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Differential forms, Maxwell equations and compensated compactness in Carnot groups

Annalisa Baldi, Bruno Franchi and Maria Carla Tesi

A Ermanno con amicizia e stima

Abstract[1]. In this paper we introduce a class of Maxwell equations in Heisenberg groups $\mathbb{H}^n$, and we show they are equivalent – as in the Euclidean case – to a system of equations for (intrinsic) differential forms, that can be written, in fact, in any Carnot group. A crucial feature of the problem relies on the fact that – unlike the Euclidean case – the natural differential on intrinsic forms in Heisenberg groups may be an operator of order 2. Then we prove an homogenization theorem for Maxwell equations via compensated compactness in Heisenberg groups. In this case we cannot apply our previous results for differential forms in Carnot groups obtained in collaboration with N. Tchou ([1]), sketched here, due to the lack of a geometric property of the classes of intrinsic 2-forms in $\mathbb{H}^1 \times \mathbb{R}$. Nevertheless, the Hodge decomposition proved in [1] can be adapted to fit the new setting.

1. Intrinsic forms in Carnot groups

Let $(\mathbb{G}, \cdot)$ be a *Carnot group of step* κ identified to $\mathbb{R}^n$ through exponential coordinates (see [4] for details). By definition, the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$(1) \qquad \mathfrak{g} = V_1 \oplus ... \oplus V_\kappa \ , \quad [V_1, V_i] = V_{i+1} \ , \quad V_\kappa \neq \{0\} \ , \quad V_i = \{0\} \quad \text{if } i > \kappa \ ,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Choose a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, i.e. such that

$$e_1, \dots, e_{m_1} \quad \text{is a basis of } V_1$$

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[1] Authors' address: A. Baldi, Università degli Studi di Bologna, Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40127 Bologna, Italy; e-mail: `baldi@dm.unibo.it` .

B. Franchi, Università degli Studi di Bologna, Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40127 Bologna, Italy; e-mail: `franchib@dm.unibo.it` .

M.C. Tesi, Università degli Studi di Bologna, Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40127 Bologna, Italy; e-mail: `tesi@dm.unibo.it` .

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and, accordingly,

$$e_{m_{j-1}+1},\dots,e_{m_j} \quad \text{is a basis of } V_j \text{ for each } j=2,\dots,\kappa\ .$$

Let $X=\{X_1,\dots,X_n\}$ be the family of left invariant vector fields such that $X_i(0)=e_i$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle\cdot,\cdot\rangle$, making $\{X_1,\dots,X_n\}$ an orthonormal basis.

Two important families of automorphism of $\mathbb{G}$ are given by left translations $p\mapsto\tau_q p:=q\cdot p$ and group dilations δ_λ for $\lambda>0$.

As customary, we fix a smooth homogeneous norm $|\cdot|$ in $\mathbb{G}$ such that the gauge distance $d(x,y):=|y^{-1}x|$ is a left-invariant true distance, equivalent to the Carnot-Carathéodory distance in $\mathbb{G}$ (see [14], p.638). We set $B(p,r)=\{q\in\mathbb{G};\ d(p,q)<r\}$.

The Haar measure of $\mathbb{G}=(\mathbb{R}^n,\cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$. If $A\subset\mathbb{G}$ is $\mathcal{L}$-measurable, we write also $|A|:=\mathcal{L}(A)$.

We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q:=\sum_{i=1}^{\kappa} i\,\dim(V_i)\ .$$

Since for any $x\in\mathbb{G}$ $|B(x,r)|=|B(e,r)|=r^Q|B(e,1)|$, Q is the Hausdorff dimension of the metric space $(\mathbb{G},d)$.

By (1), the subset $X_1,\dots,X_{m_1}$ generates by commutations all the other vector fields. Therefore, the subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by $X_1,\dots,X_{m_1}$ plays a particularly important role in the theory, and it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x=\mathrm{span}\{X_1(x),\dots,X_{m_1}(x)\}\ ,\qquad x\in\mathbb{G}\ .$$

From now on, for sake of simplicity, sometimes we set $m:=m_1$.

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle\cdot,\cdot\rangle_x$ making the basis $X_1(x),\dots,X_m(x)$ an orthonormal basis. The sections of $H\mathbb{G}$ are called *horizontal sections*, and a vector of $H\mathbb{G}_x$ is an *horizontal vector*.

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1\mathfrak{g}$. The basis of $\bigwedge^1\mathfrak{g}$, dual of the basis $X_1,\cdots,X_n$, is the family of covectors $\{\theta_1,\cdots,\theta_n\}$. We indicate by $\langle\cdot,\cdot\rangle$ also the inner product in $\bigwedge^1\mathfrak{g}$ that makes $\theta_1,\cdots,\theta_n$ an orthonormal basis.

Following Federer (see [7] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1\mathfrak{g}$ are the graded algebras indicated as $\bigwedge_*\mathfrak{g}=\bigoplus_{k=0}^{n}\bigwedge_k\mathfrak{g}$ and $\bigwedge^*\mathfrak{g}=\bigoplus_{k=0}^{n}\bigwedge^k\mathfrak{g}$ where $\bigwedge_0\mathfrak{g}=\bigwedge^0\mathfrak{g}=\mathbb{R}$ and, for $1\le k\le n$,

$$\begin{aligned}\textstyle\bigwedge_k\mathfrak{g}\ &:=\mathrm{span}\{X_{i_1}\wedge\cdots\wedge X_{i_k}:1\le i_1<\cdots<i_k\le n\}\ ,\\ \textstyle\bigwedge^k\mathfrak{g}\ &:=\mathrm{span}\{\theta_{i_1}\wedge\cdots\wedge\theta_{i_k}:1\le i_1<\cdots<i_k\le n\}\ .\end{aligned}$$

The elements of $\bigwedge_k\mathfrak{g}$ and $\bigwedge^k\mathfrak{g}$ are called *k-vectors* and *k-covectors*.

We denote by Θ^k the basis $\{\theta_{i_1}\wedge\cdots\wedge\theta_{i_k}:1\le i_1<\cdots<i_k\le n\}$ of $\bigwedge^k\mathfrak{g}$.

The dual space $\bigwedge^1(\bigwedge_k\mathfrak{g})$ of $\bigwedge_k\mathfrak{g}$ can be naturally identified with $\bigwedge^k\mathfrak{g}$. The action of a k-covector φ on a k-vector v is denoted as $\langle\varphi,|v\rangle$.

The inner product $\langle\cdot,\cdot\rangle$ extends canonically to $\bigwedge_k\mathfrak{g}$ and to $\bigwedge^k\mathfrak{g}$ making the bases $X_{i_1}\wedge\cdots\wedge X_{i_k}$ and $\theta_{i_1}\wedge\cdots\wedge\theta_{i_k}$ orthonormal.

Definition 1.1. We define linear isomorphisms (Hodge duality: see [7] 1.7.8)

$$*: \bigwedge_k \mathfrak{g} \longleftrightarrow \bigwedge_{n-k} \mathfrak{g} \quad \text{and} \quad *: \bigwedge^k \mathfrak{g} \longleftrightarrow \bigwedge^{n-k} \mathfrak{g}\,,$$

for $1 \le k \le n$, putting, for $v = \sum_I v_I X_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum\nolimits_I v_I (*X_I) \quad \text{and} \quad *\,\varphi := \sum\nolimits_I \varphi_I (*\theta_I)$$

where

$$*X_I := (-1)^{\sigma(I)} X_{I^*} \quad \text{and} \quad *\,\theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \cdots, i_k\}$, $1 \le i_1 < \cdots < i_k \le n$, $X_I = X_{i_1} \wedge \cdots \wedge X_{i_k}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$, $I^* = \{i_1^* < \cdots < i_{n-k}^*\} = \{1, \cdots, n\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i_ℓ^*) with $i_h > i_\ell^*$.

If $v \in \bigwedge_k \mathfrak{g}$ we define $v^\natural \in \bigwedge^k \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^\natural \in \bigwedge_k \mathfrak{g}$ for $\varphi \in \bigwedge^k \mathfrak{g}$.

Definition 1.2. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* k, and we write $w(\alpha) = k$, if $\alpha^\natural \in V_k$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

We have

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}\,, \tag{2}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h–covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of left-invariant h-covectors.

Since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p} \mathfrak{g}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p} \mathfrak{g}$. In other words, the basis $\Theta^h = \cup_p \Theta^{h,p}$ is a basis adapted to the filtration of $\bigwedge^h \mathfrak{g}$ associated with (2).

Starting from $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, we can define by left translation fiber bundles over $\mathbb{G}$ that we can still denote by $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively. To do this, for instance we identify $\bigwedge^h \mathfrak{g}$ with the fiber $\bigwedge_e^h \mathfrak{g}$ over the origin, and we define the fiber over $x \in \mathbb{G}$ pulling back $\bigwedge_e^h \mathfrak{g}$ by the left translation $\tau_{x^{-1}}$. Sections of $\bigwedge_h \mathfrak{g}$ are called h-vector fields, and sections of $\bigwedge^h \mathfrak{g}$ are called h-forms. We denote by Ω_h (Ω^h) the vector space of all smooth sections of $\bigwedge_h \mathfrak{g}$ (of $\bigwedge^h \mathfrak{g}$, respectively).

The identification of $\bigwedge^h \mathfrak{g}$ and $\bigwedge_e^h \mathfrak{g}$ yields a corresponding identification of the basis Θ^h of $\bigwedge^h \mathfrak{g}$ and Θ_e^h of $\bigwedge_e^h \mathfrak{g}$. Notice that the Lie algebra $\mathfrak{g}$ can be identified with the Lie algebra of the left invariant vector fields on $\mathbb{G} \equiv \mathbb{R}^n$. Hence, the elements of Θ_x^h can be identified with the elements of Θ^h evaluated at the point x. Through all this Note, we make systematic use of these identifications, interchanging the roles of left invariant vector fields and elements of $\bigwedge_1 \mathfrak{g}$.

Keeping in mind the decomposition (2), we can define in the same way several fiber bundles over $\mathbb{G}$, that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$). We notice also that the fiber $\bigwedge_x^h \mathfrak{g}$ (and hence the fiber $\bigwedge_x^{h,p} \mathfrak{g}$) can be endowed with a natural scalar product $\langle \cdot, \cdot \rangle_x$.

We denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p}\mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p} \, . \tag{3}$$

The following crucial property of the weight follows from Cartan identintity: see [13], Section 2.1:

Lemma 1.3. *We have $d(\bigwedge^{h,p}\mathfrak{g}hp) = \bigwedge^{h+1,p}\mathfrak{g}$, i.e., if $\alpha \in \bigwedge^{h,p}\mathfrak{g}$ is a left invariant h-form of weight p with $d\alpha \neq 0$, then $w(d\alpha) = w(\alpha)$.*

Let now $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. Then we can write $d\alpha = d_0\alpha + d_1\alpha + \cdots + d_{Q-p}\alpha$, where $d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$ does not increase the weight, $d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$ increases the weight of 1, and so on. In particular, d_0 is an algebraic operator. Analogously, δ_0, the L^2–adjoint of d_0 in Ω^* is also an algebraic operator preserving the weight.

The following definition of intrinsic covectors (and therefore of intrinsic forms) is due to M. Rumin ([13], [12]).

Definition 1.4. If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap \ker \delta_0 = \ker d_0 \cap (\mathrm{Im}\, d_0)^\perp \subset \Omega^h$$

In the sequel, we refer to the elements of E_0^h as to *intrinsic h-forms on* $\mathbb{G}$. Since the construction of E_0^h is left invariant, this space of forms can be viewed as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

Moreover, there exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (2). Without loss of generality, we can take $\xi_j = \theta_j$ for $j = 1, \dots, m$.

Once the basis Θ^h is chosen, the spaces of sections $\mathcal{E}(\Omega, E_0^h)$, $\mathcal{D}(\Omega, E_0^h)$ (for instance) can be identified with $\mathcal{E}(\Omega)^{\dim E_0^h}$, $\mathcal{D}(\Omega)^{\dim E_0^h}$, respectively.

Proposition 1.5 ([13]). *If $0 \le h \le n$, then*

$$*E_0^h = E_0^{n-h} \, .$$

Theorem 1.6 ([13]). *There exists a differential operator $d_c : E_0^h \to E_0^{h+1}$ such that*

- i) *$d_c^2 = 0$;*
- ii) *the complex $E_0 := (E_0^*, d_c)$ is exact;*
- iii) *the differential d_c acting on h-forms can be identified, with respect to the bases Ξ_0^h and Ξ_0^{h+1}, with a matrix-valued differential operator $L^h := \big(L_{i,j}^h\big)_{i,j}$. If ξ_j and ξ_i have respectively weight p and q, then $L_{i,j}^h$ is a homogeneous left invariant differential operator of order $q-p$ in the horizontal derivatives if $q - p \ge 1$, and $L_{i,j}^h = 0$ otherwise.*

In particular, if $h = 0$ and $f \in E_0^0 = \mathcal{E}(\mathbb{G})$, then $d_c f = \sum_{i=1}^m (X_i f)\theta_i^1$ is the *horizontal differential of f*.

As usual, δ_c denotes the L^2 adjoint of d_c, and again in E_0^h we have

$$\delta_c = (-1)^{n(h+1)+1} * d_c * \ .$$

Definition 1.7. If $\Omega \subset \mathbb{G}$ is an open set, we say that T is a h-current on Ω if T is a continuous linear functional on $\mathcal{D}(\Omega, E_0^h)$ endowed with the usual topology. We write $T \in \mathcal{D}'(\Omega, E_0^h)$.

The definition of $\mathcal{E}'(\Omega, E_0^h)$ is given analogously.

Proposition 1.8. *If $\Omega \subset \mathbb{G}$ is an open set, and $T \in \mathcal{D}'(\Omega)$ is a (usual) distribution, then T can be identified canonically with a n-current $\tilde{T} \in \mathcal{D}'(\Omega, E_0^n)$ through the formula*

$$\langle \tilde{T}, |\alpha\rangle := \langle T, | * \alpha\rangle \tag{4}$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^n)$. Reciprocally, by (4), any n-current $\tilde{T}$ can be identified with an usual distribution $T \in D'(\Omega)$.

Proof. See [6], Section 17.5, and [2], Proposition 4.

□

Following [7], 4.1.7, we give the following definition.

Definition 1.9. If $T \in \mathcal{D}'(\Omega, E_0^n)$, and $\varphi \in \mathcal{E}(\Omega, E_0^k)$, with $0 \le k \le n$, we define $T \llcorner \varphi \in D'(\Omega, E_0^{n-k})$ by the identity

$$\langle T \llcorner \varphi, |\alpha\rangle := \langle T, |\alpha \wedge \varphi\rangle$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{n-k})$.

The following result is taken from [2], Propositions 5 and 6, and Definition 10, but we refer also to [6], Sections 17.3 17.4 and 17.5.

Proposition 1.10. *Let $\Omega \subset \mathbb{G}$ be an open set. If $1 \le h \le n$, $\Xi_0^h = \{\xi_1^h, \dots \xi_{\dim E_0^h}^h\}$ is a left invariant basis of E_0^h and $T \in \mathcal{D}'(\Omega, E_0^h)$, then*

- i) *there exist (uniquely determined) $T_1, \dots, T_{\dim E_0^h} \in \mathcal{D}'(\Omega)$ such that we can write*

$$T = \sum_j \tilde{T}_j \llcorner (*\xi_j^h) \ ;$$

- ii) *if $\alpha \in E(\Omega, E_0^h)$, then α can be identified canonically with a h-current T_α through the formula*

$$\langle T_\alpha, |\beta\rangle := \int_\Omega *\alpha \wedge \beta$$

for any $\beta \in \mathcal{D}(\Omega, E_0^h)$. Moreover, if $\alpha = \sum_j \alpha_j \xi_j^h$ then

$$T_\alpha = \sum_j \tilde{\alpha}_j \llcorner (*\xi_j^h) \ ;$$

- iii) *we say that T is smooth in $\mathcal{U}$ when $T_1, \dots, T_{\dim E_0^h}$ are (identified with) smooth functions. This is clearly equivalent to say that there exists $\beta \in E(\Omega, E_0^h)$ such that*

$$\langle T, |\alpha\rangle = \int_{\mathcal{U}} \langle \beta, \alpha\rangle \, dV$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^h)$ (in fact, we choose $\beta = \sum_j T_j \xi_j^h$).

2. Function spaces

Let $\{X_1, \dots, X_m\}$ be the fixed basis of the horizontal layer $\mathfrak{g}_1$ of $\mathfrak{g}$. We denote by $\Delta_{\mathbb{G}}$ the nonnegative horizontal sublaplacian

$$\Delta_{\mathbb{G}} := -\sum_{j=1}^{m} X_j^2 \, .$$

If $1 < s < \infty$ and $a \in \mathbb{C}$, we define $\Delta_{\mathbb{G}}^a$ in $L^s(\mathbb{G})$ following [8]. If in addition $m \geq 0$, again as in [8], we denote by $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ the domain of the realization of $\Delta_{\mathbb{G}}^{m/2}$ in $L^s(\mathbb{G})$ endowed with the graph norm. In fact, since $s \in (1, \infty)$ is fixed through all the paper, to avoid cumbersome notations, we do not stress the explicit dependence on s of the fractional powers $\Delta_{\mathbb{G}}^{m/2}$ and of its domain.

Definition 2.1. Let $m \geq 0$, $1 < s < \infty$ be fixed indices. Let $\Omega \subset \mathbb{G}$ be a given open set w$\mathcal{L}^n(\partial\Omega) = 0$. We denote by $\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega)$ the completion in $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ of $\mathcal{D}(\Omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{G}$ be an open set. If $m \geq 0$ and $1 < s < \infty$, $W^{-m,s}_{\mathbb{G}}(\Omega)$ is the dual space of $\overset{\circ}{W}{}^{k,s'}_{\mathbb{G}}(\Omega)$, where $1/s + 1/s' = 1$. It is well known that, if $m \in \mathbb{N}$ and Ω is bounded, then

$$W^{-m,s}_{\mathbb{G}}(\Omega) = \{ \sum_{d(I)=k} X^I f_I, \ f_I \in L^s(\Omega) \text{ for any } I \text{ such that } d(I) = k \} \, ,$$

and

$$\|u\|_{W^{-m,s}_{\mathbb{G}}(\Omega)} \approx \inf\{ \sum_I \|f_I\|_{L^s(\Omega)} \, ; \, d(I) = k, \ \sum_{d(I)=k} X^I f_I = u \} \, .$$

Proposition 2.3. *If* $1 < s < \infty$ *and* $m, m' \geq 0$, $m' < m$, *then*

$$W^{m,s}_{\mathbb{G}}(\mathbb{G}) \hookrightarrow W^{m',s}_{\mathbb{G}}(\mathbb{G}) \quad \text{and} \quad W^{-m',s}_{\mathbb{G}}(\mathbb{G}) \hookrightarrow W^{-m,s}_{\mathbb{G}}(\mathbb{G})$$

algebraically and topologically.

In addition, if Ω *is a bounded open set,* $1 < s < \infty$ *and* $m, m' \geq 0$, $m' < m$, *then*

$$\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega) \quad \text{is compactly embedded in } W^{m',s}_{\mathbb{G}}(\Omega)$$

and

$$W^{-m',s}_{\mathbb{G}}(\Omega) \quad \text{is compactly embedded in } W^{-m,s}_{\mathbb{G}}(\Omega) \, .$$

We need now to introduce a class of Sobolev spaces of intrinsic differential forms in $\mathbb{G}$ that is adapted to the lack of homogeneity of Rumin's complex. We set

$$\mathcal{I}_0^h := \{p \, ; \, I_{0,p}^h \neq \emptyset\} \quad \text{and} \quad |\mathcal{I}_0^h| = \operatorname{card} \mathcal{I}_0^h \, . \tag{5}$$

Let

$$\underline{m} = (m_{N_h^{\min}}, \dots, m_{N_h^{\max}})$$

be a $|\mathcal{I}_0^h|$–dimensional vector where the components are indexed by the elements of $\mathcal{I}_0^h$ (i.e. by the possible weights) taken in increasing order. We stress that, since weights p such that $I_{0,p}^h = \emptyset$ can exist, then some consecutive indices in $\underline{m}$ can be missed. In the sequel we shall say that $\underline{m}$ is a *h–vector weight*. We say that $\underline{m} \geq 0$ if $m_p \geq 0$ for $p \in \mathcal{I}_0^h$, and that $\underline{m} \geq \underline{n}$ if $m_p \geq n_p$ for all $p \in \mathcal{I}_0^h$. We say also that $\underline{m} > \underline{n}$ if $m_p > n_p$ *for all* $p \in \mathcal{I}_0^h$. Finally, if m_0 is a real number, we identify m_0 with the h–vector weight $m_0 = (m_0, \dots, m_0)$. In particular, we set $\underline{m} - m_0 := (m_{N_h^{\min}} - m_0, \dots, m_{N_h^{\max}} - m_0)$.

Definition 2.4. A special h–vector weight that we shall use in the sequel is the h-vector weight $\underline{N}_h = (m_{N_h^{\min}}, \dots, m_{N_h^{\max}})$ with

$$m_p = p \quad \text{for all } p \in I_0^h \ .$$

If all h-forms have *pure weight* N_h, i.e. if $N_h^{\min} = N_h^{\max} := N_h$, then a h-vector weight has only one component, i.e. $\underline{m} = (m_{N_h})$.

Definition 2.5. If $\underline{m} \geq 0$ is a h-vector weight, $0 \leq h \leq n$, and $s > 1$, we say that a measurable section α of E_0^h, $\alpha := \sum_p \sum_{j \in I_{0,p}^h} \alpha_j \xi_j^h$ belongs to $W_{\mathbb{G}}^{\underline{m},s}(\mathbb{G}, E_0^h)$ if, for all $p \in \mathcal{I}_0^h$, i.e. for all p, $N_h^{\min} \leq p \leq N_h^{\max}$, such that $I_{0,p}^h \neq \emptyset$,

$$\alpha_j \in W_{\mathbb{G}}^{m_p,s}(\mathbb{G})$$

for all $j \in I_{0,p}^h$, endowed with the natural norm.

The spaces $W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$, where Ω is an open set in $\mathbb{G}$, as well as the local spaces $W_{\mathbb{G},\mathrm{loc}}^{\underline{m},s}(\Omega, E_0^h)$ are defined in the obvious way.

Since

$$W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h) \quad \text{is isometric to } \prod_{p \in \mathcal{I}_0^h} \left(W_{\mathbb{G}}^{m_p,s}(\mathbb{G}) \right)^{\operatorname{card} \mathcal{I}_{0,p}^h} \ ,$$

then

- $W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$ is a reflexive Banach space (remember $s > 1$);
- $\mathbf{C}^\infty(\Omega, E_0^h) \cap W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$ is dense in $W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$.

The spaces $\overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$ are defined in the obvious way.

Coherently with our notations, instead of $W_{\mathbb{G}}^{(m,m,\dots,m),s}(\Omega, E_0^h)$, we write $W_{\mathbb{G}}^{m,s}(\Omega, E_0^h)$.

We can define and characterize the dual spaces of Sobolev spaces of forms.

Proposition 2.6. *If* $1 < s < \infty$, $1/s + 1/s' = 1$, $0 \leq h \leq n$, $\underline{m}$ *is a* h*–vector weight, and* $\Omega \subset \mathbb{G}$ *is a bounded open set, then the dual space* $\left(\overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m}.s'}(\Omega, E_0^h) \right)^*$ *coincides with the set of all currents* $T \in \mathcal{D}'(\Omega, E_0^h)$ *of the form*

$$T = \sum_p \sum_{j \in I_{0,p}^h} \tilde{T}_j \llcorner (*\xi_j^h) \tag{6}$$

with $T_j \in W_{\mathbb{G}}^{-m_p,s}(\Omega)$ *for all* $j \in I_{0,p}^h$ *and for* $p \in \mathcal{I}_0^h$*. The action of* T *on the form* $\alpha = \sum_p \sum_{j \in I_{0,p}^h} \alpha_j \xi_j^h \in \overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h)$ *is given by the identity*

$$T(\alpha) = \sum_p \sum_{j \in I_{0,p}^h} \langle T_j, |\alpha_j \rangle \ . \tag{7}$$

In particular, it is natural to set

$$W_{\mathbb{G}}^{-\underline{m},s}(\Omega, E_0^h) := \left(\overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h) \right)^* \ .$$

Moreover, if T *is as in* (6)

$$\|T\|_{W_{\mathbb{G}}^{-\underline{m},s}(\Omega, E_0^h)} \approx \sum_p \sum_{j \in I_{0,p}^h} \|T_j\|_{W_{\mathbb{G}}^{-m_p,s}(\Omega)} \ .$$

3. HODGE DECOMPOSITION AND COMPENSATED COMPACTNESS

The following Hodge decomposition theorem for intrinsic forms in Carnot groups is the core of [1] (see Theorem 4.1).

Theorem 3.1. *Let* $s > 1$ *and* $h = 1, \dots, n$ *be fixed, and suppose* h*-forms have pure weight* N_h. *Let* $\Omega \subset\subset \mathbb{G}$ *a given open set, and let* $\alpha^\varepsilon \in L^s(\mathbb{G}, E_0^h) \cap \mathcal{E}'(\Omega, E_0^h)$ *be compactly supported differential* h*-forms such that*

$$\alpha^\varepsilon \rightharpoonup \alpha \quad \textit{as } \varepsilon \to 0 \quad \textit{weakly in } L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$$

and

$$\{d_c\alpha^\varepsilon\} \quad \textit{is pre-compact in } W^{-(\underline{N}_{h+1}-N_h),s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^h)\ .$$

Then there exist h*–forms* $\omega^\varepsilon \in E_0^h$ *and* $(h-1)$*–forms* $\psi^\varepsilon \in E_0^{h-1}$ *such that*

i) $\omega^\varepsilon \to \omega$ *strongly in* $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$ *;*
ii) $\psi^\varepsilon \to \psi$ *strongly in* $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^{h-1})$ *;*
iii) $\alpha^\varepsilon = \omega^\varepsilon + d_c\psi^\varepsilon$.

In addition, we can choose ω^ε *and* ψ^ε *supported in a fixed suitable neigborhood of* Ω*, which are smooth forms if the* α^ε *are also smooth.*

From Theorem 3.1, a compensated compactness theorem follows ([1], Theorem 4.13).

Theorem 3.2. *If* $1 < s_i < \infty$, $0 \le h_i \le n$ *for* $i = 1, 2$, *and* $0 < \varepsilon < 1$, *assume that* $\alpha_i^\varepsilon \in L^{s_i}_{\mathrm{loc}}(\mathbb{G}, E_0^{h_i})$ *for* $i = 1, 2$, *where* $(1/s_1) + (1/s_2) = 1$ *and* $h_1 + h_2 = n$. *Suppose* h_1*-forms have pure weight* N_{h_1} *(by Hodge duality, this implies that also* h_2*-forms have pure weight* N_{h_2}*). Assume that, for any open set* $\Omega_0 \subset\subset \mathbb{G}$,

$$\alpha_i^\varepsilon \to \alpha_i \textit{ weakly in } L^{s_i}(\Omega_0, E_0^{h_i})\ , \tag{8}$$

and that

$$\{d_c\alpha_i^\varepsilon\} \quad \textit{is pre-compact in } W^{-(\underline{N}_{h_i+1}-N_{h_i}),s_i}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^{h_i}) \tag{9}$$

for $i = 1, 2$.

Then

$$\int_{\mathbb{G}} \varphi\, \alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \to \int_{\mathbb{G}} \varphi\, \alpha_1 \wedge \alpha_2 \tag{10}$$

for any $\varphi \in \mathcal{D}(\mathbb{G})$.

A crucial feature of Theorem 3.1, and hence of Theorem 3.2 is the assumption that all forms in E_0^h have the same weight. This geometric condition is satisfied in several situations (see Section 7 of [1]) and, first of all, is always satisfied when $h = 1$, so that it was possible to prove in [1], Section 5, a curl-div theorem for horizontal vector fields in any Carnot group $\mathbb{G}$, and then eventually a compactness result for the H-convergence of elliptic operators on groups. But, as we shall see in Section 4, this condition may fail to hold when we try to attack the homogenization of Maxwell equations in groups. In particular, it fails to hold in the "simplest" case of the first Heisenberg group $\mathbb{H}^1$. In order to introduce the content of Section 4, let us remind the basic steps of the proof of Theorem 3.1.

Here we must rely on (a somehow simplified version of) the pseudodifferential calculus in homogeneous groups of [5] (see also [1], Section 6 for a short review).

If $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Z}^+ := \mathbb{N} \cup \{0\}$, then we denote by $\mathbf{K}^\alpha$ the set of the distributions in $\mathbb{G}$ that are smooth away from the origin and homogeneous of degree α, whereas, if $\alpha \in \mathbb{Z}^+$, we say that $K \in \mathcal{D}'(\mathbb{G})$ belongs to $\mathbf{K}^\alpha$ if has the form

$$K = \tilde{K} + p(x) \ln |x| \ ,$$

where $\tilde{K}$ is smooth away from the origin and homogeneous of degree α, and p is a homogeneous polynomial of degree α.

Definition 3.3. If $\alpha \in \mathbb{R}$, we say that $\mathcal{K}$ is a pseudodifferential operator of order α on $\mathbb{G}$ with core K if

1) $K \in \mathcal{D}'(\mathbb{G} \times \mathbb{G})$.
2) Let $\beta := -Q - \alpha$. There exist $K^m = K^m_x \in \mathbf{K}^{\beta+m}$ depending smoothly on $x \in \mathbb{G}$ such that for each $N \in \mathbb{N}$ there exists $M \in \mathbb{Z}^+$ such that, if we set
$$K_x - \sum_{m=0}^{M} K^m_x := E_M(x, \cdot) \ ,$$
then $E_M \in \mathbf{C}^N(\mathbb{G} \times \mathbb{G})$.
3) For some finite $R \geq 0$, supp $K_x \subset B(e, R)$ for all $x \in \mathbb{G}$.
4) If $u \in \mathcal{D}(\mathbb{G})$ and $x \in \mathbb{G}$, then
$$\mathcal{K}u(x) = (u * K_x)(x) \ .$$

We write $K \sim \sum_m K^m$, $\mathcal{K} = \mathcal{O}(K)$, and $r(K) = r(\mathcal{K}) = \inf\{R > 0$ such that 3) holds$\}$.

We let

$$\mathcal{OC}^\alpha(\mathbb{G}) := \{\text{pseudodifferential operators of order } \alpha \text{ on } \mathbb{G}\} \ .$$

Clearly, if $\mathcal{K} \in \mathcal{OC}^\alpha(\mathbb{G})$, then $\mathcal{K} : \mathcal{D}(\mathbb{G}) \to \mathcal{E}(\mathbb{G})$. Moreover, $\mathcal{K}$ can be extended to an operator $\mathcal{K} : \mathcal{E}'(\mathbb{G}) \to \mathcal{D}'(\mathbb{G})$. Notice also that, if supp $u \subset B(e, r)$, the supp $\mathcal{K}u \subset B(e, r + r(\mathcal{K}))$.

Theorem 3.4 (see [5], p.63 (3))**.** *If* $1 < s < \infty$ *and* $K \in \mathcal{OC}^0(\mathbb{G})$*, then* $\mathcal{O}(K) : L^p_{\mathrm{loc}}(\mathbb{G}) \to L^p_{\mathrm{loc}}(\mathbb{G})$ *is continuous.*

Theorem 3.5 ([5], Theorem 2.5)**.** *We have:*

(a) *If* $\mathcal{K} := \mathcal{O}(K) \in \mathcal{OC}^\alpha(\mathbb{G})$*, then there exists a core* K^* *such that* $\mathcal{O}(K^*) \in \mathcal{OC}^\alpha(\mathbb{G})$ *and*
$$\langle v, \mathcal{K}u \rangle_{L^2(\mathbb{G})} = \langle \mathcal{O}(K^*)v, u \rangle_{L^2(\mathbb{G})}$$
for all $u, v \in \mathcal{D}(\mathbb{G})$.
(b) *If* $\mathcal{K} \in \mathcal{OC}^\alpha(\mathbb{G})$, $V \subset \mathbb{G}$ *is an open set, and* $u \in \mathcal{E}'(\mathbb{G})$ *is smooth on* V*, then* $\mathcal{K}u$ *is smooth on* V.
(c) *If* $\mathcal{K}_i \in \mathcal{OC}^\alpha_i(\mathbb{G})$, $K_i \sim \sum_m K^m_i$, $i = 1, 2$, *then* $\mathcal{K} := \mathcal{K}_2 \circ \mathcal{K}_1$ *belongs to* $\mathcal{OC}^{\alpha_1+\alpha_2}(\mathbb{G})$.

In other words, the operators in $\mathcal{OC}^\alpha(\mathbb{G})$ behave very much like usual properly supported pseudifferential operators in $\mathbb{R}^n$, except for the fact the commutator of $\mathcal{K}_i \in \mathcal{OC}^\alpha_i(\mathbb{G})$, $i = 1, 2$ may fail to belong to $\mathcal{OC}^{\alpha_1+\alpha_2-1}(\mathbb{G})$ because of the non-commutativity of the group convolution of the two cores, that appears in the leading part of the core of the composition. However, it is possible to show that these pseudodifferential operators well behave with respect to our scale of Sobolev spaces (see [1], Proposition 6.19).

We say that a convolution operator $u \to u * E(x,\cdot)$ from $\mathcal{E}'$ to $\mathcal{D}'$ belongs to $\mathcal{OC}^{-\infty}(\mathbb{G})$ if E is smooth on $\mathbb{G}\times\mathbb{G}$. We notice that, properly speaking, $\mathcal{OC}^{-\infty}(\mathbb{G})$ is not contained in $\mathcal{OC}^{\alpha}(\mathbb{G})$ for $\alpha\in\mathbb{R}$, since $E(x,\cdot)$ is not assumed to be compactly supported.

If $\mathcal{T},\mathcal{S}\in\mathcal{OC}^{\ell}(\mathbb{G})$, we say that $\mathcal{S}=\mathcal{T} \mod \mathcal{OC}^{-\infty}$ if $\mathcal{S}-\mathcal{T}\in\mathcal{OC}^{-\infty}(\mathbb{G})$.

In the sequel, if $m\in\mathbb{R}$ and $1<s<\infty$, we use repeatedly a family of pseudodifferential operators $\Delta_{\mathbb{G},R}^{m/2}u\in\mathcal{OC}^{m}(\mathbb{G})$ with $r(\Delta_{\mathbb{G},R}^{m/2})=R$ for $R>>1$ that differ from $\Delta_{\mathbb{G}}^{m/2}$ by a regularizing operator.

Lemma 3.6. *We have*

$$\Delta_{\mathbb{G},R}^{m/2}\circ\Delta_{\mathbb{G},R}^{-m/2}=Id \mod \mathcal{OC}^{-\infty}\ ,$$

and

$$\Delta_{\mathbb{G},R}^{-m/2}\circ\Delta_{\mathbb{G},R}^{m/2}=Id \mod \mathcal{OC}^{-\infty}\ .$$

Definition 3.7. Let $T\in\mathcal{E}'(\mathbb{G},E_0^h)$ be a compactly supported h-current on $\mathbb{G}$ of the form

$$T=\sum_p\sum_{j\in I_{0,p}^h}\tilde{T}_j\llcorner(*\xi_j^h)\quad\text{with } T_j\in\mathcal{E}'(\mathbb{G}) \text{ for } j=1,\dots,\dim E_0^h\ .$$

Let $\underline{m}$ be a h–vector weight, and let $R>0$ be fixed. We set

$$\Delta_{\mathbb{G},R}^{\underline{m}/2}T:=\sum_p\sum_{j\in I_{0,p}^h}(\widetilde{\Delta_{\mathbb{G},R}^{m_p/2}T_j})\llcorner(*\xi_j^h)\ .$$

In particular, if T can be identified with a compactly supported h-form $\alpha=\sum_p\sum_{j\in I_{0,p}^h}\alpha_j\xi_j^h$, then our previous definition becomes

$$\Delta_{\mathbb{G},R}^{\underline{m}/2}\alpha=\sum_p\sum_{j\in I_{0,p}^h}(\alpha_j*(P_{m_p})_R)\xi_j^h\ .$$

Definition 3.8. Let $R>0$ be fixed. If $0\le h\le n$, following Rumin we define the "0-order differential" acting on compactly supported h-currents belonging to $\mathcal{E}'(B(e,R),E_0^h)$ by

$$\tilde{d}_c:=\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\ ,$$

where $\underline{N}_h$ is defined in Definition 2.4. The definition is well posed, and

$$\tilde{d}_c:\mathcal{E}'(B(e,R),E_0^h)\to\mathcal{E}'(B(e,3R),E_0^h)\ .$$

Analogously, we define the following "0-order codifferential" acting on compactly supported $(h+1)$-currents belonging to $\mathcal{E}'(B(e,R),E_0^{h+1})$:

$$\tilde{\delta}_c:=\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\ .$$

Again the definition is well posed, and

$$\tilde{\delta}_c:\mathcal{E}'(B(e,R),E_0^{h+1})\to\mathcal{E}'(B(e,3R),E_0^h)\ .$$

By Theorem 3.5 (a),

$$\tilde{\delta}_c=(\tilde{d}_c)^*\ .$$

Notice also that

$$\tilde{d}_c^2=0,\quad\tilde{\delta}_c^2=0\quad(\mod\mathcal{OC}^{-\infty})\ .$$

We set

$$\Delta^{(0)}_{\mathbb{G},R} := \widetilde{\delta}_c \tilde{d}_c + \tilde{d}_c \widetilde{\delta}_c \ .$$

It follows from Rumin's results and from [5] that the operator $\Delta^{(0)}_{\mathbb{G},R}$ has a parametrix (both right and left), that commutes (up to a smoothing operator) with $\tilde{d}_c$ and $\widetilde{\delta}_c$.

Theorem 3.9. *For any $R > 0$ there exists a (matrix-valued) pseudodifferential operator in the sense of* [5] $(\Delta^{(0)}_{\mathbb{G},R})^{-1}$ *such that*

$$(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{(0)}_{\mathbb{G},R} = Id \quad on \quad \mathcal{E}'(\mathbb{G}, E_0^h) \quad (\mathrm{mod}\, \mathcal{OC}^{-\infty}) \ , \tag{11}$$

and

$$\Delta^{(0)}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1} = Id \quad on \quad \mathcal{E}'(\mathbb{G}, E_0^h) \quad (\mathrm{mod}\, \mathcal{OC}^{-\infty}) \ . \tag{12}$$

Proposition 3.10. *For any $R > 0$*

$$(\Delta^{(0)}_{\mathbb{G},R})^{-1}\tilde{d}_c = \tilde{d}_c(\Delta^{(0)}_{\mathbb{G},R})^{-1} \quad on \quad \mathcal{E}'(\mathbb{G}, E_0^h) \quad (\mathrm{mod}\, \mathcal{OC}^{-\infty}) \ , \tag{13}$$

and

$$(\Delta^{(0)}_{\mathbb{G},R})^{-1}\widetilde{\delta}_c = \widetilde{\delta}_c(\Delta^{(0)}_{\mathbb{G},R})^{-1} \quad on \quad \mathcal{E}'(\mathbb{G}, E_0^h) \quad (\mathrm{mod}\, \mathcal{OC}^{-\infty}) \ . \tag{14}$$

Now we can sketch the proof of Theorem 3.1. This will make our subsequent arguments easier to follow.

Without loss of generality, suppose $\alpha^\varepsilon \in \mathcal{D}(\Omega, E_0^h)$; keeping in mind Theorem 3.9 and the explicit form of $\Delta^{(0)}_{\mathbb{G},R}$, we obtain

$$\begin{aligned}
\alpha^\varepsilon = \qquad\qquad & \qquad (15)\\
= \Delta_{\mathbb{G},R}^{\underline{N}_h/2}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_h/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon + &\\
+\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\,d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_{h-1}/2}\Delta_{\mathbb{G},R}^{\underline{N}_{h-1}/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon + S_0\alpha^\varepsilon = &\\
= \Delta_{\mathbb{G},R}^{\underline{N}_h/2}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_h/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon + S_0\alpha^\varepsilon + &\\
+d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_{h-1}/2}\Delta_{\mathbb{G},R}^{\underline{N}_{h-1}/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon := &\\
:= \omega^\varepsilon + d_c\psi^\varepsilon \ , &
\end{aligned}$$

where S_0 is a properly supported smoothing operator.

We want to show that $(\psi^\varepsilon)_{\varepsilon>0}$ and $(\omega^\varepsilon)_{\varepsilon>0}$ converge strongly in $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^{h-1})$ and $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$, respectively. As for $(\psi^\varepsilon)_{\varepsilon>0}$, $(\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon)_{\varepsilon>0}$ converges weakly in $W_{\mathbb{G}}^{\underline{N}_h,s}(\mathbb{G}, E_0^h)$. On the other hand, also $\Big((\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon\Big)_{\varepsilon>0}$ converges weakly in $W_{\mathbb{G}}^{\underline{N}_h,s}(\mathbb{G}, E_0^h)$. Thus, also $\Big(\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon\Big)_{\varepsilon>0}$ converges weakly in
$W_{\mathbb{G}}^{2\underline{N}_h,s}(\mathbb{G}, E_0^h)$. We remind that all intrinsic h-forms have the same weight N_h, so that all the components of a form in E_0^h belonging to $W_{\mathbb{G}}^{2\underline{N}_h,s}(\mathbb{G}, E_0^h)$ belong to the same Sobolev space $W_{\mathbb{G}}^{2N_h,s}(\mathbb{G}, E_0^h)$.

For sake of simplicity, denote now by β_j^ε, $j \in I^h_{0,N_h}$, a generic component of

$$\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon$$

that converges weakly in $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G}, E_0^{h-1})$. If $i \in I^{h-1}_{0,q}$ $(q < N_h)$, then the i-th component of $\delta_c\beta_j^\varepsilon$ is given by ${}^tL_{j,i}\beta_j^\varepsilon$. Keeping in mind that $L_{j,i}$ is a homogeneous differential operator in the horizontal vector fields of order N_h-q, then $\big({}^tL_{j,i}\beta_j^\varepsilon\big)_{\varepsilon>0}$ converges weakly in $W^{N_h+q,s}_{\mathbb{G}}(\mathbb{G})$, so that, eventually, the i-th component of $(\psi^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{N_h-q,s}_{\mathbb{G}}(\mathbb{G})$. Then the assertion follows by Rellich theorem (Proposition 2.3), since supp ψ^ε is contained is a fixed neighborhood of Ω, and $q < N_h$.

Let us consider now $(\omega^\varepsilon)_{\varepsilon>0}$. Obviously, we can forget the smoothing operator S_0. By Proposition 3.10, we can write

$$\begin{aligned}
&\Delta_{\mathbb{G},R}^{\underline{N}_h}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon =\\
&= \Delta_{\mathbb{G},R}^{\underline{N}_h/2}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon + S_0\alpha^\varepsilon =\\
&= \Delta_{\mathbb{G},R}^{\underline{N}_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon + S_0\alpha^\varepsilon\ .
\end{aligned}\tag{16}$$

Moreover

$$\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon \quad \text{is pre-compact in } W^{\underline{N}_{h+1}+N_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^h)\ .$$

Arguing as above, denote now by β_j^ε, $j \in I^{h+1}_{0,p}$, a generic component of

$$\beta^\varepsilon := \Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon\ .$$

We know that β_j^ε is pre-compact in $W^{p+N_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^{h+1})$. Moreover notice that $\delta_c\beta_\varepsilon$ is a h-form, and therefore, by assumption, has pure weight N_h. If $i \in I^h_{0,N_h}$ $(N_h < p)$, then the i-th component of $\delta_c\beta_j^\varepsilon$ is given by ${}^tL_{j,i}\beta_j^\varepsilon$. Keeping in mind that $L_{j,i}$ is a homogeneous differential operator in the horizontal vector fields of order $j-i = p-N_h$, then $(\delta_c\beta_j^\varepsilon)_i$ is pre-compact in $W^{2N_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$. Thus, $\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$ is pre-compact in $W^{2\underline{N}_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^h)$. Again, $\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$ is pre-compact in $W^{\underline{N}_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^h)$. As above, we can rely now on the fact that all components of $\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$ have the same weight and hence belong to the same Sobolev space, to conclude that

$$(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$$

is pre-compact in $W^{\underline{N}_h,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^h)$. Then, the proof of the assertion follows.

4. Maxwell equations in Carnot groups

Consider now in particular the first Heisenberg group $\mathbb{G} = \mathbb{H}^1$, with variables x, y, t. Set $X := \partial_x + 2y\partial_t$, $Y := \partial_y - 2x\partial_t$, $T := \partial_t$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y\}$ and $V_2 = \text{span}\,\{T\}$. We have $X^\natural = dx$, $Y^\natural = dy$, $T^\natural = \theta$ (the contact form of $\mathbb{H}^1$). In this case

$$\begin{aligned} E_0^1 &= \text{span}\,\{dx, dy\}\ ; \\ E_0^2 &= \text{span}\,\{dx\wedge\theta, dy\wedge\theta\}\ ; \\ E_0^3 &= \text{span}\,\{dx\wedge dy\wedge\theta\}\ . \end{aligned}$$

The action of d_c on E_0^1 is given by ([11], [9], [3])

$$d_c(\alpha_1 dx + \alpha_2 dy) =$$

$$= -\frac{1}{4}(X^2\alpha_2 - 2XY\alpha_1 + YX\alpha_1)dx\wedge\theta - \frac{1}{4}(2YX\alpha_2 - Y^2\alpha_1 - XY\alpha_2)dy\wedge\theta :=$$

$$:= P_1(\alpha_1,\alpha_2)dx\wedge\theta + P_2(\alpha_1,\alpha_2)dy\wedge\theta\ .$$

Coherently, if $\vec{V} = (V_1, V_2)$ is a horizontal vector field, following [9], [3], [1], we set

$$\text{curl}_{\mathbb{H}}\,\vec{V} = (*d_c(\vec{V}^\natural))^\natural\ ,$$

or, in explicit form

$$\text{curl}_{\mathbb{H}}\,\vec{V} = (P_2(V_1,V_2), -P_1(V_1,V_2))\ .$$

As usual, we put also

$$\text{div}_{\mathbb{H}}\,\vec{V} = XV_1 + YV_2\ .$$

We want to attack Maxwell systems in $\mathbb{H}^1$ mimicking the standard approach in the Euclidean setting, consisting of transforming the system for time-depending vector fields in $\mathbb{R}^3$ into a suitable system for differential 2-forms in $\mathbb{R}^4$ (Faraday differential forms). To this end, consider now $\mathbb{G} := \mathbb{H}^1\times\mathbb{R}$, and denote by (x,y,t) the variables in $\mathbb{H}^1$ and by s the "time" variable in $\mathbb{R}$. Set X, Y, T as above, and $S := \partial_s$. We have $X^\natural = dx$, $Y^\natural = dy$, $S^\natural = ds$, $T^\natural = \theta$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1\oplus V_2$, where $V_1 = \text{span}\,\{X, Y, S\}$ and $V_2 = \text{span}\,\{T\}$. In this case

$$\begin{aligned} E_0^1 &= \text{span}\,\{dx, dy, ds\}\ ; \\ E_0^2 &= \text{span}\,\{dx\wedge ds, dy\wedge ds, dx\wedge\theta, dy\wedge\theta\}\ ; \\ E_0^3 &= \text{span}\,\{dx\wedge dy\wedge\theta, dx\wedge ds\wedge\theta, dy\wedge ds\wedge\theta\}\ . \end{aligned} \tag{17}$$

Lemma 4.1 (see [1])**.** *If*

$$\alpha = \alpha_{13}dx\wedge ds + \alpha_{23}dy\wedge ds + \alpha_{14}dx\wedge\theta + \alpha_{24}dy\wedge\theta \in E_0^2\ ,$$

then

$$\begin{aligned} d_c\alpha &= (X\alpha_{24} - Y\alpha_{14})\,dx\wedge dy\wedge\theta - \\ &- (P_1(\alpha_{13},\alpha_{23}) + S\alpha_{14})\,dx\wedge ds\wedge\theta - \\ &- (P_2(\alpha_{13},\alpha_{23}) + S\alpha_{24})\,dy\wedge ds\wedge\theta\ , \end{aligned} \tag{18}$$

where

$$P_1(V_1,V_2) = -\frac{1}{4}\left(X^2V_2 - 2XYV_1 + YXV_1\right)$$

and

$$P_2(V_1,V_2) = \frac{1}{4}\left(Y^2V_1 - 2YXV_2 + XYV_2\right)\ .$$

Since (E_0^*, d_c) is a complex, we can associate with it a formal system of “Maxwell equations”. Let us start by the counterpart in $\mathbb{H}^1$ of their classical formulation, in terms of vector fields (that in Carnot groups have to be *horizontal* vector fields) and curl operator.

Given the horizontal vector fields in $\mathbb{H}^1$ $\vec{B} = (B_1, B_2)$, $\vec{E} = (E_1, E_2)$, $\vec{D} = (D_1, D_2)$, $\vec{H} = (H_1, H_2)$, $\vec{J} = (J_1, J_2)$, we write

$$\frac{\partial \vec{B}}{\partial s} + \mathrm{curl}_{\mathbb{H}}\, \vec{E} = 0\ , \quad \mathrm{div}_{\mathbb{H}}\, \vec{B} = 0;\ , \tag{19}$$

$$\frac{\partial \vec{D}}{\partial s} - \mathrm{curl}_{\mathbb{H}}\, \vec{H} = -\vec{J}, \quad \mathrm{div}_{\mathbb{H}}\, \vec{D} = \rho\ , \tag{20}$$

For sake of simplicity, we assume all physical constants to be 1.

If, in particular, $\rho \equiv 0$ and $\vec{J} \equiv 0$, i.e. in absence of charges and currents, if we choose the constitutive relations $\vec{D} = \vec{E}$ and $\vec{B} = \vec{H}$, equations (19) and (20) become

$$\frac{\partial \vec{B}}{\partial s} + \mathrm{curl}_{\mathbb{H}}\, \vec{E} = 0\ , \quad \mathrm{div}_{\mathbb{H}}\, \vec{B} = 0\ , \tag{21}$$

$$\frac{\partial \vec{E}}{\partial s} - \mathrm{curl}_{\mathbb{H}}\, \vec{B} = 0\ , \quad \mathrm{div}_{\mathbb{H}}\, \vec{E} = 0\ . \tag{22}$$

Replacing (22) in (21) and then (21) in (22), we get

$$\frac{\partial^2 \vec{B}}{\partial s^2} = -\mathrm{curl}_{\mathbb{H}}(\mathrm{curl}_{\mathbb{H}}\, \vec{B}) \tag{23}$$

$$\frac{\partial^2 \vec{E}}{\partial s^2} = -\mathrm{curl}_{\mathbb{H}}(\mathrm{curl}_{\mathbb{H}}\, \vec{E}). \tag{24}$$

In order to write explicitly $\mathrm{curl}_{\mathbb{H}}(\mathrm{curl}_{\mathbb{H}}\, \vec{B})$ and $\mathrm{curl}_{\mathbb{H}}(\mathrm{curl}_{\mathbb{H}}\, \vec{E})$, it is easier to work now with the forms $(\vec{E})^\natural, (\vec{B})^\natural \in E_0^1$. We use this “traditional” notation $\vec{B}$ (and therefore necessarily also $\vec{E}$) to distinguish between the vector field $\vec{B}$ in (19) and the 2-form B that appears below in the “geometric” formulation and which is related to $\vec{B}$ by the identity $B := *(\vec{B})^\natural$.

Keeping in mind that $0 = -\mathrm{div}_{\mathbb{H}}\, \vec{B} = \delta_c \vec{B}^\natural$, we have

$$\Big(\mathrm{curl}_{\mathbb{H}}(\mathrm{curl}_{\mathbb{H}}\, \vec{B})\Big)^\natural = *d_c(*d_c(\vec{B})^\natural) = \delta_c d_c(\vec{B})^\natural = \big(\delta_c d_c + (d_c\delta_c)^2\big)\,(\vec{B})^\natural =$$

$$= \Delta_{\mathbb{H},1}(\vec{B})^\natural\ ,$$

where $\Delta_{\mathbb{H},1}$ is the (4th order) Rumin’s Laplacian on intrinsic 1-forms (see, e.g., [11], [3]). Accordingly, we can define $\Delta_{\mathbb{H},1}$ acting on horizontal vector fields as $\Delta_{\mathbb{H},1}\vec{B} := \Big(\Delta_{\mathbb{H},1}(\vec{B})^\natural\Big)^\natural$. We stress that this is a positive operator and that, unlike the usual Laplacian on forms, it is not diagonal. Thus, equations (21) in (22) yield

$$\frac{\partial^2 \vec{B}}{\partial s^2} = -\Delta_{\mathbb{H},1}\vec{B} \tag{25}$$

$$\frac{\partial^2 \vec{E}}{\partial s^2} = -\Delta_{\mathbb{H},1}\vec{E}\ . \tag{26}$$

As in the Euclidean setting (see for instance [15]), we can reformulate now the system (19) and (20) in terms of differential forms. We set

$$\begin{aligned} F &= E_1\, dx\wedge ds + E_2\, dy\wedge ds + B_1\, dy\wedge\theta - B_2\, dx\wedge\theta := \\ &:= E\wedge ds + B\ (\in E_0^2) \end{aligned} \tag{27}$$

$$\begin{aligned} M &= H_1\, dx\wedge ds + H_2\, dy\wedge ds - D_1\, dy\wedge\theta + D_2\, dx\wedge\theta := \\ &:= H\wedge ds - D\ (\in E_0^2) \end{aligned} \tag{28}$$

$$\mathcal{J} = J_2\, dx\wedge ds\wedge\theta - J_1 dy\wedge ds\wedge\theta - \rho\, dx\wedge dy\wedge\theta\ (\in E_0^3)\ . \tag{29}$$

In other words, we put $E := \vec{E}^\natural$, $H := \vec{H}^\natural$, $D := -(*\vec{D}^\natural)$.

Proposition 4.2. *Maxwell system* (19) *and* (20) *for horizontal vector fields in* $\mathbb{H}^1$ *is equivalent to the following system for intrinsic 2-forms in* $\mathbb{H}^1\times\mathbb{R}$*:*

$$d_c F = 0 \quad \textit{and} \quad d_c M = \mathcal{J}\ .$$

Proof. We have

$$d_c F = (XB_1 + YB_2)\, dx\wedge dy\wedge\theta -$$
$$-(P_1(E_1,E_2) - SB_2)dx\wedge ds\wedge\theta - (P_2(E_1,E_2) + SB_1)dy\wedge ds\wedge\theta\ .$$

Therefore $d_cF = 0$ if and only if

$$\mathrm{div}_{\mathbb{H}}\, \vec{B} = 0$$

and

$$\frac{\partial B_1}{\partial s} + P_2(E_1,E_2) = 0 \quad \frac{\partial B_2}{\partial s} - P_1(E_1,E_2) = 0\ ,$$

i.e.

$$\frac{\partial \vec{B}}{\partial s} + \mathrm{curl}_{\mathbb{H}}\, \vec{E} = 0\ .$$

In other words, (19) holds if and only if $d_cF = 0$. Analogously (20) is equivalent to

$$d_c M = -(XD_1 + YD_2)\, dx\wedge dy\wedge\theta -$$
$$-(P_1(H_1,H_2) + SD_2)dx\wedge ds\wedge\theta + (-P_2(H_1,H_2) + SD_1)dy\wedge ds\wedge\theta =$$
$$= -\rho\, dx\wedge dy\wedge\theta + J_2 dx\wedge ds\wedge\theta - J_1 dy\wedge ds\wedge\theta = \mathcal{J}\ .$$

$\square$

Remark 4.3. In fact, Proposition 4.2 is not correctly formulated (and proved) when the vector fields $\vec{B}, \vec{E}, \vec{D}, \vec{H}$ just belong to $L^2_{\mathrm{loc}}(\mathbb{H}^1)$ and distributional derivatives are involved. Unfortunately, this is precisely the case we are dealing with in our homogenization theorem below. However, everything can be correctly stated and proved in the setting of the intrinsic currents on $\mathbb{H}^1\times\mathbb{R}$, that provides a good notion of "differential form with distribution coefficients".

We can state now our homogenization result for a Maxwell system in $\mathbb{H}^1$ (see [10] Lemma 2.2 in the Euclidean setting). We stress that it cannot be derived from the general statement of Theorem 3.2 since the intrinsic 2-forms F and M have no pure weight. This follows from (17), but, on the other hand, it is not unexpected, since, in $\mathbb{G} := \mathbb{H}^1\times\mathbb{R}$, $n = 4$ so that $E_0^2 = *E_0^2$. This means that, if $\alpha\in E_0^2$, then $*\alpha\in E_0^2$. Therefore, if all forms in E_0^2 had the same weight, α and $*\alpha$ should have the same weight, that is impossible, since $w(*\alpha) = Q - w(\alpha)$, and $Q = 5$ is odd.

Theorem 4.4. *Let* $\Omega \subset \mathbb{H}^1$ *be a bounded open set, and let* $\vec{B}^\varepsilon$, $\vec{E}^\varepsilon$, $\vec{D}^\varepsilon$, $\vec{H}^\varepsilon$ *and* $\vec{J}^\varepsilon$ *be horizontal vector fields in* $L^2(\mathbb{H}^1, H\mathbb{H}^1) \cap \mathcal{E}'(\Omega, H\mathbb{H}^1)$ *weakly converging to* $\vec{B}$, $\vec{E}$, $\vec{D}$, $\vec{H}$ *and* $\vec{J}$, *respectively. Suppose also that*

$$(J_1^\varepsilon)_{\varepsilon>0}\,,\ (J_2^\varepsilon)_{\varepsilon>0}\,,\ (\rho^\varepsilon)_{\varepsilon>0} \quad \textit{are precompact in } W^{-1,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$$

If

$$d_c F^\varepsilon = 0 \quad \textit{and} \quad d_c M^\varepsilon = \mathcal{J}^\varepsilon\ , \tag{30}$$

then

$$\begin{aligned} \langle \vec{B}^\varepsilon, \vec{H}^\varepsilon\rangle - \langle \vec{D}^\varepsilon, \vec{E}^\varepsilon\rangle &\to \langle \vec{B}, \vec{H}\rangle - \langle \vec{D}, \vec{E}\rangle \\ \langle \vec{B}^\varepsilon, \vec{E}^\varepsilon\rangle &\to \langle \vec{B}, \vec{E}\rangle \\ \langle \vec{D}^\varepsilon, \vec{H}^\varepsilon\rangle &\to \langle \vec{D}, \vec{H}\rangle \end{aligned} \tag{31}$$

in the sense of distribution, as $\varepsilon \to 0$.

Proof. In the sequel, S will always denote a smoothing operator belonging to $\mathcal{OC}^{-\infty}$ that may change from formula to formula, and, with the same convention, we shall denote by S_0 an operator of the form φS, with $S \in \mathcal{OC}^{-\infty}$ and $\varphi \in \mathcal{D}(\mathbb{G})$. Moreover, without loss of generality, we may assume $F^\varepsilon, M^\varepsilon \in \mathcal{D}(\Omega, E_0^2)$. Finally, take $R > 0$ such that $\Omega \subset B(e, R)$.

We start by showing that a decomposition as in Theorem 3.1 holds both for F^ε and M^ε.

The statement for F^ε follows easily arguing as in the proof of Theorem 3.1, Indeed, keeping in mind (30) and (14), we have

$$\begin{aligned} F^\varepsilon &= \Delta_{\mathbb{G},R}^{\underline{N}_2/2}\Delta_{\mathbb{G},R}^{\underline{N}_2/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_3/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_3/2}\,d_c\,F^\varepsilon + \\ &+ d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_1/2}\Delta_{\mathbb{G},R}^{\underline{N}_1/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}F^\varepsilon + S_0F^\varepsilon = \\ &= d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_1/2}\Delta_{\mathbb{G},R}^{\underline{N}_1/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}F^\varepsilon + S_0F^\varepsilon = \\ &= d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_1/2}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta_{\mathbb{G},R}^{\underline{N}_1/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}F^\varepsilon + S_0F^\varepsilon := \\ &:= d_c\,\psi^\varepsilon + \omega^\varepsilon\ . \end{aligned} \tag{32}$$

Clearly, $(\omega^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2(\Omega, E_0^2)$, since S_0 is a properly supported smoothing operator. On the other hand, the components of weights 2 and 3 of $\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}F^\varepsilon$ converge weakly respectively in $W^{4,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ and in $W^{6,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$, so that the components of $\delta_c\ \Delta_{\mathbb{G},R}^{-\underline{N}_2/2}\Delta_{\mathbb{G},R}^{-\underline{N}_2/2}F^\varepsilon$ are sums of terms weakly convergent in $W^{3,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ and in $W^{4,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$, and therefore they converge weakly in $W^{3,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$. Thus eventually ψ^ε converge weakly in $W^{1,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^1)$ and hence strongly in $L^2_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^1)$.

As for M^ε, arguing as above we can write

$$\begin{aligned} M^\varepsilon \quad &= \Delta_{\mathbb{G},R}^{N_2/2}\Delta_{\mathbb{G},R}^{N_2/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_3/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_3/2}\,d_c\,M^\varepsilon + S_0M^\varepsilon + \\ &+ d_c\,\Delta_{\mathbb{G},R}^{N_1/2}\Delta_{\mathbb{G},R}^{N_1/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_2/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_2/2}M^\varepsilon = \\ &= \Delta_{\mathbb{G},R}^{N_2/2}\Delta_{\mathbb{G},R}^{N_2/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_3/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_3/2}\,(\mathcal{J}^\varepsilon) + S_0M^\varepsilon + \\ &+ d_c\,\Delta_{\mathbb{G},R}^{N_1/2}\Delta_{\mathbb{G},R}^{N_1/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_2/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_2/2}M^\varepsilon := \\ &:= \omega^\varepsilon + d_c\psi^\varepsilon\ . \end{aligned} \tag{33}$$

We can show that ψ^ε converge strongly in $L^2_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^1)$ by repeating verbatim the arguments used for F^ε. As for ω^ε, we can reduce again ourselves to prove the L^2-convergence of $\Delta_{\mathbb{G},R}^{N_2/2}\Delta_{\mathbb{G},R}^{N_2/2}\delta_c\Delta_{\mathbb{G},R}^{-N_3/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_3/2}(\mathcal{J}^\varepsilon)$. Thus, $\Delta_{\mathbb{G},R}^{-N_3/2}(\mathcal{J}^\varepsilon)$ is precompact in $W^{3,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^3)$, yielding the (pre)compactness in $W^{3,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^3)$ of $(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_3/2}(\mathcal{J}^\varepsilon)$ and eventually that of $\Delta_{\mathbb{G},R}^{N_2/2}\delta_c\Delta_{\mathbb{G},R}^{-N_3/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_3/2}(\mathcal{J}^\varepsilon)$ in the space $W^{3,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^2)$, since $\Delta_{\mathbb{G},R}^{N_2/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_3/2}$ is an operator of order 0. Thus, we achieve the proof of the L^2-convergence of ω^ε by noticing that $\Delta_{\mathbb{G},R}^{N_1/2}$ is an diagonal operator with components of orders 2 and 3.

Finally, we argue as in the proof of Theorem 3.2 to show that

$$M^\varepsilon\wedge M^\varepsilon\ ,\ M^\varepsilon\wedge F^\varepsilon\ ,\ F^\varepsilon\wedge F^\varepsilon$$

converge in the sense of distributions, that is nothing but (31).

□

Remark 4.5. Hypotheses of Theorem4.4 are not as sharp as one could wish, since formula (18) would suggest a weaker set of assumptions, i.e.

i) $(\rho^\varepsilon)_{\varepsilon>0}$ is precompact in $W^{-1,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$;

ii) $(J_i^\varepsilon)_{\varepsilon>0}$ is precompact in $W^{-2,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ for $i=1,2$.

Unfortunately, we have not been able to prove a better statement, essentially because of the presence of 2-forms of different weights. However, we point out that equations of the form (30) in $\mathbb{H}^n$ with $n>1$ do not present the same difficulties, since all intrinsic 2-forms in $\mathbb{H}^n\times\mathbb{R}$ have weight 2 when $n>1$ (see [1], Example 7.6), and therefore Theorem 4.13 of [1] applies. More precisely, the following result holds.

Theorem 4.6. *Suppose $n>1$ and let $\Omega\subset\mathbb{H}^n$ be a bounded open set. If $\varepsilon>0$, let now $E^\varepsilon, *D^\varepsilon$ and $B^\varepsilon, *H^\varepsilon$ be respectively couples of families of intrinsic 1-forms and 2-forms of $\mathbb{H}^n$ supported in Ω. Let us define the two following families of intrinsic forms in $\mathbb{H}^n\times\mathbb{R}$:*

$$F^\varepsilon := E^\varepsilon\wedge ds + B^\varepsilon\in E_0^2 \quad \textit{and} \quad M^\varepsilon := H^\varepsilon\wedge ds - D^\varepsilon \in E_0^{2n}\ .$$

Analogously, let $\mathcal{J}^\varepsilon = J^\varepsilon\wedge ds - \rho^\varepsilon$ be a family of closed intrinsic $(2n+1)$-form in $\mathbb{H}^n\times\mathbb{R}$, where J^ε and $\rho^\varepsilon\in$ are respectively families of intrinsic $2n$-forms and $(2n+1)$-forms of $\mathbb{H}^n$ (so that ρ^ε is a volume form on the surfaces $s=$ const.). Suppose $F^\varepsilon\to F$, $H^\varepsilon\to H$, $B^\varepsilon\to B$, $D^\varepsilon\to D$, $J^\varepsilon\to J$, $\rho^\varepsilon\to\rho$ as $\varepsilon\to 0$ weakly in $L^2(\Omega,E_0^*)$. Suppose also that

$$(J^\varepsilon)_{\varepsilon>0}\ \textit{and}\ (\rho^\varepsilon)_{\varepsilon>0} \quad \textit{are precompact in}\ W^{-1,2}_{\mathbb{G},\mathrm{loc}}(\mathbb{G},E_0^*)\ .$$

If

$$d_c F^\varepsilon = 0 \quad \textit{and} \quad d_c M^\varepsilon = \mathcal{J}^\varepsilon \ ,$$

then

$$F^\varepsilon \wedge M^\varepsilon \to F \wedge M$$

in the sense of distribution, as $\varepsilon \to 0$*.*

We want to stress that we can define a general set of "geometric Maxwell equations" in any Carnot group $\mathbb{G}$ ($\mathbb{G}$ is always identified with $\mathbb{R}^n$ through exponential coordinates). For sake of simplicity, we restrict ourselves to the case $E^\varepsilon = *D^\varepsilon$ and $B^\varepsilon = *H^\varepsilon$. Let us show preliminarily the following structure lemma for intrinsic forms in $\mathbb{G} \times \mathbb{R}$. From now on, we denote by x a "space" point in $\mathbb{G}$ and by $s \in \mathbb{R}$ the "time", and we choose in $\mathbb{G} \times \mathbb{R}$ as canonical volume form $dV \wedge ds$, where dV is the canonical volume form in $\mathbb{G}$. Moreover, we keep the notation E_0^* for the intrinsic forms on $\mathbb{G} \times \mathbb{R}$; no specific notation will be used for intrinsic forms on $\mathbb{G}$.

Lemma 4.7. *If* $1 \le h \le n$ *then a* h*-form* α *in* $\mathbb{G} \times \mathbb{R}$ *is an intrinsic form in* E_0^h *if and only if we can write*

$$\alpha = \beta \wedge ds + \gamma \ , \tag{34}$$

where β *and* γ *are respectively intrinsic* $(h-1)$*-forms and* h*-forms in* $\mathbb{G}$ *with coefficients depending on* x *and* s*.*

Proof. Without loss of generality, we can restrict ourselves to prove an analogous decomposition for left invariant covectors $\alpha, \gamma \in \bigwedge^h \mathfrak{g}$ and $\beta \in \bigwedge^{h-1} \mathfrak{g}$, where $\mathfrak{g}$ is as above the Lie algebra of the left invariant covectors of $\mathbb{G}$.

First of all, notice that, if $\sigma \in \bigwedge^h \mathfrak{g}$ is an arbitrary left invariant h-covector of $\mathbb{G}$, then $d_0(\sigma \wedge ds) = d(\sigma \wedge ds) = d\sigma \wedge ds = d_0\sigma \wedge ds$ (notice the two d_0 in the previous identity must be meant acting on forms of two different degrees). Thus, if $d_0\beta = 0$ and $d_0\gamma = 0$, then $d_0\alpha = 0$.

On the other hand, let $\alpha' = \beta' \wedge ds + \gamma'$ be arbitrarily given in $\bigwedge^{h-1} \mathfrak{g}$. Then, keeping in mind that covectors in $\bigwedge^\ell \mathfrak{g}$ are orthogonal to covectors of the form $\sigma \wedge ds$ with $\sigma \in \bigwedge^{\ell-1} \mathfrak{g}$, by the very definition of the scalar product, we have

$$\begin{aligned} \langle \alpha, d_0\alpha' \rangle \ &= \langle \beta \wedge ds + \gamma, d_0\beta' \wedge ds + d_0\gamma' \rangle = \\ &= \langle \beta \wedge ds, d_0\beta' \wedge ds \rangle + \langle \beta \wedge ds, d_0\gamma' \rangle + \langle \gamma, d_0\beta' \wedge ds \rangle + \langle \gamma, d_0\gamma' \rangle = \\ &= \langle \beta, d_0\beta' \rangle + \langle \gamma, d_0\gamma' \rangle = 0 \ , \end{aligned}$$

since both β and γ belong to E_0^* and hence are orthogonal to the range of d_0. Thus also α is orthogonal to the range of d_0 and eventually $\alpha \in E_0^h$.

Suppose now α belongs to E_0^h. We can always write it as $\alpha = \beta \wedge ds + \gamma$, with $\beta \in \bigwedge^{h-1} \mathfrak{g}$ and $\gamma \in \bigwedge^h \mathfrak{g}$, and $0 = d_0\alpha = d_0\beta \wedge ds + d_0\gamma$. But $d_0\beta \wedge ds$ and $d_0\gamma$ are orthogonal, and hence $d_0\beta = 0$ and $d_0\gamma = 0$. Let now $\gamma' \in \bigwedge^h \mathfrak{g}$ be given. Since $d_0\gamma'$ is orthogonal to any h-covector of the form $\beta' \wedge ds$ with $\beta' \in \bigwedge^{h-1} \mathfrak{g}$, we have $0 = \langle \alpha, d_0\gamma' \rangle = \langle \beta \wedge ds + \gamma, d_0\gamma' \rangle = \langle \gamma, d_0\gamma' \rangle$, i.e. γ is orthogonal to the range of d_0 and then $\gamma \in E_0^h$. Analogously, if $\beta' \in \bigwedge^{h-1} \mathfrak{g}$, then $0 = \langle \alpha, d_0(\beta' \wedge ds) \rangle = \langle \alpha, d_0\beta' \wedge ds \rangle = \langle \beta \wedge ds, d_0\beta' \wedge ds \rangle + \langle \gamma, d_0\beta' \wedge ds \rangle = \langle \beta, d_0\beta' ds \rangle$. Thus β is orthogonal to the range of d_0 and then $\beta \in E_0^{h-1}$.

$\square$

Let now $\mathcal{J}$ be a fixed closed intrinsic n-form in $\mathbb{G}\times\mathbb{R}$ (a source form). By Lemma 4.7, $\mathcal{J} = J \wedge ds - \rho$, where $\rho(\cdot, s)$ is a volume form on $\mathbb{G}$ for any fixed $s \in \mathbb{R}$.

If $1 \le \ell \le n$, by Lemma 4.7, keeping in mind again that forms in E_0^ℓ are orthogonal to forms of the form $\sigma \wedge ds$ with $\sigma \in E_0^{\ell-1}$, we can define a new (non definite) scalar product $\langle\cdot,\cdot\rangle_L$ in E_0^h as

$$\langle \beta \wedge ds + \gamma, \beta' \wedge ds + \gamma' \rangle_L := \langle \gamma, \gamma' \rangle - \langle \beta, \beta' \rangle \ .$$

Denote by $*_L$ the associated Hodge operator such that $\alpha \wedge *_L\beta = \langle\alpha,\beta\rangle_L$. If $\alpha = \beta \wedge ds + \gamma \in E_0^h$, $1 \le h \le n$, we have

$$*_L\alpha = *\gamma \wedge ds + (-1)^{n-h} * \beta \ .$$

Indeed, if $\sigma = \sigma_1 \wedge ds + \sigma_2$, keeping into account that σ_2 is a h-form in $\mathbb{G}$ and that $*\beta$ is a $(n-h+1)$-form in $\mathbb{G}$, so that $\sigma_2 \wedge *\beta = 0$, then

$$\sigma \wedge (*\gamma \wedge ds + (-1)^{n-h} * \beta) = (-1)^{n-h}\sigma_1 \wedge ds \wedge *\beta + \sigma_2 \wedge *\gamma \wedge ds =$$
$$= \Big(\langle \sigma_2, \gamma + (-1)^{2(n-h)+1} \rangle \, \langle \sigma_1, \beta \rangle \Big) \, dV \wedge ds = \langle \sigma, \alpha \rangle_L \ .$$

If $A \in E_0^1$ (the electromagnetic potential) is given, we write

$$F := d_cA := E \wedge ds + B$$

and

$$*_LF = *B \wedge ds + (-1)^{n-2} * E := H \wedge ds + (-1)^{n-2} D \ .$$

We call *Maxwell equations in* $\mathbb{G}$ the system in E_0^*

$$d_cF = 0 \quad \text{and} \quad d_c(*_LF) = \mathcal{J} \ . \tag{35}$$

Notice that – as usual – the first equation derives trivially from the fact that $F = d_cA$, and that the second one is consistent with $d_c\mathcal{J} = 0$. We notice also that the second equation can be viewed as the Euler-Lagrange equation for the critical points of the functional

$$\mathcal{F}(A) := -\frac{1}{2}\int_{\mathbb{G}} d_cA \wedge *_L d_cA + \int_{\mathbb{G}} A \wedge \mathcal{J} \ .$$

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