/\sqrt{\lambda}) = q'\overline{u}^2(\cdot/\sqrt{\lambda}) = q'\tilde{u}^2 \, .$$

We prove that $(\tilde{u}, \tilde{\phi})$ satisfies the first equation of the system (57)

$$-\Delta\tilde{u} = -\frac{1}{\lambda}\, \Delta\overline{u}(\cdot/\sqrt{\lambda}) = -\frac{1}{\lambda}\, q\phi_{\overline{u}}(\cdot/\sqrt{\lambda})\overline{u}(\cdot/\sqrt{\lambda}) + g(\overline{u}(\cdot/\sqrt{\lambda})) = -q'\tilde{\phi}\tilde{u} + g(\tilde{u}) \, .$$

□

Now we are ready to complete the proof of the Theorem 7.6. Let $(u, \phi)$ be a solution found by Theorem 7.1. The symmetry properties derive from the natural constraint where we have studied the functional of the action and (55). Now, observe that $u$ can be assumed nonnegative in the semispace $x_3 > 0$ and nonpositive in the semispace $x_3 < 0$. In fact, if $\overline{u}$ is a minimizer obtained as in Theorem 7.7, we can replace it with the function

$$v = \begin{cases} |\overline{u}| & \text{on } \mathbb{R}^2 \times ]0, +\infty[\ ; \\ -|\overline{u}| & \text{on } \mathbb{R}^2 \times ]-\infty, 0[\ . \end{cases}$$

Obviously $v \in H^1_{cyl,o}(\mathbb{R}^3)$ and since $J_q$ and $G$ are even, $v$ is also a minimizer of $J_q|_{\mathcal{M}}$. Now we can apply the strong maximum principle in the second equation, and obtain that $\phi > 0$, and in the first equation obtaining that $u$ can vanish only on the plane $x_3 = 0$. The same considerations on the sign hold for $(\tilde{u}, \tilde{\phi})$, and are true everywhere, since by a standard regularity argument, we can prove that $\tilde{u}$ and $\tilde{\phi}$ are in $C^{2,\alpha}_{loc}(\mathbb{R}^3)$, with $\alpha \in (0, 1)$.

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