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Conformal Killing vector fields and Rellich type identities on Riemannian manifolds, I

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This paper is dedicated to Professor Ermanno Lanconelli on the occasion of his 65th birthday

Abstract[1]. We establish generalizations to Riemannian manifolds of the Rellich's identity for pairs of functions, the Hardy inequality and the Pohozaev's identity for classical solutions of elliptic Hamiltonian systems of semilinear differential equations. The applied method is based on the use of conformal Killing vector fields. We also discuss the relationship between the Rellich's and the Pohozaev's identities.

1. Introduction

In his pioneering work [10] Stanislav Ivanovich Pohozaev obtained the following identity

$$\int_\Omega \left[\frac{n-2}{2}\, uf(u) - nF(u)\right] dx = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x,\nu)\, ds \,, \tag{1}$$

for solutions of the problem

$$\begin{cases} \Delta u + f(u) = 0 \quad \text{in } \Omega \,, \\ \\ \quad u = 0 \quad \text{on } \partial\Omega \,, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain, $F(u) = \int_0^u f(z)\, dz$ and ν is the outward unit normal vector to $\partial\Omega$. In another fundamental paper [11] he obtained integral identities and nonexistence results for solutions of general variational problems. In [12] P. Pucci and J. Serrin discuss variants of the Pohozaev's identity which enabled them to prove various nonexistence results.

The purpose of this paper is three-fold. Our first aim is to establish a generalization to Riemannian manifolds of the Pohozaev's identity for classical *solutions* of elliptic Hamiltonian systems of semilinear differential equations. The other purpose is to find a Rellich type identity for *functions* on Riemannian manifolds and to

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discuss the close relationship between the Pohozaev's and the Rellich's identities. We shall also obtain a Hardy inequality on Riemannian manifolds. The main point of the used method is common for the three types of results and can be described as follows.

To begin with, we emphasize that the corner stone of the original Pohozaev's approach is the use of conformal vector fields. In this regard R. Schoen and S.-T. Yau have observed that 'One can use conformal vector fields to derive certain identities for some special differential equations. Such a fact was first discovered by S. I. Pohozaev [10], who made use of $X = r\,\partial/\partial r$ on $\mathbb{R}^n$. ([16], p. 196). Indeed, the radial vector field

$$\xi = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} = r\frac{\partial}{\partial r}\ ,$$

used by Pohozaev, is a conformal Killing vector field satisfying $div\,(\xi) = n$ and the corresponding to it point transformation is a dilation in $\mathbb{R}^n$. Actually this observation is our starting point and main motivation to write the present paper in which we shall widely use conformal Killing vector fields.

Let M be an $n-$dimensional oriented compact manifold, $n \geq 3$, endowed with a Riemannian metric $g = (g_{ij})$. We assume that M has a boundary $\partial M \neq \emptyset$ of class C^∞. We shall study the following system on M consisting of $2m$ equations:

$$(2)\qquad \begin{cases} -\Delta_g u^1 &= H_{v^1}\ , \\ -\Delta_g v^1 &= H_{u^1}\ , \\ \cdots & \\ -\Delta_g u^m &= H_{v^m}\ , \\ -\Delta_g v^m &= H_{u^m}\ , \end{cases}$$

where Δ_g is the Laplace-Beltrami operator associated to the metric g, the function $H = H(u,v) = H(u^1, \cdots, u^m, v^1, \cdots, v^m) \in C^1(\mathbb{R}^{2m})$ and $H_{u^1} = \partial H/\partial u^1$, etc.

Note that throughout the paper we shall use the Einstein summation convention, that is, summation from 1 to n over repeated Latin indices and from 1 to m over repeated Greek indices is understood.

The first result in this paper is the following

Theorem 1. *Suppose that (M,g) admits a conformal Killing vector field $\xi = \xi^i \partial/\partial x^i$ which is not an isometry of M, that is, ξ satisfies*

$$(3)\qquad \nabla^k \xi^s + \nabla^s \xi^k = \mu(x) g^{ks} = \frac{2}{n}\, div\,(\xi)\, g^{ks}\ ,$$

where ∇^k is the covariant derivative corresponding to the Levi-Civita connection, uniquely determined by g, div is the covariant divergence operator and $\mu = \mu(x) \not\equiv 0$. Assume also that $H(0,0) = 0$.

Then the classical solutions of the Hamiltonian system (2) *with homogeneous Dirichlet boundary conditions satisfy the identity*

$$(4)\qquad \int_M \mu \left[\frac{n-2}{4}\,(u^\alpha H_{u^\alpha} + v^\alpha H_{v^\alpha}) - \frac{n}{2}\,H\right] dV + \frac{n-2}{4}\int_M \Delta_g \mu (u^\alpha v^\alpha)\, dV = = -\int_{\partial M} (g^{ij} u^\alpha_i v^\alpha_j)(\xi,\nu)\, dS\ ,$$

where dV and dS are the volume and surface measures with respect to the metric g, the vector ν is the outward unit normal to ∂M, $u_i^\alpha := \partial u^\alpha/\partial x_i$, $v_j^\alpha := \partial v^\alpha/\partial x_j$ and $(\xi,\nu) = g_{ij}\xi^i\nu^j = \xi^i\nu_i$.

In fact, the curvature of M appears in the identity (4) as can be seen from

Corollary 1. *The identity* (4) *is equivalent to*

$$\int_M \mu\left[\frac{n-2}{4}(u^\alpha H_{u^\alpha} + v^\alpha H_{v^\alpha}) - \frac{n}{2}\,H\right] dV - \tag{5}$$
$$-\frac{n-2}{4(n-1)}\int_M (\mathcal{L}_\xi R + \mu R)(u^\alpha v^\alpha)\,dV =$$
$$= -\int_{\partial M} (g^{ij}u_i^\alpha v_j^\alpha)(\xi,\nu)\,dS\ ,$$

where $\mathcal{L}_\xi R$ is the Lie derivative of the scalar curvature R with respect to the vector field ξ.

Now let us suppose that M admits a conformal Killing vector field

$$\xi = \xi^i\frac{\partial}{\partial x^i}$$

such that

$$\nabla^k\xi^s + \nabla^s\xi^k = c\,g^{ks} = \frac{2}{n}\,div\,(\xi)\,g^{ks}\ , \tag{6}$$

where $c \neq 0$ is a constant. That is, we suppose that M admits a homothety which is not an infinitesimal isometry of M. In this case we may assume that $c = 2$ in (6) and hence $div\,(\xi) = n$. (Otherwise, since $c \neq 0$, we could consider $2\xi/c$ instead of ξ.) With this at hand, we state another Pohozaev's identity for the Hamiltonian system (2) as follows

Theorem 2. *Suppose that (M, g) admits a conformal Killing vector field ξ such that*

$$div\,(\xi) = n\ . \tag{7}$$

Assume also that $H(0,0) = 0$. Then the classical solutions of the Hamiltonian system (2) *with homogeneous Dirichlet boundary conditions satisfy the identity*

$$\int_M\left[\frac{n-2}{2}\,(a^\alpha u^\alpha H_{u^\alpha} + b^\alpha v^\alpha H_{v^\alpha}) - n\,H\right] dV = -\int_{\partial M}(g^{ij}u_i^\alpha v_j^\alpha)(\xi,\nu)\,dS\ , \tag{8}$$

where a^α and b^α are constants such that $a^\alpha + b^\alpha = 2$, $\alpha = 1,\cdots,m$.

Since the system (2) has a variational structure the proof of Theorems 1 and 2 is by an application of the general Noetherian approach to Pohozaev's identities which we proposed and discussed in [3]. The main point of this method is the observation that the Pohozaev's identity for *solutions* of differential equations and systems can be obtained from the Noether's identity for *functions* after integration and application of the divergence theorem, taking into account the boundary conditions. In order not to increase the volume of this paper as well as to avoid some repetitions we direct the interested reader to [3] for details and applications of the Noether approach to Pohozaev's identities. We merely point out that this approach applies to more general Hamiltonian systems of type:

$$\begin{cases} Lu &= H_v(u,v), \\ L^*v &= H_u(u,v) \end{cases}$$

where L is a linear higher order elliptic operator in divergence form and L^* is its formally adjoint operator. Clearly the system (2) is of this kind.

A direct consequence of (8) and the maximum principle is the following

Corollary 2. *Suppose that (M, g) admits a conformal Killing vector field ξ such that the relation (7) holds. Let $H = H(s,t) = H(s^1, \dots, s^m, t^1, \dots, t^m) \in C^1(\mathbb{R}^{2m})$ satisfy the conditions*

(1) $H(0,0) = \dfrac{\partial H}{\partial s^\alpha}(0,0) = \dfrac{\partial H}{\partial t^\alpha}(0,0) = 0$, $\alpha = 1, \cdots, m$;

(2) *if* $s^\alpha, t^\alpha \geq 0$, $\alpha = 1, \cdots, m$, *then*

$$\frac{\partial H}{\partial s^\alpha}(s,t) \geq 0 \quad \text{and} \quad \frac{\partial H}{\partial t^\alpha}(s,t) \geq 0\ , \quad \alpha = 1, \dots, m\ ;$$

(3) *there exist constants $c_1 \geq 2n/(n-2)$ and $c_2 \in (0,2)$ such that for any $s \in \mathbb{R}^m$ and $t \in \mathbb{R}^m$:*

$$c_1 H(s,t) \leq c_2 s^\alpha H_{s^\alpha}(s,t) + (2 - c_2) t^\alpha H_{t^\alpha}(s,t)\ . \tag{9}$$

In addition, suppose that $(\xi, \nu) > 0$ on ∂M.

Then there is no nontrivial classical solution (that is $C^2(M) \cap C^1(\bar{M})$) of the Hamiltonian system (2)) with homogeneous Dirichlet boundary conditions.

This result is the analog to Riemannian manifolds of the nonexistence result by Mitidieri [8] and its proof follows the same argument presented in [8]. See section 3.

Another consequence of Theorems 1 and 2 is the *exact* generalization to Riemannian manifolds of the 1965 Pohozaev's identity (1). Namely:

Corollary 3. *Suppose that (M, g) admits a conformal Killing vector field ξ such that the relation (7) holds. Then the classical solutions u of the problem*

$$\begin{cases} \Delta_g u + f(u) = 0 \quad \text{in } M\ , \\[1em] \quad u = 0 \quad \text{on } \partial M\ , \end{cases} \tag{10}$$

satisfy

$$\int_M \left[\frac{n-2}{2}\, u f(u) - n F(u)\right] dV = -\frac{1}{2} \int_{\partial M} (g^{ij} u_i u_j)(\xi, \nu)\, dS\ . \tag{11}$$

More generally, we have

Corollary 4. *Suppose that (M, g) admits a conformal Killing vector field ξ such that the relation (3) holds. Then the classical solutions u of the problem (10) satisfy*

$$\begin{aligned} \int_M \mu \left[\frac{n-2}{4}\, u f(u) - \frac{n}{2}\, F(u)\right] dV - \frac{n-2}{8(n-1)} \int_M (\mathcal{L}_\xi R + \mu R) u^2\, dV = \\ = -\frac{1}{2} \int_{\partial M} (g^{ij} u_i u_j)(\xi, \nu)\, dS\ , \end{aligned} \tag{12}$$

where $\mathcal{L}_\xi R$ is the Lie derivative of the scalar curvature R with respect to the vector field ξ.

Obviously these results follow directly from Theorems 1 and 2. Although the identity (11) should be well-known it is presented here for sake of completeness since we have not seen it in the literature explicitly stated in this way. In fact, the form of the identity (11) is the same as that of the identity (1) - just the inner norm (in terms of the metric g) of the gradient of u appears in the right-hand side of (11) in the place of the usual Euclidean norm as well as the scalar product $(\xi,\nu) = g_{ij}\xi^i\nu^j = g^{ij}\xi_i\nu_j = \xi^i\nu_i$.

We observe that the vector field

$$X = \xi^i \frac{\partial}{\partial x^i} + au\frac{\partial}{\partial u}$$

where a is a constant, is a variational symmetry of the equation in (10) if and only if $a = (2-n)/2$ and $f(u) = u^p$, $p = (n+2)/(n-2)$ being the critical Sobolev exponent. Thus, in the sense of [4], the equation

$$\Delta_g u + u^{(n+2)/(n-2)} = 0$$

is the unique critical Poisson equation on M. In regard to the invariance properties of the latter equation see [7] and the references therein. For a complete group classification of nonlinear Poisson equations on Riemannian manifolds and the relations between their Lie point symmetries and the corresponding conformal group see [2].

In fact, the nonlinear Poisson equation in (10) has been previously studied by various authors with regard to some geometrical problems. In particular, A. Ratto and M. Rigoli obtained in [13] a priori estimates and Liouville theorems for this equation on complete Riemannian manifolds. See also [7]. In the context of the geometry of the considered Riemannian manifold a variant of the Pohozaev's identity was established by R. Schoen in [15]. An identity which generalizes that of R. Schoen [15] was obtained by M. Gursky in [6]. Another remarkable Pohozaev type identity for a nonlinear eigenvalue equation involving the Dirac operator on Riemannian manifolds with boundary was obtained in [1].

We point out that the method in this paper based on the use of conformal Killing vector fields can be applied to higher order differential equations on Riemannian manifolds. Just for an illustration, we shall establish here a Pohozaev type identity for semilinear differential equations involving the biharmonic operator.

Corollary 5. *Suppose that (M,g) admits a conformal Killing vector field ξ such that*

$$div\,(\xi) = n\ .$$

Then the classical solutions of the biharmonic equation on M

$$\Delta_g^2 u = f(u) \tag{13}$$

with Navier boundary conditions

$$u = \Delta_g u = 0 \tag{14}$$

on ∂M satisfy the identity

$$\int_M \left[\frac{n-4}{2}\,uf(u) - nF(u)\right] dV = \int_{\partial M} g^{ij} u_i (\Delta_g u)_j (\xi,\nu)\, dS\ . \tag{15}$$

We would like to emphasize again that the Pohozaev's identity is an identity for *solutions* of the problem under investigation. Another powerful tool in the study of differential equations and systems are the so-called Rellich type identities [8], generalizing to a pair of *functions* the classical Rellich identity [14] for a single function. That is, the Rellich type identity is an integral identity which concerns *functions*, without any reference to the equation(s) or boundary condition(s) which they may satisfy. This makes it an important a priori instrument for obtaining among other things, nonexistence results [8] and sharp Hardy type inequalities [9].

The form of the Rellich's and Pohozaev's identities suggests to conjecture that there should be a relationship between them. In a subsequent paper [5] we shall show that general Rellich type identities can be obtained from the Noether's identity [7, 3] for arbitrary differential functions. Here we shall establish a Rellich type identity for a pair of functions defined on oriented compact Riemannian manifolds following an argument similar to that presented in [8].

Theorem 3. *Let $u, v \in C^2(M)$ be two given functions and*

$$h = h^i(x)\frac{\partial}{\partial x^i}$$

a C^1 (contravariant) vector field. Then the following identity holds:

$$\begin{aligned}\int_M \{\Delta_g u\,(h,\nabla v)\,dV + \Delta_g v\,(h,\nabla u)\}\,dV = \int_M div\,h\,(\nabla u,\nabla v)\,dV - \\ -\int_M \mathcal{L}_h g_{ik}\,u^i v^k\,dV + \\ +\int_{\partial M}\left\{\frac{\partial u}{\partial \nu}(h,\nabla v) + \frac{\partial v}{\partial \nu}(h,\nabla u) - (\nabla u,\nabla v)(h,\nu)\right\}dS\end{aligned} \tag{16}$$

where $\mathcal{L}_h g_{ik}$ is the Lie derivative of the metric (g_{ik}) with respect to the vector field h, $u^i = g^{is}u_s$, $v^k = v^{kl}v_l$.

In the Euclidean case ($M \subset \mathbb{R}^n$, a bounded domain, $g_{ij} = \delta_{ij}$) with $h^i = x^i$ this is exactly the identity (2.4) obtained in [8], p. 128.

Corollary 6. *If h is a conformal Killing vector field, that is,*

$$\mathcal{L}_h g_{ik} = \nabla_i h_k + \nabla_k h_i = \frac{2}{n}\,(div\,h)g_{ik}$$

then

$$\begin{aligned}\int_M \{\Delta_g u\,(h,\nabla v)\,dV + \Delta_g v\,(h,\nabla u)\}\,dV = \frac{n-2}{n}\int_M div\,h\,(\nabla u,\nabla v)\,dV + \\ +\int_{\partial M}\left\{\frac{\partial u}{\partial \nu}(h,\nabla v) + \frac{\partial v}{\partial \nu}(h,\nabla u) - (\nabla u,\nabla v)(h,\nu)\right\}dS\end{aligned} \tag{17}$$

for $u, v \in C^2(M)$.

If $M = \Omega$ is a bounded domain in $\mathbb{R}^n$, $g_{ij} = \delta_{ij}$ – the Euclidean metric and $h = x^i\partial/\partial x^i$, then we obtain the Rellich type identity (2.5) established in [8], pp. 128-129. The latter has been used in [9] in the proof of sharp Hardy type inequalities. In fact, in that paper ([9]) two approaches to Hardy identities are proposed. The first one can be generalized in the context of conformal Killing vector fields on Riemannian manifolds as follows.

Theorem 4. *Let $n > p > 1$ and $u \in W^{1,p}(M)$. Suppose that M admits a C^1 conformal Killing vector field h such that $div\, h > 0$. Then*

$$\frac{|u|^p}{|h|^p} \in L^1(M)$$

and

$$\left(\frac{n-p}{np}\right)^p \int_M \frac{div\, h}{|h|^p}\, |u|^p\, dV \leq \int_M (div\, h)^{1-p}\, |\nabla u|^p\, dV\ . \tag{18}$$

Clearly, if $div\, h$ equals to a positive constant (in particular, if $div\, h = n$), then (18) becomes the exact generalization of the well-known Hardy Inequality in $\mathbb{R}^n$ with $h = x^i\partial/\partial x^i$, the radial vector field. Observe that we do not impose conditions neither on the gradient of the weight function nor on its second derivatives (e.g. certain kind of superharmonicity).

The second method devised in [9] is based on a version of the classical Rellich identity whose generalization to Riemannian case is given in (17). As pointed out in [9] that 'identity implicitly contains the main instrument for proving Hardy inequalities for operators (derivatives) of higher orders with sharp constants'. Such applications of (17) will be treated elsewhere.

This paper is organized as follows. In section 2 we present some preliminary facts. In sections 3 and 4 we prove the presented results concerning the Pohozaev's identity in the Riemannian context. In section 5 we establish a Rellich type identity for compact Riemannian manifolds. In section 6 we prove the Hardy type inequality (18) and briefly comment on higher order generalizations based on the use of identity (17).

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2. Preliminaries

We recall that the Laplace-Beltrami operator of M is defined by

$$\Delta_g\varphi = \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\,\frac{\partial\varphi}{\partial x^j}\right) = \nabla_i\nabla^i\varphi = \nabla^i\nabla_i\varphi$$

where $g = \det(g_{kl}) > 0$ and φ is a function on M. The volume form of the Riemannian manifold M is given by

$$dV = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{g}\, dx\ .$$

We shall make use of the following two lemmas, presented without their respective proofs.

Lemma 1. *The system (2) has a variational structure and it is (formally) the Euler-Lagrange equation of a functional $\int_M L\, dx$, where the Lagrangian*

$$L = g^{ks}\sqrt{g}\, u^\alpha_k v^\alpha_s - H(u,v)\sqrt{g}\ . \tag{19}$$

Now let

$$X = \xi^i(x,u,v)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u,v)\frac{\partial}{\partial u^\alpha} + \phi^\alpha(x,u,v)\frac{\partial}{\partial v^\alpha}$$

be a partial differential operator on $M \times \mathbb{R}^{2m}$ and $X^{(1)}$–its first order prolongation. (See [7] for the corresponding definitions).

Lemma 2. *If* (u^α, v^α), $\alpha = 1, \cdots, m$, *is a solution of the Hamiltonian system* (2), *the Noether identity* ([7, 3]) *assumes the form*

$$X^{(1)}L + L\frac{\partial \xi^i}{\partial x^i} = \frac{\partial}{\partial x^i}\left[L\xi^i + (\eta^\alpha - \xi^j u_j^\alpha)\frac{\partial L}{\partial u_i^\alpha} + (\phi^\alpha - \xi^j v_j^\alpha)\frac{\partial L}{\partial v_i^\alpha}\right] . \tag{20}$$

Then, following [3, 4], the Pohozaev's identity is obtained by choosing an appropriate vector field X and integrating (20).

3. An integral identity for elliptic Hamiltonian systems on Riemannian manifolds and some consequences

In this section we prove some of the main results of the paper.

Proof of Theorem 1. Let

$$X = \xi^i\frac{\partial}{\partial x^i} + \frac{2-n}{4}\mu(x)u^\alpha\frac{\partial}{\partial u^\alpha} + \frac{2-n}{4}\,\mu(x)v^\alpha\frac{\partial}{\partial v^\alpha}\ ,$$

where the conformal Killing vector field $\xi = \xi^i\frac{\partial}{\partial x^i}$ satisfies

$$\nabla^k\xi^s + \nabla^s\xi^k = \mu(x)g^{ks}\ . \tag{21}$$

Here, for notational simplicity, we have denoted the conformal factor by

$$\mu = \mu(x) = \frac{2}{n}\,div\,(\xi)\ .$$

By a straightforward calculation we obtain the first order prolongation

$$X^{(1)} = X + \left[\frac{2-n}{4}\,\mu_i\,u^\alpha + \left(\frac{2-n}{2}\,\mu\,\delta_i^j - \xi^j_{\ ,i}\right)\right]u_j^\alpha\frac{\partial}{\partial u_i^\alpha} +$$
$$+\left[\frac{2-n}{4}\,\mu_i\,v^\alpha + \left(\frac{2-n}{2}\,\mu\,\delta_i^j - \xi^j_{\ ,i}\right)\right]v_j^\alpha\frac{\partial}{\partial v_i^\alpha}$$

where ‘,’ means partial derivative: $\xi^j_{\ ,i} = \partial\xi^j/\partial x^i$ and δ_i^j is the Kronecker symbol. Our purpose is to first calculate the left-hand side of the Noether identity (20). For we apply $X^{(1)}$ to L given by (19) and then change some of the indices in the obtained expression. In this way we get that

$$\begin{aligned}X^{(1)}L + L\,\frac{\partial\xi^i}{\partial x^i} &= \Bigg[\xi^i(g^{ks}\sqrt{g})_{,i} + \left(\frac{2-n}{4}\,\mu\delta_i^k - \xi^k_{\ ,i}\right)g^{is}\sqrt{g} + \\ &+\left(\frac{2-n}{4}\,\mu\delta_i^s - \xi^s_{\ ,i}\right)g^{ki}\sqrt{g} + \xi^i_{\ ,i}\,g^{ks}\sqrt{g}\Bigg]u_k^\alpha v_s^\alpha + \\ &+\frac{2-n}{4}\,\mu_i g^{ij}u^\alpha v_j^\alpha\sqrt{g} + \frac{2-n}{4}\,\mu_i g^{ij}v^\alpha u_j^\alpha\sqrt{g} + \\ &+\frac{n-2}{4}\,\mu\,(u^\alpha H_{u^\alpha} + v^\alpha H_{v^\alpha})\sqrt{g} - \xi^i(\sqrt{g})_{,i}H - \xi^i_{\ ,i}\sqrt{g}\,H\ .\end{aligned} \tag{22}$$

Further we shall make use of the formulae

$$(g^{ks})_{,i} = -g^{sl}\Gamma^k_{li} - g^{kl}\Gamma^s_{li}\ ,\quad (\sqrt{g})_{,i} = \Gamma^k_{ik}\sqrt{g}\ , \tag{23}$$

where Γ's are the Christoffel symbols. Then by the definition of the covariant derivative operators ∇^i, corresponding to the Levi-Civita connection ∇, and the second formula in (23), the last two terms in (22) can be written as

$$-\xi^i(\sqrt{g})_{,i}H - \xi^i_{\ ,i}\sqrt{g}\,H = -div\,(\xi)H\sqrt{g} = -n\mu\,H\sqrt{g}/2 \tag{24}$$

since the vector field ξ satisfies (21). (We recall that $div\,(\xi) = \nabla_i\xi^i$ is the covariant divergence of ξ.)

Now we denote by A the expression in the right-hand side of (22) containing $u^\alpha_k v^\alpha_s$. Using (23) we obtain that

$$\begin{aligned} A \;&=\; \Big\{\xi^i\sqrt{g}\left[-g^{sl}\Gamma^k_{li} - g^{kl}\Gamma^s_{li} + g^{ks}\Gamma^l_{il}\right] + \frac{2-n}{4}\,\mu g^{ks}\sqrt{g} - \\ &-\; g^{is}\sqrt{g}\,\xi^k_{\ ,i} + \frac{2-n}{4}\,\mu g^{ks}\sqrt{g} - g^{ki}\sqrt{g}\,\xi^s_{\ ,i} + g^{ks}\sqrt{g}\,\xi^i_{\ ,i}\Big\}\,u^\alpha_k v^\alpha_s = \\ &=\; \Big\{-(g^{is}\xi^k_{\ ,i} + g^{sl}\Gamma^k_{li}\xi^i) - (g^{ki}\xi^s_{\ ,i} + g^{kl}\Gamma^s_{li}\xi^i) + \\ &+\; g^{ks}(\xi^i_{\ ,i} + \Gamma^l_{il}\xi^i) + \frac{2-n}{2}\,\mu g^{ks}\Big\}\,\sqrt{g}\,u^\alpha_k v^\alpha_s\,. \end{aligned} \tag{25}$$

From the definition of the covariant derivative we have that

$$\begin{aligned} \nabla^s\xi^k &= g^{is}\xi^k_{\ ,i} + g^{sl}\Gamma^k_{li}\xi^i\,, \\ \nabla^k\xi^s &= g^{ki}\xi^s_{\ ,i} + g^{kl}\Gamma^s_{li}\xi^i\,, \\ \nabla_i\xi^i &= \xi^i_{\ ,i} + \Gamma^l_{il}\xi^i\,. \end{aligned}$$

We substitute these formulae into (25). Thus

$$A = \left[-\nabla^s\xi^k - \nabla^k\xi^s + g^{ks}div\,(\xi) + (1-n/2)\mu\, g^{ks}\right]\sqrt{g}\,u^\alpha_k v^\alpha_s\,.$$

But ξ is a conformal Killing vector field satisfying (21). Hence $A = 0$. Thus from (22) and (24) we obtain

$$\begin{aligned} &X^{(1)}L + L\frac{\partial\xi^i}{\partial x^i} = \\ &= \mu\left[\frac{n-2}{4}(u^\alpha H_{u^\alpha} + v^\alpha H_{v^\alpha}) - \frac{n}{2}\,H\right]\sqrt{g} + \frac{2-n}{4}\,\mu^j(u^\alpha v^\alpha)_{,j}\sqrt{g}\,, \end{aligned} \tag{26}$$

where $\mu^j = g^{ij}\mu_i$. Further we integrate the Noether identity (20) with L given in (19), use (26) and apply the divergence theorem:

$$\begin{aligned} &\int_M \mu\,\left[\frac{n-2}{4}(u^\alpha H_{u^\alpha} + v^\alpha H_{v^\alpha}) - \frac{n}{2}\,H\right]dV + \frac{2-n}{4}\int_M \mu^j(u^\alpha v^\alpha)_{,j}\,dV = \\ &= \int_{\partial M}[g^{ks}u^\alpha_k v^\alpha_s - H(u,v)]\xi^i\nu_i\,dS + \\ &+ \int_{\partial M}\left\{\left(\frac{n-2}{2}\,a^\alpha u^\alpha - u^\alpha_j\xi^j\right)g^{ks}v^\alpha_s\nu_k + \left(\frac{n-2}{2}\,b^\alpha v^\alpha - v^\alpha_j\xi^j\right)g^{ks}u^\alpha_k\nu_s\right\}dS\,. \end{aligned}$$

Hence the identity (4) follows easily from another application of the divergence theorem to the second term in the first line above, $H(0,0)=0$ and the facts that on ∂M we have $u^\alpha = v^\alpha = 0$,

$$u^\alpha_j \nu_k = u^\alpha_k \nu_j \quad \text{and} \quad v^\alpha_j \nu_s = v^\alpha_s \nu_j \tag{27}$$

(see [12] for the latter property).

Proof of Corollary 1. Since the conformal vector field satisfies (3)

$$\Delta_g \mu = -\frac{1}{n-1}(\mathcal{L}_\xi R + \mu R) \ . \tag{28}$$

See [17], p. 160. Then substitution of (28) into (4) yields (5) immediately.

Proof of Theorem 2. Let

$$X = \xi^i \frac{\partial}{\partial x^i} + \frac{2-n}{2}\, a^\alpha u^\alpha \,\frac{\partial}{\partial u^\alpha} + \frac{2-n}{2}\, b^\alpha v^\alpha \,\frac{\partial}{\partial v^\alpha} \ ,$$

where the conformal Killing vector field $\xi = \xi^i \partial/\partial x^i$ satisfies (7) and a^α, b^α are constants such that $a^\alpha + b^\alpha = 2$, $\alpha = 1, \cdots, m$. By a straightforward calculation we obtain the first order prolongation

$$X^{(1)} = X + \left(\frac{2-n}{2}\, a^\alpha \delta^j_i - \xi^j_{\ ,i}\right) u^\alpha_j \,\frac{\partial}{\partial u^\alpha_i} + \left(\frac{2-n}{2}\, b^\alpha \delta^j_i - \xi^j_{\ ,i}\right) v^\alpha_j \frac{\partial}{\partial v^\alpha_i} \ .$$

The rest of the proof follows the same lines and arguments as that of Theorem 1.

Proof of Corollary 2. From the given conditions and the strong maximum principle we conclude that $u^\alpha > 0$, $v^\alpha > 0$ in M and that $(\partial u^\alpha)/(\partial \nu) < 0$, $(\partial v^\alpha)(\partial \nu) < 0$ on ∂M, $\alpha = 0, \cdots, m$. Then on ∂M we have

$$0 < \frac{\partial u^\alpha}{\partial \nu}\,\frac{\partial v^\alpha}{\partial \nu} = g^{ik} u^\alpha_k \nu_i \, g^{js} v^\alpha_s \nu_j = g^{ik} g^{js} u^\alpha_j \nu_k v^\alpha_s \nu_i = (g^{ik}\nu_i\nu_k)\,(g^{js} u^\alpha_j v^\alpha_s) =$$

$$= g^{ij} u^\alpha_i v^\alpha_j \ ,$$

where we have used (27) and $|\nu|^2 = g^{ik}\nu_i\nu_k = 1$. Thus $g^{ij}u^\alpha_i v^\alpha_j > 0$ on ∂M. Therefore the right-hand side of (8) is negative since $(\xi, \nu) > 0$ on ∂M. Hence

$$\frac{2n}{n-2}\int_M H\, dV > \int_M [a^\alpha u^\alpha H_{u^\alpha} + b^\alpha v^\alpha H_{v^\alpha}]\, dV$$

which contradicts (9).

Proof of Corollaries 3 and 4. Obviously the equation in (10) is a particular case of the system (2) with $m=1$, $u = u^1 = v^1$ and $H(u,u) = 2F(u)$. Then the identities (11) and (12) follow easily from (8) and (4) respectively.

For a direct proof we could consider the vector field

$$X = \xi^i \frac{\partial}{\partial x^i} + \frac{2-n}{2}\,\mu\, u \frac{\partial}{\partial u} \ .$$

Then we apply its first order prolongation $X^{(1)}$ to the Lagrangian

$$L = \frac{\sqrt{g}}{2}\, u_i^2 - F(u)\sqrt{g} \ .$$

Again the rest of the proof follows the same lines as that of Theorem 1.

4. A integral identity for biharmonic equations on Riemannian Manifolds

Proof of Corollary 5. It is clear that the biharmonic equation (13) can be equivalently written in the form of system (2):

$$\begin{cases} -\Delta_g u &= H_v = v \ , \\ -\Delta_g v &= H_u = f(u) \ , \end{cases}$$

where $H = F(u) + v^2/2$. The Dirichlet boundary conditions for (2) $u = v = 0$ on ∂M become the Navier boundary conditions (14).

Let $m = 1$, $u^1 = u$, $v^1 = v = -\Delta_g u$, $a^1 = (n-4)/(n-2)$ and $b^1 = n/(n-2)$. Obviously $a^1 + b^1 = 2$. Then by substituting the above data into (8) we obtain the identity (15).

We would like to observe that (15) can be directly obtained following the argument in the proof of Theorem 2, starting with the well known function of Lagrange

$$L = \frac{\sqrt{g}}{2}(\Delta_g u)^2 - F(u)\sqrt{g} \ .$$

Such a proof, however, contains various differential geometric details which make it lengthy and for this reason it is not presented here.

5. A Rellich's identity on Riemannian manifolds

Let $u, v \in C^2(M)$ be two given functions. Consider another function

$$F = F(x, u, v, \nabla u, \nabla v) =$$

$$= F(x^1, \cdots, x^n, u, v, u_1, \cdots, u_n, v_1, \cdots, v_n) \in C^1(M \times \mathbb{R}^{2+2n}, \mathbb{R}) \ .$$

We define

$$F_p := (\frac{\partial F}{\partial u_1}(x, u, v, \nabla u, \nabla v), \cdots, \frac{\partial F}{\partial u_n}(x, u, v, \nabla u, \nabla v)) = \frac{\partial F}{\partial u_i}\frac{\partial}{\partial x^i}$$

and

$$F_q := (\frac{\partial F}{\partial v_1}(x, u, v, \nabla u, \nabla v), \cdots, \frac{\partial F}{\partial v_n}(x, u, v, \nabla u, \nabla v)) = \frac{\partial F}{\partial v_i}\frac{\partial}{\partial x^i} \ .$$

Then the covariant divergences of these vector fields are given by

$$div\, F_p = \nabla_i F_{u_i} \ , \quad div\, F_q = \nabla_i F_{v_i} \ .$$

We also have

$$(h, F_p) = h^j F_{u_j} \ , \quad (h, F_q) = h^j F_{v_j} \ , \quad (F_p, F_q) = g_{ik} F_{u_i} F_{v_k}$$

where

$$h = h^i(x)\frac{\partial}{\partial x^i}$$

is a C^1 (contravariant) vector field on M .

In order to prove the Theorem 3 we need a technical result, namely

Proposition 1. *Let h, u, v and F be as above. Then the following identity holds:*

$$
\begin{aligned}
&\int_M \{div\, F_p\ (h, F_q)\, dV + div\, F_q\ (h, F_p)\}\, dV = \\
&= \int_M div\, h\ (F_p, F_q)\, dV - \int_M \mathcal{L}_h g_{ik}\ F_{u_i} F_{v_k}\, dV + \\
&+ \int_M F_{u_i}[g_{ik} h^j \nabla_j F_{v_k} - h_k \nabla_i F_{v_k}]\, dV + \\
&+ \int_M F_{v_i}[g_{ik} h^j \nabla_j F_{u_k} - h_k \nabla_i F_{u_k}]\, dV + \\
&+ \int_{\partial M} \{(F_p, \nu)(h, F_q) + (F_p, \nu)(h, F_q) - (h, \nu)(F_p, F_q)\}\, dS
\end{aligned}
\tag{29}
$$

where $\mathcal{L}_h$ is the Lie derivative operator with respect to the vector field h.

Proof. We begin with the following obvious identity

$$
\begin{aligned}
&div\, F_p\ (h, F_q) = h_k F_{v_k} \nabla_i F_{u_i} - \\
&-\frac{1}{2}\, h^j\ \nabla_j (g_{ik} F_{u_i} F_{v_k}) + \frac{1}{2}\, h^j g_{ik} F_{u_i} \nabla_j F_{v_k} + \frac{1}{2}\, h^j g_{ik} F_{v_k} \nabla_j F_{u_i}
\end{aligned}
\tag{30}
$$

which follows from the fact that the covariant derivative satisfies the Leibniz rule. Then from (30) and integration by parts in the first two terms of its right-hand side we obtain

$$
\begin{aligned}
&\int_M div\, F_p\ (h, F_q)\, dV = \\
&= \frac{1}{2} \int_M div\, h\ (F_p, F_q)\, dV - \int_M (F_p, \nabla(h, F_q))\, dV + \\
&+ \frac{1}{2} \int_M h^j g_{ik} F_{u_i} \nabla_j F_{v_k}\, dV + \frac{1}{2} \int_M h^j g_{ik} F_{v_k} \nabla_j F_{u_i}\, dV + \\
&+ \int_{\partial M} \{(F_p, \nu)(h, F_q) + (F_p, \nu)(h, F_q) - (h, \nu)(F_p, F_q)\}\, dS\ .
\end{aligned}
\tag{31}
$$

Interchanging F_p and F_q we obtain a similar identity, which we sum up with (31) to get

$$
\begin{aligned}
&\int_M \{div\, F_p\ (h, F_q)\, dV + div\, F_q\ (h, F_p)\}\, dV = \\
&= \int_M div\, h\ (F_p, F_q)\, dV - \int_M \{(F_p, \nabla(h, F_q)) + (F_q, \nabla(h, F_p))\}\, dV + \\
&+ \int_M h^j g_{ik} F_{u_i} \nabla_j F_{v_k}\, dV + \int_M h^j g_{ik} F_{v_k} \nabla_j F_{u_i}\, dV + \\
&+ \int_{\partial M} \{(F_p, \nu)(h, F_q) + (F_p, \nu)(h, F_q) - (h, \nu)(F_p, F_q)\}\, dS\ .
\end{aligned}
$$

Further we differentiate in the second line above:

$$\int_M \{div\, F_p\ (h, F_q)\, dV + div\, F_q\ (h, F_p)\}\, dV =$$

$$= \int_M div\, h\ (F_p, F_q)\, dV - \int_M \nabla_i h_k\ F_{u_i} F_{v_k}\, dV - \int_M h_k F_{u_i} \nabla_i F_{v_k}\, dV -$$

$$- \int_M \nabla_i h_k\ F_{u_k} F_{v_i}\, dV - \int_M h_k F_{v_i} \nabla_i F_{u_k}\, dV +$$

$$+ \int_M h^j g_{ik} F_{u_i} \nabla_j F_{v_k}\, dV + \int_M h^j g_{ik} F_{v_k} \nabla_j F_{u_i}\, dV +$$

$$+ \int_{\partial M} \{(F_p, \nu)(h, F_q) + (F_p, \nu)(h, F_q) - (h, \nu)(F_p, F_q)\}\, dS\ .$$

By grouping and interchanging some indices (e.g. i and k) we get that

$$\int_M \{div\, F_p\ (h, F_q)\, dV + div\, F_q\ (h, F_p)\}\, dV =$$

$$= \int_M div\, h\ (F_p, F_q)\, dV - \int_M \{\nabla_i h_k + \nabla_k h_i\} F_{u_i} F_{v_k}\, dV +$$

$$+ \int_M F_{u_i} [g_{ik} h^j \nabla_j F_{v_k} - h_k \nabla_i F_{v_k}]\, dV + \tag{32}$$

$$+ \int_M F_{v_i} [g_{ik} h^j \nabla_j F_{u_k} - h_k \nabla_i F_{u_k}]\, dV +$$

$$+ \int_{\partial M} \{(F_p, \nu)(h, F_q) + (F_p, \nu)(h, F_q) - (h, \nu)(F_p, F_q)\}\, dS\ .$$

We observe that in the second line of (32) the Lie derivative of the metric with respect to the vector field h appears, namely

$$\mathcal{L}_h g_{ik} = \nabla_i h_k + \nabla_k h_i\ . \tag{33}$$

Then the Proposition 1 follows from (32) and (33).

After this preparatory work we are ready for the

Proof of Theorem 3. Let

$$F = g^{kl}(u_k u_l + v_k v_l)/2\ .$$

Then we substitute

$$F_{u_i} = g^{is} u_s = u^i\ , \quad F_{v_k} = g^{ks} v_s = v^k\ , \quad (h, F_p) = h^k u_k\ , \quad (h, F_q) = h^k v_k\ ,$$

$$div\, F_p = \nabla_i u^i = \Delta_g u\ , \quad div\, F_q = \nabla_i v^i = \Delta_g v$$

into identity (29). In this way, taking into account the fact that the second covariant derivatives of a function commute (since the torsion of the Levi-Civita connection is zero), we obtain (16).

Corollary 6 is an obvious consequence of Theorem 3.

6. Some Hardy inequalities on Riemannian manifolds

In this section we prove Theorem 4. Then we shall briefly discuss possible higher order generalizations of the Hardy Inequality on Riemannian manifolds. To begin with, we need the following

Lemma 3. *If h is a conformal Killing vector field such that*

$$\nabla^i h^k + \nabla^k h^i = \frac{2}{n}(div\, h)\, g^{ik} = \mu\, g^{ik} \tag{34}$$

then

$$div\left(\frac{h}{|h|^{\theta+2}}\right) = \frac{n-\theta-2}{2}\ \frac{\mu}{|h|^{\theta+2}}\ , \tag{35}$$

where $\theta \in \mathbb{R}$. If $\theta > -2$, the relation (35) *holds on $M \setminus \{zeros\ \ of\ \ h\}$; otherwise, it holds on the whole of M.*

Proof. From (34) we have

$$\nabla_k\left(\frac{h}{|h|^{\theta+2}}\right) = \frac{n}{2}\ \frac{\mu}{|h|^{\theta+2}} - (\theta+2)|h|^{-\theta-4} h_j h^k \nabla_k h^j\ . \tag{36}$$

But

$$h_j h^k \nabla_k h^j = h_j h_k \nabla^k h^j = h_j h_k(-\nabla^j h^k + \mu\, g^{kj}) = -h_j h^k \nabla_k h^j + \mu\, |h|^2\ .$$

(Above we have changed the indices k and j, and we have used (34)). Hence

$$h_j h^k \nabla_k h^j = \frac{\mu}{2}\ |h|^2\ . \tag{37}$$

Then the relation (35) follows from (36) and (37).

Now we are ready for the

Proof of Theorem 4. Without loss of generality we may assume that $u \in C_0^\infty(M)$. Then integrating by parts and using (35) with $n > \theta + 2$ we obtain that

$$\int_M \frac{|u|^{p-1}(h, \nabla u)}{|h|^{\theta+2}}\, dV = -\frac{n-\theta-2}{2p}\ \int_M \frac{\mu}{|h|^{\theta+2}}\ |u|^p\, dV\ .$$

Thus

$$\frac{n-\theta-2}{2p}\ \int_M \frac{\mu}{|h|^{\theta+2}}\ |u|^p\, dV \le \int_M \frac{|u|^{p-1}|\nabla u|}{|h|^{\theta+1}}\, dV =$$

$$= \int_M \frac{|u|^{p/q}\mu^{1/q}}{|h|^{(\theta+2)/q}}\ \frac{|\nabla u|\mu^{-1/q}}{|h|^{\theta/2+1-2/q}}\, dV \le$$

$$\le \left(\int_M \frac{\mu}{|h|^{\theta+2}}\ |u|^p\, dV\right)^{1/q}\ \left(\int_M \frac{\mu^{1-p}}{|h|^{\theta+2-p}}\ |\nabla u|^p\, dV\right)^{1/p}$$

by the Hölder inequality with $q = p/(p-1)$. Hence

$$\left(\frac{n-\theta-2}{2p}\right)^p \int_M \frac{\mu}{|h|^{\theta+2}}\ |u|^p\, dV \le \int_M \frac{\mu^{1-p}}{|h|^{\theta+2-p}}\ |\nabla u|^p\, dV\ . \tag{38}$$

Then the inequality (18) follows from (38) with $\theta = p-2$ and $\mu = 2(div\, h)/n$ (see (34)).

Now we would like to discuss a possible application of the identity (17) to obtaining an higher order version of Hardy inequalitiy (18) on Riemannian manifolds.

Let us suppose that M admits a conformal Killing vector field $h = h^i\partial/\partial x^i$ such that

$$\nabla^k h^s + \nabla^s h^k = 2\, g^{ks} = \frac{2}{n}\, div\,(h)\, g^{ks}\ ,$$

that is

$$div\,(h) = n\ .$$

Then the Rellich type identity (17) for $u \in C_0^\infty(M)$, a positive function, gives

$$\int_M \{\Delta_g u\ (h, \nabla v)\, dV + \Delta_g v\ (h, \nabla u)\}\, dV = (n-2)\int_M (\nabla u, \nabla v)\, dV\ .$$

Let θ be a real number such that $n > \theta + 2$. Now we choose

$$v = (|h|^\theta + \varepsilon)^{-1}$$

and substitute this into above identity with u^p in the place of u. After some work, letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}\int_M \frac{\Delta_g u\, u^{p-1}}{|h|^\theta}\, dV + (p-1)\int_M \frac{u^{p-2}|\nabla u|^2}{|h|^\theta}\, dV + \frac{\theta}{p}\int_M \frac{u^p\cdot\varphi}{|h|^{\theta+2}}\, dV - \\ -\frac{1}{p}\int_M \frac{u^p}{|h|^{\theta+2}}\,(h, \nabla\varphi)\, dV = 0\ ,\end{aligned} \tag{39}$$

where

$$\varphi = -(\theta+2)|\nabla|h||^2 + |\nabla h|^2 - Ric(h,h)\ ,$$

$|\cdot|$ is the inner norm of a tensor (e.g. $|h|^2 = g_{ij}h^ih^j$) and $Ric(h,h) = R_{ij}h^ih^j$, R_{ij} being the Ricci tensor.

Note that in the Euclidean case $\varphi = n-\theta-2$ and the identity (39) becomes the same as in [9]. This fact suggests to suppose for a moment that

$$\varphi = -(\theta+2)|\nabla|h||^2 + |\nabla h|^2 - Ric(h,h) = \lambda > 0\ ,$$

where λ is a constant. Then from (39) and the identity

$$-\frac{1}{p}\int_M \frac{u^p}{|h|^{\theta+2}}\,(h,\nabla\varphi)\, dV = \frac{n-\theta-2}{p}\int_M \frac{u^p\cdot\varphi}{|h|^{\theta+2}}\, dV + \int_M \frac{u^{p-1}(h,\nabla u)}{|h|^{\theta+2}}\,\varphi\, dV\ ,$$

following the same argument as in [9] we obtain

$$\left[\frac{\lambda\theta}{p} + (p-1)\left(\frac{n-\theta-2}{p}\right)^2\right]^p \int_M \frac{u^p}{|h|^{\theta+2}}\, dV \le \int_M \frac{|\Delta_g u|^p}{|h|^{\theta+2-2p}}\, dV\ .$$

If $M = \mathbb{R}^n$ and $h(x) = x$, then the constant

$$\left[\frac{(n-\theta-2)\theta}{p} + (p-1)\left(\frac{n-\theta-2}{p}\right)^2\right]^p\ ,$$

appearing in the above inequality is sharp.

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