

Harnack inequalities and lifting procedure for evolution hypoelliptic equations

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Abstract¹. We consider, for any given $k \in \mathbb{N}$, the degenerate hypoelliptic Partial Differential Equation

$$\partial_t u = \partial_{x_1}^2 u + x_1^k \partial_{x_2} u, \quad (x, t) \in \mathbb{R}^3,$$

and we prove a Harnack type inequality which is expressed in terms of the integral curves of the vector fields ∂_{x_1} and $\partial_t - x_1^k \partial_{x_2}$. The novelty of our result is in that, as $k > 1$, we cannot assume the existence of any Lie group in \mathbb{R}^3 such that ∂_{x_1} and $\partial_t - x_1^k \partial_{x_2}$ are left-invariant. As an application of the Harnack inequality, we prove a lower bound for the fundamental solution of the operator $\partial_{x_1}^2 + (x_1^k \partial_{x_2} - \partial_t)$.

1. INTRODUCTION

We are interested in Harnack type inequalities for positive solutions to hypoelliptic evolution Partial Differential Equations. As a prototype of the class of PDE's we are interested in, we consider the following example

$$(1.1) \quad \mathcal{H}u := \partial_{x_1}^2 u + x_1^k \partial_{x_2} u - \partial_t u, \quad (x, t) \in \mathbb{R}^3,$$

where k is any given positive integer. We prove our main results for (1.1), and we aim, in a future study, to consider a more general class of equations in the following form:

$$(1.2) \quad \mathcal{L}u(x, t) := \sum_{j=1}^m X_j^2 u(x, t) + X_0 u(x, t) - \partial_t u(x, t) = 0,$$

where $(x, t) = (x_1, \dots, x_N, t)$ denotes the point in \mathbb{R}^{N+1} , and $1 \leq m \leq N$. The X_j 's in (1.2) are smooth vector fields on \mathbb{R}^N , i.e.

$$X_j(x) = \sum_{k=1}^N b_k^j(x) \partial_{x_k}, \quad j = 0, \dots, m,$$

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and any b_k^j is a C^∞ function. We are interested in the operators satisfying the well-known Hörmander's hypoellipticity condition. Denote by $z = (x, t)$ the point in \mathbb{R}^{N+1} , and

$$(1.3) \quad Y = X_0 - \partial_t .$$

We say that \mathcal{L} satisfies the Hörmander's condition if

$$(1.4) \quad \text{rank}(\text{Lie}\{X_1, \dots, X_m, Y\}(z)) = N + 1, \quad \text{for every } z \in \mathbb{R}^{N+1} .$$

The Harnack inequalities play a central role in the theory of Partial Differential Equation. Among the main fields of their applications, we recall the abstract potential theory, the local regularity theory, the study of the bounds of the fundamental solutions. Several different versions of the Harnack inequality are available in literature. We first recall the statement due to Bony [3]. It applies to the operator \mathcal{L} in the form (1.2), that is *non totally degenerate*, in the following sense

$$\sum_{k=1}^N \sum_{j=1}^m (b_k^j(x))^2 > 0, \quad \text{for every } x \in \mathbb{R}^N .$$

The statement of the Bony's inequality reads as follows. *Consider the non totally degenerate operator \mathcal{L} satisfying the Hörmander's condition. Let $u : \Omega \rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$ in $\Omega \subset \mathbb{R}^{N+1}$. Then, for every compact set $K \subset \Omega$ there exist a positive constant M , and a finite set of points $z_1, \dots, z_n \in \Omega$ such that*

$$(1.5) \quad \sup_K u \leq M (u(z_1) + \dots + u(z_n)) .$$

We remark that the above result applies to the operator \mathcal{K} in (1.1). Our aim is to prove a Harnack inequality that gives the explicit dependence of the constant M and of the points z_1, \dots, z_n with respect to the set K . This kind of result has been proved by Kogoj and Lanconelli in [6] for a wide class of operators \mathcal{L} in the form (1.2). Specifically, in [6] the following conditions are assumed.

[H.1] there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ, (\delta_\lambda)_{\lambda>0})$ such that

i) X_1, \dots, X_m and Y are left-invariant with respect to the composition law of \mathbb{G} , i.e.

$$X_j(u((\xi, \tau) \circ \cdot)) = (X_j u)((\xi, \tau) \circ \cdot), \quad j = 1, \dots, m,$$

$$Y(u((\xi, \tau) \circ \cdot)) = (Y u)((\xi, \tau) \circ \cdot),$$

for every function $u \in C^\infty(\mathbb{R}^{N+1})$, and for any $(\xi, \tau) \in \mathbb{R}^{N+1}$;

ii) X_1, \dots, X_m are δ_λ -homogeneous of degree one and Y is δ_λ -homogeneous of degree two:

$$X_j(u(\delta_\lambda(x, t))) = \lambda(X_j u)(\delta_\lambda(x, t)), \quad j = 1, \dots, m,$$

$$Y(u(\delta_\lambda(x, t))) = \lambda^2(Y u)(\delta_\lambda(x, t)),$$

for every function $u \in C^\infty(\mathbb{R}^{N+1})$, and for any $(x, t) \in \mathbb{R}^{N+1}$, $\lambda > 0$;

We say that \mathcal{L} is *Lie-invariant* with respect to \mathbb{G} if it satisfies [H.1]. The family of dilations $(\delta_\lambda)_{\lambda>0}$ acts on \mathbb{R}^{N+1} as follows:

$$\delta_\lambda(x_1, x_2, \dots, x_N, t) = (\lambda^{\sigma_1} x_1, \lambda^{\sigma_2} x_2, \dots, \lambda^{\sigma_N} x_N, \lambda^2 t), \quad \text{for every } (x, t) \in \mathbb{R}^{N+1},$$

where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N) \in \mathbb{N}^N$ is a multi-index.

[H.2] for every $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$ with $t > \tau$, there exists an \mathcal{L} -admissible path $\gamma : [0, T] \rightarrow \mathbb{R}^{N+1}$ such that $\gamma(0) = (x, t)$, $\gamma(T) = (\xi, \tau)$.

We say that a curve $\gamma : [0, T] \rightarrow \mathbb{R}^{N+1}$ is \mathcal{L} -admissible if it is continuous and is a sum of a finite number of diffusion and drift trajectories. A *diffusion trajectory* is a curve $\gamma \in C^1(I)$, for some open interval $I \subset [0, T]$, satisfying

$$(1.6) \quad \gamma'(s) = \sum_{k=1}^m \omega_k(s) X_k(\gamma(s)), \quad \text{for every } s \in I, \quad \text{with } \omega_1, \dots, \omega_m \in L^\infty(I).$$

A *drift trajectory* is any positively oriented integral curve of Y . We explicitly remark that [H.2] is a controllability condition, and we recall that [H.1] and [H.2] imply the well known Hörmander condition (1.4).

The statement of the Harnack inequality proved in [6] reads as follows. *Let \mathcal{L} as in (1.2) be an operator satisfying [H.1] and [H.2]. Let $u : \Omega \rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$ in $\Omega \subset \mathbb{R}^{N+1}$. Then there exist two constants $M > 0$ and $\theta \in]0, 1[$ such that*

$$(1.7) \quad \sup_{S_{\theta r}(z_0)} u \leq Mu(z_0),$$

for every $z_0 \in \Omega$ and $r > 0$ such that $H_r(z_0) \subset \Omega$. Here $H_r(z_0) = z_0 \circ \delta_r(H_1)$ and $S_r(z_0) = z_0 \circ \delta_r(S_1)$, with

$$H_1 = \{(x, t) \in \mathbb{R}^{N+1} : \|(x, t)\|_{\mathbb{G}} \leq 1, t \leq 0\},$$

$$S_1 = \{(x, t) \in H_1 : 1/4 \leq -t \leq 3/4\},$$

and $\|\cdot\|_{\mathbb{G}}$ denotes any norm which is homogeneous with respect to the dilation of \mathbb{G} (see Theorem 7.1 in [6]).

Let us compare the above statement with (1.5). In (1.7) the constant M don't depend on the compact set $K = S_{\theta r}(z_0)$, which is defined in terms of the operations of \mathbb{G} . The statement (1.7) then provides us with an useful information about the asymptotic behavior as $x \rightarrow \infty$ of the non-negative solutions to $\mathcal{L}u = 0$. Indeed, it is the starting point of the main result of the paper [4] by Boscain and Polidoro, where an accurate lower bound of the fundamental solution of \mathcal{L} is proved (see Theorem 1.2 in [4]).

We recall that, as $k = 1$, the operator \mathcal{K} in (1.1) agrees with the simplest case of degenerate Kolmogorov operator introduced by Kolmogorov in [8], $\mathcal{K}u := \partial_{x_1}^2 u + x_1 \partial_{x_2} u - \partial_t u$, and studied by Lanconelli and Polidoro in [9]. In [9] is proved that the Kolmogorov operator is Lie-invariant with respect to a suitable homogeneous Lie group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ, (\delta_\lambda)_{\lambda > 0})$, and that the non-negative solutions of $\mathcal{K}u = 0$ satisfy a Harnack inequality that is invariant with respect to the translations and to the dilation of \mathbb{K} .

We explicitly remark that a Lie group $(\mathbb{R}^3, \circ, (\delta_\lambda)_{\lambda > 0})$ such that the operator \mathcal{K} in (1.1) is Lie-invariant does not exist for any $k > 1$. Indeed, if by contradiction there were such a Lie group, then the dimension of its Lie algebra should be equal to the dimension of the domain \mathbb{R}^3 (see [2]). On the other hand, a direct computation shows that $\dim(\text{Lie}\{\partial_{x_1}, x_1^k \partial_{x_2} - \partial_t\}) = k + 2 \neq 3$. Our aim in this paper is to prove a statement analogous to (1.7) even in the case of non-existence of any Lie group. We recall that Harnack type inequalities for operators in the form (1.2), in the case of non-existence of Lie group, have been proved in the paper [7] by Kogoj and Lanconelli. The paper [7] is concerned with hypoelliptic operators \mathcal{L} in the form (1.2), satisfying hypothesis [H.2] and the following one

[H.1'] there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ, (\delta_\lambda)_{\lambda>0})$ such that X_1, \dots, X_m are δ_λ -homogeneous of degree one and Y is δ_λ -homogeneous of degree two:

$$X_j(u(\delta_\lambda(x, t))) = \lambda(X_j u)(\delta_\lambda(x, t)), \quad j = 1, \dots, m,$$

$$Y(u(\delta_\lambda(x, t))) = \lambda^2(Yu)(\delta_\lambda(x, t)),$$

for every function $u \in C^\infty(\mathbb{R}^{N+1})$, and for any $(x, t) \in \mathbb{R}^{N+1}$, $\lambda > 0$.

Under the hypotheses [H.1'] and [H.2] Lanconelli and Kogoj in [7] prove the existence of a fundamental solution for \mathcal{L} , which is invariant with respect to the dilation group in [H.1'], and the following Harnack inequality: *there exist two constants $M > 0$ and $\theta \in]0, 1[$ such that, if $u : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is a non-negative solution to $\mathcal{L}u = 0$, then*

$$(1.8) \quad \sup_{S_{\theta r}(0,0)} u \leq Mu(0, r^2) \quad r > 0,$$

(recall that $S_{\theta r}(0,0)$ is the set introduced in (1.7), see Section 2 in [7]). Note that the above Harnack inequality is invariant with respect to the dilation, but on the right hand side the function u is evaluated only at the point $(0, r^2)$. With the aim to allow a general point of \mathbb{R}^{N+1} to appear at the right hand side of (1.8), we lift the operator \mathcal{L} to a suitable Lie-invariant operator $\tilde{\mathcal{L}}$ acting on \mathbb{R}^{N+k+1} . Specifically, we assume that \mathcal{L} satisfies the following condition:

[H.3] there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{N+k+1}, \circ, (\tilde{\delta}_\lambda)_{\lambda>0})$ and a set of smooth vector fields $\tilde{X}_0, \tilde{X}_1, \dots, \tilde{X}_{m+k}$ on \mathbb{R}^{N+k+1} such that the operator

$$(1.9) \quad \tilde{\mathcal{L}}v(x, y, t) := \sum_{j=1}^{m+k} \tilde{X}_j^2 v(x, y, t) + \tilde{Y}v(x, y, t), \quad \tilde{Y} = \tilde{X}_0 - \partial_t,$$

satisfies [H.1] and [H.2]. Moreover \mathcal{L} and $\tilde{\mathcal{L}}$ are related by the following conditions:

i) if $\delta_\lambda(x, t) = (d_\lambda x, \lambda^2 t)$ then $\tilde{\delta}_\lambda(x, y, t) = (d_\lambda x, \tilde{d}_\lambda y, \lambda^2 t)$, for any $(x, y, t) \in \mathbb{R}^{N+k+1}$;

ii) if we denote $v(x, y, t) = u(x, t)$ for any $u \in C^\infty(\mathbb{R}^{N+1})$, then

$$\tilde{X}_j v(x, y, t) = X_j u(x, t), \quad \text{for } j = 1, \dots, m, \quad \tilde{Y}v(x, y, t) = Yu(x, t),$$

$$\tilde{X}_j v(x, y, t) = 0, \quad \text{for } j = m+1, \dots, m+k.$$

We point out that, if $v(x, y, t) = u(x, t)$, then ii) of [H.3] implies that $\tilde{\mathcal{L}}v(x, y, t) = \mathcal{L}u(x, t)$. In the sequel we will say that the operator $\tilde{\mathcal{L}}$ is a *lifting* of \mathcal{L} . We emphasize that the classical lifting procedure based on the Campbell-Hausdorff formula (see for instance Rothschild and Stein [10] and Bonfiglioli, Lanconelli and Uguzzoni [2]) is not suitable in our setting. Indeed, in Proposition 3.2 we consider the operator \mathcal{X} with $k = 3$, and we lift its drift term $x_1^3 \partial_{x_2} - \partial_t$ to \tilde{Y} according to that procedure. Even in this simple case, we see that it is not easy to check the requirement in [H.2] about the trajectories of \tilde{Y} . Thus, we prefer to rely on a different lifting procedure. As a consequence of our choice, some extra vector fields $\tilde{X}_{m+1}, \dots, \tilde{X}_{m+k}$ appear in our lifted operator $\tilde{\mathcal{L}}$.

In this note we explicitly construct the lifted operator for \mathcal{K} in (1.1). For more general operators in the form (1.2) we assume [H.3] as an hypothesis, and we plan to prove in a future study the existence of a lifted operator $\widetilde{\mathcal{L}}$ in a more general setting. Our main result is

Theorem 1.1. *Let \mathcal{L} be an operator in the form (1.2), satisfying assumptions [H.1'], [H.2], [H.3], and let $u : \mathbb{R}^n \times]T_1, T_2[\rightarrow \mathbb{R}$ be a non-negative solution of $\mathcal{L}u = 0$. For every $T_0 \in]T_1, T_2[$, there exist two positive constants M and h , only depending on the constants appearing in [H.1'], [H.2], [H.3] and on $T_0 - T_1$, such that the following statement holds.*

Let $(x, t), (\xi, \tau) \in \mathbb{R}^N \times]T_1, T_2[$ with $T_0 \leq \tau < t \leq T_2$. Consider a path $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^N$ such that

$$(1.10) \quad \begin{cases} \gamma'(s) = \sum_{j=1}^m \omega_j(s) X_j(\gamma(s)) + X_0(\gamma(s)), & \text{for every } s \in [0, t - \tau], \\ \gamma(0) = x, \quad \gamma(t - \tau) = \xi, \end{cases}$$

with $\omega_1, \dots, \omega_m \in L^\infty([0, t - \tau])$. If we set $\mathcal{C}(\gamma) = \int_0^{t-\tau} (\omega_1(s)^2 + \dots + \omega_m(s)^2) ds$, then

$$(1.11) \quad u(\xi, \tau) \leq M^{1+\mathcal{C}(\gamma)/h} u(x, t).$$

As an immediate consequence, we get the following local Harnack inequality

Corollary 1.2. *Under the same assumptions of Theorem 1.1 we have $u(\xi, \tau) \leq M^2 u(x, t)$, as soon as*

$$\int_0^{t-\tau} (\omega_1(s)^2 + \dots + \omega_m(s)^2) ds \leq h.$$

Remark 1.3. Even if we assume that the operator \mathcal{L} can be lifted to an operator $\widetilde{\mathcal{L}}$ which is invariant with respect some Lie group \mathbb{G} , we don't need to know the composition law of \mathbb{G} . Indeed, unlike (1.7), the Harnack type inequalities in Theorem 1.1 and in Corollary 1.2 are expressed in terms of the integral paths of X_1, \dots, X_m and Y .

We next focus on the operator \mathcal{K} in (1.1) for every $k \in \mathbb{N}$. It satisfies [H.1'], with $\delta_\lambda(x_1, x_2, t) = (\lambda x_1, \lambda^{k+2} x_2, \lambda^2 t)$. On the other hand, even if it satisfies the Hörmander condition (1.4), the condition [H.2] is verified if, and only if, k is odd. Indeed, Theorem 10, Chapter 5 in [5] tells us that k odd is a sufficient condition for the controllability. On the other hand, for even k , every \mathcal{K} -admissible curve is increasing in the x_2 axis, since the drift term has the non-negative coefficient x_1^k . Then, it is impossible to find any \mathcal{K} -admissible curve $\gamma : [0, T] \rightarrow \mathbb{R}^3$ such that $\gamma(0) = (x_1, x_2, t)$ and $\gamma(T) = (\xi_1, \xi_2, \tau)$, with $t > \tau$ and $x_2 > \xi_2$. In Proposition 3.1 we prove that the operator (1.1) satisfies [H.3], for every positive, odd, integer k . Then Theorem 1.1 and Corollary 1.2 apply to \mathcal{K} . From Theorem 1.1 we obtain the following lower bound of the fundamental solution Γ of \mathcal{K} (whose existence is proved in [7]).

Proposition 1.4. *For every positive, odd, integer k , the fundamental solution Γ of the operator \mathcal{K} satisfies the following lower bound*

$$\Gamma(x, t, 0, 0) \geq \frac{c_k}{t^{(3+k)/2}} \exp \left(-C_k \left(\frac{x_1^2}{t} + \frac{x_2^{2/k}}{t^{1+2/k}} \right) \right),$$

for every $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$, and for suitable positive constants c_k, C_k .

This note is organized as follows: in Section 2 we recall a statement of the Harnack inequality previously used in [4], then we prove Theorem 1.1. In Section 3 we show that the operator \mathcal{K} satisfies the hypothesis [H.3]. We also compare our lifting procedure with the classical one based on the Campbell-Hausdorff formula.

2. PROOF OF THE MAIN RESULTS

The proof of Theorems 1.2 and 1.1 mainly relies on the Harnack inequality (1.7) by Kogoj and Lanconelli. We first recall a consequence of (1.7) proved by Boscaïn and Polidoro (see Proposition 1.1 in [4]) for Lie-invariant operators.

Proposition 2.1. *Let $\widetilde{\mathcal{L}}$ be an operator in the form (1.9), satisfying assumptions [H.1]-[H.2]. Then there exist three constants $\theta \in]0, 1[$, $h > 0$ and $M > 1$, only depending on $\widetilde{\mathcal{L}}$, such that the following statement is true. Let $v : \mathbb{R}^{N+K} \times]T_1, T_2] \rightarrow \mathbb{R}$ be a positive solution to $\widetilde{\mathcal{L}}v = 0$, let $(x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^{N+K} \times]T_1, T_2]$ be such that $T_2 - \theta^2(T_2 - T_1) \leq \tau < t \leq T_2$, and let $\tilde{\gamma} : [0, t - \tau] \rightarrow \mathbb{R}^{N+K}$ be a path such that*

$$(2.1) \quad \begin{cases} \tilde{\gamma}'(s) = \sum_{j=1}^{m+k} \omega_j(s) \tilde{X}_j(\tilde{\gamma}(s)) + \tilde{X}_0(\tilde{\gamma}(s)), & \text{for every } s \in [0, t - \tau], \\ \tilde{\gamma}(0) = (x, y), \quad \tilde{\gamma}(t - \tau) = (\xi, \eta), \end{cases}$$

with $\omega_1, \dots, \omega_m \in L^\infty([0, t - \tau])$. If we set $\mathcal{C}(\tilde{\gamma}) = \int_0^{t-\tau} (\omega_1(s)^2 \dots + \omega_{m+k}(s)^2) ds$, then

$$v(\xi, \eta, \tau) \leq M^{1+\mathcal{C}(\tilde{\gamma})/h} v(x, y, t).$$

Proof of Theorem 1.1. Let $u : \mathbb{R}^N \times]T_1, T_2] \rightarrow \mathbb{R}$ be a non-negative solution of $\mathcal{L}u = 0$, and let $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^N$ be a solution of (1.10). Consider the lifted operator $\widetilde{\mathcal{L}}$ and define the function $v : \mathbb{R}^{N+K} \times]T_1, T_2] \rightarrow \mathbb{R}$ by setting $v(x, y, t) = u(x, t)$. Then v is a non-negative solution to $\widetilde{\mathcal{L}}v = 0$. We next denote by $\tilde{\gamma} : [0, t - \tau] \rightarrow \mathbb{R}^{N+K}$ the solution of the Cauchy problem

$$\begin{cases} \tilde{\gamma}'(s) = \sum_{j=1}^m \omega_j(s) \tilde{X}_j(\tilde{\gamma}(s)) + \tilde{X}_0(\tilde{\gamma}(s)), & \text{for every } s \in [0, t - \tau], \\ \tilde{\gamma}(0) = (x, 0), \end{cases}$$

where $\omega_1, \dots, \omega_m$ are the functions appearing in (1.10). Note that $\tilde{\gamma}$ is a solution of (2.1), with $\omega_{m+1} \equiv 0, \dots, \omega_{m+k} \equiv 0$ and $\tilde{\gamma}(t - \tau) = (\xi, \eta)$ for some $\eta \in \mathbb{R}^K$. Moreover, $\gamma = \pi_N \circ \tilde{\gamma}$ (here $\pi_N : \mathbb{R}^{N+K} \rightarrow \mathbb{R}^N$ stands for the canonical projection), and $\mathcal{C}(\gamma) = \mathcal{C}(\tilde{\gamma})$. By applying Proposition 2.1 with $\theta = \sqrt{(T_2 - T_0)/(T_2 - T_1)}$, we then find

$$u(\xi, \tau) = v(\xi, \eta, \tau) \leq M^{1+\mathcal{C}(\tilde{\gamma})/h} v(x, 0, t) = M^{1+\mathcal{C}(\gamma)/h} u(x, t).$$

This accomplishes the proof. □

3. THE OPERATOR $\partial_{x_1}^2 + x_1^k \partial_{x_2} - \partial_t$

In this section we focus on the model operator (1.1). As noticed in the Introduction, it satisfies hypothesis [H.1'], with the dilation

$$(3.1) \quad \delta_\lambda(x_1, x_2, t) = (\lambda x_1, \lambda^{k+2} x_2, \lambda^2 t) ,$$

and hypothesis [H.2] if, and only if, k is odd. The main task for this operator is the proof that it satisfies condition [H.3].

Proposition 3.1. *For every positive, odd, integer k , the operator (1.1) satisfies [H.3].*

Proof. As previously noticed, the required homogeneous Lie group \mathbb{K} exists for $k = 1$. We next consider the operator (1.1) with $k = 3$: $\mathcal{K}u = \partial_{x_1}^2 u + x_1^3 \partial_{x_2} u - \partial_t u$. We start our *lifting procedure* by adding an extra variable $y \in \mathbb{R}^2$ as follows: if $v = v(x, y, t) : \mathbb{R}^5 \rightarrow \mathbb{R}$ is smooth, then

$$\tilde{X}_1 v = \partial_{x_1} v, \quad \tilde{Y} v = x_1 \partial_{y_1} v + x_1^2 \partial_{y_2} v + x_1^3 \partial_{x_2} v - \partial_t v .$$

Note that $\tilde{X}_1^2 + \tilde{Y}$ has the properties *i)-ii)* of [H.3] and it is homogeneous with respect to the dilation

$$(x_1, x_2, y_1, y_2, t) \mapsto (\lambda x_1, \lambda^5 x_2, \lambda^3 y_1, \lambda^4 y_2, \lambda^2 t) .$$

On the other hand, $\tilde{X}_1^2 + \tilde{Y}$ is not a lifting of \mathcal{K} since it does not satisfy [H.2], being non-negative the coefficient x_1^2 of ∂_{y_2} . We then introduce some extra vector fields in order to restore the controllability in the direction of the new variables y_1 and y_2 by using diffusion trajectories. Specifically, we introduce the variables $y_{1,0}, y_{1,1}, y_{2,0}, y_{2,1}, y_{2,2}$ and the following vector fields:

$$(3.2) \quad \begin{aligned} \tilde{X}_2 &= \partial_{y_{2,0}} + x_1 \partial_{y_{2,1}} + x_1^2 \partial_{y_{2,2}} + x_1^3 \partial_{y_2} + \frac{3}{4} x_1^4 \partial_{x_2} , \\ \tilde{X}_3 &= \partial_{y_{1,0}} + x_1 \partial_{y_{1,1}} + x_1^2 \partial_{y_1} + \frac{2}{3} x_1^3 \partial_{y_2} + \frac{1}{2} x_1^4 \partial_{x_2} . \end{aligned}$$

We set $y = (y_{1,0}, y_{1,1}, y_1, y_{2,0}, y_{2,1}, y_{2,2}, y_2) \in \mathbb{R}^7$. The lifted operator $\tilde{\mathcal{K}}$ acts on $\tilde{u} = \tilde{u}(x, y, t) : \mathbb{R}^{10} \rightarrow \mathbb{R}$ as follows

$$\tilde{\mathcal{K}}\tilde{u} = \tilde{X}_1^2 \tilde{u} + \tilde{X}_2^2 \tilde{u} + \tilde{X}_3^2 \tilde{u} + \tilde{Y} \tilde{u} .$$

As we will see, the terms $(3/4)x_1^4 \partial_{x_2}$ and $(2/3)x_1^3 \partial_{y_2} + (1/2)x_1^4 \partial_{x_2}$ are useful in the construction of the homogeneous Lie group.

We next show that there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{10}, \circ, (\tilde{\delta}_\lambda)_{\lambda>0})$ such that $\tilde{\mathcal{K}}$ satisfies [H.3]. We first note that $\tilde{\mathcal{K}}$ satisfies *i)-ii)* of [H.3], with

$$\tilde{\delta}_\lambda(x, y, t) = (\lambda x_1, \lambda^5 x_2, \lambda y_{1,0}, \lambda^2 y_{1,1}, \lambda^3 y_1, \lambda y_{2,0}, \lambda^2 y_{2,1}, \lambda^3 y_{2,2}, \lambda^4 y_2, \lambda^2 t) .$$

In order to prove the existence of \mathbb{G} , we need to show that $\dim(\text{Lie}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{Y}\}) = 10$ and that $\text{rank}(\text{Lie}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{Y}\}(x, y, t)) = 10$ at every point $(x, y, t) \in \mathbb{R}^{10}$. A general result by Bonfiglioli and Lanconelli [1] then provides us with the existence of a homogeneous Lie group as required. We next compute all the commutators of $\text{Lie}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{Y}\}$:

$$\begin{aligned} [\tilde{X}_1, \tilde{Y}] &= \partial_{y_1} + 2x_1 \partial_{y_2} + 3x_1^2 \partial_{x_2} , & [\tilde{X}_1, [\tilde{X}_1, \tilde{Y}]] &= 2\partial_{y_2} + 6x_1 \partial_{x_2} , \\ [\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, \tilde{Y}]]] &= 6\partial_{x_2} ; \end{aligned}$$

$$\begin{aligned}
[\tilde{X}_1, \tilde{X}_2] &= \partial_{y_{2,1}} + 2x_1\partial_{y_{2,2}} + 3x_1^2\partial_{y_2} + 3x_1^3\partial_{x_2}, \\
[\tilde{X}_1, [\tilde{X}_1, \tilde{X}_2]] &= 2\partial_{y_{2,2}} + 6x_1\partial_{y_2} + 9x_1^2\partial_{x_2}, \\
[\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_2]]] &= 6\partial_{y_2} + 18x_1\partial_{x_2}, \quad [\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_2]]]] = 18\partial_{x_2}; \\
[\tilde{X}_1, \tilde{X}_3] &= \partial_{y_{1,1}} + 2x_1\partial_{y_1} + 2x_1^2\partial_{y_2} + 2x_1^3\partial_{x_2}, \quad [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_3]] = 2\partial_{y_1} + 4x_1\partial_{y_2} + 6x_1^2\partial_{x_2}, \\
[\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_3]]] &= 4\partial_{y_2} + 12x_1\partial_{x_2}, \quad [\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_3]]]] = 12\partial_{x_2}.
\end{aligned}$$

We note that only 6 of these commutators are linearly independents in $C^\infty(\mathbb{R}^{10}, \mathbb{R}^{10})$ since, thanks to the terms $(3/4)x_1^4\partial_{x_2}$ and $(2/3)x_1^3\partial_{y_2} + (1/2)x_1^4\partial_{x_2}$ appearing in (3.2), we have

$$[\tilde{X}_1, [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_2]]] = 3[\tilde{X}_1, [\tilde{X}_1, \tilde{Y}]], \quad [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_3]] = 2[\tilde{X}_1, \tilde{Y}].$$

Hence $\dim(\text{Lie}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{Y}\}) = 10$. On the other hand, it is easy to check that

$$\text{rank}\left(\text{Lie}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{Y}\}(x, y, t)\right) = 10,$$

at every point $(x, y, t) \in \mathbb{R}^{10}$. This proves that $\tilde{\mathcal{H}}$ satisfies [H.1].

We finally show that $\tilde{\mathcal{H}}$ satisfies [H.2]. Let $(x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^{10}$, $t > \tau$ be arbitrarily given. We first consider the drift trajectory $\gamma_1 : [0, t - \tau] \rightarrow \mathbb{R}^{10}$ starting from (x, y, t) . We have $\gamma_1(t - \tau) = (x', y', \tau)$ for some $(x', y') \in \mathbb{R}^9$. Since $\text{rank}(\text{Lie}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}) = 9$ at any point of \mathbb{R}^{10} , the Carathéodory-Chow's Theorem states that there exists a piecewise regular path $\tilde{\gamma} : [0, T] \rightarrow \mathbb{R}^{10}$, whose components are piecewise integral curves of a vector field in $\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}$, such that $\tilde{\gamma}(0) = (x', y', \tau)$, $\tilde{\gamma}(T) = (\xi, \eta, \tau)$. This concludes the proof of [H.3] for $k = 3$.

We next repeat the previous construction for the operator in (1.1) with an odd integer $k > 3$

$$\mathcal{H}u = \partial_{x_1}^2 u + x_1^k \partial_{x_2} u - \partial_t u.$$

We introduce $k - 1$ new real variables $(y_1, \dots, y_{k-1}) \in \mathbb{R}^{k-1}$ and the following vector fields

$$\tilde{X}_1 = \partial_{x_1}, \quad \tilde{Y} = \sum_{j=1}^{k-1} x_1^j \partial_{y_j} + x_1^k \partial_{x_2} - \partial_t,$$

that extend to \mathbb{R}^{k+2} the vector fields X_1 and Y , respectively. Note that

$$\text{Lie}\{\tilde{X}_1, \tilde{Y}\} = \text{span}\left\{\tilde{X}_1, \tilde{Y}, \text{ad}_{\tilde{X}_1}(\tilde{Y}), \dots, \text{ad}_{\tilde{X}_1}^k(\tilde{Y})\right\},$$

where

$$\begin{aligned}
\text{ad}_{\tilde{X}_1}^n(\tilde{Y}) &:= \underbrace{[\tilde{X}_1, [\tilde{X}_1, \dots, [\tilde{X}_1, \tilde{Y}] \dots]]}_n = \sum_{j=n}^{k-1} \frac{j!}{(j-n)!} x_1^{j-n} \partial_{y_j} + \\
&\quad + \frac{k!}{(k-n)!} x_1^{k-n} \partial_{x_2}, \quad 1 \leq n \leq k-1, \\
\text{ad}_{\tilde{X}_1}^k(\tilde{Y}) &:= \underbrace{[\tilde{X}_1, [\tilde{X}_1, \dots, [\tilde{X}_1, \tilde{Y}] \dots]]}_k = k! \partial_{x_2},
\end{aligned}$$

so that $\dim(\text{Lie}\{\tilde{X}_1, \tilde{Y}\}) = k+2$. As in the case $k = 3$, we have to define some extra vector fields $\tilde{X}_2, \dots, \tilde{X}_k$, since $\tilde{X}_1^2 + \tilde{Y}$ does not satisfy [H.2]. We then introduce $(1/2)(k^2 + k - 2)$ new real variables $y_{n,j}$, with $n = 1, \dots, k-1$, and $j = 0, \dots, n$. For $1 \leq h \leq (1/2)(k-1)$, we set

$$\begin{aligned}\tilde{X}_{2h} &= \sum_{j=0}^{2h} x_1^j \partial_{y_{2h,j}} + x_1^{2h+1} \partial_{y_{2h}} + \sum_{j=2h+2}^k \frac{2h+1}{j} x_1^j \partial_{y_{j-1}} + \frac{2h+1}{k+1} x_1^{k+1} \partial_{x_2}, \\ \tilde{X}_{2h+1} &= \sum_{j=0}^{2h-1} x_1^j \partial_{y_{2h-1,j}} + x_1^{2h} \partial_{y_{2h-1}} + \sum_{j=2h+1}^k \frac{2h}{j} x_1^j \partial_{y_{j-1}} + \frac{2h}{k+1} x_1^{k+1} \partial_{x_2}.\end{aligned}$$

We denote by $y = (y_{1,0}, y_{1,1}, y_1, y_{2,0}, y_{2,1}, y_{2,2}, y_2, \dots, y_{k-1,0}, \dots, y_{k-1,k-1}, y_{k-1})$ the point of $\mathbb{R}^{(1/2)(k^2+3k-4)}$. The vector fields $\tilde{X}_1, \dots, \tilde{X}_k$ and \tilde{Y} are homogeneous with respect to the dilation

$$\begin{aligned}\tilde{\delta}_\lambda(x, y, t) &= \tilde{\delta}_\lambda(x_1, x_2, y_{1,0}, \dots, y_{n,0}, \dots, y_{n,j}, \dots, y_{n,n}, y_n, \dots, y_{k-1}, t) = \\ &= (\lambda x_1, \lambda^{k+2} x_2, \lambda y_{1,0}, \dots, \lambda y_{n,0}, \dots, \lambda^{j+1} y_{n,j}, \dots, \lambda^{n+1} y_{n,n}, \\ &\quad \lambda^{n+2} y_n, \dots, \lambda^{k+1} y_{k-1}, \lambda^2 t).\end{aligned}$$

The lifted operator $\tilde{\mathcal{K}}$ acts on $\tilde{u} = \tilde{u}(x, y, t) : \mathbb{R}^{(1/2)(k^2+3k+2)} \rightarrow \mathbb{R}$ as follows

$$\tilde{\mathcal{K}} \tilde{u} = \sum_{j=1}^k \tilde{X}_j^2 \tilde{u} + \tilde{Y} \tilde{u}.$$

It is easy to see that $\tilde{\mathcal{K}}$ is related to \mathcal{K} by *i-ii* of [H.3]. We next show that there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{(1/2)(k^2+3k+2)}, \circ, (\tilde{\delta}_\lambda)_{\lambda>0})$ such that $\tilde{\mathcal{K}}$ is Lie invariant. To that aim, we show that

$$(3.3) \quad \dim \left(\text{Lie}\{\tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}\} \right) = \frac{1}{2}(k^2 + 3k + 2)$$

and that

$$(3.4) \quad \text{rank} \left(\text{Lie}\{\tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}\}(x, y, t) \right) = \frac{1}{2}(k^2 + 3k + 2)$$

for every $(x, y, t) \in \mathbb{R}^{(1/2)(k^2+3k+2)}$.

In order to prove (3.3), we show that $\text{Lie}\{\tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}\}$ is the vector space generated by a set of exactly $(1/2)(k^2 + 3k + 2)$ linearly independent vector fields. We first remark the following relations among the commutators:

$$\text{ad}_{\tilde{X}_1}^{2h+1}(\tilde{X}_{2h}) = (2h+1) \text{ad}_{\tilde{X}_1}^{2h}(\tilde{Y}), \quad \text{ad}_{\tilde{X}_1}^{2h}(\tilde{X}_{2h+1}) = 2h \text{ad}_{\tilde{X}_1}^{2h-1}(\tilde{Y}),$$

for $1 \leq h \leq (1/2)(k-1)$. As a consequence, $\text{Lie}\{\tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}\}$ is the vector space generated by the following list of linearly independent vector fields. We find

- $k+1$ vector fields: $\tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}$;
- k commutators: $\text{ad}_{\tilde{X}_1}^n(\tilde{Y})$, for $1 \leq n \leq k$;
- $(1/4)(k^2 - 1)$ commutators: $\text{ad}_{\tilde{X}_1}^n(\tilde{X}_{2h})$, for $1 \leq n \leq 2h$ and $1 \leq h \leq (1/2)(k-1)$;
- $(1/4)(k^2 - 2k + 1)$ commutators: $\text{ad}_{\tilde{X}_1}^n(\tilde{X}_{2h+1})$, for $1 \leq n \leq 2h-1$ and $1 \leq h \leq (1/2)(k-1)$.

This clearly proves (3.3). From the above list it is easy to see that also (3.4) holds. Then, as a consequence of the result by Bonfiglioli and Lanconelli, there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{(1/2)(k^2+3k+2)}, \circ, (\tilde{\delta}_\lambda)_{\lambda>0})$ such that $\tilde{\mathcal{K}}$ is Lie-invariant, thus [H.1] is verified.

We finally show that $\tilde{\mathcal{K}}$ satisfies [H.2]. Let $(x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^{(1/2)(k^2+3k+2)}$, $t > \tau$ be arbitrarily given. We first consider the drift trajectory $\gamma_1 : [0, t - \tau] \rightarrow \mathbb{R}^{(1/2)(k^2+3k+2)}$ starting from (x, y, t) . We have $\gamma_1(t - \tau) = (x', y', \tau)$ for some $(x', y') \in \mathbb{R}^{(1/2)(k^2+3k)}$. Since $\text{rank}(\text{Lie}\{\tilde{X}_1, \dots, \tilde{X}_k\}) = (1/2)(k^2 + 3k)$ at any point of $\mathbb{R}^{(1/2)(k^2+3k+2)}$, the Carathéodory-Chow's Theorem states that there exists a piecewise regular path $\tilde{\gamma} : [0, T] \rightarrow \mathbb{R}^{(1/2)(k^2+3k+2)}$, whose components are piecewise integral curves of a vector field in $\{\tilde{X}_1, \dots, \tilde{X}_k\}$, such that $\tilde{\gamma}(0) = (x', y', \tau)$, $\tilde{\gamma}(T) = (\xi, \eta, \tau)$. This concludes the proof of Proposition 3.1. \square

We next compare the lifting procedure used above with the usual one, which is based on the Campbell-Hausdorff formula, with the aim to point out the difficulties related to the drift term. By modifying the standard lifting procedure, we also provide a couple of vector fields more suitable for our purposes.

Proposition 3.2. *The usual lifting procedure based on the Campbell-Hausdorff formula, applied to the operator $\mathcal{K} = \partial_{x_1}^2 + x_1^3 \partial_{x_2} - \partial_t$, provides us with the vector fields \tilde{X}_1 and \tilde{Y} defined for $(x, y, t) \in \mathbb{R}^5$ as follows*

$$(3.5) \quad \begin{aligned} \tilde{X}_1 &= \partial_{x_1} + \frac{1}{2} t \partial_{y_1} + \left(\frac{1}{12} x_1 t - \frac{1}{2} y_1 \right) \partial_{y_2}, \\ \tilde{Y} &= x_1^3 \partial_{x_2} - \partial_t + \frac{1}{2} x_1 \partial_{y_1} + \frac{1}{12} x_1^2 \partial_{y_2}. \end{aligned}$$

By modifying the standard lifting procedure, we get the following vector fields

$$(3.6) \quad \tilde{X}_1 = \partial_{x_1} - \frac{1}{3} y_1 \partial_{y_2}, \quad \tilde{Y} = x_1^3 \partial_{x_2} - \partial_t + x_1 \partial_{y_1} + \frac{1}{6} x_1^2 \partial_{y_2}.$$

Remark 3.3. We prefer to deal with the operator defined in terms of the vector fields (3.6), since their coefficients don't depend on t . Indeed, in this case, we can rely on the result proved by Kogoj and Lanconelli in [6].

However, even if the vector fields (3.6) are in the form required in [6], we are left with two problems. Firstly, it is not clear whether the method used in the construction of the vector fields (3.6) extends to more general operators. Secondly, even in the simplest case of $k = 3$, we find the non-negative term $(1/6)x_1^2$ in \tilde{Y} , so that it is not easy to check the oriented controllability required in condition [H.2].

Proof of Proposition 3.2. The Lie algebra \mathfrak{a} , generated by $X_1 = \partial_{x_1}$ and $Y = x_1^3 \partial_{x_2} - \partial_t$, is nilpotent of step four, and we have

$$\mathfrak{a} = \text{Lie}\{X_1, Y\} = \text{span}\{X_1, Y, \text{ad}_{X_1}^2(Y), \text{ad}_{X_1}^3(Y)\}.$$

The truncated to step four Campbell-Hausdorff operation on \mathfrak{a} :

$$\begin{aligned} W_1 \diamond W_2 &:= W_1 + W_2 + \frac{1}{2} [W_1, W_2] + \frac{1}{12} [W_1, [W_1, W_2]] - \\ &\quad - \frac{1}{12} [W_2, [W_1, W_2]] - \frac{1}{24} [W_1, [W_2, [W_1, W_2]]] \end{aligned}$$

for any $W_1, W_2 \in \mathfrak{a}$, defines a Lie group structure (\mathfrak{a}, \diamond) . We identify \mathfrak{a} with \mathbb{R}^5 via the map

$$\phi : \mathbb{R}^5 \rightarrow \mathfrak{a} \quad (\xi_1, \dots, \xi_5) \mapsto \xi_1 X_1 + \xi_2 Y + \xi_3 \operatorname{ad}_{X_1}(Y) + \xi_4 \operatorname{ad}_{X_1}^2(Y) + \xi_5 \operatorname{ad}_{X_1}^3(Y),$$

and we define a group law $*$ in \mathbb{R}^5 by setting $a * b := \phi^{-1}(\phi(a) \diamond \phi(b))$. Explicitly, we have

$$\begin{aligned} a * b = & \left(a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}(a_1 b_2 - a_2 b_1), \right. \\ & a_4 + b_4 + \frac{1}{2}(a_1 b_3 - a_3 b_1) + \frac{1}{12}(a_1^2 b_2 - a_1 a_2 b_1 - a_1 b_1 b_2 + a_2 b_1^2), \\ & a_5 + b_5 + \frac{1}{2}(a_1 b_4 - a_4 b_1) + \frac{1}{12}(a_1^2 b_3 - a_1 a_3 b_1 - a_1 b_1 b_3 + a_3 b_1^2) - \\ & \left. - \frac{1}{24}(a_1^2 b_1 b_2 - a_1 a_2 b_1^2) \right), \end{aligned}$$

for any $a, b \in \mathbb{R}^5$. Then, the Jacobian basis of the Lie algebra \mathfrak{g} of $(\mathbb{R}^5, *)$ is defined as

$$(Z_j \varphi)(\xi) := \left. \frac{d}{dt} \varphi(\xi * t e_j) \right|_{t=0}, \quad \text{for any } \varphi \in C^\infty(\mathbb{R}^5),$$

where e_j is the j -th vector of the canonical basis of \mathbb{R}^5 . A direct computation shows that

$$\begin{aligned} Z_1 &= \partial_{\xi_1} - \frac{1}{2} \xi_2 \partial_{\xi_3} - \left(\frac{1}{2} \xi_3 + \frac{1}{12} \xi_1 \xi_2 \right) \partial_{\xi_4} - \left(\frac{1}{2} \xi_4 + \frac{1}{12} \xi_1 \xi_3 \right) \partial_{\xi_5}, \\ Z_2 &= \partial_{\xi_2} + \frac{1}{2} \xi_1 \partial_{\xi_3} + \frac{1}{12} \xi_1^2 \partial_{\xi_4}, \quad Z_3 = \partial_{\xi_3} + \frac{1}{2} \xi_1 \partial_{\xi_4} + \frac{1}{12} \xi_1^2 \partial_{\xi_5}, \\ Z_4 &= \partial_{\xi_4} + \frac{1}{2} \xi_1 \partial_{\xi_5}, \quad Z_5 = \partial_{\xi_5}. \end{aligned}$$

Note that $\mathfrak{g} = \operatorname{Lie}\{Z_1, Z_2\}$. We finally perform a change of coordinates on \mathbb{R}^5 by introducing new variables, setting

$$(x_1, x_2, t, y_1, y_2) = (\xi_1, 6\xi_5 + 3\xi_1 \xi_4 + \xi_1^2 \xi_3 + \frac{1}{4} \xi_1^3 \xi_2, -\xi_2, \xi_3, \xi_4).$$

We obtain \tilde{X}_1 and \tilde{Y} defined in (3.5) by expressing Z_1 and Z_2 , respectively, in the new coordinates. The vector fields \tilde{X}_1 and \tilde{Y} lift X_1 and Y in the sense of [2, formula (17.12)].

We modify this standard lifting procedure in the following way. We consider the map

$$\psi : \mathbb{R}^5 \rightarrow \mathbb{R}^5 \quad (a_1, \dots, a_5) \mapsto \left(a_1, a_2, a_3 + \frac{1}{2} a_1 a_2, a_4 + \frac{1}{6} a_1 a_3, a_5 \right),$$

and we define $a \star b := \psi(\psi^{-1}(a) * \psi^{-1}(b))$, for every $a, b \in \mathbb{R}^5$. Explicitly:

$$\begin{aligned} a \star b = & \left(a_1 + b_1, a_2 + b_2, a_3 + b_3 + a_1 b_2, \right. \\ & a_4 + b_4 + \frac{1}{3} \left(2a_1 b_3 - a_3 b_1 - a_1 b_1 b_2 + \frac{1}{2} a_1^2 b_2 \right), \\ & \left. a_5 + b_5 + \frac{1}{2}(a_1 b_4 - a_4 b_1) + \frac{1}{12}(a_1^2 b_3 + a_3 b_1^2 + a_1 b_1^2 b_2 - a_1^2 b_1 b_2 - 2a_1 b_1 b_3) \right). \end{aligned}$$

Repeating the previous construction, we finally obtain the vector fields in (3.6).

This modification of the lifting procedure is inspired to the fact that the standard Heisenberg group is isomorphic to the so-called *polarized* Heisenberg group, whose operation is defined as $(x, y, s) \cdot (x', y', s') := (x + x', y + y', s' + s - xy')$ (recall that the left-invariant vector fields in the polarized Heisenberg group are ∂_x and $\partial_y - x\partial_s$, see [11]). \square

Proof of Proposition 1.4. Let k be an odd integer number such that $k \geq 3$. Consider any point $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$. We next build a path $\gamma : [0, t/2] \rightarrow \mathbb{R}^2$ such that

$$(3.7) \quad \dot{\gamma}(s) = \begin{pmatrix} \omega(s) \\ \gamma_1^k(s) \end{pmatrix}, \quad \gamma(0) = x, \quad \gamma\left(\frac{t}{2}\right) = 0,$$

where $\omega(s) = c_1$ for $s \in [0, t/4]$, $\omega(s) = c_2$ for $s \in [t/4, (3/8)t]$, $\omega(s) = -c_2$ for $s \in [(3/8)t, t/2]$, where c_1 and c_2 are constant that will be fixed in the sequel. Note that γ is a path as required in (1.10). A simple computation shows that, if we set $c_1 = -4x_1/t$, we have

$$\gamma\left(\frac{t}{4}\right) = \begin{pmatrix} 0 \\ x_2 + \frac{t}{4} \frac{x_1^k}{k+1} \end{pmatrix}$$

We next choose c_2 such that

$$\frac{c_2^k}{k+1} \frac{t^{k+1}}{2^{3k+2}} + x_2 + \frac{t}{4} \frac{x_1^k}{k+1} = 0$$

and we obtain $\gamma(t/2) = 0$. With this choice of the function ω we then find

$$\mathcal{C}(\gamma) = \int_0^{t/2} \omega(s)^2 ds = \frac{t}{4} (c_1^2 + c_2^2) = \frac{4x_1^2}{t} + \frac{16}{t} \left(\frac{4(k+1)x_2}{t} + x_1^k \right)^{2/k}.$$

Since \mathcal{K} verifies the hypothesis of Theorem 1.1, with $T_1 = 0$, $T_2 = t$ and $(\xi, \tau) = (0, t/2)$, from (1.11) it follows that

$$\Gamma(x, t, 0, 0) \geq M^{-1-\mathcal{C}(\gamma)/h} \Gamma\left(0, \frac{t}{2}, 0, 0\right).$$

From an elementary computation we obtain

$$\frac{1}{h} \left(\frac{4x_1^2}{t} + \frac{16}{t} \left(\frac{4(k+1)x_2}{t} + x_1^k \right)^{2/k} \right) \leq \tilde{C}_k \left(\frac{x_1^2}{t} + \frac{x_2^{2/k}}{t^{1+2/k}} \right),$$

for any $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$ and for some positive constant \tilde{C}_k . Then

$$\Gamma(x, t, 0, 0) \geq M^{-1} \Gamma\left(0, \frac{t}{2}, 0, 0\right) \exp\left(-C_k \left(\frac{x_1^2}{t} + \frac{x_2^{2/k}}{t^{1+2/k}} \right)\right).$$

We next use the δ_λ -homogeneity of Γ :

$$\Gamma\left(0, \frac{t}{2}, 0, 0\right) = \Gamma\left(\delta_{\sqrt{t/2}}(0, 1), 0, 0\right) = \left(\frac{2}{t}\right)^{(3+k)/2} \Gamma(0, 1, 0, 0),$$

(see property (v) of Γ in [6]) and we conclude the proof. \square

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