Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 8(2009), pp. 69–100.

Vector valued holomorphic and CR functions

Elisabetta Barletta and Sorin Dragomir

Abstract[1]. We survey several results on holomorphic functions with values in a complex Fréchet space. This is the extended version of the paper [4].

1. Introduction

The main purpose of the present paper is to survey some of the known results on holomorphic functions $f : \Omega \to \mathfrak{X}$ where $\Omega \subset \mathbb{C}^n$ is an open set ($n \geq 1$) and $\mathfrak{X}$ a complex topological vector space. Vector-valued holomorphic functions are useful in the theory of 1-parameter semigroups (cf. e.g. W. Arendt et al., [2]) and in analytic functional calculus (cf. e.g. F-H. Vasilescu, [32]). Except for the treatment in [32] (dealing with holomorphic functions of several complex variables, with values in a complex Fréchet space, as related to the functional calculus associated to a system of several commuting operators, cf. J.L. Taylor, [28]-[29]) the body of the present day literature is confined to functions of one complex variable mostly with values in a complex Banach space. In the spirit of functional analysis there are essentially two approaches to analyticity of vector-valued functions, through the notions of a *weakly holomorphic* and a (*strongly*) *holomorphic* function $f : \Omega \to \mathfrak{X}$ (cf. Section 2 for definitions) the first of the two being easier to check in practical examples. Holomorphic functions are always weakly holomorphic. It is then an important question (addressed by K.G. Grosse-Erdmann, [12]-[13], and W. Arendt & N. Nikolski, [1]) to determine the minimal assumptions under which a weakly holomorphic function is strongly holomorphic as well. Let $\Omega \subset \mathbb{C}$ be an open set, $\mathfrak{X}$ a complex Banach space, and $W \subset \mathfrak{X}^*$ a subset. Let $\sigma(\mathfrak{X}, W)$ be the W-topology of $\mathfrak{X}$ (the weak topology on $\mathfrak{X}$ induced by W, cf. e.g. [24], p. 62). By a result in [1] a $\sigma(\mathfrak{X}, W)$-holomorphic function $f : \Omega \to \mathfrak{X}$ is holomorphic if and only if each $\sigma(\mathfrak{X}, W)$-bounded set in $\mathfrak{X}$ is bounded. If $f : \Omega \to \mathfrak{X}$ is additionally assumed to be locally bounded then the mere knowledge that $\Lambda \circ f \in \mathcal{O}(\Omega)$ for any $\Lambda \in W$ and some separating subspace $W \subset \mathfrak{X}^*$ implies that $f : \Omega \to \mathfrak{X}$ is holomorphic (cf. Theorem 15 below). A generalization of this result (to the case where $\mathfrak{X}$ is a

---

[1] Authors' address: E. Barletta, Università degli Studi della Basilicata, Dipartimento di Matematica e Informatica, Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy; e-mail: `elisabetta.barletta@unibas.it` .

S. Dragomir, Università degli Studi della Basilicata, Dipartimento di Matematica e Informatica, Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy; e-mail: `sorin.dragomir@unibas.it` .

Keywords. Vector-valued holomorphic function, α-differentiability, vector-valued CR function.
AMS Subject Classification. 30A05, 32V10, 46A04.

locally convex space assumed to be locally complete, cf. [25]) was obtained by F.G. Grosse-Erdmann, [13] (cf. Theorem 16 below).

The exposition is organized as follows. In Section 2 we review the result that weakly holomorphic functions (of one complex variable) with values in a locally convex space $\mathfrak{X}$ are (strongly) continuous and extend the result to the case of functions of several complex variables (cf. Theorem 2). If additionally $\mathfrak{X}$ is a complex Fréchet space then each weakly holomorphic function $f : \Omega \subset \mathbb{C}^n \to \mathfrak{X}$ ($n \geq 1$) is shown to be holomorphic (cf. Theorem 4). This is a classical result by A. Grothendieck, [14]. Our arguments follow the proof of the classical Hartogs theorem on separate analyticity. Sections 3 to 5 describe a selection of results in the theory of holomorphic functions with values in a complex Fréchet or Banach space and report on results by W. Arendt & N. Nikolski, [1], F.G. Grosse-Erdmann, [13], F-H. Vasilescu, [31], and P. Vieten, [33]. Section 6 presents the extension (to the case of vector-valued CR functions) of a result by M.S. Baouendi & F. Treves, [3], on uniform approximation of CR functions of class C^1 with holomorphic functions. The result (cf. Theorem 19 in Section 6) was recently generalized to the case of CR functions of Teodorescu class B^1 (cf. S. Nishikawa et al., [11]). Appendix A is devoted to a review of the Bochner integral (in the theory of integration of functions with values in a topological vector space) as needed through this survey. Appendix B is about power series in Fréchet spaces (most results are elementary and perhaps well known yet spread through the mathematical literature). Appendix C transposes a few results by N. Teodorescu, [30], to the case of functions with values in a complex Fréchet space (in the spirit of the work by L-J. Nicolescu, [21], and I. Ciorănescu, [8]).

While writing the present survey the Authors kept in mind three possible developments of the theory that is i) recovering results in complex analysis in several complex variables, ii) allowing for more general complex topological vector spaces $\mathfrak{X}$ as target spaces (e.g. Fréchet spaces instead of Banach spaces), and iii) building a theory of vector-valued CR functions (an open problem so far).

As a distinctive feature of the present paper (with respect to the announcement [4]) we give rather detailed proofs of the main theorems. As well as its shorter version [4] this work is dedicated to the 70th birthday of Professor Stere Ianuş.

2. HOLOMORPHIC FUNCTIONS

Let $\Omega \subset \mathbb{C}^n$ be an open subset ($n \geq 1$) and $\mathfrak{X}$ a complex topological vector space. A function $f : \Omega \to \mathfrak{X}$ is *weakly holomorphic* in Ω if $\Lambda \circ f \in \mathcal{O}(\Omega)$ i.e. $\Lambda \circ f : \Omega \to \mathbb{C}$ is a holomorphic function for any $\Lambda \in \mathfrak{X}^*$. Also $f : \Omega \to \mathfrak{X}$ is (*strongly*) *holomorphic* if for any $a \in \Omega$ there is a neighborhood $a \in U \subset \Omega$ and a series $\sum_{|\alpha| \geq 0}(z-a)^\alpha x_\alpha$ with $x_\alpha \in \mathfrak{X}$ such that $\sum_{|\alpha|=0}^\infty (z-a)^\alpha x_\alpha = f(z)$ for any $z \in U$.

As each $\Lambda \in \mathfrak{X}^*$ is continuous, strongly holomorphic functions are weakly holomorphic as well. As to the converse one may state (cf. Theorem 3.31 in [24], p. 82)

Theorem 1. *Let $\mathfrak{X}$ be a locally convex space. Let $\Omega \subset \mathbb{C}$ be an open set. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function. Then*

i) f is strongly continuous in Ω.

Let us additionally assume that the closed convex hull of $f(\Gamma)$ is a compact subset of $\mathfrak{X}$ for any $\Gamma \in \mathbf{\Gamma}(\Omega)$. Then *ii) if* $\Gamma \in \mathbf{\Gamma}(\Omega)$ *is a curve such that* $\operatorname{Ind}_\Gamma(z) = 0$ *for any* $z \in \mathbb{C} \setminus \Omega$ *then*

$$\int_\Gamma f(\zeta)\, d\zeta = 0 \ , \tag{1}$$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma (\zeta - z)^{-1} f(\zeta)\, d\zeta \quad , \quad z \in \Omega \ , \ \operatorname{Ind}_\Gamma(z) = 1 \ . \tag{2}$$

Moreover

$$\int_{\Gamma_1} f(\zeta)\, d\zeta = \int_{\Gamma_2} f(\zeta)\, d\zeta \ , \tag{3}$$

for any $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}(\Omega)$ *such that* $\operatorname{Ind}_{\Gamma_1}(z) = \operatorname{Ind}_{\Gamma_2}(z)$ *for each* $z \in \mathbb{C} \setminus \Omega$*. Let us additionally assume that* $\mathfrak{X}$ *is a complex Fréchet space. Then*

iii) f *is* $\mathbb{C}$*-differentiable at each* $z_0 \in \Omega$ *that is the limit*

$$\lim_{z \to z_0} (z - z_0)^{-1} [f(z) - f(z_0)]$$

exists in the topology of $\mathfrak{X}$ *for any* $z_0 \in \Omega$*.*

Here $\mathbf{\Gamma}(\Omega)$ denotes the set of all closed rectifiable curves $\Gamma = \{\gamma(t) \ : \ a \leq t \leq b\}$ in $\Omega \subset \mathbb{C}$ and

$$\operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z} \quad , \quad \Gamma \in \mathbf{\Gamma}(\Omega) \quad , \quad z \in \mathbb{C} \ .$$

Formulae (1)-(2) are respectively Cauchy's theorem and Cauchy's formula for $\mathfrak{X}$-valued weakly holomorphic functions (cf. e.g. Theorem 1.5 and formula (1.59) in [27], p. 42-49, for the scalar valued counterpart). The proof of strong continuity of weakly holomorphic functions relies essentially on Cauchy's formula for ordinary (scalar valued) holomorphic functions (cf. [24], p. 83-84).

We recall that a subset $E \subset \mathfrak{X}$ is *bounded* if for any neighborhood V of $0 \in \mathfrak{X}$ there is a number $s > 0$ such that $E \subset tV$ for any $t \geq s$. Also E is *weakly bounded* if $\Lambda(E) \subset \mathbb{C}$ is a bounded set for any $\Lambda \in \mathfrak{X}^*$. The assumption that $\mathfrak{X}$ is a locally convex space (in statement (i) of Theorem 1 above) is exploited in two ways. First in a locally convex space weakly bounded subsets are (strongly) bounded (cf. Theorem 3.18 in [24], p. 70). Second for any locally convex space $\mathfrak{X}$ the dual $\mathfrak{X}^*$ separates points (cf. Corollary to Theorem 3.4 in [24], p. 59-60).

When $n \geq 2$ Cauchy's formula (for holomorphic functions of one complex variable) plays a similar role. Indeed let $\Omega \subset \mathbb{C}^n$ be an open set ($n \geq 2$) and let $a \in \Omega$. Let $\rho = (\rho_1, \cdots, \rho_n)$ be a polyradius ($\rho_j > 0$) such that the polydisc $\overline{P}(a, 2\rho) = \{z \in \mathbb{C}^n \ : \ |z_j - a_j| \leq 2\rho_j \ , \quad 1 \leq j \leq n\}$ is contained in Ω. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function and let $\Lambda \in \mathfrak{X}^*$. Let us set $a^{(j)} = (a_1, \cdots, a_j) \in \mathbb{C}^j$ for any $1 \leq j \leq n$. As $\Lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic for each $z \in P(a, 2\rho)$ (by applying twice Cauchy's formula)

$$\Lambda[f(z)] - \Lambda[f(a)] =$$

$$= \sum_{j=1}^{n} \left\{ \Lambda \left[f \left(a^{(j-1)}, z_j, \cdots, z_n \right) \right] - \Lambda \left[f \left(a^{(j)}, z_{j+1}, \cdots, z_n \right) \right] \right\} =$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{n} \left[\int_{|\zeta_j - a_j| = 2\rho_j} \frac{\Lambda \left[f \left(a^{(j-1)}, \zeta_j, z_{j+1}, \cdots, z_n \right) \right]}{\zeta_j - z_j} \, d\zeta_j - \right.$$

$$- \int_{|\zeta_j - a_j| = 2\rho_j} \frac{\Lambda \left[f \left(a^{(j-1)}, \zeta_j, z_{j+1}, \cdots, z_n \right) \right]}{\zeta_j - a_j} \, d\zeta_j \Bigg] =$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{n} (z_j - a_j) \int_{|\zeta_j - a_j| = 2\rho_j} \frac{\Lambda \left[f \left(a^{(j-1)}, \zeta_j, z_{j+1}, \cdots, z_n \right) \right]}{(\zeta_j - z_j)(\zeta_j - a_j)} \, d\zeta_j \; .$$

Let $M(\Lambda) = \sup \left\{ |\Lambda[f(\zeta)]| \; : \; \zeta \in \overline{P}(a, 2\rho) \right\}$. If $z \in \overline{P}(a,\rho) \setminus \{a\}$ then

$$|\Lambda[f(z)] - \Lambda[f(a)]| \leq$$

$$\leq \frac{M(\Lambda)}{4\pi} \sum_{j=1}^{n} \frac{|z_j - a_j|}{\rho_j} \int_{|\zeta_j - a_j| = 2\rho_j} \frac{d\zeta_j}{|\zeta_j - z_j|} \leq$$

$$\leq M(\Lambda) \, |z - a| \sum_{j=1}^{n} \frac{1}{\rho_j}$$

because of

$$|\zeta_j - z_j| \geq |\zeta_j - a_j| - |z_j - a_j| = 2\rho_j - |z_j - a_j| \geq \rho_j \; .$$

Consequently the set

$$\left\{ |z - a|^{-1} \left[f(z) - f(a) \right] \; : \; z \in \overline{P}(a, \rho) \setminus \{a\} \right\} \tag{4}$$

is weakly bounded in $\mathfrak{X}$ hence strongly bounded, as well. Let $V \subset \mathfrak{X}$ be an open neighborhood of the origin $0 \in \mathfrak{X}$. As (4) is bounded there is $s > 0$ such that

$$f(z) - f(a) \in t|z - a| V \quad , \quad z \in \overline{P}(a, \rho) \setminus \{a\} \quad , \quad t \geq s \; .$$

A subset $B \subset \mathfrak{X}$ of a complex topological vector space is *balanced* if for any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ one has $\alpha B \subset B$. As every topological vector space has a balanced local base of neighborhoods of the origin (cf. e.g. [24], p. 13) it follows that f is (strongly) continuous in a. We have shown that

Theorem 2. *Let $\mathfrak{X}$ be a locally convex space and $\Omega \subset \mathbb{C}^n$ an open set ($n \geq 2$). Any weakly holomorphic holomorphic function $f : \Omega \to \mathfrak{X}$ is strongly continuous.*

Let us go back to the discussion of Theorem 1. Once the continuity of weakly holomorphic functions $f : \Omega \to \mathfrak{X}$ is proved one may use Theorem 3.27 in [24], p. 78, to conclude that under the assumptions of Theorem 1 the integrals $\int_\Gamma f(\zeta) \, d\zeta$ and $\int_\Gamma (\zeta - z)^{-1} f(\zeta) \, d\zeta$ are well defined elements of $\mathfrak{X}$ (as Bochner integrals i.e. in the sense of Definition 3.26 in [24], p. 77). Then (1)-(3) hold by the classical Cauchy's formula and Cauchy's theorem applied to the holomorphic functions $\Lambda \circ f$ for any $\Lambda \in \mathfrak{X}$. Similarly for functions of $n \geq 2$ complex variables one has

Theorem 3. *Let $\mathfrak{X}$ be a locally convex space and $\Omega \subset \mathbb{C}^n$ an open set ($n \geq 2$). Let $a \in \Omega$ and $\rho = (\rho_1, \cdots, \rho_n)$ a polyradius such that $\overline{P}(a, \rho) \subset \Omega$. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function such that the closed convex hull of $f\left[\partial_0 P(a,\rho)\right]$ is a compact subset of $\mathfrak{X}$. Then for each $z \in P(a, \rho)$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 P(a,\rho)} f(\zeta_1, \cdots, \zeta_n) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} \, d\zeta_1 \cdots d\zeta_n \; . \tag{5}$$

Here $\partial_0 P(a,\rho) = \prod_{j=1}^n S^1(a_j, \rho_j)$ is the essential boundary of $P(a,\rho)$ and $S^1(z_0, r) = \{z \in \mathbb{C} \; : \; |z - z_0| = r\}$ with $z_0 \in \mathbb{C}$ and $r > 0$.

Proof. By Theorem 2 the function $f : \Omega \to \mathfrak{X}$ is continuous. Then (by Theorem 3.27 in [24]) the integral

$$\int_{\partial_0 P(a,\rho)} f(\zeta) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} \, d\zeta \in \overline{\text{co}} \left[f\left(\partial_0 P(a,\rho) \right) \right]$$

is well defined and

$$\Lambda \left(\int_{\partial_0 P(a,\rho)} f(\zeta) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} \, d\zeta \right) =$$

$$= \int_{\partial_0 P(a,\rho)} \frac{\Lambda[f(\zeta)] \, d\zeta}{\prod_{j=1}^{n} (\zeta_j - z_j)} \quad , \quad \Lambda \in \mathfrak{X}^* \; .$$

Then (5) follows from Cauchy's formula for $\Lambda \circ f$ and the fact that $\mathfrak{X}^*$ separates points.

□

Part (iii) in Theorem 1 is stated under the assumption that $\mathfrak{X}$ is a Fréchet space. This guarantees that $\overline{\text{co}}[F(\Gamma)]$ is a compact set, where $F(z) = z^{-2} f(z)$, $z \in \Gamma = \{\zeta \in \mathbb{C} \, : \, |\zeta| = 2r\}$, hence both Cauchy's theorem (1) and Cauchy's integral formula (2) hold for the holomorphic function F (and the proof in [24], p. 84, applies).

Let $\mathfrak{X}$ be a complex Fréchet space and $f : \Omega \to \mathfrak{X}$ a weakly holomorphic function. For $a \in \mathbb{C}^n$ we set

$$\Omega_{j,a} = \{ z \in \mathbb{C} \, : \, (a_1, \cdots, a_{j-1}, z, a_{j+1}, \cdots, a_n) \in \Omega \}$$

and $f_{j,a}(z) = f(a_1, \cdots, a_{j-1}, z, a_{j+1}, \cdots, a_n)$ for any $z \in \Omega_{j,a}$. Each $f_{a,j}$ is weakly holomorphic in $\Omega_{j,a}$ hence (by Theorem 1) strongly holomorphic in $\Omega_{j,a}$. Is then $f : \Omega \to \mathfrak{X}$ holomorphic? Equivalently does Hartogs' theorem hold for $\mathfrak{X}$-valued functions possessing this property (cf. [20], p. 43, for $\mathfrak{X} = \mathbb{C}$)? We establish

Theorem 4. *Let $\mathfrak{X}$ be a complex Fréchet space, $\Omega \subset \mathbb{C}^n$ an open set $(n \geq 1)$, and $f : \Omega \to \mathfrak{X}$ a weakly holomophic function. Then f is strongly holomorphic.*

To establish Theorem 4 we follow the arguments in the proof of the classical Hartogs' theorem (cf. e.g. [20], p. 43-44). The proof is considerably easier because weakly holomorphic functions are readily continuous (while Hartogs' theorem assumes but separate analyticity to start with). To prove Theorem 4 it suffices to show

Theorem 5. *Let $\mathfrak{X}$ be a complex Fréchet space. Let Ω be the polydisc $\{z \in \mathbb{C}^n \, : \, |z_j| < R \; , \; \; 1 \leq j \leq n\}$ with $R > 0$. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function. Then there is $0 < r < R$ and a power series $\sum_{|\alpha| \geq 0} z^\alpha x_\alpha$ with $x_\alpha \in \mathfrak{X}$ converging uniformly on $P(0, \mathbf{r})$ such that $f(z) = \sum_{|\alpha|=0}^{\infty} z^\alpha x_\alpha$ for any $z \in P(0, \mathbf{r})$ (here $\mathbf{r} = (r, \cdots, r)$).*

Proof. Let $D_r(0) = \{z \in \mathbb{C} \, : \, |z| < r\}$. As $f : \Omega \to \mathfrak{X}$ is continuous (cf. Theorem 2) it is Bochner integrable on the product of the circles

$$T_\rho = \prod_{j=1}^{n} \{ \zeta_j \in \mathbb{C} \, : \, |\zeta_j| = \rho \} \quad , \quad 0 < \rho < R \; ,$$

i.e. $f \in L^1(T_\rho, \mathfrak{X}, d\zeta)$. Let $z' = (z_1, \cdots, z_{n-1})$ such that $|z_j| < R$ for any $1 \le j \le n-1$ and note that $D_\rho(0) \subset \Omega_{n,z'}$. As argued above $f_{n,z'}$ is holomorphic in $\Omega_{n,z'}$ and in particular in $D_\rho(0)$ hence (by Theorem 1) for $|z_n| < \rho$

$$f(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta_n| = \rho} (\zeta_n - z_n)^{-1} f(z', \zeta_n)\, d\zeta_n \; .$$

For fixed $z_1, \cdots, z_{n-2}$ with $|z_j| < R$, $1 \le j \le n-2$, and fixed $\zeta_n \in D_\rho(0)$ the function $f(z_1, \cdots, z_{n-1}, \zeta_n)$ is holomorphic in the disc $|z_{n-1}| < \rho$ hence we may repeat the procedure above. In the end for any $|z_j| < \rho$, $1 \le j \le n$, one has

$$(2\pi i)^n f(z_1, \cdots, z_n) = \tag{6}$$

$$= \int_{|\zeta_n| = \rho} d\zeta_n \int_{|\zeta_{n-1}| = \rho} d\zeta_{n-1} \cdots \int_{|\zeta_1| = \rho} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f(\zeta_1, \cdots, \zeta_n)\, d\zeta_1 \; .$$

Let $\overline{P}(0, \mathbf{r}) = \{z \in \mathbb{C}^n \; : \; |z_j| \le r\}$ where $\mathbf{r} = (r, \cdots, r)$ and $0 < r < \rho$. Let $z \in \overline{P}(0, \mathbf{r})$ and $\zeta \in T_\rho$. Then

$$\prod_{j=1}^{n} (\zeta_j - z_j)^{-1} = \sum_{|\alpha|=0}^{\infty} \frac{z^\alpha}{\zeta^{\alpha + \mathbf{1}}} \tag{7}$$

where $\alpha + \mathbf{1} = (\alpha_1 + 1, \cdots, \alpha_n + 1)$ and the series in the right hand side of (7) converges uniformly for $\zeta \in T_\rho$ and $z \in \overline{P}(0, \mathbf{r})$. Let $f_\alpha(z, \zeta) = \left(z^\alpha / \zeta^{\alpha + \mathbf{1}} \right) f(\zeta)$. Then

Lemma 1. *For any $z \in \overline{P}(0, \mathbf{r})$ the series $\sum_{|\alpha| \ge 0} f_\alpha(z, \zeta)$ is convergent in the topology of $\mathfrak{X}$ uniformly with respect to $\zeta \in T_\rho$.*

Indeed let $\mathcal{B}$ be a balanced local base (of neighborhoods of the origin in $\mathfrak{X}$). As each $\Lambda \circ f$ ($\Lambda \in \mathfrak{X}^*$) is continuous the set $f(T_\rho)$ is weakly bounded and then bounded in $\mathfrak{X}$. Hence for any $V \in \mathcal{B}$ there is $s > 0$ such that $f(T_\rho) \subset tV$ for any $t \ge s$. Let $s_\nu(z, \zeta) = \sum_{|\alpha|=0}^{\nu} z^\alpha / \zeta^{\alpha + \mathbf{1}}$ and $S_\nu(z, \zeta) = \sum_{|\alpha|=0}^{\nu} f_\alpha(z, \zeta)$. As $\sum_{|\alpha| \ge 0} z^\alpha / \zeta^{\alpha + \mathbf{1}}$ is convergent (uniformly for $\zeta \in T_\rho$ and $z \in \overline{P}(0, \mathbf{r})$) there exists $N \ge 1$ such that $|R_{\nu\mu}(z, \zeta)| < 1/s$ for any $\nu \ge \mu > N$ and any $\zeta \in T_\rho$ and $|z_j| \le r$, where $R_{\nu\mu} = s_\nu - s_\mu$. Next (as V is balanced)

$$R_{\nu\mu}(z, \zeta) f(\zeta) \in R_{\nu\mu}(z, \zeta)\, f(T_\rho) \subset R_{\nu\mu}(z, \zeta) s V \subset V$$

for any $\nu \ge \mu > N$ and any $\zeta \in T_\rho$, $z \in \overline{P}(0, \mathbf{r})$. Hence for each $z \in \overline{P}(0, \mathbf{r})$ the sequence $\{S_\nu(z, \zeta)\}_{\nu \ge 0}$ is Cauchy in $\mathfrak{X}$ uniformly for $\zeta \in T_\rho$. The topology of $\mathfrak{X}$ (as a Fréchet space) is compatible to an invariant metric d so that the adopted description of Cauchy sequences is equivalent to the ordinary (metric) one. Hence (as d is complete) $\{S_\nu(z, \zeta)\}_{\nu \ge 0}$ is convergent in $\mathfrak{X}$ uniformly for $\zeta \in T_\rho$. Lemma 1 is proved.

By Lemma 1 we may integrate $\sum_{|\alpha| \ge 0} f_\alpha(z, \zeta)$ term-by-term (with respect to $\zeta \in T_\rho$) and obtain (by (6))

$$f(z) = \sum_{|\alpha|=0}^{\infty} z^\alpha x_\alpha \quad , \quad z \in \overline{P}(0, \mathbf{r}) \; , \;\; 0 < r \le \rho \; ,$$

$$x_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_n| = \rho} d\zeta_n \int_{|\zeta_{n-1}| = \rho} d\zeta_{n-1} \cdots \int_{|\zeta_1| = \rho} \left(1 / \zeta^{\alpha + \mathbf{1}} \right) \, f(\zeta)\, d\zeta_1 \in \mathfrak{X} \; .$$

To end the proof of Theorem 5 we need to show

Lemma 2. *The series* $\sum_{|\alpha|\geq 0} z^\alpha x_\alpha$ *converges in* $\mathfrak{X}$ *uniformly for* $z \in \overline{P}(0,r)$.

Proof. Let $\mathcal{P}$ be a separating family of seminorms for the topology of $\mathfrak{X}$. We set $V(p,k) = \{x \in \mathfrak{X} : p(x) < 1/k\}$ for any $p \in \mathcal{P}$ and $k \in \mathbb{Z}$, $k \geq 1$. The collection $\mathcal{B}$ of all finite intersections of the sets $V(p,k)$ is a convex balanced local base of $\mathfrak{X}$. Let $p \in \mathcal{P}$ and $k \in \mathbb{Z}$, $k \geq 1$. As $f(T_\rho)$ is bounded in $\mathfrak{X}$ there is $s > 0$ such that $f(T_\rho) \subset s\, V(p,k)$ i.e. $p[f(\zeta)] < s/k$ for any $\zeta \in T_\rho$. Then (by Theorem 20 in Appendix A to this paper)

$$p\,(x_\alpha) \leq \frac{1}{(2\pi)^n} \int_{|\zeta_n|=\rho} d\zeta_n \int_{|\zeta_{n-1}|=\rho} d\zeta_{n-1} \cdots \int_{|\zeta_1|=\rho} \frac{p[f(\zeta)]}{|\zeta^{\alpha+\mathbf{1}}|}\, d\zeta_1 \leq$$

$$\leq \frac{s}{k(2\pi)^n} \int_{T_\rho} \frac{d\zeta}{|\zeta^{\alpha+\mathbf{1}}|} = \frac{s}{k}\, \rho^{-|\alpha|}$$

hence

$$p\left(\sum_{|\alpha|=\mu+1}^{\nu} z^\alpha x_\alpha\right) \leq \sum_{|\alpha|=\mu+1}^{\nu} |z^\alpha|\; p\,(x_\alpha) \leq$$

$$\leq \frac{s}{k} \sum_{|\alpha|=\mu+1}^{\nu} \prod_{j=1}^{n} |z_j|^{\alpha_j} \rho^{-|\alpha|} \leq \frac{s}{k} \sum_{|\alpha|=\mu+1}^{\nu} \left(\frac{r}{\rho}\right)^{|\alpha|} < \frac{1}{k}$$

for some $N \geq 1$ and any $\nu > \mu \geq N$. Lemma 2 is proved.

Theorem 4 goes back to A. Grothendieck, [14]. Actually the following more general result holds.

Proposition 1. *Let* $\mathfrak{X}$ *be a complete locally convex space and* $\Omega \subset \mathbb{R}^N$ *an open set. Let* P *be a linear differential operator and* $f \in C^\infty(\Omega, \mathfrak{X})$. *Then* $Pf = 0$ *if and only if* $\Lambda \circ f \in \operatorname{Ker}(P) \subset C^\infty(\Omega)$ *for each* $\Lambda \in \mathfrak{X}^*$.

Proposition 1 is an easy consequence of the result by A. Grothendieck (cf. [14], Part I, p. 39, and Part II, p. 82) that for any complete locally convex space $\mathfrak{X}$ a function $f : \Omega \to \mathfrak{X}$ is C^∞ if and only if $\Lambda \circ f \in C^\infty(\Omega)$ for any $\Lambda \in \mathfrak{X}^*$.

The following Liouville type theorem may be proved.

Theorem 6. *Let* $\mathfrak{X}$ *be a complex topological vector space such that* $\mathfrak{X}^*$ *separates points. If* $f : \mathbb{C} \to \mathfrak{X}$ *is weakly holomorphic and* $f(\mathbb{C})$ *is a weakly bounded subset of* $\mathfrak{X}$ *then* f *is constant.*

The proof doesn't depend on Theorem 1. Under the assumptions of Theorem 6 the function $\Lambda \circ f$ is entire and bounded for each $\Lambda \in \mathfrak{X}^*$. Hence $\Lambda(f(z)) = \Lambda(f(0))$ for any $z \in \mathbb{C}$ (by Liouville's theorem for scalar-valued functions). As $\mathfrak{X}^*$ separates points it must be that $f(z) = f(0)$ for any $z \in \mathbb{C}$.

□

3. α-Differentiability *versus* areolar derivatives

The scope of this section is to discuss α-differentiability of functions $f : \Omega \subset \mathbb{C} \to \mathfrak{X}$ with values in a Fréchet space (cf. F-H. Vasilescu, [31]) as related to areolar derivatives (cf. D. Pompeiu, [22]-[23], N. Teodorescu, [30]). Let $\alpha : [0, 2\pi] \times [0, +\infty) \to \mathbb{C}$ be a continuous function such that

$$\lim_{r \to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta, r) d\,\theta = 0\ . \tag{8}$$

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$ an open neighborhood of z_0. Let $\mathfrak{X}$ be a complex topological vector space such that $\mathfrak{X}^*$ separates points and let $f : \Omega \to \mathfrak{X}$ be a continuous function. For sufficiently small $r > 0$ we consider the function $F_r : [0, 2\pi] \to \mathfrak{X}$ given by $F_r(\theta) = \alpha(\theta, r) f(z_0 + re^{i\theta})$ for any $0 \leq \theta \leq 2\pi$. If i) there is $r_0 > 0$ such that $\overline{\mathrm{co}}\,[F_r([0, 2\pi])]$ is a compact subset of $\mathfrak{X}$ for any $0 < r \leq r_0$ and ii) the limit

$$\lim_{r \to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta, r) f(z_0 + re^{i\theta})\, d\theta \tag{9}$$

exists in the topology of $\mathfrak{X}$ then f is said to be *α-differentiable* in z_0 and the limit (9) is denoted by $(\partial_\alpha f)(z_0)$. Let $B_\alpha(\Omega, z_0, \mathfrak{X})$ denote the set of all functions $f : \Omega \to \mathfrak{X}$ which are α-differentiable at z_0. The map $B_\alpha(\Omega, z_0, \mathfrak{X}) \to \mathfrak{X}$ given by $f \mapsto (\partial_\alpha f)(z_0)$ is $\mathbb{C}$-linear.

Let $\mathfrak{X}$ be a locally convex space. It may be easily shown (cf. F.H. Vasilescu, [31], or Proposition 5.2 in [32], p. 21-22) that any constant map $f : \Omega \to \mathfrak{X}$ is an element of $B_\alpha(\Omega, z_0, \mathfrak{X})$ and $(\partial_\alpha f)(z_0) = 0$. Let us assume additionally that $\mathfrak{X}$ is a Fréchet space. If $f \in B_\alpha(\Omega, z_0, \mathfrak{X})$ then $\varphi f \in B_\alpha(\Omega, z_0, \mathfrak{X})$ and

$$[\partial_\alpha(\varphi f)]\,(z_0) = (\partial_\alpha \varphi)(z_0) f(z_0) + \varphi(z_0)(\partial_\alpha f)(z_0)$$

for any function $\varphi \in B_\alpha(\Omega, z_0, \mathbb{C})$ such that $\varphi(z_0 + re^{i\theta}) - \varphi(z_0) = O(r)$ uniformly in $\theta \in [0, 2\pi]$ as $r \to 0$.

Let $\alpha : [0, 2\pi] \times [0, +\infty) \to \mathbb{C}$ be a continuous function with the property (8). Let $\mathfrak{X}$ be a Fréchet space. Let $\Omega \subset \mathbb{C}$ be an open set and $f \in C^\infty(\Omega, \mathfrak{X})$. Then

$$(\partial_\alpha f)(z_0) = \alpha_1 \frac{\partial f}{\partial x}(z_0) + \alpha_2 \frac{\partial f}{\partial y}(z_0) \tag{10}$$

where $2\pi\alpha_1 = \int_0^{2\pi} \cos\theta\ \alpha(\theta, 0)\, d\theta$ and $2\pi\alpha_2 = \int_0^{2\pi} \sin\theta\ \alpha(\theta, 0)\, d\theta$. Consequently the restriction $\partial_{\alpha,\infty} : C^\infty(\Omega, \mathfrak{X}) \subset C(\Omega, \mathfrak{X}) \to C(\Omega, \mathfrak{X})$ of ∂_α to $C^\infty(\Omega, \mathfrak{X})$ is a preclosed operator (cf. F-H. Vasilescu, [31], Lemma 4.1, p. 1030) hence it admits a unique minimal closed extension $\overline{\partial}_\alpha = (\partial_{\alpha,\infty})^- : \mathcal{D}\left[(\partial_{\alpha,\infty})^-\right] \subset C(\Omega, \mathfrak{X}) \to C(\Omega, \mathfrak{X})$ (the closure of $\partial_{\alpha,\infty}$). We set $B^0_\alpha(\Omega, \mathfrak{X}) = C(\Omega, \mathfrak{X})$ and $B^1_\alpha(\Omega, \mathfrak{X}) = \mathcal{D}\left(\overline{\partial}_\alpha\right)$. Let $B^p_\alpha(\Omega, \mathfrak{X})$ be defined by recurrence as $B^p_\alpha(\Omega, \mathfrak{X}) = \{f \in B^1_\alpha(\Omega, \mathfrak{X}) : \overline{\partial}_\alpha f \in B^{p-1}_\alpha(\Omega, \mathfrak{X})\}$ (with $p \geq 2$). Let $f \in B^p_\alpha(\Omega, \mathfrak{X})$. It may be easily shown that $B^{p+1}_\alpha(\Omega, \mathfrak{X}) \subset B^p_\alpha(\Omega, \mathfrak{X})$ for any $p \geq 1$ and if $\varphi \in C^\infty(\Omega)$ then $\varphi f \in B^p_\alpha(\Omega, \mathfrak{X})$ and $\overline{\partial}_\alpha(\varphi f) = (\overline{\partial}_\alpha \varphi) f + \varphi(\overline{\partial}_\alpha f)$ (cf. Lemma 4.2 in [31], p. 1031). Let μ be the Lebesgue measure on $\mathbb{R}^2$. We may state (cf. Lemma 4.3 in [31], p. 1031).

Theorem 7. *Let $\mathfrak{X}$ be a Fréchet space and $\Omega \subset \mathbb{C}$ an open subset. Let $f,\ g \in C(\Omega, \mathfrak{X})$. The following statements are equivalent*

a) $f \in B^1_\alpha(\Omega, \mathfrak{X})$ and $\overline{\partial}_\alpha f = g$,
b) for any $\varphi \in C^\infty_0(\Omega)$

$$\int_\Omega (\partial_\alpha \varphi)\,(z)\, f(z)\, d\mu(z) = -\int_\Omega \varphi(z)\, g(z)\, d\mu(z)\ . \tag{11}$$

Theorem 7 describes the weak solutions to $\overline{\partial}_\alpha f = g$ with $g \in C(\Omega, \mathfrak{X})$. The statement in [31] is however weaker than the proved result: there one assumes *a priori* that $f \in B^1_\alpha(\Omega, \mathfrak{X})$ while one actually shows that any continuous function f satisfying (11) is of class B^1_α. An ingredient in the proof is the fact that $\overline{\partial}_\alpha : B^1_\alpha(\Omega, \mathfrak{X}) \subset C(\Omega, \mathfrak{X}) \to C(\Omega, \mathfrak{X})$ is a closed operator (motivating the need to work with the closure of $\partial_{\alpha,\infty}$). As a corollary $C^\infty_0(\Omega, \mathfrak{X})$ is dense in $B^p_\alpha(\Omega, \mathfrak{X})$.

Note that the function $\alpha(\theta) = 2\left(\alpha_1 \cos\theta + \alpha_2 \sin\theta\right)$, $0 \le \theta \le 2\pi$ obeys to (8) and leads to the same expression (10) for $(\partial_\alpha f)(z_0)$. We work with this choice of α from now on. We may state (cf. Theorem 6.5 in [32], p. 24)

Theorem 8. *Let $\mathfrak{X}$ be a complex Fréchet space, $\Omega \subset \mathbb{C}$ an open set and $f \in C(\Omega, \mathfrak{X})$.* i) *If $(2\pi r)^{-1} \int_0^{2\pi} \alpha(\theta) f(z + re^{i\theta})\, d\theta$ converges uniformly on the compact subsets of Ω as $r \to 0$ then $f \in B^1_\alpha(\Omega, \mathfrak{X})$. Also* ii) *for any $f \in B^1_\alpha(\Omega, \mathfrak{X})$*

$$\left(\overline{\partial}_\alpha f\right)(z) = \lim_{r \to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) f(z + re^{i\theta})\, d\theta \quad , \quad z \in \Omega\,. \tag{12}$$

Proof. We start by assuming that the right hand side of (12) converges as $r \to 0$ uniformly on the compact subsets of Ω. We shall apply Theorem 7. Let $\varphi \in C^\infty_0(\Omega)$ and $0 < r < \operatorname{dist}(\Gamma, \mathbb{C} \setminus \Omega)$ where $\Gamma = \operatorname{supp}(\varphi) \subset \Omega$. Then

$$\int_\Omega \varphi(z) \left\{ \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) f\left(z + re^{i\theta}\right) d\theta \right\} d\mu(z) =$$

$$= \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) \left\{ \int_\Omega \varphi(z) f\left(z + re^{i\theta}\right) d\mu(z) \right\} d\theta =$$

$$= \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) \left\{ \int_\Omega \varphi\left(\zeta - re^{i\theta}\right) f(\zeta)\, d\mu(\zeta) \right\} d\theta =$$

$$= \int_\Omega \left\{ \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) \varphi\left(\zeta - re^{i\theta}\right) d\theta \right\} f(\zeta)\, d\zeta$$

so that for $r \to 0$

$$\int_\Omega \varphi(z) \left\{ \lim_{r \to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) f(z + re^{i\theta})\, d\theta \right\} d\mu(z) =$$

$$= - \int_\Omega \left(\partial_\alpha \varphi\right)(\zeta) f(\zeta)\, d\mu(\zeta)$$

hence (by Theorem 7) $f \in B^1_\alpha(\Omega, \mathfrak{X})$.

Moreover let us show that (12) holds for any $f \in B^1_\alpha(\Omega, \mathfrak{X})$. If this is the case let $f_\nu \in C^\infty(\Omega, \mathfrak{X})$ such that $f_\nu \to f$ as $\nu \to \infty$ and $\{\partial_\alpha f_\nu\}_{\nu \ge 1}$ is convergent in $C(\Omega, \mathfrak{X})$. Let ω be a relatively compact open set such that $\overline{\omega} \subset \Omega$ and $\partial\omega$ is a finite union of Jordan rectifiable curves. Then (by Green's formula)

$$\int_\omega \left(\partial_\alpha f_\nu\right)(z)\, d\mu(z) = \int_{\partial\omega} f_\nu \left(\alpha_1 dy - \alpha_2 dx\right)\,. \tag{13}$$

Let $z_0 \in \Omega$ and $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$. Let $\nu \to \infty$ in (13) for $\omega = D_r(z_0)$. As $\overline{\partial}_\alpha f = \lim_{\nu \to \infty} \partial_\alpha f_\nu$ (in the topology of $C(\Omega, \mathfrak{X})$)

$$\int_{D_r(z_0)} \left(\overline{\partial}_\alpha f\right)(z)\, d\mu(z) = \int_{\partial D_r(z_0)} f\left(\alpha_1 dy - \alpha_2 dx\right)$$

or

$$\int_0^r \rho d\rho \int_0^{2\pi} \left(\overline{\partial}_\alpha f\right)\left(z_0 + \rho e^{i\theta}\right) \, d\theta = \frac{r}{2} \int_0^{2\pi} \alpha(\theta) f\left(z_0 + r e^{i\theta}\right) \, d\theta \, .$$

Finally (by l'Hôspital's rule)

$$\lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) f(z_0 + re^{i\theta}) \, d\theta =$$

$$= \lim_{r\to 0} \frac{1}{\pi r^2} \int_0^r \rho \, d\rho \int_0^{2\pi} \left(\overline{\partial}_\alpha f\right)(z_0 + \rho e^{i\theta}) \, d\theta =$$

$$= \lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} r \left(\overline{\partial}_\alpha f\right)(z_0 + re^{i\theta}) \, d\theta = \left(\overline{\partial}_\alpha f\right)(z_0) \, .$$

The notion of areolar derivative may be extended to vector valued functions (cf. L-J. Nicolescu, [21], I. Ciorănescu, [8], F-H. Vasilescu, [31]) as follows. Let $\Omega \subset \mathbb{C}$ be a domain and $z_0 \in \Omega$ a point. Let $\mathfrak{X}$ be a topological vector space such that $\mathfrak{X}^*$ separates points. Let $f \in C(\Omega, \mathfrak{X})$ be a continuous function such that $\overline{\text{co}}[f(\partial\omega)]$ is a compact subset of $\mathfrak{X}$ for any domain $\omega \subset \mathbb{C}$ with simple rectifiable boundary and such that $z_0 \in \omega$ and $\overline{\omega} \subset \Omega$. We set

$$F(\omega) = \frac{1}{2i} \int_{\partial\omega} f(z) \, dz \, . \tag{14}$$

We adopt the following definition. Let $\mathfrak{X}$ be a locally convex space. If the following limit exists

$$\varphi(z_0) = \lim_{\text{diam}(\omega)\to 0} \frac{F(\omega)}{|\omega|} \in \mathfrak{X} \tag{15}$$

then f is said to be (*strongly*) *monogeneous* at z_0 and $\varphi(z_0)$ is referred to as the (*strong*) *areolar derivative* of f at z_0. Here $|\omega| = \mu(\omega)$ is the Lebesgue measure of ω. One adopts the traditional notation $\varphi(z_0) = (Df/D\omega)(z_0)$. Let $\mathcal{P}$ be a separating family of seminorms inducing the topology of $\mathfrak{X}$ as a locally convex space. By the existence of $\varphi(z_0) = \lim_{\text{diam}(\omega)\to 0} F(\omega)/|\omega|$ one means that for any $p \in \mathcal{P}$ and any positive integer $k \geq 1$ there is $r > 0$ such that

$$p\left(\frac{F(\omega)}{|\omega|} - \varphi(z_0)\right) < \frac{1}{k}$$

for any domain $\omega \subset \mathbb{C}$ such that $\partial\omega$ is a simple rectifiable curve, $z_0 \in \omega$ and $\overline{\omega} \subset D_r(z_0)$. Let $f \in C(\Omega, \mathfrak{X})$ such that the integral (14) is well defined. A vector $\varphi(x_0) \in \mathfrak{X}$ is the *weak areolar derivative* of f at z_0 if (15) holds in the weak sense i.e. for any $\Lambda \in \mathfrak{X}^*$ and any $\epsilon > 0$ there is $r > 0$ such that

$$\left|\Lambda\left(\frac{F(\omega)}{|\omega|} - \varphi(z_0)\right)\right| < \epsilon$$

for any domain $\omega \subset \mathbb{C}$ with simple rectifiable boundary $\partial\omega$ such that $z_0 \in \omega$ and $\overline{\omega} \subset D_r(z_0)$. A function $f : \Omega \to \mathfrak{X}$ admitting a weak areolar derivative at $z_0 \in \Omega$ is referred to as *weakly monogeneous* at z_0. As $\Lambda[F(\omega)] = (2i)^{-1} \int_{\partial\omega} \Lambda[f(z)] \, dz$ a function $f : \Omega \to \mathfrak{X}$ admitting a (strong) areolar derivative at z_0 has a weak areolar derivative at that point as well.

Let $\omega = D_r(z_0)$ be a ball in $\mathbb{C}$ such that $\overline{\omega} \subset \Omega$. Let $\mathfrak{X}$ be a Fréchet space and $f \in C(\Omega, \mathfrak{X})$. If the areolar derivative of f at z_0 exists then

$$\frac{Df}{D\omega}(z_0) = \lim_{r\to 0} \frac{F(D_r(z_0))}{|D_r(z_0)|} = \lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} e^{i\theta} f(z_0 + re^{i\theta})\, d\theta \; .$$

Therefore f is α-differentiable with $\alpha(\theta) = e^{i\theta}$ and

$$(\partial_\alpha f)(z_0) = \frac{Df}{D\omega}(z_0) \; .$$

On the other hand to the choice $\alpha(\theta) = e^{i\theta}$ there corresponds (cf. (10)) the operator $\partial_{\alpha,\infty} = 1/2\,(\partial/\partial x + i\,\partial/\partial y)$. We maintain this choice of α for the reminder of this section and drop the index α everywhere i.e. we adopt the notations $\overline{\partial} = \overline{\partial}_\alpha$ and $B^p(\Omega, \mathfrak{X}) = B^p_\alpha(\Omega, \mathfrak{X})$. We say that $f \in B^1(\Omega, \mathfrak{X})$ is *analytic* in Ω if $\overline{\partial} f = 0$. Let $A(\Omega, \mathfrak{X})$ be the set of all analytic functions $f \in B^1(\Omega, \mathfrak{X})$. The following analog to Weierstrass' theorem is due to F-H. Vasilescu, [32]

Theorem 9. *Let $\mathfrak{X}$ be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $\{f_\nu\}_{\nu\geq 1} \subset A(\Omega, \mathfrak{X})$ converge in $C(\Omega, \mathfrak{X})$ as $\nu \to \infty$ to $f \in C(\Omega, \mathfrak{X})$. Then $f \in A(\Omega, \mathfrak{X})$.*

Proof. Let $f_\nu \in B^1(\Omega, \mathfrak{X})$ such that $\overline{\partial} f_\nu = 0$ and $f_\nu \to f$ in $C(\Omega, \mathfrak{X})$ as $\nu \to \infty$. As $\overline{\partial}$ is a closed operator then $f \in \mathcal{D}(\overline{\partial}) = B^1(\Omega, \mathfrak{X})$ and $\overline{\partial} f = 0$.

The following version of the Cauchy-Pompeiu formula (for vector-valued functions of class B^1) holds (cf. Theorem 7.1 in [32], p. 26)

Theorem 10. *Let $\mathfrak{X}$ be a complex Fréchet space, $\Omega \subset \mathbb{C}$ an open set and $f \in B^1(\Omega, \mathfrak{X})$. Let $\omega \subset \mathbb{C}$ be a relatively compact open set such that $\overline{\omega} \subset \Omega$ and $\partial\omega$ is a finite union of Jordan rectifiable curves. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} (\zeta - z)^{-1} f(\zeta)\, d\zeta + \frac{1}{2\pi i} \int_\omega (\zeta - z)^{-1} \left(\overline{\partial} f\right)(\zeta)\, d\zeta \wedge d\overline{\zeta} \tag{16}$$

for any $z \in \omega$.

In particular

Corollary 1. *Let $f \in A(\Omega, \mathfrak{X})$ and let $\omega \subset \mathbb{C}$ be a relatively compact open subset such that $\overline{\omega} \subset \Omega$ and $\partial\omega$ is a finite union of Jordan rectifiable curves. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} (\zeta - z)^{-1} f(\zeta)\, d\zeta \tag{17}$$

for any $z \in \omega$.

By differentiating indefinitely many times in (17) one proves easily the regularity of vector-valued analytic functions

Corollary 2. *Let $\mathfrak{X}$ be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Then $A(\Omega, \mathfrak{X}) \subset C^\infty(\Omega, \mathfrak{X})$.*

In particular any analytic function $f \in A(\Omega, \mathfrak{X})$ satisfies the Cauchy-Riemann equation $f_{\overline{z}} = 0$ in Ω (equivalently f is $\mathbb{C}$-differentiable in Ω). Here $f_{\overline{z}} = 1/2(\partial f/\partial x + i\,\partial f/\partial y)$ for any $f \in C^\infty(\Omega, \mathfrak{X})$. Then (cf. Lemma 8.6 in [32], p. 29)

Proposition 2. *Let $\mathfrak{X}$ be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $f \in A(\Omega, \mathfrak{X})$. Let $z \in \Omega$ and $r > 0$ such that $\overline{D}_r(z) \subset \Omega$. Then*

$$f(\zeta) = \sum_{\nu=0}^{\infty} \frac{(\zeta - z)^\nu}{\nu!} (\partial^\nu f)(z) \quad , \quad \zeta \in D_r(z) \, , \tag{18}$$

and the series in the right hand side of (18) *is uniformly convergent on any compact subset of $D_r(z)$. In particular each $f \in A(\Omega, \mathfrak{X})$ is* (*strongly*) *holomorphic in Ω.*

The basics on power series in Fréchet spaces are given in Appendix B. Let $f \in A(\Omega, \mathfrak{X})$. A point $z_0 \in \mathbb{C}$ is an *isolated singularity* of f if $z_0 \notin \Omega$ and there is $r > 0$ such that $D_r(z_0) \setminus \{z_0\} \subset \Omega$. An isolated singularity z_0 of $f \in A(\Omega, \mathfrak{X})$ is *removable* if there is $F \in A(\Omega \cup \{z_0\}, \mathfrak{X})$ such that $F|_\Omega = f$. As an application of the Cauchy integral formula (17) we prove

Theorem 11. *Let $\mathfrak{X}$ be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f \in A(\Omega, \mathfrak{X})$. Then z_0 is removable if and only if f is bounded in some neighborhood of z_0.*

Proof. Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f \in A(\Omega, \mathfrak{X})$ and let $r > 0$ such that $D_{2r}(z_0) \setminus \{z_0\} \subset \Omega$. If z_0 is removable let F be a holomorphic extension of f to $\Omega \cup \{z_0\}$. Then $\Lambda \circ F \in C\left(\overline{D}_r(z_0)\right)$ for each $\Lambda \in \mathfrak{X}^*$ hence there is $M(\Lambda) > 0$ such that $|\Lambda[F(z)]| \leq M(\Lambda)$ for any $|z - z_0| \leq r$. That is the set $F\left[D_r(z_0)\right]$ is weakly bounded in $\mathfrak{X}$ and then strongly bounded (as $\mathfrak{X}$ is a locally convex space). Therefore $f\left(D_r(z_0) \setminus \{z_0\}\right)$ is bounded.

Viceversa let us assume that $E = f\left(D_r(z_0) \setminus \{z_0\}\right)$ is bounded in $\mathfrak{X}$. Let $0 < \rho < r$ be fixed and let $\Gamma_\rho = \partial D_\rho(z_0)$. Next let $F : \Omega \cup \{z_0\} \to \mathfrak{X}$ be given by

$$F(z) = \begin{cases} f(z) \, , & \text{if } z \neq z_0 \, , \\ \displaystyle\int_{\Gamma_\rho} (\zeta - z_0)^{-1} f(\zeta) \, d\zeta \, , & \text{if } z = z_0 \, . \end{cases}$$

Then F is a holomorphic extension of f to $\Omega \cup \{z_0\}$. Indeed let $0 < \epsilon < \rho$ and $\Gamma_\epsilon = \partial D_\epsilon(z_0)$. Let $z_\epsilon \in \Gamma_\epsilon$ and $z_\rho \in \Gamma_\rho$ be fixed. Let $\Omega_\epsilon = D_\rho(z_0) \setminus \overline{D}_\epsilon(z_0)$ so that $\partial\Omega_\epsilon = \Gamma_\epsilon \cup \Gamma_\rho$. We consider the piecewise C^1 closed curve $\Gamma = \Gamma_\rho \cdot (z_\rho z_\epsilon) \cdot \Gamma_\epsilon^- \cdot (z_\epsilon z_\rho)$. Then (by (17)) for each $z \in \Omega_\epsilon$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma (\zeta - z)^{-1} f(\zeta) \, d\zeta = \tag{19}$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\rho} (\zeta - z)^{-1} f(\zeta) \, d\zeta - \frac{1}{2\pi i} \int_{\Gamma_\epsilon} (\zeta - z)^{-1} f(\zeta) \, d\zeta \, .$$

Let $\mathcal{P}$ be a separating family of semi-norms determining the topology of $\mathfrak{X}$ as a locally convex space. As E is bounded for any $p \in \mathcal{P}$ and any positive integer $k \geq 1$ there is $M = M(p, k) > 0$ such that $E \subset M \cdot V(p, k)$ that is

$$p[f(\zeta)] < \frac{M}{k} \quad , \quad \zeta \in D_r(z_0) \setminus \{z_0\} \, .$$

Then

$$p\left(\int_{\Gamma_\epsilon} (\zeta - z)^{-1} f(\zeta) \, d\zeta\right) \leq \frac{2\pi\epsilon M}{k} \sup_{\zeta \in \Gamma_\epsilon} |\zeta - z|^{-1}$$

and $\lim_{\epsilon \to 0^+} \epsilon \sup_{\zeta \in \Gamma_\epsilon} |\zeta - z|^{-1} = 0$. Thus there is $\epsilon_0 = \epsilon_0(p, k) > 0$ such that $\epsilon \sup_{\zeta \in \Gamma_\epsilon} |\zeta - z|^{-1} < 1/(2\pi M)$. Therefore

$$\int_{\Gamma_\epsilon} (\zeta - z)^{-1} f(\zeta)\, d\zeta \in V(p, k) \quad , \quad 0 < \epsilon \leq \epsilon_0 \, .$$

Passing to the limit in (19) as $\epsilon \to 0$ leads to

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} (\zeta - z)^{-1} f(\zeta)\, d\zeta \quad , \quad z \in D_\rho(z_0) \setminus \{z_0\} \, ,$$

so that F is a well defined holomorphic function and $F|_\Omega = f$.

□

Further details on the theory of monogeneous functions (following [21] and [8]) compared to the presentation in Section 3 are given in Appendix C.

4. Holomorphic functions with values in Banach spaces

Let $\mathfrak{X}$ be a complex Banach space and $\Omega \subset \mathbb{C}$ an open set. If $f : \Omega \to \mathfrak{X}$ is $\mathbb{C}$-differentiable then $f : \Omega \to \mathfrak{X}$ is weakly holomorphic hence Cauchy's integral formula (2) holds (cf. Theorem 1 above) so that (by the proof of Theorem 4 for $n = 1$) given $z_0 \in \Omega$ and $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$

$$f(z) = \sum_{\nu=0}^{\infty} (z - z_0)^\nu x_\nu \quad , \quad x_\nu = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{\nu+1}}\, d\zeta \, ,$$

converges absolutely for $z \in D_r(z_0)$ [i.e. $f : \Omega \to \mathfrak{X}$ is (strongly) holomorphic]. The following version of the identity theorem for holomorphic functions with values in a Banach space is known (cf. e.g. [2], p. 456)

Theorem 12. *Let $\mathfrak{X}$ be a complex Banach space and $Y \subset \mathfrak{X}$ a closed subspace. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathfrak{X}$ a holomorphic function. Let us assume that there is a convergent sequence $\{z_\nu\}_{\nu \geq 1} \subset \Omega$ such that $\lim_{\nu \to \infty} z_n \in \Omega$ and $f(z_\nu) \in Y$ for any $\nu \geq 1$. Then $f(z) \in Y$ for any $z \in \Omega$.*

Proof. Let $Y^\perp = \{\Lambda \in \mathfrak{X}^* : \operatorname{Ker}(\Lambda) \supseteq Y\}$ be the conormal space associated to Y. Also if $Z \subset \mathfrak{X}^*$ we set ${}^\perp Z = \{x \in \mathfrak{X} : \Lambda(x) = 0 \; , \; \Lambda \in Z\}$. For each $\Lambda \in Y^\perp$ one has $\Lambda\left[f\left(z_\nu\right)\right] = 0$ for any $\nu \geq 1$. Yet $\Lambda \circ f$ is holomorphic in Ω in the ordinary sense hence (by the identity theorem in the scalar case) $\Lambda\left[f(z)\right] = 0$ for any $z \in \Omega$. Thus $f(z) \in {}^\perp\left(Y^\perp\right) = \overline{Y} = Y$ for any $z \in \Omega$ (cf. Theorem 4.7 in [24], p. 96).

□

A subset $N \subset \mathfrak{X}^*$ is *norming* if

$$\|x\|_1 = \sup_{\Lambda \in N} |\Lambda(x)| \quad , \quad x \in \mathfrak{X} \, ,$$

is a norm on $\mathfrak{X}$ equivalent to the original norm $\| \cdot \|$. A function $f : \Omega \to \mathfrak{X}$ is *locally bounded* if $\sup_{z \in K} \|f(z)\| < \infty$ for every compact set $K \subset \Omega$. Then (cf. Proposition A.3 in [2])

Theorem 13. *Let $\Omega \subset \mathbb{C}$ be an open set and $N \subset \mathfrak{X}^*$ a norming set. Let $f : \Omega \to \mathfrak{X}$ be a locally bounded function such that*

$$\Lambda \circ f \in \mathcal{O}(\Omega) \quad , \quad \Lambda \in N \, . \tag{20}$$

Then $f : \Omega \to \mathfrak{X}$ is holomorphic.

Then statement (iii) in Theorem 1 (rephrased for a Banach space $\mathfrak{X}$) is of course less general than Theorem 13 (the requirement of weak holomorphy has been weakened down to (20)).

Corollary 3. *Let $\Omega \subset \mathbb{C}$ be a domain and $\omega \subset \Omega$ an open subset. Let $f : \omega \to \mathfrak{X}$ be a holomorphic function. Let us assume that there is a norming set $N \subset \mathfrak{X}^*$ such that for each $\Lambda \in N$ there is a holomorphic extension $F_\Lambda : \Omega \to \mathbb{C}$ of $\Lambda \circ f : \omega \to \mathbb{C}$. If*

$$\sup_{\Lambda \in N,\ z \in \Omega} |F_\Lambda(z)| < \infty$$

then $f : \omega \to \mathfrak{X}$ admits a unique holomorphic extension $F : \Omega \to \mathfrak{X}$.

Proof. We may assume without loss of generality that $\|x\|_1 = \|x\|$ for any $x \in \mathfrak{X}$. Let Y consist of all families $y = (y_\Lambda)_{\Lambda \in N}$, $y_\Lambda \in \mathbb{C}$, such that $\|y\|_\infty := \sup_{\Lambda \in N} |y_\Lambda| < \infty$. Let $F : \Omega \to Y$ be defined by

$$F(z) = (F_\Lambda(z))_{\Lambda \in N} \quad , \quad z \in \Omega \, .$$

By the proof of Theorem 13 the function $F : \Omega \to Y$ is holomorphic. On the other hand $x \in \mathfrak{X} \mapsto (\Lambda(x))_{\Lambda \in N} \in Y$ is an isometric embedding of $\mathfrak{X}$ into Y. As $F(z) \in \mathfrak{X}$ for any $z \in \omega$ the identity theorem (Theorem 12 above) yields $F(z) \in \mathfrak{X}$ for any $z \in \Omega$.

□

Let $\Omega \subset \mathbb{C}$ be a domain and let $A \subset \Omega$ be a subset which contains an accumulation point in Ω. Vitali's theorem asserts that given a locally bounded sequence $\{f_\nu\}_{\nu \geq 1}$ of holomorphic functions on Ω such that $\{f_\nu(z)\}_{\nu \geq 1}$ converges for each $z \in A$ there is a holomorphic function $f \in \mathcal{O}(\Omega)$ such that $\{f_\nu\}_{\nu \geq 1}$ converges to f in the compact open topology. Montel's theorem states that each locally bounded sequence $\{f_\nu\}_{\nu \geq 1}$ of holomorphic functions on Ω admits a subsequence which converges in the compact open topology (cf. e.g. [27]). In the language of functional analysis Montel's theorem asserts that each bounded subset of $\mathcal{O}(\Omega)$ (endowed with the compact-open topology) is relatively compact. Vitali's and Montel's theorems are known to be equivalent. The picture is rather different for vector-valued functions and, in contrast with the scalar case, Montel's theorem doesn't hold for holomorphic functions with values in a Banach space. Nevertheless, a version of Vitali's theorem was proved by W. Arendt & N. Nikolski, [1].

Theorem 14. *Let $\mathfrak{X}$ be a complex Banach space. Let $\Omega \subset \mathbb{C}$ be a domain and $f_\nu : \Omega \to \mathfrak{X}$ a sequence of holomorphic functions such that*

$$\sup_{\nu \geq 1,\ z \in D_r(z_0)} \|f_\nu(z)\| < \infty \quad , \quad \overline{D}_r(z_0) \subset \Omega \, .$$

Let us assume that the set $\Omega_0 = \{z \in \Omega \, : \, \lim_{\nu \to \infty} f_\nu(z) \ \ \text{exists}\}$ has an accumulation point in Ω. Then there is a holomorphic function $f : \Omega \to \mathfrak{X}$ such that $f^{(k)}(z) = \lim_{\nu \to \infty} f_\nu^{(k)}(z)$ uniformly on the compact subsets of Ω for any $k \in \mathbb{Z}$, $k \geq 0$.

Another generalization of Vitali's theorem to the vector-valued case (where holomorphic functions are replaced by an appropriate sheaf of smooth functions on $\Omega \subset \mathbb{R}^n$) was produced by E. Jordá Mora, [17].

Let $\mathfrak{X}$ be a topological vector space. A subset $W \subset \mathfrak{X}^*$ is *separating* if $\Lambda(x) = 0$ for each $\Lambda \in W$ implies $x = 0$. We may state (cf. again [1]) the following criterion of analyticity

Theorem 15. *Let $\mathfrak{X}$ be a complex Banach space. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathfrak{X}$ a locally bounded function. Let $W \subset \mathfrak{X}^*$ be a separating subspace such that $\Lambda \circ f \in \mathcal{O}(\Omega)$ for any $\Lambda \in W$. Then $f : \Omega \to \mathfrak{X}$ is holomorphic.*

The proof follows from Vitali's theorem (Theorem 14 above), Krein-Šmulian's theorem (cf. Theorem 2.7.11 in [19]) and Theorem 13.

A more general version of Theorem 15 was established by F.G. Grosse-Erdmann, [13]. Let $\mathfrak{X}$ be a locally convex space. A subset $E \subset \mathfrak{X}$ is *absolutely convex* if $\alpha E + \beta E \subset E$ for any $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$. Let $B \subset \mathfrak{X}$ be a closed, absolutely convex and bounded subset. Let $\mathfrak{X}_B$ denote the linear span over $\mathbb{C}$ of B in $\mathfrak{X}$. The *Minkowski functional* of B is given by $\mu_B(x) = \inf\{t > 0 : t^{-1}x \in B\}$ for any $x \in \mathfrak{X}$. The space $\mathfrak{X}$ is *locally complete* if $(\mathfrak{X}_B \,,\, \mu_B)$ is a Banach space for any B as above. It may be shown (cf. S.A. Saxon & L.M. Sánchez Ruiz, [25]) that a locally convex space $\mathfrak{X}$ is locally complete if and only if $\sum_{\nu \geq 1} a_\nu x_\nu$ converges in $\mathfrak{X}$ for every bounded sequence $\{x_\nu\}_{\nu \geq 1} \subset \mathfrak{X}$ and every sequence $\{a_\nu\}_{\nu \geq 1} \subset \mathbb{C}$ with $\sum_{\nu=1}^{\infty} |a_\nu| < \infty$.

Let $\mathfrak{X}$ be a topological vector space and $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \to \mathfrak{X}$ is *locally bounded* if for each $z \in \Omega$ there is a neighborhood $z \in U \subset \Omega$ such that f is bounded on U. We may state (cf. Theorem 1 in [13], p. 399)

Theorem 16. *Let $\mathfrak{X}$ be a locally complete space. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathfrak{X}$ a function. If*

i) there is a separating subset $W \subset \mathfrak{X}^*$ such that $\Lambda \circ f \in \mathcal{O}(\Omega)$ for any $\Lambda \in W$, and

ii) f is locally bounded,

then $f : \Omega \to \mathfrak{X}$ is holomorphic.

It should be mentioned that Theorem 16 has already been applied in an array of situations e.g. to the summability of power series (cf. K.G. Grosse-Erdmann, [12]), to weighted spaces of vector-valued holomorphic functions (cf. K.D. Bierstedt & S. Holtmanns, [5]), to Tauberian convergence theorems (cf. [1]) and to the extension of vector-valued meromorphic functions (cf. E. Jordá Mora, [16]).

Let $\mathfrak{X}$ be a locally complete space and $\Omega \subset \mathbb{C}$ a domain. Let $f : \Omega \to \mathfrak{X}$ be a function. There are several instances where the assumption (i) in Theorem 16 is readily satisfied (though in none of these cases the assumption (ii) may be dropped, in general). For example

I) $\mathfrak{X}$ is a *sequence space* (i.e. each element of $\mathfrak{X}$ is a sequence $\{\lambda_\nu\}_{\nu \geq 1} \subset \mathbb{C}$) with continuous coordinate functionals $\{\lambda_\nu\}_{\nu \geq 1} \mapsto \lambda_\mu$, $\mu \in \mathbb{Z}$, $\mu \geq 1$, and $f = \{f_\nu\}_{\nu \geq 1}$ is *componentwise holomorphic* i.e. $f_\nu \in \mathcal{O}(\mathbb{C})$ for any $\nu \geq 1$ (where $f_\nu = \lambda_\nu \circ f$).

II) $\mathfrak{X}$ is a *function space* on some set E (i.e. each element of $\mathfrak{X}$ is a function $\varphi : E \to \mathbb{C}$) with continuous evaluation functionals $\varphi \mapsto \varphi(t)$, $t \in E$, $\varphi \in \mathfrak{X}$, and f is *pointwise holomorphic* i.e. each function $z \mapsto f(z)(t)$, $t \in E$, is holomorphic in Ω.

III) $\mathfrak{X} = Y^*$ is the *dual of a locally convex space* Y with a topology that is stronger than the weak*-topology $\sigma(Y^*, Y)$ and f is *weak*-holomorphic* i.e. each function $z \mapsto f(z)(y)$, $y \in Y$, is holomorphic in Ω.

IV) $\mathfrak{X} = L(Y, Z)$ is the *space of continuous linear operators* between locally convex spaces Y and Z with a topology that is stronger than the weak operator topology (WOT) and f is *WOT-holomorphic* i.e. each function $z \mapsto \Lambda[f(z)y]$, $y \in Y$, $\Lambda \in Z^*$ is holomorphic in Ω.

5. BOUNDARY BEHAVIOR OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS

Let $\mathfrak{X}$ be a complex Banach space and the half space $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. Let $H_p(\mathbb{C}_+, \mathfrak{X})$ consist of all holomorphic functions $f : \mathbb{C}_+ \to \mathfrak{X}$ such that

$$\|f\|_{H_p(\mathbb{C}_+,\mathfrak{X})} = \sup_{x>0} \left(\int_{-\infty}^{+\infty} \|f(x+iy)\|^p \, dy \right)^{1/p} < \infty \, . \tag{21}$$

Next we consider the sector $\Sigma_\alpha = \{re^{i\theta} \in \mathbb{C} : r > 0, \ |\theta| < \alpha\}$ and the space $H_p(\Sigma_\alpha, \mathfrak{X})$ consisting of all holomorphic functions $f : \Sigma_\alpha \to \mathfrak{X}$ such that

$$\|f\|_{H_p(\Sigma_\alpha,\mathfrak{X})} = \sup_{|\theta|<\alpha} \left(\int_0^{\infty} \|f(re^{i\theta})\|^p \, dr \right)^{1/p} < \infty \, . \tag{22}$$

By a result of A.M. Sedlecki, [26], the spaces $H_p(\mathbb{C}_+, \mathbb{C})$ and $H_p(\Sigma_{\pi/2}, \mathbb{C})$ are isomorphic for all $0 < p < \infty$. In the vector-valued case, as shown by P. Vieten, [33], the spaces $H_p(\mathbb{C}_+, \mathfrak{X})$ and $H_p(\Sigma_{\pi/2}, \mathfrak{X})$ are isomorphic for any $1 \le p < \infty$ and any complex Banach space $\mathfrak{X}$ and the norms (21)-(22) are equivalent. As an application P. Vieten studied (cf. *op. cit.*) the boundary behavior of holomorphic functions $f \in H_p(\Sigma_\alpha, \mathfrak{X})$. To report on his findings we need some preparation. Let $L^p(\mathbb{R}, \mathfrak{X}, dt)$ be the Bochner L^p-space (cf. e.g. [2], p. 14). A function $u \in L^p(\mathbb{R}, \mathfrak{X}, dt)$ is the *boundary values* of f if f is the Poisson integral of u i.e.

$$f(z) = \int_{-\infty}^{+\infty} P_z(t) u(t) \, dt \quad , \quad z \in \mathbb{C}_+ \, ,$$

where P_z is the Poisson kernel

$$P_z(t) = \frac{1}{\pi} \, \frac{x}{x^2 + (y-t)^2} \quad , \quad t \in \mathbb{R}, \ \ z = x + iy \in \mathbb{C}_+ \, .$$

Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Let $Y_q(\mathbb{R}) = L^q(\mathbb{R}, \mathbb{C})$ for any $1 < q \le \infty$ and let $Y_\infty(\mathbb{R})$ be the space of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity. Let $1 < p \le \infty$. An operator $T : Y_q(\mathbb{R}) \to \mathfrak{X}$ is *p-bounded* if there is $u \in L^p(\mathbb{R}, \mathbb{C})$ such that

$$\|Tv\| \le \int_{-\infty}^{+\infty} |v(t)| u(t) \, dt \quad , \quad v \in Y_q(\mathbb{R}) \, .$$

Cf. J. Diestel & J.J. Uhl, [9], for the description of p-bounded operators. An operator $T : Y_\infty(\mathbb{R}) \to \mathfrak{X}$ is *1-bounded* if there is a function of bounded variation $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\|Tv\| \le \int_{-\infty}^{+\infty} |v(t)| \, d\phi(t) \quad , \quad v \in Y_\infty(\mathbb{R}) \, .$$

Cf. [2], p. 49-50, for the Riemann-Stieltjes integral of a $\mathfrak{X}$-valued function. Let $h_p(\mathbb{C}_+, \mathfrak{X})$ consist of all harmonic functions $f : \mathbb{C}_+ \to \mathfrak{X}$ such that $\|f\|_{h_p(\mathbb{C}_+,\mathfrak{X})} < \infty$ where

$$\|f\|_{h_p(\mathbb{C}_+,\mathfrak{X})} = \begin{cases} \sup\limits_{x>0} \left(\displaystyle\int_{-\infty}^{+\infty} \|f(x+iy)\|^p \, dy \right)^{1/p} , & \text{if } 1 \le p < \infty, \\[2ex] \sup\limits_{z \in \mathbb{C}_+} \|f(z)\| \, , & \text{if } p = \infty \, . \end{cases}$$

A complex Banach space has the *Radon-Nikodym property* if each Lipschitz function $f : [0, +\infty) \to \mathfrak{X}$ is differentiable a.e. in $[0, +\infty)$ (cf. Proposition 1.2.4 in [2], p. 19, for an equivalent description of spaces with the Radon-Nikodym property). We may state (cf. Theorem 2 in [33])

Theorem 17. *Let $1 \le p \le \infty$. A harmonic function $f : \mathbb{C}_+ \to \mathfrak{X}$ belongs to $h_p(\mathbb{C}_+, \mathfrak{X})$ if and only if there is a p-bounded operator $T : Y_q(\mathbb{R}) \to \mathfrak{X}$ such that*

$$f(z) = T(P_z) = \lim_{s \to 0^+} \int_{-\infty}^{+\infty} P_z(t) f(s + it)\, dt \quad , \quad z \in \mathbb{C}_+ \, .$$

If $1 < p \le \infty$ and $\mathfrak{X}$ has the Radon-Nikodym property there exists $u \in L^p(\mathbb{R}, \mathfrak{X})$ which is the boundary values of f.

The statements about functions in $h_p(\mathbb{C}_+, \mathfrak{X})$ apply to functions in $H_p(\mathbb{C}_+, \mathfrak{X})$ as well. To state a similar result on functions $f \in H_p(\Sigma_\alpha, \mathfrak{X})$ let us set

$$f_\eta(t) = f\,[|t| \, \exp(i \, \mathrm{sign}(t) \, \eta)] \quad , \quad t \in \mathbb{R}, \;\; 0 < \eta < \alpha \, .$$

A function $u \in L^p(\mathbb{R}, \mathfrak{X})$ is the *boundary values* of $f \in H_p(\Sigma_\alpha, \mathfrak{X})$ if

$$\lim_{\eta \to \alpha^-} \int_{-\infty}^{+\infty} v(t) f_\eta(t)\, dt = \int_{-\infty}^{+\infty} v(t) u(t)\, dt \quad , \quad v \in Y_q(\mathbb{R}) \, .$$

Then (cf. Theorem 3 in [33])

Theorem 18. *Let $1 < p \le \infty$ and $0 < \alpha < \pi$. For each $f \in H_p(\Sigma_\alpha, \mathfrak{X})$ there is a p-bounded operator $T : Y_q(\mathbb{R}) \to \mathfrak{X}$ such that*

$$T(v) = \lim_{\eta \to \alpha^-} \int_{-\infty}^{+\infty} v(t) f_\eta(t)\, dt \quad , \quad v \in Y_q(\mathbb{R}) \, .$$

If $1 < p \le \infty$ and $\mathfrak{X}$ has the Radon-Nikodym property there exists a function $v \in L^p(\mathbb{R}, \mathfrak{X})$ which is the boundary values of f.

Boundary values of vector-valued holomorphic functions defined on a half plane in $\mathbb{C}$ were previously studied by M. Itano, [15], by solving inhomogeneous Cauchy-Riemann equations (building on the ideas of A. Martineau, [18]). The recent work by P. Domański & M. Langenbruch, [10], develops the theory of hyperfunctions with values in a locally convex (not necessarily metrizable) space $\mathfrak{X}$ and looks at the natural limits to such a theory i.e. characterizes the locally convex spaces $\mathfrak{X}$ for which a reasonable theory[2] of $\mathfrak{X}$-valued hyperfunctions exists. Vector-valued hyperfunctions can be interpreted as boundary values of vector-valued harmonic or holomorphic functions. The existence of $\mathfrak{X}$-valued hyperfunctions is closely related to the solvability of the Laplace equation. A locally convex space $\mathfrak{X}$ is (*weakly*) *N-admissible* ($N \in \mathbb{Z}$, $N \ge 1$) if for any (bounded) open set $\Omega \subset \mathbb{R}^N$ the N-dimensional Laplace operator $\Delta : C^\infty(\Omega, \mathfrak{X}) \to C^\infty(\Omega, \mathfrak{X})$ is surjective. By a result in [10] if $\mathfrak{X}$ is $(N+1)$-admissible then a reasonable theory of N-dimensional $\mathfrak{X}$-valued hyperfunctions may be built.

---

[2] One that produces a flabby sheaf whose set of sections supported by a compact set $K \subset \mathbb{R}^N$ equals the space $L(\mathcal{A}(K), \mathfrak{X})$ of all $\mathfrak{X}$-valued linear continuous operators on the space of germs of analytic functions on K (cf. [10], p. 1098).

6. VECTOR VALUED CR FUNCTIONS

Let $(M, T_{1,0}(M))$ be a CR manifold and $\Omega \subset M$ an open subset. Let $\mathfrak{X}$ be a complex topological vector space. A function $f \in C^1(\Omega, \mathfrak{X})$ is said to be a *CR function* if $\overline{Z}(f) = 0$ for any $Z \in \Gamma^\infty(\Omega, T_{1,0}(M))$. Let $\mathrm{CR}^1(\Omega, \mathfrak{X})$ denote the set of all CR functions $f : \Omega \to \mathfrak{X}$. Let $M \subset \mathbb{C}^n$ be a CR submanifold. Given an open subset $\Omega \subset M$ the *CR extension problem* is to look for an open set $D \subset \mathbb{C}^n$ such that $\Omega \subset D$ and the sequence $\mathcal{O}(D, \mathfrak{X}) \xrightarrow{r} \mathrm{CR}^1(\Omega, \mathfrak{X}) \to 0$ is exact, where $\mathcal{O}(D, \mathfrak{X})$ is the space of all holomorphic functions $F : D \to \mathfrak{X}$ and r is the restriction morphism. The known approaches [i.e. the analytic disc (cf. [7], p. 206-221) and the Fourier transform (cf. [7], p. 229-244) techniques] to the solution to the CR extension problem in the scalar case (i.e. $\mathfrak{X} = \mathbb{C}$) make use of a fundamental result by M.S. Baouendi & F. Treves, [3]. It is our purpose in the present section to extend the quoted result (cf. also Theorem 1 in [7], p. 191) to the case of CR functions with values in a given complex topological vector space $\mathfrak{X}$, as a first step towards the solution to the CR extension problem (in the vector valued case). We obtain

Theorem 19. *Let $M \subset \mathbb{C}^n$ be a real hypersurface of class C^2 and $p \in M$. Let $\mathfrak{X}$ be a complex topological vector space such that $\mathfrak{X}^*$ separates points. For any open neighborhood $\Omega \subset M$ of p there is an open set $\omega \subset M$ with $p \in \omega \subset\subset \Omega$ such that for each CR function $f \in C^1(\Omega, \mathfrak{X})$ with $\overline{\mathrm{co}}\,[f(\overline{\omega})]$ a compact subset of $\mathfrak{X}$ there is a sequence $\{F_k\}_{k\geq 1}$ of holomorphic functions $F_k \in \mathcal{O}(\mathbb{C}^n, \mathfrak{X})$ such that $F_k \to f$ uniformly on ω as $k \to \infty$.*

Our motivation springs from analytic functional calculus (cf. e.g. [29]). The analytic functional calculus is an algebra homomorphism from the algebra of germs of holomorphic functions on a neighborhood of the joint spectrum of a commutative system of operators, with values in a Banach algebra (cf. also [32]). We expect that the accomplishment of our program (as to the solution to the CR extension problem in the vector valued case) will allow for the construction of a “CR functional calculus”.

Let M be a real hypersurface of $\mathbb{C}^n$ of class C^2. Let $p \in M$. By a result in [7] (cf. Lemma 1, p. 103) there is a system of holomorphic coordinates for $\mathbb{C}^n$ such that p is the origin and

$$M = \{(z = x + iy, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ y = h(x, w)\}$$

where $h : \mathbb{R} \times \mathbb{C}^{n-1} \to \mathbb{R}$ is a function of class C^2 such that $h(0) = 0$ and $Dh(0) = 0$. We set $w = u + iv \in \mathbb{C}^{n-1}$, $t = (x, u) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$ and $s = (y, v) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$. Also let $\zeta = t + is \in \mathbb{C}^n$. Next we consider

$$H : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1} \ ,$$

$$H(t, v) = (h(x, u + iv), \, v) \quad , \quad t = (x, u) \ ,$$

so that $H(0) = 0$ and $(\partial H/\partial t)(0) = 0$. Let $\delta > 0$ and $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \eta(t) \leq 1$, $\eta(t) = 1$ for $|t| < \delta$, and $\eta(t) = 0$ for $|t| \geq 2\delta$. Let $\varphi(z) = \eta(|z|)$ for any $z \in \mathbb{C}^n$. We are only interested in the geometry of $M \cap D$ for a small neighborhood $D \subset \mathbb{C}^n$ of the origin hence from now we replace M by $M_\varphi = \{(x + iy, w) : y = h_\varphi(x, w)\}$ where $h_\varphi = \varphi h$. To keep notation simple we drop φ yet we may assume that $\mathrm{supp}(h) \subset T \times V$ for some neighborhoods of the origin

$T \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{n-1}$. For small enough T and V

$$\left\| \frac{\partial H}{\partial t}(t,v) \right\| \leq \frac{1}{2} \quad , \quad (t,v) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \ . \tag{23}$$

Here for each linear operator L of normed spaces we set as customary $\|L\| = \sup_{|x|=1} |Lx|$. One has $M \cap D = \{t + iH(t,v) \, : \, t \in T, \quad v \in V\}$ hence $M \cap D$ carries a foliation $\mathcal{F}$ such that $(M \cap D)/\mathcal{F} = \{M_v(T) \, : \, v \in V\}$ where $M_v(T) = \{t + iH(t,v) \, : \, t \in T\}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\Gamma = \mathrm{supp}(\varphi) \subset T$ and $\varphi(t) = 1$ for any $t \in T'$ and some open set $T' \subset \mathbb{R}^n$ with $0 \in T' \subset\subset T$. Let $g(t + is) = \varphi(t)$ for any $t + is \in \mathbb{C}^n$. Also we set $\langle \zeta \rangle = \sum_{j=1}^n \zeta_j^2$ for each $\zeta \in \mathbb{C}^n$. Let us consider the map $\zeta : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{C}^n$ given by

$$\zeta(t,v) = t + iH(t,v) \quad , \quad t \in \mathbb{R}^n \ , \ \ v \in \mathbb{R}^{n-1} \ .$$

Lemma 3. *Let $\mathfrak{X}$ be a complex topological vector space such that $\mathfrak{X}^*$ separates points and $\Omega = M \cap D$. Let $K_v = \zeta(\Gamma, v)$ and $f \in C(\Omega, \mathfrak{X})$ such that $\overline{\mathrm{co}}[f(K_v)]$ is a compact subset of $\mathfrak{X}$. Then for any $\zeta \in M_v\,(T')$, $v \in V$*

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^{-n}}{\pi^{n/2}} \int_{\xi \in M_v(T)} g(\xi) \exp\left\{-\epsilon^{-2} \langle \zeta - \xi \rangle\right\} f(\xi) \, d\xi_1 \wedge \cdots \wedge d\xi_n = f(\zeta) \ .$$

The limit is uniform in $v \in V$ and $\zeta \in M_v\,(T')$.

Proof. For each $\Lambda \in \mathfrak{X}^*$ and $(t,v) \in T \times V$ we consider the complex valued differential form $\omega_\Lambda^\epsilon \in \Omega^{n,0}(\mathbb{C}^n)$ given by

$$\omega_\Lambda^\epsilon(\xi) = g(\xi) \exp\left\{-\epsilon^{-2} \langle \zeta(t,v) - \xi \rangle\right\} \Lambda\left[f(\xi)\right] \, d\xi_1 \wedge \cdots \wedge d\xi_n$$

for any $\xi \in \mathbb{C}^n$. Then (by Lemma 1 in [7], p. 192-193)

$$\lim_{\epsilon \to 0^+} \epsilon^{-n} \int_{M_v(T)} \omega_\Lambda^\epsilon = \pi^{n/2} \, \Lambda[f(\zeta(t,v))] \tag{24}$$

where from Lemma 3 follows (as Λ is continuous and $\mathfrak{X}^*$ separates points). For the sake of completeness we give a direct proof of (24) as well. Let $\zeta_v : T \to \mathbb{C}^n$ be the injection given by $\zeta_v(\tau) = \zeta(\tau, v)$ for any $\tau \in T$. Then $\zeta_v(T) = M_v(T)$ and

$$\int_{M_v(T)} \omega_\Lambda^\epsilon = \int_{\mathbb{R}^n} \zeta_v^* \, \omega_\Lambda^\epsilon =$$

$$= \int_{\tau \in \mathbb{R}^n} \varphi(\tau) \exp\left\{-\epsilon^{-2} \langle \zeta(t,v) - \zeta(\tau,v) \rangle\right\} \Lambda[f(\zeta(\tau,v))] \det\left[\frac{\partial \zeta_j}{\partial t_i}(\tau,v)\right] d\tau_1 \wedge \cdots \wedge d\tau_n \ .$$

Under a change of variables $\tau = t - \epsilon\, s$

$$\int_{M_v(T)} \omega_\Lambda^\epsilon = \epsilon^n \int_{s \in \mathbb{R}^n} \varphi(t - \epsilon s) \Lambda[f(\zeta(t - \epsilon s, v))] \times \tag{25}$$

$$\times \exp\left\{-\epsilon^{-2} \langle \zeta(t,v) - \zeta(t - \epsilon s, v) \rangle\right\} \det\left[\frac{\partial \zeta_j}{\partial t_i}(t - \epsilon s, v)\right] ds$$

where $ds = ds_1 \wedge \cdots \wedge ds_n$. Let $G_\epsilon(t,s,v)$ be the integrand in (25). Note that $\lim_{\epsilon \to 0^+} \varphi(t - \epsilon s) = 1$ for any $t \in T'$. By the Taylor formula

$$\zeta_j(t - \epsilon s, v) = \zeta_j(t,v) - \sum_{i=1}^n \frac{\partial \zeta_j}{\partial t_i}(t,v) s_i \epsilon + O(\epsilon^2)$$

hence

$$\epsilon^{-2}\langle\zeta(t,v)-\zeta(t-\epsilon s,v)\rangle = \left\langle \sum_{i=1}^{n} \frac{\partial \zeta}{\partial t_i}(t,v)s_i \right\rangle + O(\epsilon)\ .$$

Thus

$$G_\epsilon(t,s,v) \to G(t,s,v) =$$

$$= \Lambda[f(\zeta(t,v))]\exp\left\{ -\left\langle \sum_{i=1}^{n} \frac{\partial \zeta}{\partial t_i}(t,v)s_i \right\rangle \right\} \det\left[\frac{\partial \zeta_j}{\partial t_i}(t,v)\right]$$

as $\epsilon \to 0^+$ (the convergence is pointwise in s and uniform in $t \in T'$, $v \in V$). Clearly

$$\varphi(t-\epsilon s)\Lambda[f(\zeta(t-\epsilon s,v))]\det\left[\frac{\partial \zeta_j}{\partial t_i}(t-\epsilon s,v)\right]$$

is bounded as a function of $s \in \mathbb{R}^n$. Moreover

$$\left|\exp\left\{-\epsilon^{-2}\left\langle\zeta(t,v)-\zeta(t-\epsilon s,v)\right\rangle\right\}\right| =$$

$$= \exp\left\{-\epsilon^{-2}\mathrm{Re}\langle\zeta(t,v)-\zeta(t-\epsilon s,v)\rangle\right\} =$$

$$= \exp\left\{-|s|^2+\epsilon^{-2}|H(t,v)-H(t-\epsilon s,v)|^2\right\}\ .$$

If $x, y \in \mathbb{R}^n \times \mathbb{R}^{n-1}$ we set $[x,\ y] = \{x+\gamma(y-x) : 0 \le \gamma \le 1\}$. The function H_j is Gateaux differentiable at each point of $[(t-\epsilon s,v),\ (t,v)]$ in the direction e_i (where $\{e_1,\cdots,e_n\} \subset \mathbb{R}^n$ is the canonical linear basis). By the mean value theorem there is $0<\gamma<1$ such that

$$H_j(t,v)-H_j(t-\epsilon s,v) = \epsilon\sum_{i=1}^{n} s_i\,\frac{\partial H_j}{\partial t_i}(t+(\gamma-1)\epsilon s,v)\ .$$

Hence (by (23))

$$|H(t,v)-H(t-\epsilon s,v)| \le \frac{\epsilon}{2}\,|s|\ .$$

Gathering the information got so far

$$|G_\epsilon(t,s,v)| \le C\exp\left(-\frac{1}{4}\,|s|^2\right) \in L^1(\mathbb{R}^n)$$

for some $C>0$. Then (by Lebesgue's theorem)

$$\epsilon^{-n}\int_{M_v}\omega^\epsilon_\Lambda = \int_{s\in\mathbb{R}^n} G_\epsilon(t,s,v)\,ds \to \int_{s\in\mathbb{R}^n} G(t,s,v)\,ds, \quad \epsilon\to 0^+\ .$$

Finally

$$\int_{s\in\mathbb{R}^n} G(t,s,v)\,d\,s =$$

$$= \Lambda[f(\zeta(t,v))]\int_{s\in\mathbb{R}^n}\exp\left\{ -\left\langle \sum_{i=1}^{n} \frac{\partial \zeta}{\partial t_i}(t,v)s_i \right\rangle \right\} \det\left[\frac{\partial \zeta_j}{\partial t_i}(t,v)\right]\,ds =$$

$$= \pi^{n/2}\,\Lambda[f(\zeta(t,v))]\ .$$

The identity (24) is proved. Under the assumption of Lemma 3 we also establish

Lemma 4. *Let $f \in C^1(\Omega, \mathfrak{X})$ be a CR function defined on a neighborhood Ω of the origin in M such that $\overline{\text{co}}\,[f\,(K_0)]$ is compact in $\mathfrak{X}$. There is an open set $\Omega' \subset M$ with $0 \in \Omega' \subset\subset \Omega$ such that for any $\zeta \in \Omega'$*

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^{-n}}{\pi^{n/2}} \int_{\xi \in M_0(T)} g(\xi) \exp\left\{-\epsilon^{-2}\langle \zeta - \xi \rangle\right\} f(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n = f(\zeta)$$

and the convergence is uniform in $\zeta \in \Omega'$.

Proof. For each $v \in V$ we set

$$N_v = M \cap \{\tau + iH(\tau, \gamma v) \ : \ \tau \in T\,, \quad 0 \le \gamma \le 1\}\,.$$

Then N_v is a real $(n+1)$-dimensional submanifold-with-boundary in M and

$$\partial N_v = M_0(T) \cup M_v(T) \cup \{\zeta(\tau, \gamma v) \ : \ \tau \in \partial T\,, \quad 0 \le \gamma \le 1\}\,.$$

As $g(\xi) = \varphi(\text{Re}(\xi))$ the τ-support of ω^ϵ_Λ is contained in T. Hence for each $\Lambda \in \mathfrak{X}^*$ and $(t, v) \in T' \times V$ (by Lemma 3 and Stokes' theorem)

$$\pi^{n/2}\Lambda[f(\zeta(t,v))] = \lim_{\epsilon \to 0^+} \epsilon^{-n} \int_{M_v(T)} \omega^\epsilon_\Lambda =$$

$$= \lim_{\epsilon \to 0^+} \epsilon^{-n} \left\{ \int_{N_v} d\omega^\epsilon_\Lambda + \int_{M_0(T)} \omega^\epsilon_\Lambda \right\}$$

As $f \in C^1(\Omega, \mathfrak{X})$ it follows that $\Lambda \circ f$ is an ordinary (scalar valued) CR function (in the sense of Lemma 1 in [7], p. 140) of class C^1 on Ω. Hence we may apply Theorem 2 in [7], p. 147, to extend $\Lambda \circ f$ to a C^1 function $F : \mathbb{C}^n \to \mathbb{C}$ such that $\overline{\partial} F = 0$ on $\Omega \subset M$. Let us assume that T and V in Lemma 3 are sufficiently small so that $\zeta(T \times V) \subset \Omega$. Next (as $d = \partial + \overline{\partial}$)

$$\pi^{n/2}\Lambda[f(\zeta(t,v))] = \lim_{\epsilon \to 0^+} \epsilon^{-n} \int_{M_0(T)} \omega^\epsilon_\Lambda +$$

$$+ \lim_{\epsilon \to 0^+} \epsilon^{-n} \int_{\xi \in N_v} \left(\overline{\partial} g\right)(\xi) \Lambda[f(\xi)] \exp\{-\epsilon^{-2}\langle \zeta(t,v) - \xi \rangle\}\, d\xi\,.$$

By the proof of Lemma 2 in [7], p. 195-196, the second limit is zero. Then Lemma 4 is proved and Theorem 19 follows.

Corollary 4. *Let $M \subset \mathbb{C}^n$ be a real hypersurface of class C^2 and $p \in M$. Let $\mathfrak{X}$ be a complex Fréchet space. For any open neighborhood $\Omega \subset M$ of p there is an open set $\omega \subset M$ with $p \in \omega \subset \Omega$ such that each $\mathfrak{X}$-valued CR function of class C^1 on Ω may be uniformly approximated on ω by a sequence of $\mathfrak{X}$-valued holomorphic functions on $\mathbb{C}^n$.*

Appendix A. Integration of vector valued functions

The Bochner integral (for functions of one real variable with values in a Banach space $\mathfrak{X}$) is systematically presented in the recent monograph by W. Arendt et al., [2], p. 6-15. The older yet excellent monograph by N. Boboc & G. Bucur, [6], gives a detailed treatment of the Bochner, Dunford and Pettis integrals (for functions from a measure space (Q, μ), where μ is a positive measure, with values in a Banach space $\mathfrak{X}$) cf. *op. cit.*, p. 238-272. The case where $\mathfrak{X}$ is a Fréchet space is briefly described by F-H. Vasilescu, [32], 15-17. The approach in Appendix A follows the presentation in [24].

Let (Q, μ) be a measure space and $\mathfrak{X}$ a topological vector space such that $\mathfrak{X}^*$ separates points. Let $f : Q \to \mathfrak{X}$ be a function such that $\Lambda \circ f \in L^1(Q, \mathfrak{X}, \mu)$ for any $\Lambda \in \mathfrak{X}^*$. As $\mathfrak{X}^*$ separates points there is at most one vector $I_f \in \mathfrak{X}$ such that $\Lambda\,(I_f) = \int_Q \Lambda \circ f \, d\mu$ for any $\Lambda \in \mathfrak{X}^*$. Such $I_f \in \mathfrak{X}$ is the *integral* over Q of the vector valued function $f : Q \to \mathfrak{X}$ (denoted by $I_f = \int_Q f \, d\mu$). We may state (cf. e.g. Theorems 3.27 and 3.29 in [24], p. 78-81)

Theorem 20. *Let $\mathfrak{X}$ be a topological vector space such that $\mathfrak{X}^*$ separates points. Let Q be a compact Hausdorff space and μ be a positive Borel measure on Q. Let $f : Q \to \mathfrak{X}$ be a continuous function.*

i) If $\overline{\text{co}}\,[f(Q)]$ is a compact subset of $\mathfrak{X}$ then the integral $I_f = \int_Q f \, d\mu$ exists and $I_f \in \overline{\text{co}}[f(Q)]$.

ii) Let $\mathfrak{X}$ be a Fréchet space and $\mathcal{P}$ a separating family of seminorms defining the topology of $\mathfrak{X}$. Then

$$p\left(\int_Q f \, d\mu\right) \leq \int_Q p(f(\omega)) \, d\mu(\omega) \quad , \quad p \in \mathcal{P} \, .$$

The claim (ii) in Theorem 20 is stated in [24] for functions $f : Q \to \mathfrak{X}$ with values in a Banach space $\mathfrak{X}$. If this is the case a simple proof based on the Hahn-Banach theorem is available. Indeed let $\| \cdot \|$ be the norm on $\mathfrak{X}$ and let us set $I = \int_Q f \, d\mu \in \mathfrak{X}$. Then (by the Corollary to Hahn-Banach's Theorem 3.3 in [24], p. 58-59) there is $\Lambda \in \mathfrak{X}^*$ such that $\Lambda(I) = \|I\|, \quad |\Lambda(x)| \leq \|x\|$ for any $x \in \mathfrak{X}$. In particular $|\Lambda(f(\omega)) \leq \|f(\omega)\|$ for any $\omega \in Q$. Hence $\|I\| = \Lambda(I) = \int_Q \Lambda(f(\omega)) \, d\mu(\omega) \leq \int_Q \|f(\omega)\| \, d\mu(\omega)$. To prove (ii) when $\mathfrak{X}$ is a Fréchet space one needs a more constructive approach to the integral $\int_Q f \, d\mu$. A *partition* of Q is a finite collection $\{A_j \, : \, 1 \leq j \leq N\}$ of mutually disjoint Borel subsets $A_j \subset Q$ whose union is Q. Let $\mathcal{S}(Q, \mathfrak{X})$ be the set of all functions of the form $f = \sum_{j=1}^N \chi_{A_j} x_j : Q \to \mathfrak{X}$ where $\{x_1, \cdots, x_N\} \subset \mathfrak{X}$ and $\{A_j \, : \, 1 \leq j \leq N\}$ is a partition of Q. Here χ_A is the characteristic function of the set $A \subset Q$. For any such $f = \sum_{j=1}^N \chi_{A_j} x_j$ its integral is defined by $\int_Q f \, d\mu = \sum_{j=1}^N \mu(A_j) x_j$. Let $\mathcal{I}(Q, \mathfrak{X})$ be the set of all functions defined on Q with values in $\mathfrak{X}$ which are uniform limits of sequences of functions in $\mathcal{S}(Q, \mathfrak{X})$. Let $f \in \mathcal{I}(Q, \mathfrak{X})$ and let $\{f_n\}_{n \geq 1} \subset \mathcal{S}(Q, \mathfrak{X})$ such that $f_n \to f$ as $n \to \infty$ uniformly on Q. By definition

$$\int_Q f \, d\mu = \lim_{n \to \infty} \int_Q f_n \, d\mu \, . \tag{26}$$

The limit doesn't depend upon the choice of sequence $\{f_n\}_{n \geq 1}$ approximating f. It may be easily shown that $\int_Q (\Lambda \circ f) \, d\mu = \Lambda\left(\int_Q f \, d\mu\right)$ for any $\Lambda \in \mathfrak{X}^*$ hence the integral given by (26) agrees with that furnished by (i) in Theorem 20. Also $C(Q, \mathfrak{X}) \subset \mathcal{I}(Q, \mathfrak{X})$. The constructive approach (26) allows one to give an elementary proof of the more general claim (ii). Indeed let $f = \sum_{j=1}^N \chi_{A_j} x_j \in \mathcal{S}(Q, \mathfrak{X})$. Then $p\left(\int_Q f \, d\mu\right) \leq \sum_{j=1}^N \mu(A_j) p(x_j) = \int_Q p[f(\omega)] \, d\mu(\omega)$. Next let $f \in \mathcal{I}(Q, \mathfrak{X})$ be the uniform limit of a sequence $\{f_n\}_{n \geq 1} \subset \mathcal{S}(Q, \mathfrak{X})$. Then $p\left(\int_Q f_n \, d\mu\right) \leq \int_Q p[f_n(\omega)] \, d\mu(\omega)$ for any $n \geq 1$. As $\{f_n\}_{n \geq 1}$ converges to f uniformly on Q for any $\epsilon > 0$ there is $n'_\epsilon \geq 1$ such that $p[f_n(\omega) - f(\omega)] < \epsilon/[2\mu(Q)]$ for any $n \geq n'_\epsilon$ and $\omega \in Q$. Also as $\left\{\int_Q f_n \, d\mu\right\}_{n \geq 1}$ converges to $\int_Q f \, d\mu$ in $\mathfrak{X}$ for any $\epsilon > 0$ there is $n''_\epsilon \geq 1$

such that $p\left(\int_Q f_n(\omega)\,d\mu(\omega) - \int_Q f(\omega)\,d\mu(\omega)\right) < \epsilon/2$ for any $n \geq n''_\epsilon$. Consequently for any $n \geq n_\epsilon = \max\{n'_\epsilon\,,\, n''_\epsilon\}$ one has $p\left(\int_Q f(\omega)\,d\mu(\omega)\right) < \epsilon + \int_Q p[f(\omega)]\,d\mu(\omega)$.

Appendix B. Power series arguments

1) Let $\mathfrak{X}$ be a complex topological vector space and $\Omega \subset \mathbb{C}$ an open set. If $z = x + iy$ we set as customary $\partial/\partial z = 1/2\,(\partial/\partial x - i\,\partial/\partial y)$ and $\partial/\partial \overline{z} = 1/2\,(\partial/\partial x + i\,\partial/\partial y)$. Let $f : \Omega \to \mathfrak{X}$ be $\mathbb{C}$-differentiable at $z_0 \in \Omega$. Then f admits partial derivatives at z_0. Precisely if $f'(z_0) = \lim_{z\to z_0}(z - z_0)^{-1}[f(z) - f(z_0)] \in \mathfrak{X}$ then $(\partial f/\partial x)(z_0) = f'(z_0)$ and $(\partial f/\partial y)(z_0) = i\,f'(z_0)$. In particular $(\partial f/\partial z)(z_0) = f'(z_0)$ and $(\partial f/\partial \overline{z})(z_0) = 0$. Also f is differentiable at z_0 (as a function of two real variables) and $(d_{z_0} f)h = f'(z_0)h$ for any $h \in \mathbb{C}$. Viceversa if $f : \Omega \to \mathfrak{X}$ is differentiable at z_0 and $(\partial f/\partial \overline{z})(z_0) = 0$ then f is $\mathbb{C}$-differentiable at z_0 and $\lim_{z\to z_0}(z - z_0)^{-1}[f(z) - f(z_0)] = (\partial f/\partial x)(z_0)$.

2) Let $\mathfrak{X}$ be a topological vector space and $\{x_\nu\}_{\nu\geq 0} \subset \mathfrak{X}$. *If* $\sum_{\nu\geq 0} x_\nu$ *is convergent then* $x_\nu \to 0$ *on* $\mathfrak{X}$ *as* $\nu \to \infty$. Indeed let W be a neighborhood of the origin in $\mathfrak{X}$. As the map $(x, y) \mapsto x - y$ is continuous there is a neighborhood of the origin $V \subset \mathfrak{X}$ such that $V - V \subset W$. Let $S_\mu = \sum_{\nu=0}^{\mu} x_\nu$ and $S = \lim_{\mu\to\infty} S_\mu$. There is $N = N(V) \geq 1$ such that $S_\nu - S \in V$ for any $\nu \geq N$. Hence $x_\nu = (S_\nu - S) - (S_{\nu-1} - S) \in W$ for any $\nu \geq N$.

3) Let $\mathfrak{X}$ be a Fréchet space and $\{x_\nu\}_{\nu\geq 0}$ a sequence in $\mathfrak{X}$. Let $\mathcal{P}$ be a separating family of seminorms defining the topology of $\mathfrak{X}$. *If* $\sum_{\nu=0}^{\infty} p(x_\nu) < \infty$ *for each* $p \in \mathcal{P}$ *then the series* $\sum_{\nu\geq 0} x_\nu$ *is convergent in* $\mathfrak{X}$. Indeed let $\sigma_\mu(p) = \sum_{\nu=0}^{\mu} p\,(x_\nu)$ and $S_\mu = \sum_{\nu=0}^{\mu} x_\nu$. Let $p \in \mathcal{P}$ and $k \geq 1$ be an arbitrary seminorm and a positive integer. As $\{\sigma_\nu(p)\}_{\nu\geq 0} \subset \mathbb{R}$ is a Cauchy sequence there is $N = N(p, k)$ such that $|\sigma_\mu(p) - \sigma_\nu(p)| < 1/k$ for any $\mu > \nu \leq N$. Then $p\,(S_\mu - S_\nu) \leq |\sigma_\mu(p) - \sigma_\nu(p)| < 1/k$ for any $\mu > \nu \geq N$ that is $\{S_\nu\}_{\nu\geq 0} \subset \mathfrak{X}$ is a Cauchy sequence in $\mathfrak{X}$. Yet the topology of $\mathfrak{X}$ (as a Fréchet space) is compatible to a complete invariant metric hence $\{S_\nu\}_{\nu\geq 0}$ is convergent in $\mathfrak{X}$.

4) Let $\mathfrak{X}$ be a complex Fréchet space and $\{x_\nu\}_{\nu\geq 0} \subset \mathfrak{X}$. *If there is* $z_0 \in \mathbb{C} \setminus \{0\}$ *such that* $\sum_{\nu\geq 0} z_0^\nu x_\nu$ *is convergent then* $\sum_{\nu\geq 0} z^\nu x_\nu$ *is convergent for any* $z \in D_{|z_0|}(0)$. *Also* $\sum_{\nu\geq 0} z^\nu x_\nu$ *is uniformly convergent for* $z \in D_r(0)$ *for any* $0 < r < |z_0|$. To prove the statement let $\mathcal{P}$ be a separating family of seminorms determining the topology of $\mathfrak{X}$. As $z_0^\nu x_\nu \to 0$ in $\mathfrak{X}$ as $\nu \to \infty$ for any $p \in \mathcal{P}$ and any positive integer $k \geq 1$ there is $N = N(p, k) \geq 1$ such that $|z_0|^\nu p\,(x_\nu) < 1/k$ for any $\nu \geq N$. If $z \in D_{|z_0|}(0)$ then $p\,(z^\nu x_\nu) < q^\nu/k$ where $q = |z/z_0|$ so that $0 \leq q < 1$. Therefore $\sum_{\nu=0}^{\infty} p\,(z^\nu x_\nu) < \infty$ so that $\sum_{\nu\geq 0} z^\nu x_\nu$ is convergent in $\mathfrak{X}$. Finally for each $0 < r < |z_0|$ one has

$$\sup_{|z|<r} p\,(z^\nu x_\nu) \leq \frac{1}{k}\left(\frac{r}{|z_0|}\right)^\nu .$$

The *convergence radius* of $\sum_{\nu\geq 0} z^\nu x_\nu$ is taken to be

$$R = \sup\{|z_0| \;:\; \sum_{\nu\geq 0} z_0^\nu x_\nu \;\text{ is convergent in }\; \mathfrak{X}\}\,.$$

For each $p \in \mathcal{P}$ we set

$$\ell(p) = \limsup_{\nu \to \infty} p\left(x_\nu\right)^{1/\nu} . \tag{27}$$

If $0 < \ell(p) < \infty$ let $z \in D_{1/\ell(p)}(0)$ so that $|z|\ell(p) < 1$. Then $|z|[\ell(p) + \epsilon] < 1$ for some $\epsilon > 0$. As $\ell(p) + \epsilon > \ell(p)$ there is $N \geq 1$ such that $\ell(p) + \epsilon > p(x_\nu)^{1/\nu}$ for any $\nu \geq N$. Thus

$$p\left(z^\nu x_\nu\right) = |z|^\nu p\left(x_\nu\right) < [|z|(\ell(p) + \epsilon)]^\nu$$

so that $\sum_{\nu=0}^{\infty} p\left(z^\nu x_\nu\right) < \infty$. We may state

Proposition 3.

i) If $0 < \ell(p) < a$ for some $a > 0$ and any $p \in \mathcal{P}$ we set

$$R = \inf\left\{\frac{1}{\ell(p)} \; : \; p \in \mathcal{P}\right\} .$$

Then $R > 0$ and the series $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent (respectively divergent) for any $z \in D_R(0)$ (respectively for any $z \in \mathbb{C} \setminus \overline{D}_R(0)$).

ii) If $0 < \ell(p) < \infty$ for any $p \in \mathcal{P}$ yet there is a sequence $\{p_j\}_{j \geq 1} \subset \mathcal{P}$ such that $\lim_{j \to \infty} \ell(p_j) = \infty$ or $\ell(p) = \infty$ for some $p \in \mathcal{P}$ then $\sum_{\nu \geq 0} z^\nu x_\nu$ is divergent for any $z \in \mathbb{C} \setminus \{0\}$.

iii) If $\ell(p) = 0$ for some $p \in \mathcal{P}$ let

$$\mathcal{P}_0 = \{p \in \mathcal{P} \; : \; \ell(p) = 0\} \quad , \quad R = \inf\left\{\frac{1}{\ell(p)} \; : \; p \in \mathcal{P} \setminus \mathcal{P}_0\right\} .$$

If $\sup\{\ell(p) \; : \; p \in \mathcal{P} \setminus \mathcal{P}_0\} < \infty$ then $R > 0$ and $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent for any $z \in D_R(0)$ while if $\sup\{\ell(p) \; : \; p \in \mathcal{P} \setminus \mathcal{P}_0\} = \infty$ then $R = 0$ and $\sum_{\nu \geq 0} z^\nu x_\nu$ is divergent for any $z \in \mathbb{C} \setminus \{0\}$.

Proof. i) If $R = 0$ then for any $\epsilon > 0$ there is $p_\epsilon \in \mathcal{P}$ such that $\ell\left(p_\epsilon\right) > 1/\epsilon$. Hence $\ell\left(p_\epsilon\right) \to \infty$ as $\epsilon \to 0$, a contradiction.

Let $z \in D_R(0)$. Then (by the argument above) $\sum_{\nu=0}^{\infty} p\left(z^\nu x_\nu\right) < \infty$ for any $p \in \mathcal{P}$ hence $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent. If in turn $|z| > R$ then $|z|\ell(p_0) > 1$ for some $p_0 \in \mathcal{P}$. Hence there is $\delta > 0$ such that $|z|\left[\ell(p_0) - \delta\right] > 1$. On the other hand $\ell(p_0) - \delta < \ell(p_0)$ hence for any $n \geq 1$ there is $\nu(n) \geq n$ such that $\ell(p_0) - \delta < p_0\left(x_{\nu(n)}\right)^{1/\nu(n)}$. Consequently there is a sequence $\{\nu_j\}_{j \geq 1} \subset \mathbb{Z}$ such that $1 \leq \nu_1 < \nu_2 < \cdots \uparrow \infty$ and $\ell(p_0) - \delta < p_0\left(x_{\nu_j}\right)^{1/\nu_j}$ for any $j \geq 1$. Finally

$$p_0\left(z^{\nu_j} x_{\nu_j}\right) = |z|^{\nu_j} p_0\left(x_{\nu_j}\right) > |z|^{\nu_j}\left[\ell(p_0) - \delta\right]^{\nu_j} > 1 \quad , \quad j \geq 1 .$$

Yet p_0 is continuous (cf. e.g. Theorem 1.37 in [24], p. 28) so that $\sum_{\nu \geq 0} z^\nu x_\nu$ is divergent.

ii) Let $\ell(p_j) \to \infty$ for $j \to \infty$. Then for any $A > 0$ there is $j(A) \geq 1$ such that any $j \geq j(A)$ and any $n \geq 1$ there is $\nu \geq n$ satisfying $p_j(x_\nu)^{1/\nu} > A$. Given $z \in \mathbb{C} \setminus \{0\}$ one may choose $A = 1/|z|$ and $j_0 \geq j(|z|^{-1})$. Consequently there is a sequence $1 \leq \nu_1 < \nu_2 < \cdots \uparrow \infty$ such that $p_{j_0}(z^{\nu_k} x_{\nu_k}) > 1$ for any $k \geq 1$ so that (by the continuity of p_{j_0}) the series $\sum_{\nu \geq 0} z^\nu x_\nu$ is divergent. A similar argument may be given when $\ell(p) = \infty$ for some $p \in \mathcal{P}$.

iii) Let $p \in \mathcal{P}_0$. Then for any $\epsilon > 0$ there is $n_\epsilon \geq 1$ such that $p\left(x_\nu\right)^{1/\nu} < \epsilon$ for any $\nu \geq n_\epsilon$. Let $z \in \mathbb{C} \setminus \{0\}$ and choose $0 < \epsilon < 1/|z|$. Then $p(z^\nu x_\nu) < (\epsilon|z|)^\nu$ for any $\nu \geq n_\epsilon$ so that $\sum_{\nu=0}^{\infty} p(z^\nu x_\nu) < \infty$. If $\mathcal{P}_0 = \mathcal{P}$ then $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent for any $z \in \mathbb{C}$. If $\mathcal{P} \setminus \mathcal{P}_0 \neq \emptyset$ then let $R = \inf\{1/\ell(p) : p \in \mathcal{P} \setminus \mathcal{P}_0\}$. The remainder of the proof is similar to that of (i) (when $\sup_{p\in\mathcal{P}\setminus\mathcal{P}_0} \ell(p) < \infty$) and (ii) (when $\sup_{p\in\mathcal{P}\setminus\mathcal{P}_0} \ell(p) = \infty$).

By slightly restating Proposition 3 one has

Corollary 5. *Let $\mathfrak{X}$ be a complex Fréchet space and $\{x_\nu\}_{\nu\geq 1} \subset \mathfrak{X}$.*
Let $\mathcal{P}$ be a separating family of seminorms determining the topology of $\mathfrak{X}$ and $\ell(p) = \limsup_{\nu\to\infty} p\left(x_\nu\right)^{1/\nu}$ for each $p \in \mathcal{P}$. Let $\mathcal{P}_0 = \{p \in \mathcal{P} : \ell(p) = 0\}$ and $R = \inf\{1/\ell(p) : p \in \mathcal{P} \setminus \mathcal{P}_0\}$. Then R is the radius of convergence of the series $\sum_{\nu\geq 0} z^\nu x_\nu$. Precisely a) if $\mathcal{P}_0 = \mathcal{P}$ then $R = \infty$ and b) if $\mathcal{P} \setminus \mathcal{P}_0 \neq \emptyset$ then either $\sup_{p\in\mathcal{P}\setminus\mathcal{P}_0} \ell(p) = \infty$ and then $R = 0$ or $\sup_{p\in\mathcal{P}\setminus\mathcal{P}_0} \ell(p) < \infty$ and then $0 < R < \infty$.

5) The *derivative* of $S = \sum_{\nu\geq 0} z^\nu x_\nu$ is by definition the series $S' = \sum_{\nu\geq 0} (\nu + 1)\, z^\nu\, x_{\nu+1}$. For any sequence $\{a_n\}_{n\geq 1} \subset [0, +\infty)$ one has $\limsup_{n\to\infty} \left[(n + 1)a_{n+1}\right]^{1/n} = \limsup_{n\to\infty} a_n^{1/n}$ hence S and its derivative have the same radius of convergence.

Proposition 4. *Let R be the radius of convergence of the series $S = \sum_{\nu\geq 0} z^\nu x_\nu$. If $R > 0$ let $f_S : D_R(0) \to \mathfrak{X}$ given by $f_S(z) = \sum_{\nu=0}^{\infty} z^\nu x_\nu$ for any $|z| < R$. Then f_S is $\mathbb{C}$-differentiable on $D_R(0)$ and $f_S'(z) = f_{S'}(z)$ for any $|z| < R$.*

Proof. Let $\zeta \in D_R(0)$ and let $r > 0$ such that $|\zeta| < r < R$. Then

$$(z - \zeta)^{-1} \left[f_S(z) - f_S(\zeta)\right] - f_{S'}(\zeta) = \sum_{\nu=1}^{\infty} K_\nu(z, \zeta) x_\nu \ ,$$

$$K_\nu(z, \zeta) = \frac{z^\nu - \zeta^\nu}{z - \zeta} - \nu\, \zeta^{\nu-1} \ ,$$

On the other hand it should be observed that $K_1(z, \zeta) = 0$ while for any $\nu \geq 2$

$$|K_\nu(z, \zeta)| \leq |z - \zeta| \nu^2 r^{\nu-2} \quad , \quad |z| < r \ .$$

Next for each $p \in \mathcal{P}$

$$p\left((z - \zeta)^{-1} \left[f_S(z) - f_S(\zeta)\right] - f_{S'}(\zeta)\right) \leq$$

$$\leq \sum_{\nu=2}^{\infty} |K_\nu(z, \zeta)|\, p(x_\nu) \leq |z - \zeta| \sum_{\nu=2}^{\infty} \nu^2 r^{\nu-2} p(x_\nu) = |z - \zeta| \sum_{\nu=0}^{\infty} a_\nu r^\nu$$

where $a_\nu = (\nu + 2)^2 p(x_{\nu+2})$. As $r < R$ it follows that $\ell(p) < 1/r$ for any $p \in \mathcal{P}$. Let $|\xi| < \rho$ be the domain of convergence of the power series $\sum_{\nu\geq 0} a_\nu \xi^\nu$. Then

$$\frac{1}{\rho} = \limsup_{\nu\to\infty} |a_\nu|^{1/\nu} = \limsup_{\nu\to\infty} \left[(\nu + 2)^2 p(x_{\nu+2})\right]^{1/\nu} =$$

$$= \limsup_{\nu\to\infty} p(x_\nu)^{1/\nu} = \ell(p) < \frac{1}{r}$$

hence $0 < r < \rho$ so that $s = \sum_{\nu=0}^{\infty} a_\nu r^\nu < \infty$. For each $k \geq 1$ we consider the open set

$$U = \{z \in \mathbb{C} \ : \ |z - \zeta| < \min\{1/(ks),\, \mathrm{dist}(\zeta, \partial D_R(0))\}\}$$

so that $\zeta \in U \subset D_R(0)$. Consequently $(z - \zeta)^{-1} \left[f_S(z) - f_S(\zeta)\right] - f_{S'}(\zeta) \in V(p, k)$ for any $z \in U$.

□

An alternative approach to Proposition 4 was devised by F-H. Vasilescu (cf. Lemma 8.5 in [32], p. 29)

Proposition 5. *Let $\mathfrak{X}$ be a complex Fréchet space and $\{p_m\}_{m\geq 1}$ a countable family of seminorms determining the topology of $\mathfrak{X}$. Let $\{x_\nu\}_{\nu\geq 0}\subset\mathfrak{X}$ be a sequence such that*

$$\ell=\sup_{m\geq 1}\limsup_{\nu\to\infty}p_m(x_\nu)^{1/\nu}<\infty\ .$$

Then the function $f:D_{1/\ell}(0)\to\mathfrak{X}$ given by $f(z)=\sum_{\nu=0}^{\infty}z^\nu x_\nu$ for any $z\in D_{1/\ell}(0)$ is analytic that is $f\in A(D_{1/\ell}(0),\mathfrak{X})$ [with $D_{1/\ell}(0)=\mathbb{C}$ when $\ell=0$].

Appendix C. (α)-Holomorphic functions with values in Fréchet spaces

Let $\mathfrak{X}$ be a complex Fréchet space and $f:\Omega\to\mathfrak{X}$ a monogeneous function defined on an open set $\Omega\subset\mathbb{C}$. Following the terminology in [30] we say f is (α)-*holomorphic* if its areolar derivative $\varphi(z)=(Df/D\omega)(z)$ is continuous at any point $z\in\Omega$. Thus the class of (α)-holomorphic functions may be seen as a generalization of $C^1(\Omega,\mathfrak{X})$. Indeed each $f\in C^1(\Omega,\mathfrak{X})$ is (α)-holomorphic and $Df/D\omega=f_{\overline{z}}$. Is there an analog to the Cauchy-Pompeiu formula for (α)-holomorphic functions? At this stage of the exposition of the theory, although a (α)-holomorphic function is α-differentiable with $\alpha(\theta)=e^{i\theta}$, Theorem 10 doesn't apply (as it is unknown at this point whether f is $B^1(\Omega,\mathfrak{X})$ regular). To clear up this matter we transpose a few facts from [30] to the case of $\mathfrak{X}$-valued (α)-holomorphic functions. As emphasized by L-J. Nicolescu, [21], the proofs are but straightforward verifications.

A function $f:\Omega\to\mathfrak{X}$ is *weakly* (α)-*holomorphic* if $\Lambda\circ f:\Omega\to\mathbb{C}$ is (α)-holomorphic as a scalar valued function (cf. [30], p. 8) for each $\Lambda\in\mathfrak{X}^*$. Any (α)-holomorphic function is weakly (α)-holomorphic. Let $\omega\subset\mathbb{C}$ be a domain such that $\overline{\omega}\subset\Omega$ and $\Gamma=\partial\omega$ is a closed rectifiable curve. If f is (α)-holomorphic then

$$\Lambda\left[\frac{Df}{D\omega}(z)\right]=\frac{D(\Lambda\circ f)}{D\omega}(z)\quad,\quad z\in\Omega\ ,\ \ \Lambda\in\mathfrak{X}^*\ , \tag{28}$$

hence (by (19) in [30], p. 28)

$$\Lambda\left(\frac{1}{2\pi i}\int_\Gamma f(z)\,dz\right)=\frac{1}{2\pi i}\int_\Gamma\Lambda[f(z)]\,dz=$$

$$=\frac{1}{\pi}\int_\omega\frac{D(\Lambda\circ f)}{D\omega}(z)\,d\mu(z)=\Lambda\left(\frac{1}{\pi}\int_\omega\varphi(z)\,d\mu(z)\right)$$

hence

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz=\frac{1}{\pi}\int_\omega\frac{Df}{D\omega}(z)\,d\mu(z)\ . \tag{29}$$

Cf. also [21], p. 1008. As a consequence of (29) the class of (α)-holomorphic functions may be seen as a generalization of $\mathcal{O}(\Omega,\mathfrak{X})$. Indeed if $Df/D\omega=0$ then (by (29)) $\int_\Gamma f(z)\,dz=0$ for any $\Gamma\in\mathbf{\Gamma}(\Omega)$ so that $\int_\Gamma\Lambda[f(z)]\,dz=0$ for any $\Lambda\in\mathfrak{X}^*$. Thus (by the classical Cauchy theorem) f is weakly holomorphic in Ω and we may use Theorem 1 in Section 1 to conclude that f is strongly holomorphic.

Let $f:\Omega\to\mathfrak{X}$ be a (α)-holomorphic function. Then (by (22) in [30], p. 33)

$$\Lambda[f(\zeta)]=\frac{1}{2\pi i}\int_\Gamma\frac{\Lambda[f(z)]}{z-\zeta}\,dz-\frac{1}{\pi}\int_\omega\frac{\dfrac{D(\Lambda\circ f)}{D\omega}(z)}{z-\zeta}\,d\mu(z)$$

hence (by (28))

$$f(\zeta) = \frac{1}{2\pi i} \int_\Gamma (z-\zeta)^{-1} f(z)\, dz - \frac{1}{\pi} \int_\omega (z-\zeta)^{-1} \varphi(z)\, d\mu(z) \ . \tag{30}$$

Cf. also (2) in [21], p. 1009. This is the Cauchy-Pompeiu type formula we were seeking for. In particular if $\varphi(z) = 0$ for any $z \in \Omega$ the function $f : \Omega \to \mathfrak{X}$ satisfies the Cauchy formula (17). Therefore $f \in C^\infty(\Omega, \mathfrak{X})$ so that $\partial_{\alpha,\infty} f = 0$ (with $\alpha(\theta) = e^{i\theta}$) i.e. $f \in A(\Omega, \mathfrak{X})$. We may state

Theorem 21. *Let $f : \Omega \to \mathfrak{X}$ be a (α)-holomorphic function. Let $\omega \subset \mathbb{C}$ be a domain such that $\overline{\omega} \subset \Omega$ and $\Gamma = \partial\omega \in \mathbf{\Gamma}(\Omega)$.*

i) If the areolar derivative of f is a holomorphic function i.e. $Df/D\omega = h$ for some $h \in \mathcal{O}(\Omega, \mathfrak{X})$ then

$$f(\zeta) = \frac{1}{2\pi i} \int_\Gamma (z-\zeta)^{-1} \left[f(z) - \overline{z}\, h(z) \right] dz + \tag{31}$$

$$+ \frac{\overline{\zeta}}{2\pi i} \int_\Gamma (z-\zeta)^{-1} h(z)\, dz \ .$$

In particular $f \in C^\infty(\Omega, \mathfrak{X})$.

ii) If f admits continuous areolar derivatives $D^\nu f / D\omega^\nu \in C(\Omega, \mathfrak{X})$ of arbitrary order $\nu \geq 0$ which are equi-bounded in Ω then

$$f(\zeta) = \frac{1}{2\pi i} \sum_{\nu=0}^{\infty} \int_\Gamma \frac{1}{\nu!} \frac{\left(\overline{\zeta} - \overline{z}\right)^\nu}{z - \zeta} \frac{D^\nu f}{D\omega^\nu}(z)\, dz \tag{32}$$

and the convergence is uniform in $\zeta \in \omega$.

The formulae (31)-(32) for $\mathfrak{X} = \mathbb{C}$ are due to N. Teodorescu, [30], p. 13-19. When $\mathfrak{X}$ is a complex Banach space Theorem 21 was established by L-J. Nicolescu, [21].

The proof of Theorem 21 is elementary. Indeed for each $\Lambda \in \mathfrak{X}^*$ one has $D(\Lambda \circ f)/D\omega = \Lambda \circ h \in \mathcal{O}(\Omega)$. Then the identity (7) in [30], p. 15

$$\Lambda[f(\zeta)] = \frac{1}{2\pi i} \int_\Gamma \frac{\Lambda[f(z)] - \overline{z}\, \dfrac{D(\Lambda \circ f)}{D\omega}(z)}{z - \zeta}\, dz + \frac{\overline{\zeta}}{2\pi i} \int_\Gamma \frac{\dfrac{D(\Lambda \circ f)}{D\omega}(z)}{z - \overline{\zeta}}\, dz$$

yields (31). To prove statement (ii) in Theorem 21 let us observe that the set

$$\left\{ \frac{D^\nu f}{D\omega^\nu}(z) \ : \ z \in \Omega \, , \ \ \nu \geq 0 \right\} \subset \mathfrak{X} \tag{33}$$

is bounded in $\mathfrak{X}$ hence (cf. e.g. Theorem 3.18 in [24], p. 70) it is weakly bounded as well i.e. for each $\Lambda \in \mathfrak{X}^*$ there is $M(\Lambda) > 0$ such that

$$\left| \Lambda \left[\frac{D^\nu f}{D\omega^\nu}(z) \right] \right| \leq M(\Lambda) \quad , \quad z \in \Omega \, , \ \ \nu \geq 0 \ .$$

Then (by (16) in [30], p. 20)

$$\Lambda[f(\zeta)] = \frac{1}{2\pi i} \sum_{\nu=0}^{\infty} \int_\Gamma \frac{1}{\nu!} \frac{\left(\overline{\zeta} - \overline{z}\right)^\nu}{z - \zeta} \frac{D^\nu(\Lambda \circ f)}{D\omega^\nu}(z)\, dz$$

i.e. the series in the right hand side of (32) converges weakly to $f(\zeta)$ in $\mathfrak{X}$. It suffices to show that it is also strongly convergent in $\mathfrak{X}$ (and then the sum is still

$f(\zeta)$ because $\mathfrak{X}^*$ separates points). To this end let $\mathcal{P}$ be a separating family of seminorms determining the topology of $\mathfrak{X}$ as a locally convex space. As the set (33) is bounded for any $p \in \mathcal{P}$ and any integer $k \geq 1$ there is $t > 0$ such that

$$\left\{ \frac{D^\nu f}{D\omega^\nu}(z) \; : \; z \in \Omega \, , \;\; \nu \geq 0 \right\} \subset t\, V(p,k) \, .$$

Then for each $p \in \mathcal{P}$

$$p \left(\frac{1}{2\pi i} \int_\Gamma \frac{1}{\nu!} \, \frac{\left(\overline{\zeta} - \overline{z}\right)^\nu}{z - \zeta} \, \frac{D^\nu f}{D\omega^\nu}(z) \, dz \right) \leq$$

$$\leq \frac{1}{2\pi} \int_\Gamma \frac{1}{\nu!} \, \frac{\left|\overline{\zeta} - \overline{z}\right|^\nu}{|z - \zeta|} \, p \left[\frac{D^\nu f}{D\omega^\nu}(z) \right] ds(z) < \frac{t}{2\pi k \nu!} \int_\Gamma \frac{\left|\overline{\zeta} - \overline{z}\right|^\nu}{|z - \zeta|} \, ds(z) \leq$$

$$\leq \frac{t}{2\pi k} \, \frac{\rho^\nu}{\nu!} \int_\Gamma \frac{ds(z)}{|z - \zeta|} \leq \frac{t}{k} \, \frac{\rho^\nu}{\nu!}$$

where $\rho = \mathrm{diam}(\omega)$. Thus

$$\sum_{\nu=0}^\infty p \left(\frac{1}{2\pi i} \int_\Gamma \frac{1}{\nu!} \, \frac{\left(\overline{\zeta} - \overline{z}\right)^\nu}{z - \zeta} \, \frac{D^\nu f}{D\omega^\nu}(z) \, dz \right)$$

converges uniformly in $\zeta \in \omega$ for any $p \in \mathcal{P}$ and then (by Remark 3 in Appendix B) the series in (32) converges uniformly on ω.

Let $C^k_{\overline{z}}(\Omega, \mathfrak{X})$ be the class of all (α)-holomorphic functions $f : \Omega \to \mathfrak{X}$ admitting continuous areolar derivatives $D^\nu f / D\omega^\nu \in C(\Omega, \mathfrak{X})$ up to order $0 \leq \nu \leq k$. We may state

Theorem 22 (I. Ciorănescu, [8]). *Let $k \geq 1$. If $\Lambda \circ f \in C^{k+1}_{\overline{z}}(\Omega, \mathbb{C})$ for any $\Lambda \in \mathfrak{X}^*$ then $f \in C^k_{\overline{z}}(\Omega, \mathfrak{X})$.*

To prove Theorem 22 we need the following

Lemma 5. *Any weakly (α)-holomorphic function $f : \Omega \to \mathfrak{X}$ is strongly continuous in Ω.*

The proof relies on the fact that for any $g \in C^1_{\overline{z}}(\Omega, \mathbb{C})$ and any bounded set $\overline{\omega} \subset \Omega$

$$|g(z) - g(\zeta)| \leq M \; |z - \zeta| \; |\log a |z - \zeta| \, | \tag{34}$$

for some constants $M > 0$ and $a > 0$ and any $z, \zeta \in \omega$, and is similar to the proof of statement (i) in Theorem 1 (cf. [8], p. 640-641). To prove Theorem 22 we first show that each function $f : \Omega \to \mathfrak{X}$ such that $\Lambda \circ f \in C^2_{\overline{z}}(\Omega, \mathbb{C})$ for any $\Lambda \in \mathfrak{X}^*$ is monogeneous in Ω. By Lemma 5 one has $f \in C(\Omega, \mathfrak{X})$ hence (by Theorem 20) the integral $\int_\Gamma f(z)\, dz$ exists for any $\Gamma = \partial\omega \in \mathbf{\Gamma}(\Omega)$. Given another domain $\omega' \subset \mathbb{C}$ such that $\overline{\omega'} \subset \Omega$ and $\Gamma' = \partial\omega' \in \mathbf{\Gamma}(\Omega)$ we estimate (by (34))

$$\left| \Lambda \left(\frac{1}{|\omega|} \int_\Gamma f(z) \, dz - \frac{1}{|\omega'|} \int_{\Gamma'} f(z) \, dz \right) \right| = \left| \frac{\displaystyle\int_\Gamma \Lambda[f(z)] \, dz}{|\omega|} - \frac{\displaystyle\int_{\Gamma'} \Lambda[f(z)] \, dz}{|\omega'|} \right| =$$

$$= 2 \left| \frac{\displaystyle\int_\omega \frac{D(\Lambda \circ f)}{D\omega}(z)\, d\mu(z)}{|\omega|} - \frac{\displaystyle\int_{\omega'} \frac{D(\Lambda \circ f)}{D\omega}(z)\, d\mu(z)}{|\omega'|} \right| =$$

$$= 2 \left| \frac{Df}{D\omega}(z_0) - \frac{Df}{D\omega}(z_0') \right| \leq M\, |\zeta_0 - \zeta_0'|\, |\log a|\zeta_0 - \zeta_0'|\,|$$

for some $\zeta_0 \in \omega$ and $\zeta_0' \in \omega'$. Let $\omega_\nu \subset \mathbb{C}$ be a sequence of domains such that $\overline{\omega}_\nu \subset \Omega$ and $\Gamma_\nu = \partial\omega_\nu \in \mathbf{\Gamma}(\Omega)$ and $\omega_\nu \to z_0 \in \Omega$ as $\nu \to \infty$. By the estimate above there exist $\zeta_\nu \in \omega_\nu$ and $\zeta_\mu \in \omega_\mu$ such that

$$\left| \Lambda \left(\frac{1}{|\omega_\nu|} \int_{\Gamma_\nu} f(z)\, dz - \frac{1}{|\omega_\mu|} \int_{\Gamma_\mu} f(z)\, dz \right) \right| \leq M\, |\zeta_\nu - \zeta_\mu|\, |\log a|\zeta_\nu - \zeta_\mu|\,|$$

so that the set

$$\left\{ \frac{1}{(\zeta_\nu - \zeta_\mu)\ \log a|\zeta_\nu - \zeta_\mu|} \left[\frac{1}{|\omega_\nu|} \int_{\Gamma_\nu} f(z)\, dz - \frac{1}{|\omega_\mu|} \int_{\Gamma_\mu} f(z)\, dz \right] : \nu, \mu \geq 1 \right\}$$

is weakly bounded in $\mathfrak{X}$ and then strongly bounded, as well. Hence for any $p \in \mathcal{P}$ and any integer $k \geq 1$ there is $t > 0$ such that

$$\frac{1}{(\zeta_\nu - \zeta_\mu)\ \log a|\zeta_\nu - \zeta_\mu|} \left[\frac{1}{|\omega_\nu|} \int_{\Gamma_\nu} f(z)\, dz - \frac{1}{|\omega_\mu|} \int_{\Gamma_\mu} f(z)\, dz \right] \in t\, V(p, k)$$

i.e.

$$p \left(\frac{1}{|\omega_\nu|} \int_{\Gamma_\nu} f(z)\, dz - \frac{1}{|\omega_\mu|} \int_{\Gamma_\mu} f(z)\, dz \right) < \frac{t}{k}\, |\zeta_\nu - \zeta_\mu|\, |\log a|\zeta_\nu - \zeta_\mu|\,|\,.$$

For sufficiently large ν and μ

$$t\, |\zeta_\nu - \zeta_\mu|\, |\log a|\zeta_\nu - \zeta_\mu|\,| \leq 1$$

hence

$$\frac{1}{|\omega_\nu|} \int_{\Gamma_\nu} f(z)\, dz - \frac{1}{|\omega_\mu|} \int_{\Gamma_\mu} f(z)\, dz \in V(p, k)\,.$$

That is $\{\left(\int_{\Gamma_\nu} f(z)\, dz\right) / |\omega_\nu|\}_{\nu \geq 1}$ is a Cauchy sequence, hence it converges to some point $\varphi(z_0) \in \mathfrak{X}$. It may be easily shown that the limit doesn't depend upon the choice of domains $\{\omega_\nu\}_{\nu \geq 1}$ tending to z_0 (because the weak limit enjoys this property). Thus f is monogeneous at each $z_0 \in \Omega$. Let $Df/D\omega = \varphi$ be its areolar derivative. By the very assumption in Theorem 22 the function $Df/D\omega : \Omega \to \mathfrak{X}$ is weakly (α)-holomorphic hence (by Lemma 5) $Df/D\omega \in C(\Omega, \mathfrak{X})$. Consequently $f \in C^1_{\overline{z}}(\Omega, \mathfrak{X})$. The full statement in Theorem 22 may be easily got by induction over $k \geq 1$.

In particular the following analog (where areolar derivatives replace ordinary derivatives) to a result by A. Grothendieck, [14], holds (cf. I. Ciorănescu, [8], p. 843).

Corollary 6. *Let $\mathfrak{X}$ be a complex Fréchet space, $\Omega \subset \mathbb{C}$ an open set, and $f : \Omega \to \mathfrak{X}$ a continuous function. Then $f \in C^\infty_{\overline{z}}(\Omega, \mathfrak{X})$ if and only if $\Lambda \circ f \in C^\infty_{\overline{z}}(\Omega, \mathbb{C})$ for any $\Lambda \in \mathfrak{X}^*$.*

There is yet another approach to the Cauchy-Pompeiu type formula (30) closing our parallel among the work in [21], [8], and the exposition in [31]. Let $f : \Omega \to \mathfrak{X}$ be a monogeneous function with the areolar derivative φ. Let $\mathcal{P}$ be a separating family of seminorms determining the topology of $\mathfrak{X}$. Following [30], p. 26, we say $(1/|\omega|) \int_\Gamma f(z)\, dz$ converges *uniformly* to $\varphi(z_0)$ as $\omega \to z_0 \in \Omega$ if for any $p \in \mathcal{P}$ and any integer $k \geq 1$ there is $\rho = \rho(p, k)$ (independent of z_0) such that

$$\frac{1}{2\pi i |\omega|} \int_\Gamma f(z)\, dz - \frac{1}{\pi}\, \varphi(z_0) \in V(p, k) \tag{35}$$

for any domain $\omega \subset \mathbb{C}$ such that $\overline{\omega} \subset D_\rho(z_0)$ and $\Gamma = \partial\omega \in \mathbf{\Gamma}(\Omega)$. We shall show that

Theorem 23. *Let $\mathfrak{X}$ be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $f \in C^1_{\overline{z}}(\Omega, \mathfrak{X})$ be a (α)-holomorphic function and $z_0 \in \Omega$. Then $(1/|\omega|) \int_\Gamma f(z)\, dz$ converges uniformly to $\varphi(z_0)$ as $\omega \to z_0$.*

For $\mathfrak{X} = \mathbb{C}$ Theorem 23 is due to N. Teodorescu, [30], p. 28. As a consequence of Theorem 23 it follows that $(2\pi r)^{-1} \int_0^{2\pi} e^{i\theta} f(z_0 + re^{i\theta})\, d\theta$ converges uniformly in $z_0 \in \Omega$ as $r \to 0$ hence (by Theorem 8) $f \in B^1(\Omega, \mathfrak{X})$. Therefore Theorem 10 may be applied thus yielding (30). To prove Theorem 23 we need

Lemma 6. *Let $f \in C^1_{\overline{z}}(\Omega, \mathfrak{X})$ be a (α)-holomorphic function and $\varphi = Df/D\omega \in C(\Omega, \mathfrak{X})$ its areolar derivative. Then for any $p \in \mathcal{P}$ and any integer $\ell \geq 1$ there is $r = r(p, \ell) > 0$ such that for any $z_0 \in \Omega$*

$$\varphi(z) - \varphi(z_0) \in V(p, \ell) \tag{36}$$

for any $z \in \omega$ and any domain $\omega \subset \mathbb{C}$ such that $\overline{\omega} \subset D_r(z_0)$.

Proof of Theorem 23. We wish to show that (35) holds good. Let $p \in \mathcal{P}$ and let $k \geq 1$ be an integer. Let us set $\ell = [k/\pi] + 1$ (where $[a]$ denotes the integer part of $a \in \mathbb{R}$) and consider the number $r = r(p, \ell) > 0$ furnished by Lemma 6. Moreover let us consider a domain $\omega \subset \mathbb{C}$ such that $\overline{\omega} \subset D_r(z_0)$ and $\Gamma = \partial\omega \in \mathbf{\Gamma}(\Omega)$. Let us set $\varphi_r(z) = \varphi(z) - \varphi(z_0)$ for each $z \in \omega$. Then (by (29))

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz - \frac{|\omega|}{\pi}\, \varphi(z_0) = \frac{1}{\pi} \int_\omega \varphi_r(z)\, d\mu(z)\ .$$

As a consequence of (36) one has $p\,(\varphi_r(z)) < 1/\ell$ for any $z \in \omega$ hence

$$p\left(\frac{1}{2\pi i} \int_\Gamma f(z)\, dz - \frac{|\omega|}{\pi}\, \varphi(z_0)\right) \leq \frac{|\omega|}{\ell\pi} < \frac{|\omega|}{k}\ .$$

It remains that we prove Lemma 6. The proof is by contradiction. Let us assume that there is a pair (p, ℓ) with $p \in \mathcal{P}$ and $\ell \in \mathbb{Z}$, $\ell \geq 1$ such that for any $r > 0$ there is a domain $\omega \subset \mathbb{C}$ and a point $z \in \omega$ such that $\overline{\omega} \subset D_r(z_0)$ and $p\,[\varphi(z) - \varphi(z_0)] \geq 1/\ell$. In particular for $r \in \{1/\nu \,:\, \nu \in \mathbb{Z},\ \nu \geq 1\}$ there is a sequence of domains $\omega_\nu \subset \mathbb{C}$ such that $\overline{\omega}_\nu \subset D_{1/\nu}(z_0)$ and there is a sequence of points $z_\nu \in \omega_\nu$ such that $p\,[\varphi(z_\nu) - \varphi(z_0)] \geq 1/\ell$. Yet $\varphi \in C(\Omega, \mathfrak{X})$ and each seminorm p is continuous (cf. e.g. Theorem 1.37 in [24], p. 27) so that taking $\nu \to \infty$ leads to a contradiction.

REFERENCES

[1] W. Arendt & N. Nikolski, *Vector-valued holomorphic functions revisited*, Math. Z., 234(2000), 777–805.

[2] W. Arendt, C.J.K. Batty, M. Hieber & F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, Vol. 96, Birkhäuser Verlag, Basel-Boston-Berlin, 2001.

[3] M.S. Baouendi & F. Treves, *A property of functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. Math., 113(1981), 387–421.

[4] E. Barletta & S. Dragomir, *Vector valued holomorphic functions*, Bull. Math. Soc. Sci. Math. Roumanie, 100(3)(2009), 211–226.

[5] K.D. Bierstedt & S. Holtmanns, *An operator representation for weighted spaces of vector valued holomorphic functions*, Results Math., 36(1999), 9–20.

[6] N. Boboc & G. Bucur, *Măsură şi capacitate*, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1985.

[7] A. Bogges, *CR manifolds and the tangential Cauchy-Riemann complex*, Studies in Adavanced Mathematics, CRC Press, Inc., Boca Raton-Ann Arbor-Boston-London, 1991.

[8] I. Ciorănescu, *Asupra funcţiilor olomorfe* (α)*-vectoriale*, St. Cerc. Mat., 18(6)(1966), 839–844.

[9] J. Diestel & J.J. Uhl, *Vector measures*, American Mathematical Society, Providence, 1977.

[10] P. Domański & M. Langenbruch, *Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions*, Publ. RIMS, Kyoto. Univ., 44(2008), 1097–1142.

[11] S. Dragomir & S. Nishikawa, *On vector valued Cauchy-Riemann functions*, Preprint, 2009.

[12] K.G. Grosse-Erdmann, *The Borel-Okada theorem revisited*, Habilitationsschrift, Fernuniversität Hagen, Hagen, 1992.

[13] F.G. Grosse-Erdmann, *A weak criterion for vector-valued holomorphy*, Math. Proc. Camb. Phil. Soc., 136(2004), 399–411.

[14] A. Grothendieck, *Sur certaines espaces de fonctions holomorphes*, I-II, J. Reine Angew. Math., 192(1953), 35–64, 77–95.

[15] M. Itano, *On the distributional boundary values of vector-valued holomorphic functions*, J. Sci. Hiroshima Univ., Ser. A-I, 32(1968), 397–440.

[16] E. Jordá Mora, *Espacios de funciones meromorfas*, PhD. thesis, Valencia, 2001.

[17] E. Jordá Mora, *Vitali's and Harnack's type results for vector-valued functions*, J. Math. Anal. Appl., 327(2007), 739-743.

[18] A. Martineau, *Distributions et valeurs au bord des fonctions holomorphes*, *Theory of Distributions*, Proc. Internat. Summer Inst., Lisbon, 1964, Inst. Gulbenkian Ci., Lisbon, 1964, 193–326.

[19] R.E. Megginson, *An introduction to Banach space theory*, Springer-Verlag, Berlin, 1998.

[20] R. Narasimhan, *Several complex variables*, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago-London, 1971.

[21] L.-J. Nicolescu, *Derivata areolară a funcţiilor de o variabilă complexă cu valori într-un spaţiu Banach*, Com. Acad. R.P.R., 9(1959), 1007–1012.

[22] D. Pompeiu, *Sur une classe de fonctions d'une variable complexe*, Rend. Circ. Mat. Palermo, 33(1912), 108–113.

[23] D. Pompeiu, *Sur une classe de fonctions d'une variable complexe et sur certaines equations integrales*, Rend. Circ. Mat. Palermo, 35(1913).

[24] W. Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York-St Louis-San Francisco-Auckland-Bogotà-Caracas-Lisbon--London-Madrid-Mexico-Milan-Montreal-New Delhi-Paris-San Juan-Singapore-Sydney--Tokyo-Toronto, Second Edition, 1991.

[25] S.A. Saxon & L.M. Sánchez Ruiz, *Dual local completeness*, Proc. Amer. Math. Soc., 125(1997), 1063-1070.

[26] A.M. Sedlecki, *An equivalent definition of* H^p *spaces in the half-plane and some applications*, Math. U.S.S.R. Sbornik, 25(1975), 69–75.

[27] A. Sveshnikov & A. Tikhonov, *The theory of functions of a complex variable*, Mir Publishers, Moscow, 1978.

[28] J.L. Taylor, *A joint spectrum for several commuting operators*, J. Functional Analysis, 6(1970), 172–191.

[29] J.L. Taylor, *The analytic functional calculus for several commuting operators*, Acta Math., 125(1970), 1–38.

[30] N. Teodorescu, *La derivée aréolaire et ses applications à la physisque mathematique*, Thèse, Gauthier-Villars, Paris, 1931.

[31] F.-H. Vasilescu, *Funcţii analitice şi forme diferenţiale în spaţii Fréchet*, St. Cerc. Mat., 26(7)(1974), 1023–1049.

[32] F.-H. Vasilescu, *Calcul funcţional analitic multidimensional*, Editura Academiei Republicii Socialiste România, Bucureşti, 1979.

[33] P. Vieten, *Two equivalent norms for vector valued-holomorphic functions*, Tübinger Berichte zur Funktionalanalysis, 5(1996), 412–424.