Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 7(2008), pp. 1–19.

A Lagrangian method for the periodic orbit problem of Reeb vector-fields

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Dedicated to Ermanno Lanconelli on his sixty fifth birthday

Abstract[1]. We establish in this paper that the techniques of [1] hold in full generality and build a Lagrangian method for the study of periodic orbits for general three-dimensional Reeb vector-fields.

1. Introduction

Let M^3 be a three dimensional compact manifold without boundary and let α be a contact form on M with Reeb vector field ξ. Let v be a vector field in $\ker \alpha$ and let $\beta = d\alpha(v, \cdot)$. Let C_β be the space of closed curves with regularity H^1 whose tangent vector reads $\dot{x} = a\xi + bv$, a being a positive constant (varying with the curve) and b being an L^2 function.

When β is a contact form with the same orientation than α, C_β is a sub-manifold of the free loop space of M and the action functional defined at a curve x of C_β as $J(x) = \int_0^1 \alpha_x(\dot{x})$ is the natural functional for the study of periodic orbits.

When β is not a contact form, C_β can be subdivided into two regions Σ^+ and Σ^-. In Σ^+, β has the orientation of α, in Σ^-, it has the opposite orientation.

We have then replaced the functional J in [4] by the new functional

$$J(x) = \int_{\Sigma^+} \alpha_x(\dot{x}) + \frac{1}{1 + \int_{\Sigma^-} \alpha_x(\dot{x})}$$

and we have proved in [4] that this functional allowed to generalize the results of [1], [2], [3]. In particular, we could extend to this general framework our homology.

In [4], we stopped short from proving that the critical points and critical points at infinity of the generalized J on C_β were natural extensions of the critical points and critical points at infinity found in [1], [5] under the assumption that β is a contact form. Indeed, we had pointed out that some curves of C_β having sub-pieces on the dividing $\Sigma = \{x\ ,\ \beta \wedge d\beta = 0\}$ were "critical" for J. Although we strongly suspected in [4] that we could get rid of them as stationary points in our

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Keywords. 53D35, 53D10, 37J55.

AMS Subject Classification. Contact form, homology, critical points at infinity, Reeb vector-fields.

deformations, we could not really prove such a result and we simply showed that we could consider them also as critical points at infinity, although they were not a natural extension of this concept, and that the compactness results of [2] extended to them.

We improve on this result here and we prove (Theorem 1, section 4) that the functional J_∞ which extends J on the stratified space $\bigcup \Gamma_{2k}$, the stratified space of ξ-pieces of orbits alternating with $\pm v$-pieces of orbits, does not have critical points near such curves. Therefore, its only critical points are the "classical" ones, critical points and critical points at infinity and the homology defined in [1], [2], see also the more recent [6], extends readily since the unstable manifolds of the periodic orbits can be achieved in $\bigcup \Gamma_{2k}$. This result suffices for our purposes.

Short of allowing for zeros of v-we have not tried yet to get rid of this assumption-our method is now general and offers a Lagrangian/Legendrian version of the "Hamiltonian" periodic orbit problem in the framework of general three dimensional contact vector-fields.

For the sake of the completeness of the paper and of the techniques involved in our method, we have included in this present paper the first three sections of [4], slightly modified (for section 3), so that the nature of the difficulty, the techniques used to overcome it, the definition of the Sturm-Liouville linearized operator at a periodic orbit etc are included here. The present paper departs completely from [4] in the fourth section where the claim described above about the behavior of J_∞ on $\bigcup \Gamma_{2k}$ is established.

2. The vector-field v and the hypersurface Σ

Let v be a nowhere zero vector-field in $\ker \alpha$.

Considering the form $\beta = d\alpha(v, \cdot)$, we compute the quantity $p = \beta \wedge d\beta$. Generically the set $\Sigma = \{x\ ,\ p(x) = 0\}$ is a submanifold of codimension 1 in M.

Let us consider a point x_0 of Σ where the Reeb vector-field ξ of α is transverse to Σ. We pick up a set of coordinates near x_0 where Σ reads as $\{z = 0\}$ and x_0 is $(0, 0, 0)$.

After rescaling v or β appropriately, we may assume that ξ near $(0, 0, 0)$ reads as $\partial/\partial z$.

Let us consider the restriction γ of α to $T\Sigma$. Since α is a contact form, $d\gamma$ is non zero near $(0, 0, 0)$ and using an appropriate chart near x_0 in Σ, we may assume that $d\gamma = dx \wedge dy$, so that $\gamma = xdy + df$. f is a function of x, y only.

Accordingly, α reads as $xdy + df + dz$. v reads as $l\,(\partial/\partial x) + m\,(\partial/\partial y) + C\,(\partial/\partial z)$. The value of C depends on df, but this is irrelevant as far as β and p are concerned. Indeed,

$$\beta = d\alpha(v, \cdot) = -m\,dx + l\,dy$$

$$p = \frac{\beta \wedge d\beta}{dx \wedge dy \wedge dz} = lm_z - ml_z\ .$$

C does not appear in the formulae above, once l and m are given, C can be computed easily using the formula for α.

This simple fact gives us a lot of freedom which we will use in order to modify v and make $\ker \alpha$ tangent at $(0, 0, 0)$ to Σ. After a suitable modification of α, we can then reduce it to Darboux form in coordinates where $\Sigma = \{z = 0\}$. This freedom can also be used in order to create tangencies as we please of v (after modification) to Σ (also after modification). These results are established in the following sequence of Lemmata/Propositions:

Lemma 2.1 ([1]). *Let $\underline{w}$ be another vector field in the kernel of α, defined near $x_0 = (0,0,0)$, such that $d\alpha(v,\underline{w}) = 1$. We may then modify v into $v + \theta\underline{w}$, θ being C^0 small and C^1 bounded so that Σ is perturbed into $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a graph in z ($\xi = \partial/\partial z$) over Σ which is C^0-close to Σ and $\ker \alpha_{x_0} = T_{x_0}\tilde{\Sigma}$ ($x_0 \in \tilde{\Sigma}$).*

Proof. Let $\tilde{v} = v + \theta\underline{w}$. Then $\tilde{p} = -d\alpha(\tilde{v},[\xi,\tilde{v}]) = p - \theta_\xi - \theta(d\alpha(\underline{w},[\xi,v]) + d\alpha(v,[\xi,\underline{w}]) + \theta d\alpha(\underline{w},[\xi,\underline{w}])$.

Choose (x,y)-coordinates on Σ and set

$$\theta = (a_1 x + b_1 y) z w(z) \psi(x,y)$$

w and ψ have small compact supports, $w(0) = 1$, $\psi(0,0) = 1$. Choose also the support of ψ in (x,y) so small that:

$$\theta,\ \theta_z,\ \theta_{zz} = o(1)\ .$$

Calculating $\tilde{p}_z$,

$$\tilde{p}_z = p_z - \theta_{zz} - \theta_z A - \theta A_z = p_z + o(1)\ ,$$

we recognize that $\tilde{p}_z$ is far from zero near x_0. Thus, $\tilde{\Sigma}$ is a graph over Σ in the z-direction and is a C^0-perturbation of Σ as claimed. Furthermore,

$$T_{x_0}\Sigma = \{z_1 / d\tilde{p}_{(0,0,0)}(z_1) = 0\}\ .$$

We compute $d\tilde{p}_{(0,0,0)}(z_1)$:

$$d\tilde{p}_{(0,0,0)}(z_1) = (dp_{(0,0,0)} - d(a_1 x + b_1 y))(z_1)\ .$$

This shows that after a suitable choice of a_1, b_1, $T_{x_o}\tilde{\Sigma}$ and $\ker \alpha_{x_0}$ can be made to coincide.

□

Coming back to the reduction of α described above near x_0 (before Lemma 2.1), we derive that after the modification of Lemma 2.1, df is zero at $(0,0)$. The coordinates x and y near x_0 in Σ can be chosen as we please for this reduction of α as long as $d\gamma = dx \wedge dy$. We thus can choose $\partial/\partial x = v_{x_0}$. We assume in the following Lemma that we have completed this choice. We then have:

Lemma 2.2. *α can be modified so that it reduces to $xdy + dz$ in coordinates where Σ is $\{z = 0\}$ near x_0, $d\alpha$, β, Σ are left unchanged through this modification.*

Proof. As we start the proof, we may assume by the arguments above that $\ker \alpha$ is tangent to Σ at x_0, that the equation of Σ near $x_0 = (0,0,0)$ is $\{z = 0\}$, that α reads locally as $xdy + df + dz$ and that $v$reads locally as $l\,(\partial/\partial x) + m\,(\partial/\partial y) + C\,(\partial/\partial z)$, with $l(0,0,0) = 1$, $m(0,0,0) = 0$, $C(0,0,0) = 0$. f is a function of (x,y) only, C^∞, equal to $O(x^2 + y^2)$ in the C^2-sense.

Let $g(x,y)$ be a C^∞-function equal to 1 near x_0 and equal to 0 outside of a small neighborhood of x_0. We can choose g so that $d(gf)$ is $O(|x| + |y|)$. Then, $\tilde{\alpha} = \alpha - d(gf)$ is a new contact form ($d(gf)$is small) such $d\tilde{\alpha} = d\alpha$ and $\tilde{\alpha} = xdy + dz$ near x_0. Keeping l and m unchanged, but adjusting C into $\tilde{C}$, we derive a new $\tilde{v}$ in $\ker \tilde{\alpha}$. Clearly, since $\partial/\partial z$ is characteristic for $d\alpha$, $\beta = d\alpha(v,.) = d\tilde{\alpha}(\tilde{v},\cdot) = \tilde{\beta}$. Thus, Σ is unchanged and Lemma 2.2 follows.

□

Σ divides M into two pieces, denoted $\Sigma^+(p(x) \geq 0)$ and $\Sigma^-(p(x) \leq 0)$.

The two lemmas established above will be used later to prove that a "composite" functional defined on a submanifold C_β, see below, of the Legendrian curves of β, and made of the action functional for the part of the curve in Σ^+ and of (more or less, the construction is more involved, see below) - the same action functional for the part of the curve in Σ^- is a C^2-functional at the periodic orbits of ξ with a second order Sturm-Liouville linearized operator of finite index.

We now consider a piece of orbit σ of the trace of β on Σ near a point x_0 where neither v nor ξ are tangent to Σ. Let S be a piece of curve drawn on Σ and transverse to this piece of orbit at x_0. $\ker \alpha_{x_0}$ is not tangent to Σ at x_0. α reads in local coordinates $-x_0 = (0,0,0)$-as $xdy + df + dz$ where f is a function of (x,y). Σ reads as $z = 0$. v reads as $l\,(\partial/\partial x) + m\,(\partial/\partial y) + C\,(\partial/\partial z)$. Assume that the equations of S are in these coordinates:

$$\delta(x,y) = 0\ , \quad z = 0\ , \quad |x| + |y| \leq \nu\ ,$$

ν above is a small positive real, δ is a smooth function which vanishes at $(0,0)$. We may assume, without loss of generality, that its gradient is bounded away from zero on the domain of S. $\ker \beta$ is spanned by $\xi = \partial/\partial z$ and v so that its trace on Σ is spanned by $l\,(\partial/\partial x) + m\,(\partial/\partial y)$. Transversality to S reads:

$$l(x,y,0)\,\delta'_x + m(x,y,0)\delta'_y = a_1 \neq 0$$

if

$$\delta(x,y) = 0\ .$$

Let $\phi = \phi(x,y)$ be a C^∞ -function valued into $[0,1]$, equal to 1 on S and to zero outside of a small neighborhood of S in Σ. Picking up a value $\varepsilon \gneq 0$, we may ask that $\phi(x,y)$ vanishes if $|\delta(x,y)| \geq \varepsilon^2$ and also that

$$\nabla(\phi\delta) = O(1)\ .$$

Let $w = w(z)$ be also a C^∞-function valued into $[0,1]$, with $w(0) = 1$ such that $\nabla(zw(z)) = O(1)$. If ε is small enough,

$$\delta\phi zw\ , \quad (\delta\phi zw)_z\ , \quad (\delta\phi zw)_{zz}$$

are $o(1)$. Thus

$$\nabla(\phi\delta zw) = o(1)\ .$$

We will actually use only the fact that the above expression is O(1). Let now:

$$\underline{w} = -m\,\frac{\partial}{\partial x} + l\,\frac{\partial}{\partial y} + C_1\,\frac{\partial}{\partial z}\ ,$$

$$\underline{w} \in \ker \alpha$$

$$l^2 + m^2$$

is normalized to be 1.

We then have the following Lemma and Corollary:

Lemma 2.3 ([1]). *There exists a C^∞-bounded function $c(x,y)$, with geometric bounds dependent only on a_1, such that if v is replaced by $\tilde{v} = v + \theta\underline{w}$, with $\theta(x,y,z) = c(x,y)\delta(x,y)\phi(x,y)zw(z)$, then Σ is replaced by $\tilde{\Sigma}$, p is replaced by $\tilde{p}$, β by $\tilde{\beta}$ with the following properties which hold as ε and the size of the supports of w, ϕ tends to zero :*

(i) $\tilde{\Sigma}$ is C^0-close to $\Sigma, \tilde{p}$ is equal to p outside of a small neighborhood of S and $d\tilde{p}(\xi)$ is also bounded away from zero in this small neighborhood.

(ii) *S is contained in $\tilde\Sigma$ and $d\tilde p(v+\theta \underline{w})=0$ on S.*
(iii) *$\tilde\beta$ and $\tilde d\beta$ are C^0-close to β, $d\beta$ respectively.*
(vi) *The trajectories of $\ker\tilde\beta\cap\tilde\Sigma$ and the trajectories of $\ker\beta\cap\Sigma$ have C^1-close differential equations in the (x,y)-coordinates and are therefore, after reparametrization, C^0-close in the z-variable and C^1-close in the (x,y)-variables (depending on the initial conditions).*

Corollary 2.1 ([1])**.** *Given $\nu \gneq 0$, there exists a C^1-small perturbation of v, in the vicinity of Σ, such that, for the new v, every trajectory of $\ker\beta\cap T\Sigma$ of length larger than ν crosses the set $\{x\in\Sigma\ ,\ p_v(x)=0\}$ of points where v is tangent to Σ. The lines where ξ is tangent to Σ are not perturbed through this modification. The transport equations along ξ and v are only slightly perturbed in the C^0-sense.*

Proof of Lemma 2.3 and Corollary 2.1. Computing $d\alpha(\tilde v,\,[\xi,\tilde v])$, we find

$$dx\wedge dy\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}-\theta m\frac{\partial}{\partial x}+\theta\, l\frac{\partial}{\partial y}, l_z\frac{\partial}{\partial x}+m_z\frac{\partial}{\partial y}-(\theta m)_z\frac{\partial}{\partial x}+(\theta l)_z\frac{\partial}{\partial y}\right)=$$

$$=(lm_z-ml_z)(1+\theta^2)+\theta_z\ .$$

Since $\theta=\theta_z=0$ on S, S is contained in $\tilde\Sigma$. Computing $d\tilde p(v+\theta\underline{w})$ wherever $\theta=0$, we find:

$$l((lm_z-ml_z)(1+\theta^2)+\theta_z)_x+m((lm_z-ml_z)(1+\theta^2)+\theta_z)_y+C((lm_z-ml_z)(1+\theta^2)+\theta_z)_z$$

$\theta,\theta_z,\theta_{zz}$ are zero on S, this expression rereads wherever these quantities are zero:

$$l\,\theta_{zx}+m\,\theta_{zy}-h(x,y)$$

where $h(x,y)$ is a known C^∞-function of (x,y). $\delta(x,y)$ is zero on S. Wherever δ , θ, θ_z, θ_{zz} are zero, this rereads:

$$c(x,y)(l(x,y,0)\delta_x+m(x,y,0)\delta_y)-h(x,y)\ .$$

Setting this expression to be equal to zero, we derive a value for the function $c(x,y)$ wherever $l(x,y,0)\delta_x+m(x,y,0)\delta_y$ is non zero. Our assumption is that this expression is bounded away from zero in a neighborhood of S so that we find a smooth function $c(x,y)$ with bounds depending only on a_1 see above.

Coming back to $\tilde p$, now that $c(x,y)$ has been defined with the related bounds, we find:

$$d\tilde p\left(\frac{\partial}{\partial z}\right)=(ml_z-lm_z)_z+o(1)\ .$$

Since $dp(\partial/\partial z)$ is bounded away from zero near $(0,0,0)$, the statement about $\Sigma,\tilde\Sigma$ follows.

β is equal to $dx\wedge dy(v,\cdot)=ldy-mdx$. The new β, denoted $\tilde\beta$ is equal to $(l-\theta m)dy-(m+\theta l)dx$. $\theta,d\theta$ are both $o(1)$, so that $\tilde\beta$ is C^1-close to β.

A vector X in $\ker\tilde\beta$ reads

$$X=A(l-\theta m)\frac{\partial}{\partial x}+A(m+\theta l)\frac{\partial}{\partial y}+B\,\frac{\partial}{\partial z}\ .$$

We normalize A and B so that $A^2+B^2=1$. We claim that $B/A=O(1)$ if X is in addition tangent to $\tilde\Sigma$ near x_0. Indeed, for such an X, would $A/B=o(1)$, then

$$\frac{1}{B}\,d\tilde p(X)=-((lm_z-ml_z)(1+\theta^2)+\theta_z)_{\frac{\partial}{\partial z}+o(1)\frac{\partial}{\partial x}+o(1)\frac{\partial}{\partial y}}=\tilde pz+o(1)$$

since $\nabla(\theta z) = O(1)$ ($\theta_{zz} = o(1), \nabla(zw(z)) = O(1), \nabla(\phi\delta) = O(1)$). Since $\tilde{p}_z$ is bounded away from zero on $\tilde{\Sigma}$, this would yield a contradiction.

We then consider the orbits on $\tilde{\Sigma}$ of $Y = X/A$. The z-component does not matter because we know that z remains small on $\tilde{\Sigma}$. The (x, y)-differential equations corresponding to these orbits are clearly C^1-perturbations of the same equations on Σ (for the trace of β) since the function θ is small at least in the C^1-sense. Furthermore, computing $d\delta(X)$, we find

$$d\delta(X) = A((l - \theta m)\delta_x + (m + \theta l)\delta y) = A(l\delta_x + m\delta_y) + Ao(1)\ .$$

Since $l\delta_x + m\delta_y$ is bounded away from zero near S, $|d\delta(X)| \geq c|A|$, where c is a fixed positive constant. The proof of Lemma 2.3 is now complete.

The proof of Corollary 2.1 is now straightforward, it suffices to choose a family of S's scattered on Σ and repeat the above construction for each of them. The uniform estimate from below provided above on $|d\delta(X)|$ as well as the fact that the trajectories of the new trace of $\tilde{\beta}$ on $\tilde{\Sigma}$ are only C^0-perturbations of the trajectories of the trace of β on Σ yield the first claim of the corollary. For the transport equations, we observe that α is unchanged, so that $(d/dt)\alpha(z) - d\alpha(\dot{x}, z)$ is unchanged. $\gamma_1 = -d\alpha(w, \cdot)$, where w satisfies $d\alpha(v, w) = 1$ is unchanged, so that $(d/dt)\gamma_1(z) - d\gamma_1(\dot{x}, z)$ is unchanged. Finally, β is changed, but the perturbation is C^1 small so that the differential equation $(d/dt)\beta(z) - d\beta(\dot{x}, z)$ is only slightly perturbed.

Using coordinates, the representation $z = \lambda\xi + \mu v + \eta w$ changes in that v is changed into $\tilde{v} = v + \theta w$, ξ and w are unchanged, so that the value of the coordinate η changes. However, the differential equations on the various components change little formally: the two first ones, which involve α, $d\alpha$ and γ_1, $d\gamma_1$ do not change formally, only the value of η changes in them. The last one, involving β, $d\beta$ reads in coordinates:

$$\dot{\eta} = d\beta(\dot{x}, w)\eta + (\mu a - \lambda b)p\ .$$

The value of η is changed, p is changed into $\tilde{p}$ which is C^0-close to the least, β, $d\beta$ are replaced by $\tilde{\beta}$, $d\tilde{\beta}$ which are C^1-close. Up to these changes, the equation is formally unchanged.

□

3. ξ-crossings of Σ

Following [1], we define on the space of curves C_β the functional $J(x)$ equal to:

$$J(x) = \int_{\Sigma^+} \alpha_x(\dot{x}) + \frac{1}{1 + \int_{\Sigma^-} \alpha_x(\dot{x})}\ .$$

This functional is obviously discontinuous and even not well-defined. Indeed, if a curve x of C_β has a piece on Σ the above formula does not allow us to compute the value of the functional on it, it is multivalued. We have discussed this problem in [1], but here, below, we will make the approach much more precise and show that it indeed provides full generality.

For the moment being, we will ignore this issue and we will discuss the differentiability of J at a curve x containing pieces of ξ-orbits crossing Σ at various points $x_0, \cdots, x_m$. We assume that x does not cross Σ at any other point and that all these crossings are transversal. Following Lemma 2.1 and Lemma 2.2, we may assume that $\ker\alpha$ is tangent to Σ at each of the crossing points and that α can be locally reduced to $xdy + dz$ in coordinates where Σ reads locally as $z = 0$ and v at each x_i is $\partial/\partial x$. Accordingly, we may assume that w and $\underline{w}$ at each x_i are equal to $\partial/\partial y$. We then claim that:

Lemma 3.1. *J is C^2 at such a curve x. Furthermore, the second derivative can be represented along the ξ-pieces by a second order Sturm-Liouville operator of finite index. In case x is a periodic orbit, this second order operator L can be viewed as acting on a Hilbert space of periodic functions η equipped with the norm $(\int (L\eta,\eta)\, dt + M\int \eta^2)^{1/2}$, with M large enough.*

Proof. We know how to compute the differential of $\int \alpha_x(\dot{x})$ along a tangent vector v. It is equal, see [1], [2], to $\int (d/dt)\alpha(z) - d\alpha(\dot{x}, z)$. Usually, because of periodicity, the term $(d/dt)\alpha(z)$ disappears. Our functional is more complicated since it involves only parts of $\int \alpha_x(\dot{x})$, two of them: the contribution of the integrand on Σ^+ and the contribution on Σ^-. Accordingly, as we compute these first variations, we find a contribution of the trace of the variation z of the curve x onto $T\Sigma$ or Σ. This trace z' is obtained after combining z and $c\dot{x}$, after an appropriate choice of c at each x_i so that we obtain a vector out of z in $T_{x_i}\Sigma$. By assumption, $\dot{x}$ is equal to ξ at each x_i.

Observe that $\alpha(z') = 0$ since $T_{x_i}\Sigma = \ker \alpha_{x_i}$. The boundary term vanishes and the first variation of J at x becomes an appropriate combination of $-\int_{\Sigma^+} d\alpha(\dot{x}, z)$ and $\int_{\Sigma^-} d\alpha(\dot{x}, z)$. Curves x of C_β have a tangent vector $\dot{x}$ equal to $a\xi + bv$ because $\dot{x}$ is in the kernel of β (in addition, a is constant since $\alpha(\dot{x})$ is constant on C_β). Setting $z = \lambda\xi + \mu v + \eta w$, we find that $\int_{\Sigma^\pm} d\alpha(\dot{x}, z) = \int_{\Sigma^\pm} b\eta$. At the periodic orbits of x_i, b is zero. These periodic orbits are thus critical points for J.

Let us find out what is the value of the linearized operator at such curves $\tilde{x}$, at periodic orbits in particular. For this purpose, we have to come back to the formula for the first variation of J. For simplicity, let us assume that there are only two crossings of Σ, y_0 being a crossing from Σ^+ to Σ^- and $y_m = y_1$ being a crossing from Σ to Σ^+, we find that:

$$\partial J.z = (\alpha(z'_0) - \alpha(z'_1))\left(1 + \frac{1}{\left(1+\int_{\Sigma^-} a\right)^2}\right) - \int_{\Sigma^+} b\eta + \int_{\Sigma^-} b\eta \,\frac{1}{\left(1+\int_{\Sigma^-} a\right)^2}\,.$$

Near x_0, x_1, α is $xdy + dz$, $v(x_i) = \partial/\partial x$, $w(x_i) = \partial/\partial y$, $\Sigma = \{z = 0\}$. Set $z(y_i) = \lambda_i\xi + \mu_i v + \eta_i w$. The y_i's are in the vicinity of the x_i's. Then $\alpha(z'_i) = x(\eta_i + o(1))$ since z'_i is in $T\Sigma_{y_i}$. $o(1)$ near each x_i is a differentiable function of y_i. Taking the second variation of $\alpha(z'_i)$ and locating the points y_i's at the x_i's, $x_i = (0, 0, 0)$, we find that it is equal to $\mu_i\eta_i$.

On the other hand, $b = \gamma_1(\dot{\tilde{x}})$ so that its first variation is $(d/dt)\gamma_1(z) - d\gamma_1(\dot{\tilde{x}}, z)$. This is equal to $\dot{\mu} - (\mu a - \lambda b)d\gamma_1(\xi, v) - \eta d\gamma_1(\dot{\tilde{x}}, w) = D$. The linearization of $b\eta$, wherever b is zero yields $D\eta$. Since we are assuming that b is identically zero at each crossing at our base curve where we are trying to compute the linearized operator, $\int_{\Sigma^\pm} b\eta$ linearizes, near Σ to the least if $\tilde{x}$ is not an orbit of ξ and b is zero throughout, as $\int_{\Sigma^\pm} D\eta + R$. R stands for the contribution of the part of $\tilde{x}$ where b is not zero. Completing an integration by parts, we find that $\int_{\Sigma^+} D\eta = \mu_0\eta_0 - \mu_1\eta_1 - \int_{\Sigma^+}(\mu\dot{\eta} + \eta((\mu a - \lambda b)\, d\gamma_1(\xi, v) + \eta\, d\gamma_1(\dot{\tilde{x}}, w)))$.

A similar formula holds for Σ^-.

Summing up, we find:

$$\partial^2 J.z.z = \int_{\Sigma^+} (\mu\dot{\eta} + \eta((\mu a - \lambda b)\, d\gamma_1(\xi, v) + \eta\, d\gamma_1(\dot{\tilde{x}}, w))) -$$

$$- \frac{1}{\left(1+\int_{\Sigma^-} a\right)^2} \int_{\Sigma^-} (\mu\dot{\eta} + \eta((\mu a - \lambda b)\, d\gamma_1(\xi, v) + \eta\, d\gamma_1(\dot{\tilde{x}}, w))) + Q$$

Q is an additional term due to the contribution of the parts where b is not zero. At a periodic orbit, Q is zero.

Recalling now that the tangent space to C_β at $\tilde{x}$ reads:

$$\dot{\eta} = d\beta(\dot{\tilde{x}}, z)\eta + (\mu a - \lambda b)p \ ,$$

$$\dot{\lambda} = b\eta - \int_0^1 b\eta$$

we study our second derivative near a transverse crossing which we assume, without loss of generality, to happen at $t = 0$ and along a ξ-piece. Let us assume, again without loss of generality, that the curve is in Σ^+ for t small negative. Our focus is on the expression:

$$\int_0^t \mu(\dot{\eta} - A\eta)\, dx$$

where A is an appropriate function. There is a similar expression, with sign changes, for the contribution of Σ^-. Using the equation of the tangent space to C_β, we derive that, for x close to zero, since $b = 0$,

$$\mu = \frac{\dot{\eta} - ad\beta(\xi, w)\eta}{p}$$

p can be expanded in x. At first order, it reads $-cx$, with $c \lneq 0$. For the sake of simplicity, we will actually assume that it reads in this way in the vicinity of the crossing point. Our expression thus reads:

$$\int_0^t \frac{(\dot{\eta} - B\eta)(\dot{\eta} - A\eta)}{-cx}\, dx$$

where B is an appropriate function. Setting

$$x^2 = s$$

this rereads

$$2c^{-1} \int_0^{t^2} e^{\int_0^{\sqrt{s}}(A+B)} \left(\frac{d}{ds}\left(e^{-\int_0^{\sqrt{s}} B}\eta\right)\right)\left(\frac{d}{ds}\left(e^{-\int_0^{\sqrt{s}} A}\eta\right)\right)\, ds \ .$$

Introducing then:

$$y(s) = \int_0^s e^{\int_0^{\sqrt{z}}(A+B)}\, dz$$

so that:

$$dy = e^{\int_0^{\sqrt{s}}(A+B)}\, ds$$

and the inverse function is

$$s = s(y)$$

our expression rereads:

$$2c^{-1} \int_0^{y(t^2)} \left(\frac{d}{dy}\left(e^{-\int_0^{\sqrt{s(y)}} B}\eta\right)\right)\left(\frac{d}{dy}\left(e^{-\int_0^{\sqrt{s(y)}} A}\eta\right)\right)\, dy \ .$$

Let us prove that $A = B$. We identify A as $-ad\gamma_1(\xi, v)$ and B as $ad\beta(\xi, w)$. Since $\gamma_1 = d\alpha(\cdot, w)$,

$$A = a\gamma_1([\xi, v]) = ad\alpha([\xi, v], w) = -a\alpha([[\xi, v], w]) \ .$$

Since $\beta = d\alpha(v, \cdot)$,

$$B = -a\beta([\xi, w]) = -ad\alpha(v, [\xi, w]) = a\alpha([v, [v, w]]) \ .$$

Observe that $\alpha([v, w]) = -d\alpha(v, w) = -1$. Thus, $\alpha([\xi, [v, w]]) = -d\alpha(\xi, [v, w]) = 0$. On the other hand, the Jacobi identity tells us that:

$$[w, [\xi, v]] + [v, [w, \xi]] + [\xi, [v, w]] = 0 \ .$$

Thus,

$$\alpha([[\xi, v], w]) = -\alpha([v, [\xi, w]])$$

and $A = B$ as claimed.

The expression that we were studying thus reads:

$$2c^{-1} \int_0^{y(t^2)} \left(\frac{d}{dy} \left(e^{-\int_0^{\sqrt{s(y)}} A} \eta \right) \right) \left(\frac{d}{dy} \left(e^{-\int_0^{\sqrt{s(y)}} A} \eta \right) \right) \ dy \ .$$

The remainder of the second derivative near 0 reads as (with C an appropriate smooth function):

$$\int_0^t C\eta^2 = \frac{1}{2} \int_0^{t^2} C\eta^2 \ \frac{ds}{\sqrt{s}} \ .$$

□

4. Critical curves with pieces on Σ

Given a curve in C_β, we can use a local decreasing pseudogradient for the portion of curve in the interior of Σ^+ and also a local decreasing pseudogradient for the portion of curve in the interior of Σ^-. In Σ^+, the pseudogradient follows the construction of [1], [2], when β was assumed to be a contact form with the same orientation than α, a hypothesis which is gone here. However, these techniques apply in the interior of Σ^+ since this assumption holds in this open set. The same techniques apply in the interior of Σ^-, but for the functional $1/\left(1 + \int_{\Sigma^-} a\right)$ since now β has the wrong orientation. This can be checked directly, when β is a contact form with the wrong orientation, normalized so that $\beta \wedge d\beta = -\alpha \wedge d\alpha$, the "natural" regularizing pseudogradient, which has $\eta = b$ when the orientation is right see [1], [2], has now $\eta = -b$ with the same geometric properties. Only that this is not anymore a decreasing pseudogradient for $\int \alpha(\dot{x})$, but rather an increasing pseudogradient for this functional. As in [1], [2], this "natural" pseudogradient generates a semi-flow for short times. It has to be modified in order to turn it into a global flow. Except for this change in functionals, the construction of [1], [2], extends verbatim to this new framework. The issue of the verification of the Palais-Smale condition for this modified functional has been discussed, with positive answers under reasonable assumptions in [1], Chapter V.1, where a large part of the results of the present work are outlined (up to a few irrelevant misprints).

After these pseudogradients are defined and used, the extremal curves which we find might have pieces on Σ. Then, the functional J is not defined. We have followed in [4] the direction of [1], Chapter V.1, and we have established that the "functional", which is now multivalued still admits a decreasing pseudogradient in some generalized sense and that such curves can be disregarded. Full generality for the techniques of [1], [2], [3] follows.

We recall here, for the sake of the completeness of this paper and for further reference, the main steps in the understanding of the "critical curves" of C_β which have sub-pieces lying on Σ.

We start this analysis by observing that, even though the functional J is multi-valued whenever a piece of curve lies on Σ, there is a top and a bottom value for it, derived after considering that this piece of curve is in Σ^+ for the top value, we will call it J^+, and considering that this piece of curve is in Σ^- for the bottom value which we call J^-. There are also intermediate values obtained if we consider only part of the piece of curve to be in Σ^+. It is important to observe that the larger the part of the piece we consider to be in Σ^-, the lesser the value obtained.

There could be several such pieces of curve on Σ, we will have to consider this fact in our arguments. For the sake of simplicity, we assume that there is only one such piece for the moment being.

J^+ is strictly larger than J^-. There is a gap between these two values and we are going to exploit this fact: built in this decrease, there is a pseudogradient Z, obtained by "pushing the piece of curve "downwards"", from Σ^+ into Σ^-. Furthermore, "on each side", there is a decreasing pseudogradient, for J^+ and J^- respectively and both these pseudogradients are compatible, convex-combine with Z to yield a decreasing pseudogradient on both sides.

We need to make this constructions explicit. This requires that the equations of the tangent space to C_β be made explicit:

Lemma 4.1. *Let x be a curve of C_β with $\dot{x} = a\xi + bv$. Let z be a tangent vector to C_β at x, $z = \lambda\xi + \mu v + \eta w$. Then, λ, μ, η are periodic functions which verify:*

$$\eta{=}d\beta(\dot{x}, z)\eta + (\mu a - \lambda b)p \ ,$$

$$\dot{\lambda} = b\eta - \int_0^1 b\eta \ .$$

There is a "sloppy" way to use the two equations displayed above and construct tangent vectors z: given a function ϕ, which we should think of as the quantity: $\mu a - \lambda b$, η can be computed using the first equation about $\dot{\eta}$. Periodicity might require an integral condition on ϕ if $\int_0^1 d\beta(\dot{x}, z) = 0$. Once η is computed, λ can be derived from the second equation, up to a constant of integration. Observe then that a is constant, so that $\mu = (\phi + \lambda b)/a$. This does not respect regularity issues, but can be made to work.

Proof. These equations follow in a straightforward way from the requirements on z that the first variation of $\beta(\dot{x})$ along z be zero and that the first variation of $\alpha(\dot{x})$ be time independent. This yields:

$$\frac{d}{dt}\beta(z) - d\beta(\dot{x}, z) = 0$$

$$\frac{d}{dt}\alpha(z) - d\alpha(\dot{x}, z) = \int_0^1 d\alpha(\dot{x}, z) \ .$$

The above equations yield Lemma 4.1 after z is replaced by $z = \lambda\xi + \mu v + \eta w$. □

Consider the first equation of the tangent space, suppose that some piece of the curve x is on Σ, so that p is identically zero for some time interval $[a_1, b_1]$. Then, the first equation does not yield any constraint on (λ, μ). Once η is found, the second equation allows to compute λ up to a constant of integration. $\mu = (\phi + \lambda b)/a$ is free on $[a_1, b_1]$, as free as ϕ is on this interval. There is no issue of regularity here because $a\xi + bv$ represents on this time interval the trace of β on Σ , it is smooth. The only issue is the extension of this "deformation", represented for the moment being by a single tangent vector on a single (piece of) curve to an appropriate neighborhood of an appropriate set.

What appropriate set should we consider?:

Given $\varepsilon \gtrsim 0$, we can extremize the contribution to $J(x)$, whether this functional is well-defined or not, of the parts of the curve x which are in $\Sigma_\varepsilon^\pm = \{y \in M \ , \ |p(y)| \geq \varepsilon\}$. Indeed, in this set, the H_0^1-flow of [1] and [2] is available. This is obvious in Σ_ε^+ since in this set β and α have the same orientation. In Σ_ε^-, β defines the reverse orientation when compared to α, but the functional is also modified from $\int a$ to $1/(1+\int a)$. The conclusion stays the same. Small ξ-pieces are minima for the respective definitions of the functional in both cases, this is in fact the basic phenomenon.

Extremization yields, for each connected component of x in $\Sigma_\varepsilon^\pm$, a sequence of consecutive ξ and $\pm v$-pieces. If one of these v-pieces is small-that is if it is smaller than a given fixed positive constant θ_0. θ_0 in the sequel will be taken small - then it must connect, on one side or on the other side, with $\Sigma_\varepsilon^\pm$. Otherwise, we can use the flow of [1], [2] and decrease the functional. Since, with the flow of [1], [2], $\int_0^1 |b|$ is bounded depending on the initial condition, the number of the $\pm v$-pieces of size θ_0 to the least is bounded above, this yields a bound also on the number of ξ-pieces, so that the pieces of curve outside of $\Sigma_\varepsilon^\pm$ follow a precise pattern which we can modelize. There is, in addition, only a finite number of them which contain a ξ or a $\pm v$-piece of size θ_0 to the least. For all the other ones, the $\pm v$-pieces are "small", thus are at most two in number, with at most two "small" ξ-pieces as well. As ε tends to zero, the model remains unchanged, the set of "extrema" for our H_0^1 -flows defines a stratified set combining pieces of curves on Σ with small pieces of curves made of at most two ξ-pieces and at most two $\pm v$-pieces and with larger pieces made of a finite number of ξ and $\pm v$-pieces, all the $\pm v$-pieces being "sizable".

We will establish later, at the expense of a much more refined analysis, that these critical curves are subject to further conditions and that they can be assumed for this reason to be isolated, at least generically. Since this involves a deeper understanding of the extremal curves, we sidestep the related analysis at this point and proceed with the remainder of the proof.

We focus our attention on a simpler situation where an extremal curve x contains a small ξ-piece in Σ^+, preceded by a $\pm v$-piece and abuting at Σ. The curve continues then as a trajectory of $\ker \beta \cap T\Sigma$ for a while. As usual $\dot{x} = a\xi + bv$. We will assume that b does not have zeros on this portion of the critical curve and that ξ is never tangent to Σ over the trajectory on Σ. Since $p_\xi = \partial p / \partial \xi$ is negative at the beginning of the trajectory on Σ, we conclude also that bp_v is positive along this trajectory. For the sake of simplicity, we will assume that the first ξ-piece is parametrized by the time interval $[0, s_0]$ and the trajectory on Σ by the time interval $[s_0, s_1]$. We then have the following key result:

Lemma 4.2. *Either v rotates more than $\pi/2$ in the ξ-transport over the small starting ξ-piece or a negative direction can be built, using this portion of curve (ξ-piece, followed by the trajectory on Σ for the second derivative $\partial^2 J^-(x)$.*

Proof. We have made explicit above the equations of the tangent space to C_β. The first of these equations takes a special form at x along the time interval $[s_0, s_1]$ since $p = 0$ on this portion of the curve. This equation then reads:

$$\dot{\eta} = d\beta(\dot{x}, w)\eta$$

λ has to satisfy the second equation. $(\lambda, \eta) = (0, 0)$ satisfy both equations and μ, which is taken to be zero outside of $[s_0, s_1]$, is left arbitrary on this time interval. We choose for μ a non zero function ϕ, with compact support in (s_0, s_1). Assume for simplicity that b is positive on this portion of curve, so that p_v is positive. We then take ϕ to be negative, $\phi(s_0) = \phi(s_1) = 0$. $(\lambda, \mu, \eta) = (0, \phi, 0)$ defines a tangent vector Z_0 at x. Because ϕ is negative, ϕp_v is negative and Z_0 pushes the portion of x which lies on Σ into Σ^-. We extend Z_0 along "$x + \varepsilon Z_0$" so that we can compute the second derivative of J^- at x along Z_0. Observe that, since η and λ for Z_0 are zero, $\partial J^-(x)$. Z_0 is zero. We come back to the first equation of the tangent space to C_β, we differentiate it along Z_0 and we write the result at (x, Z_0). We find:

$$\dot{(Z_0.\eta)} = Z_0.(d\beta(\dot{x}, w))\eta + d\beta(\dot{x}, w)Z_0.\eta + ((Z_0.(a\mu - b\lambda))p + (a\mu - \lambda b)(Z_0.p)\ .$$

Since $\eta = \lambda = 0$ at (x, Z_0) and since $\mu = \phi$ has support where $p = 0$, this becomes:

$$\dot{(Z_0.\eta)} = d\beta(\dot{x}, w)Z_0.\eta + a\phi(Z_0.p) = d\beta(\dot{x}, w)Z_0.\eta + a\phi^2 p_v\ .$$

The equation holds on the interval $[s_0, s_1]$. Setting $Z_0.\eta(s_1)$ to be zero, we find:

$$Z_0.\eta(s_0) = -a\int_{s_0}^{s_1} e^{\int_{s_0}^{s_1} d\beta(\dot{x}, w)}\phi^2 p_v\ .$$

Thus $Z_0.\eta(s_0)$ is negative. $Z_0.\eta$ is now defined on $[s_0, s_1]$, we need to extend it outside of this interval. The main issue now is to define $Z_0.\mu$ in an appropriate way so that the extension of $Z_0\eta$ becomes possible.

We consider for this the ξ-piece of x abutting on Σ. It starts at time zero,exactly where the previous v-piece ends. This is for us the time $s = 0$. We consider the tangent vector $-C_v$ at this point, C positive and transport it, via ξ-transport, along this ξ-piece. We derive at s_0 a tangent vector (to M) which is in $\ker\alpha$, thus reads $\ell v + mw$, m nonzero. If v rotates less than $\pi/2$ along this ξ-piece in this ξ-transport, then m is negative and so is ℓ. Indeed, the transported vector is in $\ker\alpha$, has thus no λ-component and consequently its (v, w)-components (c, d) satisfy:

$$\dot{d} = d\beta(\xi, w)d + cp\ .$$

We assume in the sequel that the v-rotation along this ξ-piece is less than $\pi/2$. p is positive on this ξ-piece by assumption, c does not change sign then. It is therefore negative on $[0, s_0]$. Since d is zero at $s = 0$, $d = m$ is negative at s_1, just as $Z_0.\eta(s_0)$ is. Thus, ℓ and m are negative.

We now choose C appropriately so that $m = Z_0.\eta(s_0)$. We also choose $Z_0.\mu$ on $[s_0, s_1]$ so that $Z_0.\mu(s_0) = \ell$. $Z_0.\mu$ is taken to be zero identically near s_1. Then, $Z_0.\mu v + Z_0.\eta$ at s_0 matches $\ell v + mw$ and $Z_0.\mu$ as well as $Z_0.\eta$ find a natural extension in c and d respectively. Because we may assume that λ, μ, η are zero on $[0, s_0]$, this extension may indeed be viewed as an extension of a vector Z "tangent" to C_β^+ .

This construction, which will be in the sequel made even more precise, depends very little on the choice of w as long as w is in $\ker\alpha$ and satisfies that $d\alpha(v, w) = 1$.

In the above construction, we now impose a particular choice of w. Namely, we ask that w at s_0 should be along $\ker\alpha_{x(s_0)} \cap T_{x(s_0)}\Sigma$. This is possible since v

cannot be tangent to Σ at this point (the trajectory which is on x and which starts at $x(s_0)$ and runs on Σ has $\dot{x} = a\xi + bv$).

Which this particular choice of w, we seek A so that $A\xi(x(s_0)) + \ell v(x(s_0)) + mw(x(s_0))$ is tangent to Σ. Since p_w is zero at this point, this reads:

$$(Ap_\xi + \ell p_v)(x_{s_0}) = 0 \ .$$

This yields:

$$A = -\ell \, \frac{p_v}{p_\xi}(x(s_0)) \ .$$

We know that p_v is positive, p_ξ is negative, $-\ell$ is positive at $x(s_0)$. Thus A is negative.

We need to define $Z.\lambda$ and to check that the "germ" of vector-field thus defined "pushes" then the curve inside Σ^-. We set $Z.\lambda(s_1) = 0$ so that the vector Z as well as $Z.Z$ are zero at s_1. The vector Z pushes inside Σ^- all along (s_0, s_1), so we do not have to worry about the effect of $Z.Z$ along this open interval (as well as at s_1). On the other hand we have ($bZ.\eta$ is zero outside of (s_0, s_1)):

$$\dot{Z.\lambda} = bZ.\eta - \int_0^1 bZ.\eta = bZ.\eta - \int_{s_0}^{s_1} bZ.\eta$$

Setting $Z.\lambda(s_1) = 0$, we derive that $Z.\lambda(s_0) = (s_1 - s_0 - 1) \int_{s_0}^{s_1} bZ.\eta$. Since $bZ.\eta$ is negative on this interval, $Z.\lambda(s_0)$ is positive. Then, we have:

$$Z(s_0) = 0$$

$$(Z.\lambda\xi + .\mu v + Z.\eta w)(s_0) = Z.\lambda(s_0)\xi + \ell v + mw \ .$$

We know that p_ξ is negative, p_v is positive while ℓ is negative and p_w is zero at s_0, so that the vector $(Z.Z)(s_0)$ points inside Σ^- and or argument can proceed.

□

The above result extends readily to the symmetric case, when a piece of ξ-orbit instead of abutting at Σ starts at Σ to continue in Σ^+.

In the remaining cases, there must be a v-piece (assuming b is positive) starting or abutting to Σ in Σ^-, after or before (respectively) the piece of curve on Σ. Because the cases are symmetric and parallel arguments can be used which lead to the same conclusion, we can assume that the v-piece of orbit in Σ^- starts from Σ. We can follow this v-piece of orbit as it travels in Σ^-. It might then either stop in Σ^- and be replaced by a piece of ξ-orbit (case 1); or it might hit Σ and continue into another trajectory of $\ker\beta \cap T\Sigma$ (case 2); or it might cross into Σ^+; the piece of v-orbit can stop in Σ^+ and be followed by a ξ-piece of orbit in Σ^+. This ξ-piece of orbit then gets farther from Σ because p_ξ is positive. Would it continue as a ξ-piece, this piece of curve would then be large while we are assuming here -we will justify this later- that all these pieces of curve are small (much smaller than the initial trajectory on $\ker\beta \cap T\Sigma$). Thus, it must transform into a v-piece of orbit which has again to be small. The curve cannot become a ξ-piece again in Σ^+, it would not be critical, see below. It must then hit Σ again. This configuration builds case 3.

The last case is when the v-piece of orbit out of Σ^-, once it has entered in Σ^+ can also continue as a piece of v-orbit without transforming into a ξ-piece of orbit. Because it has to be small as we will see, it has to hit Σ again. A second piece of trajectory of $\ker\beta \cap T\Sigma$ can then follow, or the curve enters into Σ^- and is followed by a piece of ξ-orbit in Σ^- (case 4). No other case can occur because it would involve a higher vanishing of p_v which we can impede.

5. The functional J_∞ on $\bigcup \Gamma_{2k}$

We consider now the variational problem at infinity, that is the variational problem defined by the extension of the functional J from C_β to $\bigcup \Gamma_{2k}$. We have denoted this functional in our previous works J_∞.

While the arguments developed previously have brought us close to the conclusion that there were no critical point/critical point at infinity for J on C_β other than the "classical" ones and their immediate natural extension-excluding therefore the curves having pieces of them on Σ - we have not completely reached this conclusion. But this conclusion does hold for J_∞ on the stratified space $\bigcup \Gamma_{2k}$.

Observe that as long as v and ξ have no pieces of their orbits entirely contained into Σ, J_∞ is well-defined and continuous in $\bigcup \Gamma_{2k}$. We now claim that:

Theorem 5.1. *J_∞ has no critical point near the curves studied above.*

Proof of Theorem 5.1. The curves that the statement of Theorem 5.1 below is referring to are the curves studied above that have portions of them lying on Σ.

The claim of Theorem 5.1 stems from the various constructions of decreasing vector-fields at these curves which we have completed above.

Let us consider a curve of $\bigcup \Gamma_{2k}$, C^0-close to one of these "critical" curves having a piece on Σ. Such a curve has a sequence of ξ and $\pm v$-jumps close to a piece of curve on Σ. We can consider a sub-piece along which b does not change sign, hence the orientation of the $\pm v$-jumps of the curve in $\bigcup \Gamma_{2k}$ essentially does not change, since the curves have to be C^0-close.

If this sequence of ξ and $\pm v$-jumps stays in Σ^+, or in Σ^- for that matter, then the curve cannot be critical because the ξ-pieces are typically small, hence not characteristic, and the $\pm v$-jumps are small, staying in Σ^+ or Σ^-, hence cannot be jumps between conjugate points [1]. We can refine the argument and claim the ξ-pieces of this sequence cannot stay on one side of Σ, because then the $\pm v$-jumps would have to take place between coincidence points [1], [6] and they are too small for that.

We thus consider sequences having ξ-pieces crossing Σ. We may assume, without loss of generality, that b is positive, that is "most" of the $\pm v$-jumps are oriented along v, with p_v positive and p_ξ negative. We develop our arguments under these assumptions, but the generalizations to the other cases is straightforward.

Let us thus consider a small ξ-piece crossing Σ, from Σ^+ into Σ^-. It is preceded and followed by $\pm v$-jumps. At the point x_1 of intersection with Σ, $-v$ is directed inwards Σ^-. We thus consider $-v$ at the ending edge of the preceding $\pm v$-jump-which is the starting edge of our ξ-piece- and we ξ-transport along our ξ-piece, which is small. The transported vector remains close to $-v$, up to $o(1)$. At x_1, it reads $\mu_1 v + \eta_1 w$, with $\eta_1 = o(1)$, $\mu_1 + 1 = o(1)$. This vector projects, parallel to ξ on $T_{x_1}\Sigma$, into $\lambda_1 \xi + \mu_1 v + \eta_1 w$ and since $\mu_1 + 1 = o(1)$, $\eta_1 = o(1)$, $\lambda_1 = (p_v/p_\xi) + o(1)$. It follows that λ_1 is negative, far from zero, while the w-component, η of this transported vector is $o(1)$ throughout.

We continue our journey with our transported vector, having adjusted, by adjusting the length of the ξ-piece, its ξ-component at the end of this first ξ-piece to be zero, and its v-component to be zero also using the next $\pm v$-jump (at the end of this next $\pm v$-jump). Now, our vector is $o(1)$ and we continue transporting it (we will enter, before each crossing of Σ, adjustments as above, but they will keep the vector to be $o(1)$) until we reach larger portions of the curve, larger ξ and $\pm v$-jumps corresponding to portions of the curve that get "far" from Σ (we are assuming that the limiting curve is not entirely drawn on Σ. Such curves that are entirely drawn on Σ, being in C_β, do not necessarily exist. Furthermore, would they exist, they can be considered to be entirely in Σ^- and therefore J at these curves is bounded by 1).

Now, we claim that we can use our transported vector and "compensate" it by another transported vector outside of the support where our vector has been built up to now, a vector that is $o(1)$, so that the resulting total vector will be a tangent vector to our initial curve in $\bigcup \Gamma_{2k}$.

Indeed, if any of these longer ξ-pieces is not "characteristic" (now characteristic pieces need not be long or short if they cross Σ), we can use it to transport v from one of its edges to the other one, the transported vector will have a non-zero w -component at the other end. We can use such a vector now to "compensate" the external data in our former construction which was $o(1)$. Similarly, if along one of these longer $\pm v$- jumps, ξ is not transported parallel to itself, we can create such an "external" data and "compensate" our former construction.

It follows that, if "compensation" cannot be completed, starting from a given point of Σ, the "external curve" is completely determined. It has then along its last leg, ξ-piece or $\pm v$-jump, to hit again Σ and this last leg has to behave as a ξ-characteristic piece or a $\pm v$-jump across which v is transported parallel to itself. This imposes a condition on the initial point on Σ, which is therefore constrained to live on a one dimensional stratified subset of Σ. Setting the fixed point problem from this subset to itself, through the history of the curve, we find that these curves, these curves of C_β, are isolated, in finite number if the number of their ξ, $\pm v$ and portions on Σ is a priori bounded. We can define a Poincare-return map for this curve and construct tangent vectors, which are tangent to this curve as a curve of C_β. This Poincare-return map can be assumed to have both of its eigenvalues distinct from 1. Along the portions of the curve on Σ, the λ, μ -values, that is the values of the ξ, v-components of the vector which is transported along this Poincare-return map are irrelevant. Furthermore, ξ, v are transported parallel to themselves outside of these portions; we can therefore adjust outside of the portions lying on Σ the values of these components freely. The invertibility of the $dl - Id$, where l is the Poincare-return map will extend to curves of $\bigcup \Gamma_{2k}$ which are C^0-close. We will use this fact below and we will show below that, if the construction of z as above cannot be completed , then some other tangent vector can be built using this Poincare-return map and J will decrease along this tangent direction.

In order to complete the proof of our claim, we thus need now, in a first step, to establish that, if the process of "compensation" can be completed, then J decreases along the resulting tangent vector z.

The basic fact is the following: at the first crossing point x_1, the trace of the transported vector on $T_{x_1}\Sigma$ has a negative ξ-component, far from zero. In the variation of J/J_∞, the contribution of this component to $\partial J.z$ is then negative, bounded away from zero, as all our previous computations show. Let us write the transport equations along our piece of curve in $\bigcup \Gamma_{2k}$. We can express them by saying that the components of a vector which is transported along our curve do not change in the ξ, v, w-frame along z. Using α, β and $\gamma = d\alpha(\cdot, w)$, we find that the components of $z_1 = \lambda_1\xi + \mu_1 v + \eta_1 w$ satisfy the differential equations ($\dot{x} = a\xi + bv$, on our curve, the product $a.b = 0$):

$$\dot{\eta}_1 = d\beta(\dot{x}, w)\eta_1 + (\mu_1 a - \lambda_1 b)p, \dot{\lambda}_1 = b\eta_1, \dot{\mu}_1 = d\gamma(\dot{x}, \lambda_1\xi + \mu_1 v + \eta_1 w) .$$

After the second edge of our first ξ-piece, our vector $z = \lambda_1\xi + \mu_1 v + \eta_1 w$ is derived by transport of $\eta_0 w = o(1)w$ (the v-component is set to be zero at the end of the next $\pm v$-jump), taken at this edge and integrating the above transport equations along the curve, with initial condition $\lambda_0 = 0, \mu_0 = 0, \eta_0 = o(1)$. It is not difficult to see that these differential equations yield components λ, μ, η which are all $O(\eta_0) = o(1)$. η on the first ξ-piece is also $o(1)$. In fact, z is not really transported along our curve, we will be introducing a v-component $\mu = O(1)$ at some edges and a $\lambda = O(1)$-component at some other edges. Because μ, λ will remain $O(1)$, the first transport equation will imply that η is $o(1)$.

Besides the contribution of the first crossing, we have two other contributions of z in $\partial J.z$: one is $O(\int |b\eta|)$, the summation being taken over the support of z. Thus, this contribution is $o(1)$ because as we will see, despite our adjustments, η will remain $o(1)$. The other one is like the contribution of x_1 in $\partial J.z$, it is due to the various crossings and it could therefore be made of numerous contributions.

Each of these contributions is due to the ξ-component of the projection of z onto the tangent space to Σ at a crossing point. If the curve is crossing through a v-jump, the projection is along v. Otherwise, it is along ξ. Since η, λ, μ of the transported vector are $o(1)$ at the starting point (after the first crossing point, after adjustment at the starting edge of the next v-jump of the ξ-component and adjustment at the ending edge of the same $\pm v$-jump of the v-component), this ξ-component is also $o(1)$, but there could be many of them. We gather them by consecutive pairs of crossings, the first crossing at x_1 being left alone, and maybe, if needed for this grouping by pairs, one more where all data are $o(1)$. We will assume that all the $\pm v$-jumps are oriented along v. Since we can assume throughout our work an a priori bound on the number of zeros of b, we can limit ourselves to a portion of curve over which b does not change sign, e.g. positive. Crossings of Σ, when there are grouped in consecutive pairs, are then a v-crossing from Σ^- into Σ^+ followed by a ξ-crossing from Σ^+ into Σ^-. We could imagine that, after a crossing along ξ, the next v-piece of orbit does not cross into Σ^+. Then the next ξ-piece of orbit, if it is small, also stays in Σ^- and the curve is not critical.

We could disregard this fact and construct another argument which would be indifferent to the configuration where we would be assuming that a v-crossing on the curve is immediately followed by a ξ-crossing, occurring on the next ξ-piece of orbit. But this assumption makes the argument easier and therefore we will proceed with it.

Let us consider the $i^{th} - v$-crossing then and the consecutive ξ-crossing on the next ξ-piece. z at the v-crossing reads as $\lambda^-\xi + \mu^- + \eta^- w$. Its projection on the tangent space to Σ is derived by addition of a v-component. Thus λ^- is also the component of this projection on ξ. z is transported along v until the edge of the v-piece. At this edge, we are going to modify the v-component of z so that, at the next crossing, which is a ξ-crossing, the ξ-component of the projection of z on the tangent space of Σ is again λ^-. Similarly, we will modify, after the consecutive ξ-crossing, at the end of this ξ-piece, the ξ-component so that the ξ-component of the projection of z on the tangent space of Σ at the next (v)-crossing is again λ^-.

In this way, the λ's will compensate by pairs, but maybe for one, which is $o(1)$ (because λ^- does not change throughout and was at the beginning equal to $o(1)$), and except also for the first one, the contribution of which to $\partial J.z$ will imply that this quantity is negative.

Let us enter in more details into our modifications and our argument: Wouldn't we modify, after the $i^{th}v$-crossing, the μ component of the transported vector, at the ending edge of this v-piece, in order to define z, we would end up with a transported vector at the ξ-crossing equal to $\lambda^+\xi+\mu^+v+\eta^+w$. After ξ-adjustment, this vector projects on the tangent space to Σ at the crossing point into $-\left((\mu^+p_v+\eta^+p_w)/p_\xi\right)\xi$ $+\mu^+v+\eta^+w$. v, taken from the final edge of the previous v-jump and transported to the ξ-crossing, yields a vector equal to $\delta'v+\eta'w$, $\delta'=1+o(1)$, $\eta'=o(1)$. This vector projects onto (projection along ξ) $-\left((\delta'p_v+\eta'p_w)/p_\xi\right)\xi+\delta'v+\eta'w$. We thus scale v at this edge into $-\left[p_\xi/(\delta'p_v+\eta'p_w)\right]\left[\lambda^++(\mu^+p_v+\eta^+p_w)/p_\xi\right]v=\gamma v$. The projection of the adjusted z at the ξ-crossing of Σ will then read:

$$\lambda^-\xi+(\mu^++\gamma\delta')v+(\eta^++\gamma\eta')w\ .$$

The v-component does not matter, as long as it remains $O(1)$: it is going to be adjusted at the next edge. The w-component is $o(1)$ because η^+, η' are small, the ξ-component is small if λ^- is small.

Next, we consider this ξ-crossing and the next v-crossing, which is the $(i+1)^{th}$-crossing. We adjust the ξ-component of our vector at the end of the preceding ξ-piece so that the ξ-component of its projection at this v-crossing does not change, is equal to λ^-. This involves an adjustment in $\lambda=O(1)$.

We claim that the w-components will be $o(1)$ throughout this construction. λ^- is also $o(1)$ by construction. By looking at the transport equations displayed above, we remove the two last ones, related to the v, ξ-component λ, μ, since we are constantly violating them by the addition of $\gamma v,\theta\xi$ at these edges. But η obeys the first differential equation. The estimates are driven by the initial conditions, which are $o(1)$ and by the addition of the functions λ, μ. These functions, in our setting, are perturbed $O(1)$ at various edges. However, considering the projection of z onto the tangent space to Σ at each crossing point, we observe that its ξ-component is $\lambda^-=o(1)$, its w-component is $\eta=o(1)$ if $\lambda,\mu=O(1)$ throughout. Then the fact that this vector is tangent to Σ implies that its v-component is $o(1)$. Working with this vector at each step, we can see that, indeed, λ, μ remain $O(1)$. The differential transport equation driving η shows that η remains small throughout as long as p is small and the functions μ, λ are $O(1)$. Our claim follows for the tangent vector built by "compensating" z, z was initially built using the ξ-crossing of Σ.

If this compensation cannot take place, then as pointed out above, our limiting curve outside its portions on Σ is completely determined, its ξ-pieces are characteristic, its v-jumps (for the limiting curve) transport ξ parallel to itself. We know furthermore that it has a Poincare-return map with no eigenvalue equal to 1.

ξ is transported onto $\bar{\lambda}\xi$ along a v-jump. We can assume using genericity that $\bar{\lambda}\neq 1$ if the v-jump does not cross Σ. If we express criticality for J, we derive that $\bar{\lambda}$ should be equal to some other value derived from the precise form J. Using genericity again, we can assume that this does not happen, that is this limiting curve is not critical. We claim that the approaching curves from $\bigcup\Gamma_{2k}$ are not critical as well. This is clear if their broken portions that correspond to the portions of the limiting curve on Σ do not cross in the pattern discussed above, one v-crossing and the next ξ-crossing through the next ξ-piece.

If we are in this pattern, we have shown above how we can adjust the ξ- components of the consecutive projections of a "transported" vector z to be equal so that they compensate pairwise in the computation of $\partial J.z$, except for one or two ones where we will have to have $\lambda^- = o(1)$.

Let us consider our non-critical variation and try to extend to the approaching curves. On the limiting curve, the entire variation is taken at a single v-jump, picking ξ from e.g. its bottom and transporting to its top, getting $\bar{\lambda}\xi$, the entire variation is located near this v-jump. Consider the same variation on an approaching curve in $\bigcup \Gamma_{2k}$. Picking up ξ from the same bottom of the corresponding v-jump, transporting it along it, we find not $\bar{\lambda}\xi$, but $\bar{\lambda}\xi + o(1)$. We have to compensate this $o(1)$ and this can be done using the Poincare-return map of the approaching curve and its differential which are close to the corresponding ones on the limiting curve. The eigenvalues of the differential P of this Poincare-return map are in particular far from 1, $P - Id$ is invertible. Compensating this $o(1)$ using this Poincare-return map involves transporting a small vector throughout our curve, through the very broken portions, near Σ in particular. Because the vector is small, its -component is not relevant in the computation of $\partial J.z$, and a finite, a priori bounded number of Σ -crossings does not matter as well. But a larger number could have a significant contribution.

We have observed above that, for the limiting curve, the transport equations corresponding to the Poincare-return map along the portions of curve on Σ can be limited to the differential equation on η, the other ones being irrelevant.

Let us consider our curve now, more precisely its broken sub-piece close to Σ, start at the beginning of this sub-piece with a vector $\eta_0 w$ and complete the "transport" process described above, through the broken sub-piece. We can take λ^- to be zero, so that, at each crossing, the projected vector reads $\eta w + \mu v$, implying that $\mu = O(\eta)$. Along a ξ-trajectory, the η, μ -transport equations hold. Along a v-trajectory, the η, λ-transport equations hold. μ is only perturbed at the start of the ξ-piece, λ is only perturbed at the start of the v-piece. Thus, at each edge, there where a v-piece is to be followed by a ξ-piece, we can use the ξ-piece to estimate the η and the μ-components of our "transported" vector, starting from our projected vector, including at the edge and we can use the v-piece to estimate the η and λ-components of our "transported" vector, starting from the other projected vector. The "transported" vector is thereby controlled throughout. On the $i^{th}v$-piece followed by a ξ-piece, the vector is $O(|\eta_i^-| + |\eta_i^+|)$, where η_i^+, η_i^- are the two w-components of the "transported" vector.

On the other hand, because p is small, the differential equation $\dot{\eta} = d\beta(\dot{x}, w)\eta + (\mu a - \lambda b)p$, coupled with our estimate on the vector, $O(|\eta_i^-| + |\eta_i^+|)$, implies that

$$\eta_i^+ = e^{\int_{t_i^-}^{t_i^+} d\beta(\dot{x},w)} \left(1 + O\left(\int_{t_i^-}^{t_i^+} |p|\right)\right) \eta_i^- \ ,$$

where $t_i^{\pm}$ are the times of the two related consecutive crossings, the curve is parametrized along the normalized ξ, v. A similar estimate holds for η_i^+ , η_{i+1}^-. Combining, we find that, throughout the broken piece of curve, we have:

$$\eta(t) = e^{\int_0^t d\beta(\dot{x},w)} \left(1 + O\left(\int_0^t |p|\right)\right) \eta(0) \ .$$

This means, since $p = o(1)$ on this piece of curve, that $\eta(t)$ is very close to the corresponding solution on the limiting curve. This is what is needed for the compensation of the w-component of the tangent vector which we are building on this approaching curve, starting from this larger v-jump along which ξ is mapped onto $\bar{\lambda}\xi + o(1)$. Furthermore, because the $\lambda^{-}s$ are all zero, this piece of curve will only contribute $O(\eta(0))$ to $\partial J.z$.

Since on the limiting curve, $\partial J.z$ is negative and η is identically zero, the use of the Poincare-return (+ implicit function theorem in order to find the value needed for $\eta(0)$) will give us a value for $\eta(0) = o(\partial J.z)$. The ξ and v-components are adjusted at the exit of the piece(s) of curve lying on Σ: at each exit, if the curve continues with a ξ-piece, the v-component is adjusted and the ξ-component is adjusted later, either at a section, or at the next corner, if the curve continues with a v-piece of curve, the ξ-component is adjusted and the v-component is adjusted at the next corner. The transport then continues along the curve, until we reach again Σ and another piece of curve (otherwise, the transport ends into a section, along a ξ-piece, tangent to $\ker\alpha$ at the base point). At this next piece of curve, we repeat our construction. We get rid of the ξ, v-components using the two first corners; they are $o(1)$ and their contribution to $\partial J.z$ is $o(1)$. We travel along this piece of curve as above, changing λ, μ at corners as we go and as we exit, we recover using corners the values that they had through pure transport. In this way, our estimate on the η-component holds throughout the transport and λ, μ follow. This construction follows closely the transport equations on the limit curve, except possibly along the pieces of curves on Σ where the η -components are close and also possibly along one ξ-piece or one v-piece at each exit where (respectively) the λ or μ -components do not match. This does not effect in an essential way the value of the final vector in a section, neither does it a effect $\partial J.z$ more than by $o(1)$. The construction of a non-critical deformation and our claim follows.

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