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General linear supergroups and Schur superalgebras

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Abstract[1]. In this paper we present a review of some recent results and open problems about the category of polynomial supermodules of general linear supergroups. Once introduced the Schur superalgebra $S(m|n,r)$ for the general linear supergroup $GL(m|n)$, we consider the problem of what properties can and cannot be generalized from the ones satified, in the classical case, by the Schur algebra $S(n,r)$ and the general linear group $GL(n)$. For instance, one proves that in general $S(m|n,r)$ is not a quasi-hereditary algebra even if the category of $GL(m|n)$-supermodules is highest weight. Finally, we give an explicit description of the bases of a class of costandard modules $\nabla(\lambda)$ of $S(m|n,r)$ in terms of pairs of superstandard tableaux.

1. Introduction

The concept of a Schur algebra originated in two pioneering articles of Schur [25, 26]. The Schur algebra $S(n,r)$ associated to any integers $n,r>0$ is a finite dimensional quotient algebra of the group algebra KG of the general linear group $G=GL(n)$ over the field K, that has the property that the category $M(n,r)$ of polynomial G-modules that are homogeneous of degree r is equivalent to the category of $S(n,r)$-modules. In the first paper [25] Schur determined the polynomial representations of $GL(n)$ over the field of complex numbers. This was done using what is now called “Schur functor” that provides an equivalence between the category $M(n,r)$ and the category of modules over the group algebra KS_r of the symmetric group S_r. In the second paper [26] $S(n,r)$ was identified with the algebra of S_r-endomorphisms of the tensor power $E^{\otimes r}$, where E is the natural $GL(n)$-module. This result is known as the “Schur duality”.

The extension of Schur's ideas to the case of non-zero characteristic was done in the remarkable Green's book [15]. The publication of this book ended a period of neglect of the above mentioned Schur's work and the importance of Schur algebras has been realized by specialists in the representation theory of algebras, algebraic reductive groups and quantum groups (see [8, 9, 22, 23]). The paper [8] started a long series of articles of Donkin devoted to the investigation of Schur algebras

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and related algebras by means of homological techniques which culminated in the book [9]. The q-Schur algebras were introduced in [6] as endomorphism algebras of certain modules over Hecke algebras. Since then, the representation theory of quantum general linear groups, q-Schur algebras and Hecke algebras have become subjects of a significant interest and these theories have been advanced by numerous authors.

Further generalizations of Schur algebras appear when the general linear group $GL(n)$ is replaced by an algebraic (reductive) group G. For the first time the generalized Schur algebra was defined in [8] (see also [18], Chapter A). For a saturated subset π of dominant weights, the truncated category $C(\pi)$ is the category of G-modules such that the highest weights of all their composition factors belong to π. The category $C(\pi)$ is equivalent to the category of all modules over the generalized Schur algebra $S_G(\pi)$.

One of the motivations for the ground-breaking work [5] comes from the category $\mathcal{O}$ of Bernstein, Gelfand and Gelfand. A block $\mathcal{O}_\theta$ of that category is equivalent to the module category of a certain associative algebra A that is quasi-hereditary. The Verma flag for projective modules in the category $\mathcal{O}_\theta$ corresponds to the filtration of the projective modules of A by standard modules. On the other hand, the good filtration in the context of rational representation theory of reductive algebraic groups corresponds to the filtration by costandard modules. The concept of highest weight category and that of quasi-hereditary algebra provided a unifying framework for representation theory.

A quasi-hereditary structure implies a lot of desirable numerical and homological properties like the coincidence of the number of simple modules with the length of a defining chain of ideals, the vanishing of certain cohomology groups, the existence of tilting modules, the reciprocity law etc. Thus, to understand whether a given finite dimensional algebra is quasi-hereditary or whether a given category is highest weight category, is a very interesting and important problem. Schur algebras associated with semisimple algebraic groups (or, more generally, their centralizer subalgebras corresponding to some ideals of weights, see [8]) are quasi-hereditary. The quasi-hereditariness of q-Schur algebras was established in [9, 10, 23].

The principal aim of the present survey is to show how all the above ideas work for another intriguing generalizations of Schur algebras and general linear groups, namely Schur superalgebras and general linear supergroups. In particular, we present some announcement of the results we found recently [19] about an explicit description of some class of costandard modules of a Schur superalgebra.

2. Schur superalgebras

Let K be any field of characteristic different from 2. In what follows all spaces, (co)algebras and (co)modules are defined over K. Denote by Z_2 the additive cyclic group of order 2. We use the prefix "super" to indicate a Z_2-graded structure. Precisely, we call *superspace* a vector space together with a decomposition $V = V_0 \oplus V_1$. We denote by $|v|$ the Z_2-degree of an homogeneous element $v \in V$ that is $|v| = 0$ if $v \in V_0$ and $|v| = 1$ if $v \in V_1$. We say also that $|v|$ is the *parity* of v. If V is a superspace then its dual V^* is also a superspace with respect to the obvious Z_2-grading $V_0^* \oplus V_1^*$. The conjugated superspace V^c is defined by $V_0^c = V_1, V_1^c = V_0$. A linear map $\varphi : V \to W$ is called *homomorphism of superspaces of degree* $|\varphi|$ if $|\varphi(v)| = |\varphi| + |v| \pmod 2$ for any homogeneous element $v \in V$. We say briefly that φ is an homomorphism of superspaces if it is of degree 0 that is $\varphi(V_i) \subset W_i$ $(i = 0, 1)$. If $V \otimes W$ is the tensor product over the field K of the superspaces V, W then we define $|v \otimes w| = |v| + |w| \pmod 2$, where $v \in V, w \in W$ are Z_2-homogeneous elements.

A superspace A is said to be *super(co)algebra* if its (co)multiplication $m : A \otimes A \to A$ $(\delta : A \to A \otimes A)$ is a homomorphism of superspaces. In the similar way one can define *superbialgebras* and *Hopf superalgebras* (see [22, 28, 29] for more definitions). A superalgebra A is called *commutative* if $ab = (-1)^{|a||b|}ba$ for all Z_2-homogeneous elements $a, b \in A$.

An A-(co)module V over a super(co)algebra A is said to be *super(co)module* if V is a superspace and the (co)action $A \otimes V \to V$ $(V \to V \otimes A)$ is a homomorphism of superspaces. If A is a finite dimensional supercoalgebra then its dual A^* is a superalgebra and the category of all A-supercomodules with only even morphisms is equivalent to the category of locally finite A^*-supermodules with only even morphisms [14, 22].

Let $A(m|n)$ be a superalgebra defined by the generators $c_{ij}, 1 \leq i, j \leq m + n$, and the relations $c_{ij}c_{kl} = (-1)^{|c_{ij}||c_{kl}|}c_{kl}c_{ij}$, where the parity $|c_{ij}|$ is equal to $|i| + |j| \pmod 2$ and $|k| = 0$ iff $1 \leq k \leq m$, otherwise $|k| = 1$. Also, $A(m|n)$ is a superbialgebra with respect to the comultiplication $\delta(c_{ij}) = \sum_{1 \leq t \leq m+n} c_{it} \otimes c_{tj}$ and the counit $\epsilon(c_{ij}) = \delta_{ij}$. The supercoalgebra $A(m|n)$ has obvious grading by non-negative integers: $A(m|n) = \bigoplus_{r \geq 0} A(m|n, r)$. In particular, an r-homogeneous component $A(m|n, r)$ is also a supercoalgebra and its dual $S(m|n, r) = A(m|n, r)^*$ is the *Schur superalgebra* [22].

Consider the generic matrix C whose entries are the generators c_{ij} of $A(m|n)$ and write it as a block matrix

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where c_{ij} belongs to the blocks C_{11}, C_{22} (respectively C_{12}, C_{21}) only if $|c_{ij}| = 0$ $(|c_{ij}| = 1)$. The localized algebra $A(m|n)_d$, where $d = \det(C_{11})\det(C_{22})$, is a coordinate (Hopf) superalgebra of the general linear supergroup $GL(m|n)$ [4, 29]. The supergroup $GL(m|n)$ can be regarded as a group functor $A \to GL(m|n)(A)$ from the category of commutative superalgebras to the category of groups, where

$$GL(m|n)(A) = Hom_{K-superalg}(A(m|n)_d, A) =$$

$$= \left\{ R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in M(m|n)(A) \mid R \text{ invertible} \right\} .$$

Here $M(m|n)(A)$ is a matrix superalgebra consisting of all $(m + n) \times (m + n)$ matrices

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

whose entries belong to A. Besides, the entries of blocks R_{11}, R_{22} are even and the entries of blocks R_{12}, R_{21} are odd. Note that the above identification $GL(m|n)(A)$ with the multiplicative group $M(m|n)(A)^\times$ is correct since a matrix $R \in M(m|n)(A)$ is invertible iff its even blocks R_{11}, R_{22} are invertible.

Denote by $S(m|n,r)-mod$ the category of locally finite $S(m|n,r)$-supermodules with only even morphisms between them. By the above, this category is equivalent to the category of $A(m|n,r)$-supercomodules with only even morphisms again. The category of locally finite $S(m|n,r)$-modules (which is also equivalent to the category of $A(m|n,r)$-comodules) is denoted by $S(m|n,r)-mod$.

An $(m+n)$-tuples of non-negative integers $\lambda=(\lambda_1,\cdots,\lambda_{m+n})$ such that $|\lambda|=\sum_i\lambda_i=r$ is called (*polynomial*) *weight of degree* r. If $\lambda_1\geq\cdots\geq\lambda_m$ and $\lambda_{m+1}\geq\cdots\geq\lambda_{m+n}$ we call the weight λ *predominant.* In this case we denote $\lambda=(\lambda_+|\lambda_-)$ where $\lambda_+=(\lambda_1,\cdots,\lambda_m),\lambda_-=(\lambda_{m+1},\cdots,\lambda_{m+n})$ are partitions and one has that $|\lambda_+|+|\lambda_-|=r$. We denote by $\Lambda(r)=\Lambda(m|n,r)$ (respectively $\Lambda(r)^+=\Lambda(m|n,r)^+$) the set of weights (predominant weights) of degree r. By [28] any simple $S(m|n,r)$-supermodule $L=L(\lambda)$ is again simple as a $S(m|n,r)$-module and it is uniquely defined by its highest weight $\lambda\in\Lambda(r)^+$, with respect to the dominant order, and by parity of its highest vector.

From now on, we assume that the highest vector of $L(\lambda)$ is even. Then, another simple $S(m|n,r)$-supermodule with the same highest weight λ coincides with $L(\lambda)^c$. In general, the set of highest weights $\Lambda(r)^{++}$ is a proper subset in $\Lambda(r)^+$ [4, 11, 28]. In fact, if $m\geq r, char\,K=p>0$ then a predominant weight λ belongs to $\Lambda(r)^{++}$ iff p divides λ_-. The complete description of the set $\Lambda(r)^{++}$ for any $m,n\geq 1$ is obtained in [4].

Let $I(\lambda)$ be an injective envelope of a simple $S(m|n,r)$-supermodule $L(\lambda)$. Denote by $\nabla(\lambda)$ the largest submodule of $I(\lambda)$ whose composition factors are $L(\mu)$ with $\mu\leq\lambda$. Using the terminology of [5, 7], this is the *costandard module* with highest weight λ. The algebra $S(m|n,r)$ is quasi-hereditary iff any $I(\lambda)$ has a *good* or *costandard filtration* whose factors are $\nabla(\mu)$ $(\mu\geq\lambda)$ with $\nabla(\lambda)$ as the first member with multiplicity one. It is well-known that the classical Schur algebra $S(m|r)=S(m|0,r)\simeq S(0|m,r)$ is always quasi-hereditary. It is somehow surprising that the properties of Schur superalgebras are different from properties of the classical Schur algebras and some other generalizations.

Theorem 2.1 ([20]). *The superalgebra* $S(m|n,r)$ $(m,n>0)$ *is quasi-hereditary iff it is semisimple and this happens iff either* $char\,K=0$, *or* $char\,K=p>0$ *and* $p>r$, *or* $m=n=1$ *and* p *does not divide* r.

Note that any $I(\lambda)$ (and also any $\nabla(\lambda)$) can be endowed with two conjugated supermodule structures. One of them is uniquely defined by the condition that $L(\lambda)$ is the socle of $I(\lambda)$ (respectively, it is the socle of $\nabla(\lambda)$)) in the category $S(m|n,r)-mod$. In particular, one has that the category $S(m|n,r)-mod$ is a highest weight category iff $S(m|n,r)$ is semisimple.

Theorem 2.2 ([17]). *Let* $m,n\geq 1$. *The superalgebra* $S(m|n,r)$ *has finite representation type iff* $S(m|n,r)$ *is semisimple, otherwise* $p\leq r<2p$ *or* $m=n=1$ *and* p *divides* r.

The proof of the last theorem is a deep analysis of Specht filtrations of signed Young modules, recently introduced in [11]. On the contrary, the proof of the previous theorem is rather elementary and uses only some properties of complexes of projective modules over $S(m|n,r)$.

In the representation theory of reductive algebraic groups it is well-known that the tensor product of two costandard modules, with respect to the diagonal action, has also a costandard (good) filtration [12, 21]. Therefore, it is natural to ask whether this is true for costandard (super)modules over a Schur superalgebra. Note that in the classical case, that is for $n = 0$ or $m = 0$, this obviously holds. In [28] it has been proved that the answer is positive in the simplest non-classical case $m = n = 1$. Nevertheless, the answer is negative in general.

Theorem 2.3 ([19]). *Let* $m = 1, n = 2, char\, K = p > 0$. *If* $\lambda \geq 2, \mu > 0$ *and* $r = \lambda + p\mu$, *the* $S(m|n,r)$*-(super)module* $\nabla(\lambda|0,0) \otimes \nabla(0|p\mu,0)$ *does not have any costandard filtration.*

3. General linear supergroups and related topics

In the previous section we noticed that the category of polynomial $GL(m|n)$-supermodules of degree r is equivalent to the category of $S(m|n,r)$-supermodules (see [4, 11, 20, 22, 28]). Then, from Theorem 1.1 it is quite reasonable to expect that the category of $GL(m|n)$-supermodules is not a highest weight category in general. On the contrary, it has been recently proved that this category is a highest weight one [29]. In fact, one can develop some fragment of the theory of modules with good filtration and the dual theory of modules with Weyl filtration over $GL(m|n)$ which implies that all indecomposable injective $GL(m|n)$-supermodules have required good filtrations. More precisely, let $\mathbf{L}(\lambda)$ be a simple $GL(m|n)$-supermodule with highest (= predominant) weight λ, whose highest vector is even. Let $\mathbf{I}(\lambda)$ be an injective envelope of $\mathbf{L}(\lambda)$. Then $\mathbf{I}(\lambda)$ has a good filtration

$$0 \subseteq \mathbf{I}_1 \subseteq \mathbf{I}_2 \subseteq \cdots \subseteq \mathbf{I}_k \subseteq \cdots$$

such that $\mathbf{I}_k/\mathbf{I}_{k-1} \simeq H^0(\pi_k)^{t_k}, k \geq 1, \pi_1 = \lambda, t_1 = 0$, and $\pi_k > \lambda$ for all $k > 1$ with respect to the same dominance order. Here $H^0(\pi)^0 = H^0(GL(m|n)/B, K_\pi) = ind_B^{GL(m|n)} K_\pi$, B is the standard Borel supersubgroup of $GL(m|n)$ consisting of lower triangular matrices and $H^0(\pi)^1 = H^0(\pi)^c$.

Note that the induced modules $H^0(\pi)$ satisfy the same universal property as their classical counterparts. In fact, the socle of $H^0(\pi)$ coincides with $\mathbf{L}(\pi)$, all other composition factors of $H^0(\pi)$ have highest weights strictly smaller than π and any $GL(m|n)$-supermodule V which satisfies the same properties can be embedded in $H^0(\pi)$. In other words, the supermodules $H^0(\pi)$, as well as their conjugated doubles $H^0(\pi)^c$, play the role of costandard modules in the category $GL(m|n)-mod$. Their Chevalley duals $V(\pi)$ and $V(\pi)^c$ are called Weyl supermodules and play the role of standard modules in the category $GL(m|n)-mod$. Besides, it can be easily checked that $\nabla(\pi)$ is the largest polynomial supersubmodule of $H^0(\pi)$ and $L(\pi) = \mathbf{L}(\pi)$, provided that $\pi \in \Lambda(m|n,r)^{++}, r \geq 0$. It is still an open problem to describe bases of costandard $S(m|n,r)$-supermodules or even to find their formal characters. All is known at present are the following results.

Define the subset of variables $X^+ = \{c_{ij} | i, j > m\}$. Denote by $A(0|n)$ the subalgebra of $A(m|n)$ generated by X^+. The following result [19] allows to decompose the costandard modules when $m \geq r$.

Theorem 3.1. *If* $m \geq r$ *then* $\nabla(\lambda) \simeq \nabla(\lambda_+|0) \otimes \nabla(0|\lambda_-)$. *Moreover, in this case* $\lambda_- = p\mu$ *and* $\nabla(0|\lambda_-) = F(\nabla(\mu))$, *where* $\nabla(\mu) \subseteq A(n) = A(n|0)$ *is the costandard* $GL(n)$*-module with highest weight* μ *and* $F : A(n) \to A(0|n)$ *is the Frobenius map defined by* $c_{ij} \to c^p_{i+m,j+m}$.

We have to consider now the problem of describing the costandard modules $\nabla(\lambda|0)$. Let $I = (i_1, \cdots, i_q), J = (j_1, \cdots, j_q)$ be two multi-indices ($1 \leq i_k, j_k \leq$

$m+n$) of the same length and consider the monomial $c_{IJ} = c_{i_1 j_1} \cdots c_{i_q j_q}$ of $A(m|n)$. Let $(i_{k_1}, j_{k_1}), \cdots, (i_{k_s}, j_{k_s})$ be all pairs such that $c_{i_{k_u} j_{k_u}} \in X^+$ for $u = 1, 2, \cdots, s$. Denote by t_u the number of times the variable $c_{i_{k_u} j_{k_u}}$ occurs in c_{IJ} and define $d_{IJ} = t_1! \cdots t_s!$. Then, we define the $\mathbb{Z}$-module $Sup_{\mathbb{Z}}(m|n) \subseteq Sup_{\mathbb{Q}}(m|n) = A_{\mathbb{Q}}(m|n)$ as generated by the elements $\tilde{c}_{IJ} = (1/d_{IJ})\, c_{IJ}$ for any c_{IJ} in the monomial basis of $A(m|n)$. In [16] the superalgebra $Sup_{\mathbb{Z}}(m|n)$ was introduced and called *four-fold algebra*. In [19] one proves that $Sup_K(m|n) = K \otimes_{\mathbb{Z}} Sup_{\mathbb{Z}}(m|n)$ is a two-sided $A_K(m|n)$-supercomodule or, equivalently, two-sided $S_K(m|n)$-supermodule.

We define now ${}_{ij}D$ as the right superderivation of $A(m|n)$ such that $(c_{kl})_{ij}D = \delta_{li} c_{kj}$. We call ${}_{ij}D$ a *right superpolarization*. Let now α be an element disjoint by the set $\{1, 2, \cdots, m+n\}$ and assume that $|\alpha| = 1$. If I, J are two multi-indices of the same length, we put

$$(I|J) = \frac{1}{d_I d_J}\, c_{I\alpha^q}\, {}_{\alpha^q J}D$$

where $c_{I\alpha^q} = c_{i_1\alpha} \cdots c_{i_q\alpha}, {}_{\alpha^q J}D = {}_{\alpha j_q}D \cdots {}_{\alpha j_1}D$ and $d_I = d_{I\alpha^q}, d_J = d_{\alpha^q J}$. Since $|\alpha| = 1$ we have that d_I (respectively d_J) is equal to the product $\lambda_{m+1}! \cdots \lambda_{m+n}!$ where $\lambda = \lambda(I)$ $(\lambda = \lambda(J))$. Moreover note that $(1/d_J)\, c_{\alpha^q J}\, D$ is a product of divided powers of right superpolarizations. We have that $(I|J)$ is a homogeneous element of $Sup_{\mathbb{Z}}(m|n)$ (as well as a homogeneous element of $Sup_K(m|n)$) whose expansion with respect to the basis elements of such algebra have integer coefficients. The element $(I|J)$ has been introduced in [16] by means of an equivalent definition. The use of superpolarization operators to define $(I|J)$ is in [1].

Let (λ, μ) be a pair of weights. Denote by $Sup_K(m|n)_{\lambda\mu}$ the subspace of the superalgebra $Sup_K(m|n)$ spanned by the elements $\tilde{c}_{IJ}$ with contents $\lambda(I) = \lambda, \lambda(J) = \mu$. We have obviously that $Sup_K(m|n) = \bigoplus_{\lambda,\mu} Sup_K(m|n)_{\lambda\mu}$ is a grading of $Sup_K(m|n)$ over the group $\mathbb{Z}^{2(m+n)}$. The algebra $A_K(m|n)$ is graded in the same way and we can define two supersubalgebras

$$A_K(m|n)' = \bigoplus_{\lambda_- = 0} A_K(m|n)_{\lambda\mu} \qquad Sup_K(m|n)' = \bigoplus_{\lambda_- = 0} Sup_K(m|n)_{\lambda\mu}$$

of the superalgebras $A_K(m|n), Sup_K(m|n)$ respectively. It is clear that we have $Sup_K(m|n)' \approx A_K(m|n)'$ via the one-to-one correspondence $c_{IJ} \mapsto \tilde{c}_{IJ}$. From now on we identify $Sup_K(m|n)'$ with $A_K(m|n)'$.

Remark 3.1.

(i) If $\lambda(I) = (\lambda|0)$ then $(I|J) = (1/d_J)\ c_{I\alpha^q}\, {}_{\alpha^q J}D$ belongs to $A(m|n)' \subset A(m|n)$.
(ii) If $\lambda(I) = (\lambda|0), \lambda(J) = (\mu|0)$ then $(I|J) = \det(c_{i_k j_l}) \in A(m|0)$.

Let $I = (i_1, \cdots, i_q)$ be a multi-index. If $i_k \leq i_{k+1}$ and $i_k = i_{k+1}$ only if $|i_k| = 1$ (respectively $|i_k| = 0$) we say that I is a *superstandard row* (*column*). If I, J are superstandard rows of the same length we have that $(I|J) \neq 0$. Consider now a multi-index T of length r and assume that

$$T = (t_{11}, \cdots, t_{1q_1}, \cdots, t_{s1}, \cdots, t_{sq_s})$$

with $q_1 \geq \cdots \geq q_s$. In this case we denote also T as a table

$$T = \begin{array}{lllll} t_{11} & \cdots & \cdots & & t_{1q_1} \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & \cdot & & \\ t_{s1} & \cdots & t_{sq_s} & & \end{array}$$

If $\lambda = (q_1, \cdots, q_s)$ we say that T is a *tableau of shape* λ. Define $\lambda' = (q'_1, \cdots, q'_t)$ the conjugate of the partition λ ($|\lambda| = |\lambda'| = r$). Clearly $T_k = (t_{k1}, \cdots, t_{kq_k})$ for $k = 1, 2, \cdots, s$ (respectively $T'_l = (t_{1l}, \cdots, t_{q'_l l})$ for $l = 1, 2, \cdots, t$) are the *rows* (*columns*) of the tableau T.

Let T, U be two tableaux of the same shape $\lambda = (q_1, \cdots, q_s)$. Following [16] we define

$$(T|U) = (T_1|U_1) \cdots (T_s|U_s)\ .$$

Clearly $(T|U)$ is a homogeneous element of $Sup_K(m|n)$ with integer coefficients that we call *superbideterminant of the pair* (T, U). In the classical case that is when all entries of T, U are in $\{1, 2, \cdots, m\}$ the polynomial $(T|U) \in A(m|0)$ is called *bideterminant*.

Let T be a tableau. If all its rows T_k are superstandard rows and all columns T'_l are superstandard columns we say that T is a *superstandard tableau*. In the classical case we call T a *standard tableau*. We have the following fundamental result due to Grosshans, Rota, Stein [16].

Theorem 3.2. *The superalgebra* $Sup_{\mathbb{Z}}(m|n)$ *has a* $\mathbb{Z}$*-free generating set given by* $(T|U)$ *where* T, U *is any pair of superstandard tableaux of the same shape. In other words any superbideterminant* $(T|U)$ *where* T, U *tableaux of the same shape* λ *can be written in a unique way as* $(T|U) = \sum_i \alpha_i (T_i|U_i)$ *where* T_i, U_i *are superstandard tableaux of the same shape* $\lambda_i \geq \lambda$*, the tableaux* T, T_i *(respectively* U, U_i*) have the same weight and all the coefficients* α_i *are integers.*

For all details and proofs about superbideterminants we refer to [16].

Fix $\lambda = (\lambda|0)$ a predominant weight of degree r and let $\lambda' = (q_1, \cdots, q_s)$ be the conjugate partition. Consider the superstandard tableau of shape λ'

$$L_\lambda = \begin{array}{llllll} 1 & 2 & \cdots & \cdots & & q_1 \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & \cdot & & & \\ 1 & 2 & \cdots & q_s & & \end{array}$$

We call L_λ the *canonical tableau of weight* $\lambda = (\lambda|0)$. Let K be any field of characteristic different from 2. For the classical costandard modules we have the following result (see [15]).

Theorem 3.3. *The costandard module* $\nabla(\lambda) \subset A(m|0)$ *has a basis given by bideterminants* $(L_\lambda|T)$ *where* T *is any standard tableau of shape* λ'.

In [19] one obtains the following generalization.

Theorem 3.4. *The costandard module* $\nabla(\lambda|0) \subset A_K(m|n)' = Sup_K(m|n)'$ *has a basis given by superbideterminants* $(L_\lambda|T)$ *where* T *is any superstandard tableau of shape* λ'.

4. Open problems

The principal difference between $GL(n) - mod$ and $GL(m|n) - mod$ is that the set of highest weights $\Lambda(m|n)^+ = \bigcup_{r \in Z} \Lambda(m|n,r)^+$ over $GL(m|n)$ contains infinite decreasing and increasing chains of weights. In particular, there are finitely generated but infinite ideals in $\Lambda(m|n)^+$ and therefore, the category $GL(m|n) - mod$ is not locally equivalent to a category $A - mod$ for a quasi-hereditary finite dimensional algebra A. In other words, we have to generalize the notion of a quasi-hereditary algebra for infinite dimensional algebras too.

Let $\mathcal{D}$ be a highest weight category with a set of highest weights Λ. Then, for any finitely generated ideal $\Gamma \subseteq \Lambda$ the full subcategory $\mathcal{D}[\Gamma]$, consisting of all objects whose composition factors belong to Γ, is also highest weight category [5]. The category $\mathcal{D}[\Gamma]$ is equivalent to the category $R - PC$ consisting of all left pseudo-compact modules over a pseudo-compact algebra R, or to the category $Dis(R)$ of all right discrete R-modules (see [2, 27] for more details). Moreover, the algebra R has a stratifying structure which is similar to a stratifying structure of a finite dimensional quasi-hereditary algebra. More precisely, R is a pseudo-compact algebra where all indecomposable projectives are finite dimensional and R has an increasing chain of two-sided closed ideals $0 = H_0 \subseteq H_1 \subseteq \cdots$ which satisfies the following conditions for any $k \geq 1$:

1. H_k/H_{k-1} is a projective pseudo-compact left R/H_{k-1}-module;
2. $Hom_R(H_k/H_{k-1}, R/H_k) = 0$;
3. $H_k/H_{k-1}\ rad(R/H_{k-1})\ H_k/H_{k-1} = 0$;
4. The space $Hom_R(P, H_k)$ is not equal to zero for finitely many pairwise non-isomorphic indecomposable projective pseudo-compact R-modules P only.
5. For any open left ideal $I \subseteq A$ there is $t \geq 1$ such that $H_t + I = A$;

We call R a *pseudo-compact quasi-hereditary algebra*. As it could be suspected, if R is a pseudo-compact quasi-hereditary algebra then $R - PC$ or $Dis(R)$ are both highest weight categories [30].

Suppose that $\mathcal{D}_f$ has a duality $\tau : \mathcal{D}_f \to \mathcal{D}_f$ such that $\tau(L) = L$ for any simple object L. We call τ a *Chevalley duality*. If $\nabla(\lambda)$ is a costandard object in $\mathcal{D}$ then one defines its standard counterpart as $\Delta(\lambda) = \tau(\nabla(\lambda))$ [5, 30]. Following [3], one can define a *weak tilting object* T in $\mathcal{D}$ as follows. The object T has a (possible infinite) Δ-filtration and $Ext^1_{\mathcal{D}}(\Delta(\chi), T) = 0$ for any $\chi \in \Lambda$. In [30] some fragment of the corresponding weak tilting theory has been developed. We call an object M *restricted* iff any its simple composition factor has finite multiplicity and M belongs to some finitely generated ideal Γ.

Theorem 4.1. *A restricted weak tilting object T is equal to a (possible infinite) direct sum of indecomposable weak tilting objects $T(\lambda), \lambda \in \Gamma$. Moreover, any indecomposable weak tilting object $T(\lambda)$ is uniquely defined by its highest weight λ.*

Theorem 4.2. *Let T be a restricted weak tilting objects and let M be a restricted object with decreasing ∇-filtration. Then $Ext^1_{\mathcal{D}}(T, M) = 0$. Moreover, if M has a Δ-filtration and T is a finite direct sum of indecomposable weak tilting objects then $Ext^1_{\mathcal{D}}(M, T) = 0$.*

Corollary 4.1. *If T and T' are restricted weak tilting objects then $Ext^1_{\mathcal{D}}(T, T') = 0$.*

The existence of a Chevalley duality τ implies that the above pseudo-compact quasi-hereditary algebra R has a continuous anti-isomorphism $\phi : R \to R$ which preserves its stratifying structure. In particular, this anti-isomorphism defines some cellular structure on R which is very close to the structure of a finite dimensional cellular algebra [13]. From this viewpoint it is important to address the following problems.

Problem 1. Develop the theory of *pseudo-compact cellular algebras*.

Problem 2. Develop the theory of Ringel duality for highest weight categories with Chevalley duality.

The following problem is an attempt to define some *generalized Schur superalgebras* in the same way as it was done in [8]. The principal aim is to have the property of quasi-hereditariness which does not hold in general for the algebras $S(m|n, r)$.

Problem 3. Describe pseudo-compact quasi-hereditary algebras which correspond to finitely generated ideals of dominant weights in the category $GL(m|n) - mod$.

For example, in the classical case a Schur algebra $S(n, r)$ corresponds to the ideal of dominant weights generated by the weight $(r, 0, \cdots, 0)$ [8, 18]. In the supercase instead, the corresponding ideal is much more bigger than the set of all polynomial weights of degree r. Denote by $GS(m|n, r)$ the generalized Schur superalgebra which corresponds to the above ideal.

Problem 4 (generalized Schur duality)**.** Is any $GS(m|n, r)$ isomorphic to an endomorphism algebra $End_B(T)$, where T is simultaneously a $B \times GS(m|n, r)$-bimodule and a weak tilting (discrete) $GS(m|n, r)$-module?

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