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# Some properties of the semigroup generated by a subelliptic Hamilton-Jacobi equation

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**Abstract**[1]. We prove the existence of a (Lipschitz) solution for a Cauchy problem for a subelliptic Hamilton-Jacobi equation with bounded uniformly continuous (Lipschitz) initial data.

## 1. Introduction

Consider the Cauchy problem for a Hamilton- Jacobi equation

(CP) $$u_t + H(x, t, \nabla u) = 0$$

$$u(x, 0) = u_0(x)$$

where the Hamiltonian $H(x, t, p) : \mathbb{R}^N \times [0, T] \times \mathbb{R}^{N \times N} \to \mathbb{R}$ is not coercive in $p$. The lack of coerciveness of the Hamiltonian can be overcome by chancing the underlying geometry with a suitable family of vector fields. More precisely we consider the case $H(x, t, q) = H(x, t, \sigma(x)p)$ where $\sigma(x)$ is a $m \times N$ matrix, $m < N$, where H is coercive in $q = \sigma(x)p$. Here the rows of the matrix $\sigma(x)$ are considered as coefficients of vector fields satisfying the Hörmander condition , which generate a Carnot group, therefore $\sigma(x)\nabla u$ is the horizontal gradient in the Carnot group denoted by $D_h u$, see section 2.

Here we are interested in existence and uniqueness of viscosity solutions of the problem (CP) and in their Lipschitz continuity in space variables (in the group),when $u_0$ is Lipschitz continuous (in the group). We recall that the Lipschitz continuity in the group is the Lipschitz continuity for the right translations with respect to the Carnot-Carathéodory distance on the group and that the Lipschitz continuity is equivalent to the boundness of the horizontal gradient.

We recall that the notion of viscosity solution has been introduced by M.G. Crandall, P.L. Lions, [11]; existence, uniqueness and comparison result for (CP) in the coercive Euclidean case have been summarized in the paper [17] and in the books [4], [5]. We recall also the paper [13], where the existence of viscosity solutions is proved using a Perron method.

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Concerning the case of Carnot groups an existence and comparison result of (Lipschitz continuous in the group) viscosity solutions of a non evolution problem has been proved by B. Stroffolini, [18]. In the evolution case, at our knowledge, there are only results concerning the extension of the Hopf-Lax-Oleinik formula to the subelliptic framework, see [15] in the Heisenberg group case and [8] for the general case. In the Heisenberg case under the assumption that the initial data $u_0$ is compactly supported in a Carnot-Carathéodory ball, Lipschitz for the left translations on the group and Euclidean Lipschitz in the right translations in the direction of the commutator then the solution $u$ is Lipschitz (in the group) [2]. No regularity results are given under the only assumption that $u_0$ is Lipschitz (in the group). Here we prove an existence and continuous dependence on initial data result for viscosity solutions when the initial data is bounded. Moreover we prove that if the initial data is bounded Lipschitz continuous, then the viscosity solution of the Cauchy problem $u(x,t)$ is Lipschitz continuous in $t$ and bounded Lipschitz continuous in the group ( for a.e. $t \in [0,T]$). The method of proof used is founded on the results in [18] and on a discretization in time method that takes into account the results on nonlinear semigroups on general Banach spaces, [10], [7]. For a nonlinear semigroup approach to Euclidean Hamilton-Jacobi equations see [1], [19].

## 2. Carnot Groups

We consider $\mathbb{R}^N$ as a Carnot group with a group operation . and a family of dilations, compatible with the Lie structure.

A Carnot group $G$ of step $r \geq 1$ is a simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ is stratified. This means that $\mathfrak{g}$ admits a decompositions as a vector space sum

$$\mathfrak{g} = \mathfrak{g}_1 \bigoplus \mathfrak{g}_2 \bigoplus \cdots \bigoplus \mathfrak{g}_r$$

such that

$$[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$$

for $j = 1, \cdots, r$ where $\mathfrak{g}_k = \emptyset$ when $k > r$. Let $m_j = dim\ \mathfrak{g}_j$ and denote by $X_{i,j}$ a basis of $\mathfrak{g}_j$ formed by left invariant vector fields. We observe that $G$ considered as a manifold has dimension $N = m_1 + \cdots + m_r$. The horizontal tangent space at a point $\xi \in G$ is the $m_1$ dimensional subspace linearly spanned by $X_{1,1}(\xi), \cdots, X_{m_1,1}(\xi)$. In the following we will denote by $X_1, \cdots, X_m$ a frame of vector fields spanning the firs layer $\mathfrak{g}_1$. The exponential coordinates are given by the diffeomorphism $F : \mathbb{R}^N \to G$ defined by

$$F(x) = \exp\left(\sum_{j=1}^{r}\sum_{j=1}^{m_j} x_{i,j} X_{i,j}\right) .$$

If we denote by $\cdot$ the group operation. The mapping $(\xi, \eta) \to \xi \cdot \eta$ has polynomial entries, when we use the exponential coordinates.
There is a family of dilations compatible with the group operation:

$$\delta_\lambda(x_1, \cdots, x_N) = (\lambda x_{1,1}, \cdots, \lambda x_{1,m}, \lambda^2 x_{2,1}, \cdots, \lambda^2 x_{2,m_2}, \cdots, \lambda^r x_{r,m_r}) .$$

With the above notations the horizontal subspace can be identified by the left translation by $\xi$ of $G_1$.

A horizontal curve $\gamma(t)$, $t \in [0,1]$, is a piecewise smooth curve whose tangent vector $\gamma'(t)$ is in the horizontal tangent space $\gamma(t) \cdot G_1$. Given two points $\xi$ and $\eta$ we consider the set

$$\Gamma(\xi,\eta) = \{\gamma \text{ horizontal curve } : \gamma(0) = \xi, \gamma(1) = \eta\} .$$

By Chow's accessibility theorem, [6], the above set is never empty.
For convenience we fix an Riemannian metric in $\mathfrak{g}$ so that $\mathbb{X} = \{X_{i,j}\}$ is an orthonormal frame and the Riemannian volume element coincides with the Haar measure on $G$ and then with the Lebesgue measure on $\mathbb{R}^N$.

The Carnot-Carathéodory distance is defined as the infimum of the lenght of horizontal curves of the set $\Gamma$:

$$d_{CC}(\xi,\eta) = \inf_{\Gamma(\xi,\eta)} \int_0^1 |\gamma'(t)|\, dt .$$

The Carnot-Carathéodory ball of radius $R$ centered at $\xi$ is given by

$$B(\xi,R) = \{\eta \in G \ : \ d_{CC}(\xi,\eta) < R\} .$$

The Carnot-Carathéodory gauge is defined by

$$|\xi|_{CC} = d_{CC}(0,\xi) .$$

The important property that $|\xi|_{CC}$ is a a.e. solution of the horizontal eikonal equation has been proved by Monti and Serra-Cassano, [16].
A smooth gauge in $G$ is defined by

$$|\xi|_G = \left( \sum_{j=1}^{r} \left( \sum_{i=1}^{m_j} |x_{i,j}|^2 \right)^{r!/j} \right)^{1/2r!} .$$

The following result holds, [6]:

**Theorem 2.1.** *We have*

$$|\xi|_{CC} \simeq |\xi|_G \simeq \sum_{j=1}^{r} \sum_{i=1}^{m_j} |x_{i,j}|^{1/j}$$

$$\text{meas}\,(B(0,R)) \simeq R^Q$$

*where* $Q = \sum_{j=1}^{r} j m_j$ *is called the homogeneous dimension of* $G$.

As a consequence of Theorem 2.1 we have that the Lipschitz continuity (for the right translations) with respect to the Carnot-Carathéodory distance or with respect to the smooth gauge are equivalent.
Examples of Carnot groups are the Heisenberg group and the Engel group.

2.1. **The Heisenberg group.** The Heisenberg group can be identified with $\mathbb{R}^{2N+1}$ endowed with the non commutative group law

$$(x,y,t) \cdot (x',y',t') = \left( x + x', y + y', t + t' + \frac{1}{2}(\langle x, y'\rangle - \langle x', y\rangle) \right)$$

where $x, y \in \mathbb{R}^N$ and $t \in \mathbb{R}$. The Heisenberg algebra is splitted in $V_1 \bigoplus V_2$ where $V_1 = \mathbb{R}^{2N} \times \{0\}$ and $V_2 = \{0\} \times \mathbb{R}$ and it is generated by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\, y_j\, \frac{\partial}{\partial t}$$

$$Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}\, x_j\, \frac{\partial}{\partial t}\ .$$

The only non trivial commutator is

$$[X_j, Y_j] = \frac{\partial}{\partial t}$$

and the homogeneous dimension is $2N+2$.

2.2. **The Engel group.** The Engel group can be identified with $\mathbb{R}^4$ endowed with the non commutative group law

$$(x, y, t, s) \cdot (x', y', t', s') = (x + x', y + y', t + t' + Q_2, s + s' + Q_3)$$

where

$$Q_2 = \frac{1}{2}(xy' - xy)$$

$$Q_3 = \frac{1}{2}(xt' - t'y) + \frac{1}{12}(x^2y' - xx'(y + y') + y(x')^2)\ .$$

The Engel algebra is splitted in $V_1 \bigoplus V_2 \bigoplus V_3$ where $V_1 = \mathbb{R}^2 \times \{0\} \times \{0\}$, $V_2 = \{0\} \times \{0\} \times \mathbb{R} \times \{0\}$, $V_3 = \{0\} \times \{0\} \times \{0\} \times \mathbb{R}$ and it is generated by the vector fields

$$X_1 = \frac{\partial}{\partial x}$$

$$X_2 = \frac{\partial}{\partial y} + x\, \frac{\partial}{\partial s} + \frac{x^2}{2}\, \frac{\partial}{\partial t}\ .$$

The nontrivial commutators are

$$[X_1, X_2] = X_3 = \frac{\partial}{\partial s} + x\, \frac{\partial}{\partial t}$$

$$[X_1, X_3] = [X_1, [X_1, X_2]] = \frac{\partial}{\partial t}$$

and the homogeneous dimension is 7.

## 3. Viscosity solutions

In order to give a first definition of viscosity solution in $G \times (0, T)$ for the equation

$$u_t + H(\xi, t, D_h u) = 0 \tag{3.1}$$

(where $\xi \in G$, $t \in (0, T)$ and $D_h$ is the horizontal gradient in $G$) we must identify the first order jets adapted to our framework.

**Definition 3.1.** A function $u : G \to \mathbb{R}$ is of class $C^1$ (on $G$) if the horizontal derivatives $X_1 u, \cdots, X_m u$ are continuous (on $G$). A function $u : G \times (0, T) \to \mathbb{R}$ is of class $C^1$ (on $G \times (0, T)$) if the horizontal derivatives $X_1 u, \cdots, X_m u$ and $u_t$ are continuous (on $G \times (0, T)$).

We recall that if $u$ is in $C^1$ the following Taylor expansion holds

$$u(\xi, t) = u(\xi_0, t_0) + \langle D_h u(\xi_0, t_0), \overline{\xi_0^{-1} \cdot \xi}\rangle + +u_t(\xi_0, t_0)(t - t_0) + o(|\xi_0^{-1} \cdot \xi|_G)$$

where $D_h u$ is the horizontal gradient and $\overline{\xi}$ is the horizontal projection (of $\xi$).

If a function $u$ is not necessarily smooth but merely upper semicontinuous (on $G\times(0,T)$) we denote by $J^{1,+}_u(\xi_0,t_0)$, $(\xi_0,t_0)\in G\times(0,T)$ the collection of vectors $(p(\xi_0,t_0),r(\xi_0,t_0))\in\mathbb{R}^m\times\mathbb{R}$ such that

$$u(\xi,t)\le u(\xi_0,t_0)+\langle p(\xi_0,t_0),\overline{\xi_0^{-1}\cdot\xi}\rangle++r(\xi_0,t_0)(t-t_0)+o(|\xi_0^{-1}\cdot\xi|_G)\ .$$

**Definition 3.2.** An upper semicontinuous function $u$ is a viscosity subsolution of equation (3.1) if for every $(\xi_0,t_0)\in G\times(0,T)$ and for every $(p_0,r_0)\in J^{1,+}_u(\xi_0,t_0)$ we have

$$r_0+H(\xi_0,t_0,p_0)\le 0\ .$$

If a function $u$ is not necessarily smooth but merely lower semicontinuous (on $G\times(0,T)$) we denote by $J^{1,-}_u(\xi_0,t_0)$, $(\xi_0,t_0)\in G\times(0,T)$ the collection of vectors $(p(\xi_0,t_0),r(\xi_0,t_0))\in\mathbb{R}^m\times\mathbb{R}$ such that

$$u(\xi,t)\ge u(\xi_0,t_0)+\langle p(\xi_0,t_0),\overline{\xi_0^{-1}\cdot\xi}\rangle++r(\xi_0,t_0)(t-t_0)+o(|\xi_0^{-1}\cdot\xi|_G)\ .$$

**Definition 3.3.** A lower semicontinuous function $u$ is a viscosity supersolution of equation (3.1) if for every $(\xi_0,t_0)\in G\times(0,T)$ and for every $(p_0,r_0)\in J^{1,-}_u(\xi_0,t_0)$ we have

$$r_0+H(\xi_0,t_0,p_0)\ge 0\ .$$

**Definition 3.4.** A continuous function $u$ is a viscosity solution of equation (3.1) if it is both a viscosity subsolution and a viscosity supersolution of equation (3.1).

There is other two equivalent definition for viscosity subsolutions, supersolutions and solutions of equation (3.1)

**Definition 3.5.** An upper (lower) semicontinuous function $u$ is a viscosity subsolution (supersolution) of equation (3.1) if for every function $\psi\in C^1(G\times(0,T))$ and every local maximum point $(\xi_0,t_0)$ of $(u-\psi)$ on $G\times(0,T)$ we have

$$\psi_t(\xi_0,t_0)+H(\xi_0,t_0,D_h\psi(\xi_0,t_0))\le(\ge)0\ .$$

A continuous function $u$ is a viscosity solution of equation (3.1) if it is both a viscosity subsolution and a viscosity supersolution of equation (3.1).

Let $\phi$ be upper semicontinuous on $G\times(0,T)$ by $E_+(\phi)$ we denote the set $\{(y,t)\in G\times(0,T)\ ;\ \max\phi>0\}$. Let $\phi$ be lower semicontinuous on $G\times(0,T)$ by $E_-(\phi)$ we denote the set $\{(y,t)\in G\times(0,T)\ ;\ \min\phi<0\}$.

**Definition 3.6.** An upper (lower) semicontinuous function $u$ is a viscosity subsolution (supersolution) of equation (3.1) if for every for every function $\psi\in C^1(G\times(0,T))$ and $k\in\mathbb{R}$ there exists $(\xi_0,t_0)$ in $E_+(\psi(u-k))$ $(E_-(\psi(u-k)))$ such that

$$-\frac{u(\xi_0,t_0)-k}{\psi(\xi_0,t_0)}\ \psi_t(\xi_0,t_0)+H\left(\xi_0,t_0,-\frac{u(\xi_0,t_0)-k}{\psi(\xi_0,t_0)}\ D_h\psi(x_0,t_0)\right)\le(\ge)0\ .$$

A continuous function $u$ is a viscosity solution of equation (3.1) if it is both a viscosity subsolution and a viscosity supersolution of equation (3.1).

In the definitions 3.5, 3.6 the condition $\psi\in C^1(G\times(0,T))$ can be replaced by the condition $\psi\in C^\infty(G\times(0,T))$. The definitions 3.1-3.6 has been given in Euclidean setting in [11] and by [9]; the generalization of Definitions 3.1-3.5 to the Carnot group setting has been given in [18]. The equivalence of the three has been proved in Euclidean setting by [11] and by [9]; in the Carnot groups setting the equivalence is founded on the observation that the three definitions in Euclidean setting and Carnot group setting are equivalent. The following result is proved in [18].

**Proposition 3.1.** *Any bounded viscosity solution of* $|D_h u| \leq C$ *then* $u$ *is Lipschitz for the smooth gauge on* $G$, *i.e.*

$$|u(\xi \cdot \eta) - u(\xi)| \leq C|\eta|_G \ .$$

## 4. Existence of the viscosity solution

In the present section we consider the Cauchy problem

$$u_t + H(\xi, D_h u) = 0 \tag{4.1}$$

$$u(\xi, 0) = u_0(\xi) \ .$$

A viscosity solution of (4.1) is a function, which is continuous on $G \times [0,T]$ with value $u_0(\xi)$ at $t = 0$ and is a viscosity solution on $G \times (0,T)$ of the partial differential equation in (4.1).
Assume that the following hold:

$$|H(\xi', p) - H(\xi, p)| \leq m_H(|\xi^{-1} \cdot \xi'|_G (1 + p)) \tag{4.2}$$

$$|H(\xi, 0)| \leq C \tag{4.3}$$

$$\lim_{|p| \to +\infty} H(\xi, p) = +\infty \quad \text{uniformly in } \xi \tag{4.4}$$

where $m_H$ is an increasing bounded function such that $m_H(t) \to 0$ when $t \to 0$.
The following results are proved in [18]:

**Proposition 4.1.** *Consider the following Hamilton-Jacobi equations*

$$\gamma u + H(\xi, D_h u) = 0 \quad \text{on } G \tag{4.5}$$

*where gamma* $> 0$. *There exists a unique bounded viscosity solution* $u$ *of* (4.5). *Moreover* $u$ *is Lipschitz for the smooth gauge on* $G$ (*and then also for the Carnot-Carathéodory distance*).

**Proposition 4.2.** *Let* $u$ *be a bounded viscosity subsolution and* $v$ *a bounded viscosity supersolution of the equation* (4.2). *Then we have* $u \leq v$.

We denote by $BUC(G)$ the space of bounded uniformly continuous functions on $G$, we observe that $BUC(G)$ is a Banach space for the norm $\|u\|_{BUC} = \sup_G u(\xi)$.
From Proposition 4.1 and 4.2 we easily have:

**Proposition 4.3.** *Consider the following Hamilton-Jacobi equation*

$$\gamma u + H(\xi, D_h u) = f \quad \text{on } G$$

$$\gamma v + H(\xi, D_h v) = g \quad \text{on } G$$

*where* $f, g \in BUC(G)$. *Let* $u, v$ *be viscosity solutions of the above equations; then*

$$\|u - v\|_{L^\infty(G)} \leq \|f - g\|_{L^\infty(G)} \ .$$

Define the operator $A$ on $BUC(G)$ by $u \in BUC(G)$ is in $D(A)$ if there is a $g \in BUC(G)$ for which $H(\xi, D_h u) = g$ in viscosity sense and then set $Au = g$. The domain of $A$ is dense in $BUC(R^N)$. From Proposition 4.1, 4.3 the operator $A$ satisfies the following properties

(i) $R(I + \lambda A) = BUC(G)$ for every $\lambda > 0$
(ii) $J_\lambda = (I + \lambda A)^{-1}$ is a contaction on $BUC(G)$.

The properties (i), (ii) imply that $A$ is $m$-accretive in $BUC(G)$.
By the Crandall-Ligget theorem, [3] [7] [10] [12], the functions $u_n$ defined by the problems

$$u_n(t) = u_0 \quad \text{for } t \in \left[-\frac{T}{n}, 0\right] \tag{4.6}$$

$$\frac{u_n(t) - u_n(t-(T/n))}{T/n} + Au_n(t) = 0 \quad \text{for } t > 0$$

converge uniformly on $[0,T]$ in $BUC(G)$ to a function $u \in C(0,T; BUC(G))$ with $u(t) = u_0$ moreover there exists a constant $C$ such that if $u_0 \in D(A)$ we have

$$\|u_n(t) - u_n(\tau)\|_{BUC} \leq C|t-\tau| \tag{4.7}$$

$t, \tau \in [0,T]$ and $C$ depends on $\|Au_0\|_{BUC}$ (see (1.9) in [10] with $n = m$, $\mu = t/n$ and $\lambda = \tau/n$). Then $u(t)$ is also Lipschitz continuous in $BUC(G)$.

From (4.7) we have also that

$$\left\|\frac{u_n(t) - u_n(t-(T/n))}{T/n}\right\|_{BUC(G)} \leq C$$

$t \in [0,T]$ where $C$ depends on $\|Au_0\|_{BUC}$.
From [18] Theorem 3 we obtain that the sequence $u_n(\xi,t)$ is bounded in $L^\infty(0,T; L_G(G))$ where $L_G(G)$ denotes the space of bounded Lipschitz function (for the smooth gauge) on $G$; then $u(\xi,t))$ is also in $L^\infty(0,T; L_G(G))$. We recall, [3] [7] [10] [12], that $u$, $u_n$ depends continuously in $BUC(G)$ from $u_0$ then an approximation method gives that, if $u_0$ is in $L_G(G)$ then $u \in L^\infty(0,T; L_G(G))$. In conclusion if $u_0$is in $L_G$ we have that $u_t \in L^\infty(G \times [0,T])$, $D_h u \in (L^\infty(G \times [0,T]))^m$.

We now prove that $u$ is a viscosity solution of the Hamilton-Jacobi equation in (4.1).
Let $k \in \mathbb{R}$, $\psi \in C^1(G \times (0,T))$ and $E_+(\psi(u-k)) = \{(\xi_0, t_0)\}$. Set $u_n(\xi,t) = u_n(t)(\xi)$. Since $u_n$ converges to $u$ locally uniformly there will be an $(\xi_n, t_n)$ in $E_+(\psi(u_n - k))$ for $n$ large enough such that $t_0 \neq rT/n$, $r = 1, \cdots, n$. Clearly $(\xi_n, t_n) \to (\xi_0, t_0)$.
We have

$$\psi(\xi_n, t_n)(u_n(\xi_n, t_n) - k) \geq \psi(\xi, t)(u_n(\xi,t) - k) \tag{4.8}$$

Since $\xi_n \in E_+(\psi(u_n(\cdot, t_n) - k))$ by Definition 3.6 in the case independent of $t$ we have

$$\frac{u_n(\xi_n, t_n) - u_n(\xi_n, t_n - (T/n))}{T/n} + H\left(-\frac{u_n(\xi_n, t_n) - k}{\psi(\xi_n, t_n)}\, D_h\psi(\xi_n, t_n)\right) \leq 0 \tag{4.9}$$

By (4.4) we have

$$\psi(\xi_n, t_n)\left(u_n(\xi_n, t_n) - u\left(\xi_n, t_n - \frac{T}{n}\right)\right) = \tag{4.10}$$

$$= \psi(\xi_n, t_n)(u_n(\xi_n, t_n) - k) - \psi\left(\xi_n, t_n - \frac{T}{n}\right)\left(u_n\left(\xi_n, t_n - \frac{T}{n}\right) - k\right) -$$

$$- \left(\psi(\xi_n, t_n) - \psi\left(\xi_n, t_n - \frac{T}{n}\right)\right)\left(u_n\left(\xi_n, t_n - \frac{T}{n}\right) - k\right) \geq$$

$$\geq - \left(\psi(\xi_n, t_n) - \psi\left(\xi_n, t_n - \frac{T}{n}\right)\right)\left(u_n\left(\xi_n, t_n - \frac{T}{n}\right) - k\right).$$

Then

$$-\frac{1}{\psi(\xi_n,t_n)}\,\frac{\psi(\xi_n,t_n)-\psi(\xi_n,t_n-(T/n))}{T/n}\,\left(u_n\left(\xi_n,t_n-\frac{T}{n}\right)-k\right)+$$

$$+H\left(-\frac{u_n(\xi_n,t_n)-k}{\psi(\xi_n,t_n)}\,D_h\psi(\xi_n,t_n)\right)\leq 0\ .$$

When $n\to+\infty$ we find

$$-\frac{u(\xi_0,t_0)-k}{\psi(\xi_0,t_0)}\,\psi_t(x_0,t_0)+H\left(-\frac{u_0(\xi_0,t_0)-k}{\psi(\xi_0,t_0)}\,D_h\psi(\xi_0,t_0)\right)\leq 0$$

so $u$ is a viscosity subsolution of the partial differential equation in (4.1). Similarly we prove that $u$ is a supersolution of the partial differential equation in (4.1). We recall also that $u\in C(0,T;BUC(G))$ and $u(t)=u_0$; then $u$ is a viscosity solution of the Cauchy problem (4.1).
We have so proved the following result:

**Theorem 4.1.** *Let $u_0\in BUC(G)$. There exists a viscosity solution of the Cauchy problem* (4.1) *in $C(0,T;BUC(G))$. Moreover if $D_hu_0\in L^\infty(G)$ There exists a viscosity solution of the Cauchy problem* (4.1) *in $C(0,T;BUC(G))$ such that $u_t\in L^\infty(G\times[0,T])$, $D_hu\in(L^\infty(G\times[0,T])^m$*

**Remark 4.1.** It can be proved that a viscosity solution $u$ of the partial differential equation in (4.1) such that $u_t\in L^\infty(G\times[0,T])$, $D_hu\in(L^\infty(G\times[0,T])^m)$ verifies the partial differential equation a.e. in $G\times(0,T]$.

## 5. Uniqueness and continuous dependence on the initial data of the viscosity solution

In the present section we will prove the following result:

**Theorem 5.1.** *Assume that the condition* (4.2) *hold. Let $u^1$ ($u^2$) be a viscosity subsolution* (*supersolution*) *in $BUC(G\times[0,T])$ of the Hamilton- Jacobi equation in* (4.1)*, then*

$$\sup_{G\times[0,T]}(u^1-u^2)\leq\sup_G(u^1(\cdot,0)-u^2(\cdot,0))\ .$$

*Proof.* Let

$$A=\sup_G(u^1(\cdot,0)-u^2(\cdot,0))\ .$$

Since $u^1,u^2\in BUC(G\times[0,T])$ we have

$$|u^i(\xi,t)-u^i(\eta,s)|\leq Cm(|\xi^{-1}\eta|_G+|t-s|) \tag{5.1}$$

where $m(r)$ is a positive increasing bounded continuous function with $m(0)=0$ that goes to 0 as $r\to 0$ The proof is by contradiction; assume that

$$\sup_Q(u^1-u^2)>A+\sigma_0$$

where $Q=G\times[0,T]$ and $\sigma_0>0$. Consider for $\alpha$ and $\varepsilon>0$

$$\Phi(\xi,t,\eta,s)=u^1(\xi,t)-u^2(\eta,s)-\frac{|\xi^{-1}\eta|_G^r}{\varepsilon}-\frac{|t-s|}{\alpha}\ .$$

We observe that

$$|D_{h,\xi}(|\xi^{-1}\cdot\eta|_G^r)\leq C_1r|\xi^{-1}\cdot\eta|_G^{r-1}$$

$$|D_{h,\eta}(|\eta^{-1}\cdot\xi|_G^r) \le C_1 r|\eta^{-1}\cdot\xi|_G^{r-1}$$

$$D_{h,\xi}(|\xi^{-1}\cdot\eta|_G^r) = -D_{h,\eta}(|\eta^{-1}\cdot\xi|_G^r)\ .$$

From (5.1) $\Phi$ is bounded on $Q^2$. Then for every $0<\delta<\sigma_0/2$ there exisits $(\xi_0,t_0,\eta_0,s_0)$ such that

$$\Phi(\xi_0,t_0,\eta_0,s_0)+\delta > \sup_{Q^2}\Phi\ .$$

Let $\zeta:\mathbb{R}^2\to\mathbb{R}$ such that $\zeta(0,0)=1$, $0\le\zeta\le1$, $\zeta((x,y)=0$ for $|(x,y)|>\frac{3}{2}$, $|D\zeta(x,y)|\le 1$, consider now the function

$$\theta(\xi,\eta)=\zeta(|\xi^{-1}\cdot\xi_0|_G^r, |\eta^{-1}\cdot\eta_0|_G^r)\ .$$

We have

$$\theta(\xi_0,\eta_0)=1\ ,\quad 0\le\theta\le1\ ,\quad |D_{h,\xi}\theta|\le K\ ,\quad |D_{h,\eta}\theta|\le K$$

(we have used the relations $|D_{h,\xi}(|\xi^{-1}\cdot\xi_0|_G^r)\le C_1 r|\xi^{-1}\cdot\xi_0|_G^{r-1}$, $|D_{h,\eta}(|\eta^{-1}\cdot\eta_0|_G^r)\le C_1 r|\eta^{-1}\cdot\eta_0|_G^{r-1}$).
We denote

$$\Psi(\xi,t,\eta,s)=\Phi(\xi,t,\eta,s)+\delta\theta(\xi,\eta)-\sigma t$$

where $0<\sigma<\sigma_0/2T$.
There exists $(\overline{\xi},\overline{t},\overline{\eta},\overline{s})$ such that

$$\Psi(\overline{\xi},\overline{t},\overline{\eta},\overline{s})=\sup_{Q^2}\Psi\ .$$

The inequalities

$$2\Psi(\overline{\xi},\overline{t},\overline{\xi},\overline{s})\ge\Psi(\overline{\xi},\overline{t},\overline{\xi},\overline{t})+\Psi(\overline{\eta},\overline{s},\overline{\eta},\overline{s})$$

and (5.1) yield

$$\frac{|\overline{\xi}^{-1}\overline{\eta}|_G^r}{\varepsilon}\le m(|\overline{\xi}^{-1}\overline{\eta}|_G)+\delta$$

$$\frac{|\overline{t}-\overline{s}|}{\alpha}\le m(|\overline{t}-\overline{s}|)+\sigma|\overline{t}-\overline{s}|\ .$$

Then

(5.2) $$|\overline{\xi}^{-1}\overline{\eta}|_G^r\le C_2\varepsilon\ ,\quad |\overline{t}-\overline{s}|\le C_2\alpha$$

so

$$|\overline{\xi}^{-1}\overline{\eta}|_G^r\le C_2\varepsilon m(\varepsilon)+\delta\varepsilon\ .$$

Taking into account Definition 3.5 and assuming $(\overline{\xi},\overline{t}),(\overline{\eta},\overline{s})\in(0,T]$ we obtain

$$\sigma+\frac{\overline{t}-\overline{s}}{\alpha}+H\left(\overline{\xi},\frac{1}{\varepsilon}\,D_{h,\xi}(|\overline{\xi}^{-1}\overline{\eta}|_G^r)\right)+\delta D_{h,\xi}(\theta(\overline{\xi},\overline{\eta}))\le0$$

$$0\le\frac{\overline{t}-\overline{s}}{\alpha}+H\left(\overline{\xi},-\frac{1}{\varepsilon}D_{h,\eta}(|\overline{\xi}^{-1}\overline{\eta}|_G^r)\right)+\delta D_{h,\eta}(\theta(\overline{\xi},\overline{\eta}))\le0\ .$$

Taking into account the assumption (4.2) we obtain

(5.3) $$\sigma\le m_H((\varepsilon m(\varepsilon)+\delta)^{1/r}\,\frac{1}{\varepsilon}\,(\varepsilon m(\varepsilon)+\delta)^{(r-1)/r})$$

Choosing $\delta\le\varepsilon m(\varepsilon)$ and $\varepsilon$ small enough to make the right hand side of strictly less than $\sigma$ we obtain the desired contradiction. In order to conclude the proof we have to investigate the case where $\bar{t} = 0$ or $\bar{s} = 0$. We show that this case can not occur if $\alpha$ is small enough. Assume that there is a sequence $\alpha_n \to 0$ such that the corresponding $\Psi$ attains the maximum at $(\xi_n, t_n, \eta_n, s_n)$ with $t_n = 0$ or $s_n = 0$. If $t_n = 0$ we obtain as in the first part of the proof

$$u^1(\xi_n, 0) - u^2(\eta_n, s_n) \leq A + u^2(\eta_n, 0) - u^2(\eta_n, s_n) \leq A + m(C\alpha_n) \; .$$

Similarly if $s_n = 0$

$$u^1(\xi_n, t_n) - u^2(\eta_n, 0) \leq A + m(C\alpha_n) \; .$$

Then

$$\sup_Q \Psi(\xi, t, \xi, t) \leq \sup_{Q^2} \Psi \leq A + 2m(C\alpha_n) + \delta \; .$$

We have also

$$\sup_Q \Psi(\xi, t, \xi, t) \geq \sup_Q (u^1 - u^2) - \sigma T > A + \sigma_0 - \sigma T \; .$$

Choosing $n$ large enough and $\sigma, \delta$ small enough we obtain the desired contradiction and the result is proved.

**Remark 5.1.** If we consider viscosity subsolution and supersolution $u_1, u_2 \in BUC(G \times [0,T])$ with $u_1, u_2$ locally Lipschitz on $(0,T] \times G$ for the smooth gauge on G and for the usual distance on $\mathbb{R}$ we can replace the assumption (4.2) by

$$|H(\xi\prime, p) - H(\xi, p)| \leq m_H(|\xi^{-1} \cdot \xi'|_G(1+p))Q(\xi, \xi\prime, p) \tag{5.4}$$

where $m(t) \to 0$ when $t \to 0$ and $Q(\xi, \xi', p) = \max\{\Phi(H(\xi, p)), \Phi(H(\xi, p))\}$ and $\Phi$ is a continuous function from $\mathbb{R}$ in $\mathbb{R}_+$.

From remark 5.1 the following results:

**Theorem 5.2.** *Assume that the conditions* (4.2) (4.3) (4.4) *hold. Let* $u^1$ $(u^2)$ *be a viscosity solutions in* $BUC(G \times [0,T])$ *of the Cauchy problem* (4.1)*, then*

$$\sup_{G \times [0,T]} (u^1 - u^2) \leq \sup_G (u^1(\cdot, 0) - u^2(\cdot, 0)) \; .$$

*Assume that the conditions* (5.4)*)*(4.3)(4.4) *hold. Let* $u^1$ $(u^2)$ *be a viscosity solutions in* $BUC(G \times [0,T])$ *of the Cauchy problem* (4.1) *with* $u_1, u_2$ *locally Lipschitz on* $(0,T] \times G$ *for the smooth gauge on* $G$ *and for the usual distance on* $\mathbb{R}$*, then*

$$\sup_{G \times [0,T]} (u^1 - u^2) \leq \sup_G (u^1(\cdot, 0) - u^2(\cdot, 0)) \; .$$

**Remark 5.2.** The existence and uniqueness of viscosity solution of the Cauchy problem (4.1) and its identification as the limit of solution of problems (4.6) prove that in our framework the notion of viscosity solution coincides with the notion of "bonne solution" (good solution) introduced in the theory of nonlinear semigroups in Banach spaces, see [7].

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