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Regularity of minimal surfaces in the monodimensional Heisenberg group

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Dedicated to Ermanno on his 65th birthday

Abstract[1]. I'll present a regularity result of viscosity solutions of minimal surface equation in the Heisenberg group. The problem, arose in a model of visual perception, joint work with A. Sarti [10], has been studied with analytic instruments together with L. Capogna and M. Manfredini [5]. We assume that the solutions are locally represented as intrinsic graphs, so that the minimal surface equation is represented in term of non linear vector fields. We prove that viscosity solutions are smooth, in an intrinsic sense, which implies that the solutions are foliated in smooth curves.

1. Introduction

In this seminar we present a regularity result for intrinsic minimal graphs, obtained in collaboration with Capogna and Manfredini ([5]). The Heisenberg group is $\mathbb{R}^3$ whose tangent space is endowed with a stratification $V_1 \oplus V_2$, where V_1 has dimension 2, and $V_2 = [V_1, V_1]$ has dimension 1. In this setting the notion of intrinsic regular surface has been studied in [17] [18] as a zero level set of an intrinsic regular function. On the other side, in [12] and [1] it is proved that such a surface can be represented (near non-characteristic points) as the graph of a function $u : \mathbb{R}^2 \to \mathbb{R}$, of class C^1 with respect to the vector field $X_{1,u} = \partial_1 + u\partial_2$.

The notion of mean curvature has been recently introduced as the first variation of the area functional, and its expression has been independently established by different authors: [15], [9], [8], [4], [31], [22], [26], [33], [34]. In the particular case of intrinsic graphs it can be expressed in terms of the vector field $X_{1,u}$ just defined and the prescribed mean curvature equation becomes

$$Lu = X_{1,u}\left(\frac{X_{1,u}u}{\sqrt{1+|X_{1,u}u|^2}}\right) = f\ , \quad \text{for } x \in \Omega \subset \mathbb{R}^2\ . \tag{1.1}$$

Properties of regular minimal surfaces have been investigated in [19], [29], [8], [9], [20], [16], [2] and [28]. Existence of BV minimizers of the perimeter is proved in

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[19], [29] using direct methods of the calculus of variations. More recently, existence of Lipschitz continuous vanishing viscosity solutions has been studied in [9]. The definition relies on the fact that the operator in (1.1) is a second order operator, represented in term of only one vector field in R^2. Hence it is not elliptic at any point, however it can be approximated via a Riemannian mean curvature equation obtained completing the vector $X_{1,u}$ to a basis with a vector field $X^\varepsilon_{2,u} = \varepsilon\partial_2$ which tend to 0, as ε goes to 0. The operator then reads:

$$L_\varepsilon u = \sum_{i=1}^{2} X^\varepsilon_{i,u}\left(\frac{X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\right) = f\ , \quad \text{for } x\in\Omega\subset\mathbb{R}^2\ , \tag{1.2}$$

where

$$X^\varepsilon_{1,u} = X_{1,u}\ , \quad X^\varepsilon_{2,u} = \varepsilon\partial_2 \quad \text{and} \quad \nabla^\varepsilon_u = (X^\varepsilon_{1,u}, X^\varepsilon_{2,u})\ . \tag{1.3}$$

Definition 1.1. If C^1_E denotes the standard Euclidean C^1 norm, we will say that an Euclidean Lipschitz continuous function u is a *vanishing viscosity solution* of (1.1) in an open set Ω, if there exists a sequence u_ε of smooth solutions of (1.2) in Ω such that for every compact set $K\subset\Omega$

- $\|u_\varepsilon\|_{C^1_E(K)} \le C$ for every ε;
- $u_\varepsilon \to u$ as $\varepsilon\to 0$ pointwise a.e. in Ω.

Existence of viscosity solutions has been proved in [9], while the problem of regularity of minimal surfaces is still largely open. In order to state our main result we will need to define intrinsic regularity for an intrinsic graph. We will say that a graph u is of class C^1_u if $X_{1,u}u$ is continuous. We will say that u is intrinsically smooth if $X^k_{1,u}u$ exists and it is continuous for every k. In this paper we address the issue of regularity away from characteristic points. Our goal is to prove the following intrinsic regularity result

Theorem 1.2. *The Lipschitz continuous vanishing viscosity solutions of* (1.1) *are intrinsically smooth functions.*

This theorem highlight a very general idea: any positive semidefinite operator of second order regularizes in the direction of its positive eigenvalues. However, in general, this does not imply smoothness of solutions, since regularity can be expected only in the directions of the non vanishing eigenvalues. Indeed the following foliation result holds for minimal graphs:

Corollary 1.3. *Let $\{x_3 = u(x_1,x_2), (x_1,x_2)\in\Omega\}$ be a Lipschitz continuous vanishing viscosity minimal graph. The flow of the vector $X_{1,u}u$ yields a foliation of the domain Ω by polynomial curves γ of degree two. For every fixed $x_0\in\Omega$ denote by γ the unique leaf passing through that fixed point. The function u is differentiable at x_0 in the Lie sense along γ and the equation* (1.1) *reduces to* $(d^2/dt^2)(u(\gamma(t)) = 0$.

To better understand the notion of intrinsic regularity we consider to the non-smooth minimal graph $u(x_1,x_2) = x_2/(x_1 - sgn\,(x_2))$. Although this function is not C^1 in the Euclidean sense, observe that $X_1u = 0$ for every $x_1, x_2\in\Omega$. Hence, this is an example of a minimal surface which is not smooth but which can be differentiated indefinitely in the direction of the Legendrian foliation.

The regularity theory for intrinsic minimal surfaces in $\mathbb{H}^n$ with $n>1$ is quite different. In the recent paper [6] we show that any Lipschitz continuous vanishing

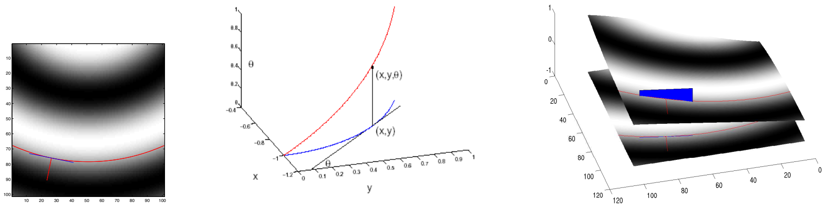


Figure 1. in the left a grey level image is represented. In the middle the lifting process of a level line, and in the right the lifting of the whole domain

viscosity minimal intrinsic graph in $\mathbb{H}^n$, $n > 1$ (defined through the Riemannian approximation scheme) is smooth in euclidean sense. The main reason is that in higher dimension the horizontal tangent bundle generates as a Lie algebra the full tangent bundle, while this does not happen in the $n = 1$ case.

2. Application to visual perception

We will apply the previous theory to visual perception. Instruments of differential geometry for the description of the visual cortex have been introduced for the first time by Hoffmann [23]. After that the cortex has been described as a Lie group with a subriemannian metric by [25] [10], and more recently new models have been proposed by [35] [30].

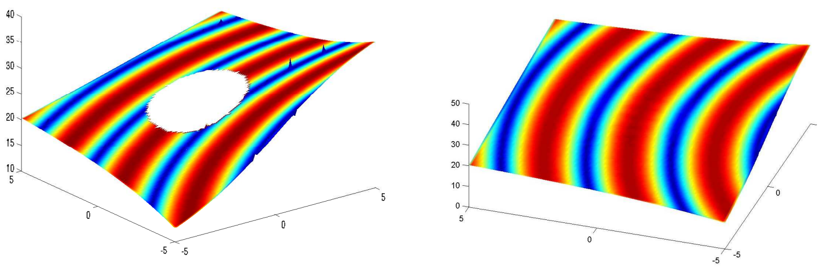
A image can represented as a C^1 function $I : D \subset R^2 \to R$. We can as well assume that in a neighborhood of a given point any level line of the image I can be represented as a function $x_2 = f(x_1)$.

There is a strong neurophisiological evidence that the cells which start the elaboration of the visual signal are the simple cells of the visual cortex. Experiments nowadays classical (see[24]) ensure that the simple cells are able to detect at every point (x_1, x_2) the direction tangent to the level lines of the image I. If we set $u(x_1, x_2) = f'(x_1)$, then the tangent vector will be represented as

$$X_1 = \partial_{x_1} + u(x_1, x_2)\partial_{x_2}$$

Each point (x_1, x_2) is lifted to the 3D point $(u(x_1, x_2), x_1, x_2)$. Any level line is lifted to a curve in the 3D space, and the whole domain D to a graph $\{(x_0, x_1, x_2) : x_0 = u(x_1, x_2)\} \subset R^3$. Let us call $Z_1 = \partial_{x_1} + x_0\partial_{x_2}$ the vector obtained from X_1 with the substitution $u = x_0$ and let us call $Z_0 = \partial_{x_0}$. By definition the tangent vector to each lifted curve lie on the plane generated buy the vector Z_1 and Z_0. The integral curve of the vector $Z_2 = \partial_{x_2} = [Z_0, Z_1]$ are not present in the lifted surface. Hence the lifted curves define a subbundle of the tangent bundle $V_1 = span(Z_0, Z_1)$, of dimension 2. Since $[V_1, V_1] = span(Z_2)$, the Lie algebra generated by Z_0 and Z_1 is an Heisenberg algebra and the graph of the function u is a regular intrinsic graph in the Heisenberg setting.

If a small part of the lifted image is missed, the brain is able to reconstruct the missing surface, with a filling in neural mechanism, which can be explained in [10] via the minimization of the Area functional, which is the natural energy functional associated to the lifted surface. In other words the surface is completed to a minimal graphs in the Heisenberg group.



Besides we lifted separately any level line: each level line has to be completed separately. Indeed by Corollary 1.3 the minimal surface is foliated in geodesics, each of which perform the completion of a single level line.

3. From *Lip* to $C^{1,\alpha}_{\bar{u}}$

In this section we will always assume that u and f are fixed smooth functions defined on an open set Ω of $\mathbb{R}^2$,and that u is a solution of the PDE $L_\varepsilon u = f$ in Ω. In particular we remark that

$$\|u\|_{L^\infty(\Omega)} + \|\nabla_\varepsilon u\|_{L^\infty(\Omega)} + \|\partial_2 u\|_{L^\infty(\Omega)} < \infty \,, \tag{3.1}$$

and we set

$$M = \|u\|_{L^\infty(\Omega)} + \|\nabla_\varepsilon u\|_{L^\infty(\Omega)} + \|\partial_2 u\|_{L^\infty(\Omega)} \,. \tag{3.2}$$

We will use the notation $W^{1,p}_\varepsilon(\Omega)$, $p > 1$ to denote the Sobolev space corresponding to the norm $\|\phi\|_{W^{1,p}_\varepsilon(\Omega)} = \|\phi\|_{L^p(\Omega)} + \|\nabla_\varepsilon \phi\|_{L^p(\Omega)}$. For simplicity, unless we want to stress the dependence on u, we will simply write X_1, X_2 instead of $X_{1,u}, X_{2,u}$. We will denote by $W^{k,p}_0(\Omega)$ the space of $L^p(\Omega)$ functions ϕ such that $X^l_1 \phi \in L^p(\Omega)$ for all $1 \le l \le k$.

Using in full strength the nonlinearity of the operator L_ε, we prove here some Caccioppoli-type inequalities for the intrinsic gradient of u, and for the derivative $\partial_2 u$. The main novelty of the proof is that putting together two intrinsic subelliptic Caccioppoli inequalities we will end up with an Euclidean Caccioppoli inequality. In this way we can obtain the Hölder-regularity of the gradient via a standard Moser procedure.

We first prove that if u is a smooth solution of equation (1.2) then its derivatives $\partial_2 u$ and $X_k u$ are solution of new second order equation, defined in terms of vector fields:

$$M_\varepsilon z = X_i \left(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z \right) \quad \text{where} \quad a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2} \ . \tag{3.3}$$

Indeed a direct computation shows that

Lemma 3.1. *the function $z = X_k u$ with $k \le 2$ is a solution of the equation*

$$M_\varepsilon z = f_1(\nabla_\varepsilon u)v^2 + f_{2,i}(\nabla_\varepsilon u)X_i v^2 + X_i\left(f_{3,i}(\nabla_\varepsilon u)v\right) \ , \tag{3.4}$$

for suitable smooth functions f_1 and $f_{j,i}$. The function $v = \partial_2 u$ is a solution of

$$M_\varepsilon z = f_1(\nabla_\varepsilon u)v^3 + f_{2,i}(\nabla_\varepsilon u)vX_i v^2 + X_i\left(f_{3,i}(\nabla_\varepsilon u)v^2\right) \ . \tag{3.5}$$

Since the operator M_ε in (3.3) is in divergence form, it is quite standard to prove the following intrinsic Caccioppoli type inequalities:

Proposition 3.2 (Intrinsic Caccioppoli type inequality)**.** *If u is a smooth solution of* (1.1)*, z denotes $X_k u$ and $v = \partial_2 u$, then for every p there exists a constant C, only dependent on p and the constant M in* (3.2) *such that for every $\phi \in C_0^\infty$*

$$\int |\nabla_u^\varepsilon z|^2 z^{p-2}\phi^2 \le C \int z^p(\phi^2 + |\nabla_u^\varepsilon \phi|^2) \ ,$$
$$\int |\nabla_u^\varepsilon v|^2 z^{p-2}\phi^2 \le C \int z^p(\phi^2 + |\nabla_u^\varepsilon \phi|^2) \ .$$

Next, we note that, from the previous Propositions one can derive a standard Euclidean Caccioppoli type inequality

Proposition 3.3 (Euclidean Caccioppoli inequality)**.** *If u is a smooth solution of* (1.1) *and $z = X_{k,u}^\varepsilon u$, with $k \le 2$, then for every $p \ne 1$ there exists a constant C, only dependent on p such that for every $\phi \in C_0^\infty$*

$$\int |\nabla_E z|^2 z^{p-2}\phi^2 \le C \int z^p(\phi^2 + |\nabla_E \phi|^2) \ . \tag{3.6}$$

Proof. It is a consequence of Proposition 3.2 and the fact that the Euclidean gradient can be estimated as

$$|\nabla_E z|^2 \le |X_{1,u}z - u\partial_{x_2}(X_{1,u}u)|^2 + |\partial_{x_2}(X_{1,u}u)|^2 \le C(|\nabla_u z|^2 + |\nabla_u v|^2 + 1) \ . \tag{3.7}$$

Using the Euclidean classical Euclidean Moser procedure we can now deduce:

Proposition 3.4. *Let u be a solution of equation* (1.2) *satisfying* (3.1)*. For every compact set $K \subset\subset \Omega$ then there exist a real number α and a constant C, only dependent on the constant M in* (3.2) *such that*

$$\|u\|_{W_\varepsilon^{2,2}(K)} + \|\partial_2 u\|_{W_\varepsilon^{1,2}(K)} + \|u\|_{C_u^{1,\alpha}(K)} \le C \ .$$

In order to boostrap the previous argument, in all the intrinsic Sobolev spaces, we will need a Caccioppoli type inequality for any solution of the equation $M_\varepsilon z = f$ (see also [5]):

Lemma 3.5. *Let $p \ge 3$ be fixed, let $f \in C^\infty(\Omega)$, let u be a function satisfying the bound* (3.1) *and let z be a smooth solution of equation $M_\varepsilon z = f$. There exist a constant C which depend on p and the constant M in* (3.2) *but are independent of ε and z such for every $\phi \in C_0^\infty(\Omega)$, $\phi > 0$,*

$$\int |\nabla_\varepsilon(|\nabla_\varepsilon z|^{(p-1)/2})|^2\phi^{2p} \le C\bigg(\int \left(|\nabla_\varepsilon \phi|^2 + \phi^2\right)^p + \int |\nabla_\varepsilon z|^{p+1/2}\phi^{2p} + \tag{3.8}$$

$$+\int |X_2(\partial_2 u)|^p \phi^{2p} + \int |f|^{2p} \left(|\nabla_\varepsilon \phi|^2 + \phi^2\right) \phi^{2p-2} + \int |\nabla_\varepsilon^2 u||\nabla_\varepsilon z|^{p-1}\phi^{2p} +$$
$$+ \int |\nabla_\varepsilon^2 u|^2 |\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla_\varepsilon^2 u||\nabla_\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_\varepsilon \phi|\Big) .$$

4. $W^{2,p}_{\rm loc}$ a priori estimates

The equation $L_\varepsilon u$ defined in (1.2) can be represented in non-divergence form

$$N_\varepsilon u = \sum_{i=1}^{2} a_{ij}(\nabla_\varepsilon u) X_i X_j u \ , \tag{4.1}$$

where a_{ij} are defined in (3.3). Following the approach in the papers [11, 14] we *linearize* the operator N_ε in the following way: While the coefficients of the vector fields X_i depend on a fixed function u, they will be applied to an arbitrary function z, sufficiently regular. The associated linear non divergence form operator is

$$N_{\varepsilon,u} z = \sum_{i=1}^{2} a_{ij}(\nabla_\varepsilon u) X_{i,u} X_{j,u} z \ . \tag{4.2}$$

The main result of this section is the following

Theorem 4.1. *Let us assume that z is a classical solution of $N_{\varepsilon,u} z = 0$.*

(i) Let us assume that $\alpha \in]0,1[$, $p > 10/3$ and for every $K \subset\subset \Omega$ there exists a constant C such that

$$\|u\|_{C^{1,\alpha}(K)} + \|\partial_2 z\|_{L^p(K)} + \|\partial_2 X_u z\|_{L^2(K)} + \|\nabla_\varepsilon^2 z\|_{L^2(K)} \le C \ .$$

Then for any compact set $K_1 \subset\subset K$, there exists a constant C_1 only dependent on K, C, and on the constant in 3.2 such that

$$\|z\|_{W^{2,10/3}_\varepsilon(K_1)} \le C_1 \ .$$

(ii) If, in addition to the previous conditions, we assume that $\alpha \ge 1/4$ and there exists a constant $\widetilde{C}$ such that

$$\|\partial_2 X_u z\|_{L^4(K)} \le \widetilde{C} \ ,$$

then for every $p > 1$ there exists a constant C_1 only dependent on C and $\widetilde{C}$ and p such that

$$\|z\|_{W^{2,p}_\varepsilon(K_1)} \le C_1 \ .$$

4.1. **Lifting and freezing.** The linear operator $N_{\varepsilon,u}$ can be interpreted as an uniformly elliptic operator, with least eigenvalue depending on ε. It is well known that this approximating operator has a fundamental solution, but its estimates strongly depend on ε. In order to obtain estimates uniform in ε we further approximate it with an Hörmander type operator, a sum of squares of vector fields, which has a similar behavior in the direction $X_{1,u}$, but for which the direction ∂_2 is the direction of one of the commutators (a step-three commutator). The idea is to use a new version of the famous Rothschild and Stein lifting theorem, only partially inspired to the procedure in [32]. We introduce a new variable s and define a new vector field $X_3 = \partial_s$. Then we lift the initial vector fields X_1, X_2, X_3 to the vectors $X_1 + s^2 X_2, X_2, X_3$ in $\Omega \times R$. Clearly the new vector fields reduce to the initial ones for $s = 0$, but the couple $X_1 + s^2 X_2, X_3$ satisfy a nonlinear Hörmander type condition at every point in $\Omega \times R$.

In order to deal with the non-smoothness of u, we will also operate a freezing: we approximate the coefficients of the vector field X_1 with their first order Taylor polynomials. If $x_0 = ((x_0)_1, (x_0)_2) \in \Omega$ is a fixed point, then for all $x \in \Omega$, $s \in \mathbb{R}$ we define exponential coordinates $(e_1(x,s), e_2(x,s), e_3(x,s))$, based at $(x_0, 0)$, via the formula

$$(x,s) = \exp_{(x_0,0)}(e_1(x,s)(X_1 + s^2 X_2) + e_2(x,s)X_2 + e_3(x,s)X_3)\ . \tag{4.3}$$

With these notation the first order Taylor polynomial of u is

$$P^1_{x_0} u\ (x) = u(x_0) + e_1(x) X_1 u(x_0, 0) + e_2(x) X_2 u(x_0, 0)\ . \tag{4.4}$$

At this point we introduce an appropriate *freezing* of the vector fields by defining

$$X_{1,x_0} = \partial_{x_1} + (P^1_{x_0} u(x) + s^2)\partial_{x_2},\ X_{2,x_0} = \varepsilon \partial_{x_2} \quad \text{and} \quad X_{3,x_0} = \partial_s\ . \tag{4.5}$$

Observe that $\{X_{1,x_0}, X_{3,x_0}\}$ is a distribution of smooth vector fields satisfying Hörmander's finite rank hypothesis with step three. We denote by $d_{x_0}(\cdot,\cdot)$ the corresponding Carnot-Carathéodory distance and remark that the homogeneous dimension of the space is 5. We also need the Riemannian distance function $d_{x_0,\varepsilon}(\cdot,\cdot)$ defined as the control distance associated to $\{X_{1,x_0}, X_{2,x_0}, X_{3,x_0}\}$. It is well known (see for instance the discussion in [7, Section 2.4] that $(\mathbb{R}^3, d_{x_0,\varepsilon})$ converges in the Gromov-Hausdorff sense to $(\mathbb{R}^3, d_{x_0})$. In particular one has that for each fixed x and s, then $d_{x_0,\varepsilon}((x,s),(x_0,0)) \to d_{x_0}((x,s),(x_0,0))$ as $\varepsilon \to 0$. Moreover the volume of the balls $B_\varepsilon((x_0,0),R)$ in the $d_{x_0,\varepsilon}$ metric converges to the volume of the limit Carnot-Caratheodory balls, i.e. $|B_\varepsilon((x_0,0),R)| \to |B_0((x_0,0),R)|$ as $\varepsilon \to 0$.

The freezing process allows to introduce "*frozen*" sub-Laplacians operators N_{ε,x_0} formally defined as $N_{\varepsilon,u}$, but in terms of the smooth vector fields X^ε_{i,x_0} instead of the original non-smooth vector fields X_i. Precisely

$$N_{\varepsilon,x_0} z = \sum_{i,j=1}^{3} a_{ij}(\nabla_\varepsilon u(x_0)) X_{i,x_0} X_{j,x_0} z\ .$$

N_{ε,x_0} is an uniformly elliptic operator with C^∞ coefficients, which can be considered the elliptic regularization of an uniformly subelliptic operator. The linear theory yields that it has a fundamental solution Γ_{ε,x_0} (see [21], [27] and [3]). Precise estimates of Γ_{ε,x_0} have been established in [27] and [3], while in [13] it is proved that it locally satisfies natural growth estimates, with constant independent of ε.

Proof of Theorem 4.1. Extend both u and z to be functions defined on $\Omega \times (-1,1)$ by letting them be constant along the s variable. For any $\phi \in C^\infty_0(\Omega \times (-1,1))$, $\xi \in \Omega$ and $s \in (-1,1)$, the product $z(\xi)\phi(\xi,s)$ can be represented as

$$z(\xi)\phi(\xi,s)) = \int_{\Omega\times(-1,1)} \Gamma((\xi,s),(\zeta,\sigma))\Big(z\, N_{\varepsilon x_0}\phi +$$

$$+ \sum_{ij=1}^{2} \bar{a}_{ij}(x_0)\left(X_{i,z_0} z X_{j,z_0}\phi + X_{j,z_0} z X_{i,z_0}\phi\right)\Big)\, d\zeta +$$

$$+ \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon z_0}((\xi,s),(\zeta,\sigma))\ g(\zeta)\,\phi(\zeta,\sigma)\, d\zeta d\sigma +$$

$$+\sum_{ij=1}^{2}\int_{\Omega\times(-1,1)}\Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\,(\bar a_{ij}(x_0)-\bar a_{ij}(\zeta))\,X_{i,u}X_{j,u}z(\zeta)\,\phi(\zeta)\,d\zeta d\sigma-$$
$$-\sum_{j=1}^{2}\bar a_{1j}(x_0)\int_{\Omega\times(-1,1)}\Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\left(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\right)\times$$
$$\times\partial_2 X_{j,u}z(\zeta)\,\phi(\zeta,\sigma)\,d\zeta d\sigma+$$
$$+\sum_{i=1}^{2}\bar a_{i1}(x_0)\int_{\Omega\times(-1,1)}X_{i,x_0}(\xi,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\left(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\right)\times$$
$$\times\partial_2 z(\zeta)\,\phi(\zeta,\sigma)\,d\zeta d\sigma+$$
$$+\sum_{i=1}^{2}\bar a_{i1}(x_0)\int_{\Omega\times(-1,1)}\Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\left(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\right)\times$$
$$\times\partial_2 z(\zeta)\,X_{i,x_0}\phi(\zeta,\sigma)\,d\zeta d\sigma\ .$$

In order to simplify notations we have set $\bar a_{ij}(\zeta)=a_{ij}(\nabla_\varepsilon u(\zeta))$. Differentiating this representation formula, and using the estimates independent of ε of the fundamental solution, we obtain the proof of the main result of the section Theorem 4.1.

5. A priori estimates for the non-linear approximating PDE

We now return to the equation

$$L_\varepsilon u=0\ .$$

Let u be a smooth solution satisfying (3.1). In view of Proposition 3.4 and Theorem 4.1 (i) we have the following statement: for every open set $\Omega_1\subset\subset\Omega$ there exists a positive constant constant C which depends on Ω_1 and on M in (3.2), but is independent of ε such that

$$\|u\|_{W^{2,10/3}_\varepsilon(\Omega_1)}+\|\partial_2 u\|_{W^{1,2}_\varepsilon(\Omega_1)}+\|u\|_{C^{1,\alpha}_E(\Omega_1)}\le C\ . \tag{5.1}$$

Let us first establish an interpolation property:

Proposition 5.1. *For every $p\ge 3$, for every function $z\in C^\infty(\Omega)$ there exists a constant C_p, dependent on p, the constant M in* (3.2) *such that and for every $\phi\in C^\infty_0(\Omega)$, and every $\delta>0$*

$$\int|X_iz|^{p+1}\phi^{2p}\le C\int\left(z^{p+1}\phi^{2p}+z^2|X_iz|^{p-1}\phi^{2p-2}|X_i\phi|^2\right)+$$
$$+C\int|X_i^2z|^2|X_iz|^{p-3}|z|^2\phi^{2p}\ ,$$

where i can be either 1 *or* 2.

Proof. We have

$$\int|X_iz|^{p+1}\phi^{2p}=\int X_iz|X_iz|^p\mathrm{sign}(X_iz)\phi^{2p}=$$

(integrating by parts and using the fact that $X_1^*=-X_1-\partial_2u$ and $X_2^*=-X_2$)

$$=-\int\partial_2uz|X_iz|^p\mathrm{sign}(X_iz)\phi^{2p}-p\int zX_i^2z|X_iz|^{p-1}\phi^{2p}- \tag{5.2}$$
$$-2p\int z|X_iz|^p\mathrm{sign}(X_iz)\phi^{2p-1}X_i\phi\le$$

(by Hölder inequality)

$$\leq \frac{C}{\delta} \int \left(z^{p+1}\phi^{2p} + z^2|Xz|^{p-1}\phi^{2p-2}|X_i\phi|^2\right) + \delta \int |X_i z|^{p+1}\phi^{2p} +$$
$$+ \frac{C}{\delta} \int |z|^2 |X_i^2 z|^2 |X_i z|^{p-3}\phi^{2p} ,$$

choosing δ sufficiently small we obtain the desired inequality.

Next step is the higher integrability of the Hessian of u. The proof rests on the estimates obtained from Theorem 4.1 and an euclidean Caccioppoli inequality (obtained putting together two intrinsic Caccioppoli type inequalities).

Lemma 5.2. *Let u be a smooth solution of equation* (1.1) *satifying* (3.1) *and denote $v = \partial_2 u$. For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 1$ there exists a positive constant C which depends on Ω_1, p, and on M in* (3.2)*, but is independent of ε such that*

$$\|u\|^p_{W^{2,p}_\varepsilon(\Omega_1)} + \|\nabla_\varepsilon v\|^4_{L^4(\Omega_1)} \leq C .$$

Proof. We can apply Lemma 3.5 with $p = 3$ to the function $v = \partial_2 u$ and deduce that

$$\int |\nabla^2_\varepsilon v|^2\phi^6 \leq C_1 + C_2 \Big(\int |\nabla_\varepsilon v|^{3+1/2}\phi^6 + \tag{5.3}$$
$$+ \int (1 + |\nabla_\varepsilon v| + |\nabla^2_\varepsilon u|)^{7/5}\phi^{23/5}(|\nabla_\varepsilon \phi| + \phi)^{7/5} +$$
$$+ \int |\nabla^2_\varepsilon u||\nabla_\varepsilon v|^2\phi^6 + \int |\nabla^2_\varepsilon u|^2|\nabla_\varepsilon v|^2\phi^6 + \int |\nabla^2_\varepsilon u||\nabla_\varepsilon v|^2\phi^5|\nabla_\varepsilon\phi|\Big) .$$

It follows that

$$\int |\nabla^2_\varepsilon v|^2\phi^6 \leq \frac{C_2}{\delta} \int |\nabla^2_\varepsilon u|^4\phi^6 + \delta \int |\nabla_\varepsilon v|^4\phi^6 + \frac{C_1}{\delta} . \tag{5.4}$$

Analogously, if we set $z = X_1 u$, or $z = X_2 u$, we have

$$\int |\nabla^2_\varepsilon z|^2\phi^6 \leq \frac{C_2}{\delta} \int |\nabla^2_\varepsilon u|^4\phi^6 + \frac{C_1}{\delta} + C_2 \int |\nabla_\varepsilon v|^3\phi^6 . \tag{5.5}$$

Using Lemma 5.1, (5.4) and (5.1), we obtain immediately

$$\int |\nabla_\varepsilon v|^4\phi^6 \leq C_1 + C_2 \int |\nabla^2_\varepsilon v|^2\phi^6 \leq C_1 + \frac{C_2}{\delta} \int |\nabla^2_\varepsilon u|^4\phi^6 + \delta \int |\nabla_\varepsilon v|^4\phi^6 .$$

Hence

$$\int |\nabla_\varepsilon v|^4\phi^6 \leq C_1 + C_2 \int |\nabla^2_\varepsilon u|^4\phi^6 . \tag{5.6}$$

Consequently, from the latter and (5.5) we deduce that

$$\int |\nabla^2_\varepsilon z|^4\phi^6 \leq C_1 + C_2 \int |\nabla^2_\varepsilon u|^4\phi^6 . \tag{5.7}$$

Next, from the intrinsic Caccioppoli inequalities (5.6) and (5.7) we deduce an Euclidean Caccioppoli inequality: note that

$$|\nabla_E X_1 z| \le |X_1^2 z| + C_2|\partial_2 X_1 z| \le |X_1^2 z| + C_2|v\partial_2 z| + C_2|X_1\partial_2 z| \le$$

(since $\partial_2 z = \partial_2 X_1 u = v^2 + X_1 v$)

$$\le |\nabla_\varepsilon^2 z| + C_2|\nabla_\varepsilon^2 v| + C_2|\nabla_\varepsilon v| + C_2 \ .$$

From the latter and (5.6) and (5.7) we infer

$$\int |\nabla_E \nabla_\varepsilon z|^2 \phi^6 \le C_2 \left(\int |\nabla_\varepsilon^2 v|^2 \phi^6 + \int |\nabla_\varepsilon^2 z|^2 \phi^6 + 1 \right) \le \tag{5.8}$$

$$\le C_2 \int |\nabla_\varepsilon z|^4 \phi^6 + C_1 \ .$$

Now we can apply the standard Euclidean Sobolev inequality in $\mathbb{R}^2$ and obtain

$$\left(\int (|\nabla_\varepsilon z|\phi^3)^6 \right)^{1/3} \le C_2 \int |\nabla_E(\nabla_\varepsilon z\phi^3)|^2 \le C_2 \int |\nabla_\varepsilon z|^4 \phi^6 + C_1 \le$$

(using Hölder inequality)

$$\le C_2 \left(\int (|\nabla_\varepsilon z|\phi^3)^6 \right)^{1/3} \left(\int_{\mathrm{supp}(\phi)} |\nabla_\varepsilon z|^3 \right)^{2/3} + C_1 \ .$$

By (5.1) and the fact that $|\nabla_\varepsilon z| \le |\nabla_\varepsilon^2 u|$, we already know that $|\nabla_\varepsilon z| \in L^3_{\mathrm{loc}}$. In fact

$$\left(\int_{\mathrm{supp}(\phi)} |\nabla_\varepsilon z|^3 \right)^{2/3} \le \left(\int_{\mathrm{supp}(\phi)} |\nabla_\varepsilon z|^{10/3} \right)^{3/5} |\mathrm{supp}(\phi)|^{1/15} \ .$$

Recall that C_2 does not depend on $|\nabla_\varepsilon \phi|$. If we choose the support of ϕ sufficiently small, we can assume that the integral $\int_{\mathrm{supp}(\phi)} |\nabla_\varepsilon z|^3$ is arbitrarily small. It follows that

$$\left(\int (|\nabla_\varepsilon z|\phi^3)^6 \right)^{1/3} \le C_1$$

and consequently, by (5.6)

$$\int |\nabla_\varepsilon v|^4 \phi^6 \le C_1 \ .$$

But this implies that $|\nabla_E(\nabla_\varepsilon u)| \le |\nabla_\varepsilon^2 u| + |\nabla_\varepsilon v| + v^2 \in L^4_{\mathrm{loc}}$. This implies, buy the standard Euclidean Sobolev Morrey inequality in $\mathbb{R}^2$ that

$$\nabla_\varepsilon u \in C_E^{1/2} \ .$$

By Theorem 4.1 (ii) it then follows that for every $r > 1$ there exists a constant $C > 0$ independent of ε such that

$$\|\nabla_\varepsilon^2 u\|_{W^{2,r}} \le C_1 \ .$$

□

Using a quite delicate bootstrap argument, we can now deduce the main result of this section which is the following a priory regularity estimates for solutions of the approximating non linear equation:

Theorem 5.3. *Let u be a smooth solution of equation* (1.1)*, satisfying* (3.1)*. For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 3$, and every integer $m \geq 2$ there exists a constant C which depends on p, m Ω_1 and on M in* (3.2)*, but is independent of ε such that the following estimates holds*

$$\|u\|_{W^{m,p}_\varepsilon(\Omega_1)} + \|\partial_2 u\|_{W^{m,p}_\varepsilon(\Omega_1)} \leq C \; . \tag{5.9}$$

6. Estimates for the viscosity solution

In this section we turn our attention to the proof of regularity for vanishing viscosity solutions u of equation (1.1). The regularity is expressed in terms of the intrinsic Sobolev spaces $W^{k,p}_0(\Omega)$ and rests on the *a priori* estimates proved in the previous section in the limit $\varepsilon \to 0$.

Let u be a vanishing viscosity solution, and (u_j) denote its approximating sequence, as defined in Definition 1.1. For each ε_j and function u_j we set $X_{1,j} = \partial_1 + u_j \partial_{x_2}$, $X_{2,j} = \varepsilon_j \partial_{x_2}$ the corresponding vector fields, and let ∇_{ε_j} and $W^{k,p}_{\varepsilon_j}(\Omega)$ denote the natural gradient and Sobolev spaces. We also let u , $X_1 = \partial_1 + u\partial_2$, and $\nabla_0 = (X_1, 0)$ denote the coefficients and vector fields associated to the limit equation and the limit solution u, while $W^{k,p}_0(\Omega)$ will be the associated Sobolev space. Note that ∇_E and $W^{k,p}_E(\Omega)$ are the usual gradient and Sobolev space.

Theorem 6.1. *Let $u \in Lip(\Omega)$ be a vanishing viscosity solution of* (1.1)*, and set $v_j = \partial_2 u_j$. For every ball $B(R) \subset\subset \Omega$ and $p > 1$ there exists a constant $C > 0$ such that*

$$\|\nabla_{\varepsilon_j} u_j\|_{W^{1,p}_E(B(R))} + \|v_j\|_{L^\infty(B(R))} + \|v_j\|_{W^{1,2}_{\varepsilon_j}(B(R))} \leq C \tag{6.1}$$

and

$$X_{1,j} u_j \to Xu \; , \quad X_{2,j} u_j \to 0 \tag{6.2}$$

as $j \to +\infty$ weakly in $W^{1,2}_{E,\mathrm{loc}}(\Omega)$. Moreover equation (1.1) *can be represented as $X^2 u = 0$ and is satisfied weakly in the Sobolev sense, and hence, pointwise a.e. in Ω, i.e.*

$$\int_\Omega X u X^* \phi = 0 \quad \textit{for all } \phi \in C^\infty_0(\Omega) \; .$$

Proof. The uniform bound on $\|v_j\|_{L^\infty(B(R))}$ follows from the definition of vanishing viscosity solution. The bound on $\|v_j\|_{W^{1,2}_{\varepsilon_j}(B(R))}$ is a consequence of (5.9). To prove the remaining estimate observe that for any function w: $\partial_2 X_{1,j} w = X_{1,j}\partial_2 w + \partial_2 u_j \partial_2 w$. Substituting $w = u_j$ and in view of (5.9) we see that there exist positive constants C_1, C_2 depending only on the uniform bound on $\|v_j\|_{L^\infty(B(R))}$ such that for any $p \geq 1$,

$$\begin{aligned} &\|\partial_1 \nabla_{\varepsilon_j} u_j\|_{L^p(B(R))} + \|\partial_2 \nabla_{\varepsilon_j} u_j\|_{L^p(B(R)} \leq \\ &\leq \|X_{1,j} \nabla_{\varepsilon_j} u_j\|_{L^p(B(R))} + \|(1 + |u_j|)\partial_2 \nabla_{\varepsilon_j} u_j\|_{L^p(B(R))} \leq \\ &\leq \|u_j\|_{W^{2,p}_\varepsilon(B(R))} + C_1 \|v_j\|_{W^{1,p}_\varepsilon(B(R))} + C_2 \leq C \; , \end{aligned} \tag{6.3}$$

for a new constant $C > 0$ independent of j. The weak regularity of u and the weak Sobolev convergence follow in a standard fashion.

Next we address the PDE: since for every j the approximating solution u_j is of class C^∞ then we can use the non divergence form of the equation

$$\sum_{h,k=1}^{2} a_{h,k}(\nabla_j u_j) X_{h,j} X_{k,j} u_j = 0 \ .$$

Here

$$a_{h,k}(\nabla_j u_j) \to a_{h,k}(\nabla_0 u) = \delta_{h1}\delta_{k1} \quad \text{in } L^p \ , \quad \text{and } X_{1,j}u_j \to Xu \ , \quad X_{2,j}u_j \to 0$$

as $j \to +\infty$ weakly in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Hence letting j go to ∞ in the non divergence form equation we conclude that $X^2 u = 0$ in the Sobolev sense.

□

An analogous result holds higher order derivatives:

Proposition 6.2. *For every $k \in N$ for every $p > 1$ and for every multiindex I of length k, the sequence $(\nabla^I_{\varepsilon_j} u_j)$ is bounded in $W^{1,p}_{E,\mathrm{loc}}(\Omega)$. Moreover*

$$X^k_{1,j} u_j \to X^k u \ , \quad \textit{and } X^k_{2,j} u_j \to 0$$

weakly in $W^{1,p}_E(\Omega)$ as $j \to \infty$. We will express this convergence in the notation

$$\nabla^I_{\varepsilon_j} u_j \to D^I_0 u \ \ \textit{as} \ \ j \to +\infty, \ \ \textit{weakly in} \ \ W^{1,p}_E(\Omega) \ .$$

We can now prove the main regularity properties of the limit function u:

Proposition 6.3. *For every k, and for every $p > 1$ the function $z = X^k u$ belongs to $W^{1,p}_{E,\mathrm{loc}}(\Omega)$ and it is an a.e. solution of $X^2 z = 0$ in O. In particular*

$$X^k u \in C^\alpha_{\mathrm{loc}}(\Omega) \quad \textit{for every } \alpha \ , \ 0 < \alpha < 1 \ . \tag{6.4}$$

Proof. Since u is a vanishing viscosity solution of $X^2 u = 0$ in Ω, then Proposition 6.2 implies $X^2 u \in W^{1,p}_{E,\mathrm{loc}}(\Omega)$ for all $p \geq 1$. As $X^2 u = 0$ a.e. in Ω, then a simple iteration shows that all the derivatives $X^2 X^k u$ vanish a.e. in Ω. The Hölder regularity (6.4) follows from the classical Morrey-Sobolev embedding theorem.

□

We can now give a new pointwise definition of derivative in the direction of vector fields X_1 and X_2.

Definition 6.4. Let X be a Lipschitz vector field on Ω and let $\xi_0 \in \Omega$ and γ be a solution to problem $\gamma' = X(\gamma)$. We say that a function $f \in C^\alpha_{\mathrm{loc}}(\Omega)$, with $\alpha \in]0,1[$, has Lie-derivative in the direction of the vector field X in ξ_0 if there exists

$$\frac{d}{dh}(f \circ \gamma)_{|h=0} \ ,$$

and we will denote its value by $Xf(\xi_0)$.

If the weak derivative of a function f is sufficiently regular, then the two notions of derivatives coincide. For the proof of the following result see [11, Remark 5.6].

Proposition 6.5. *If $f \in C^\alpha_{\mathrm{loc}}(\Omega)$ for some $\alpha \in]0,1[$ and its weak derivatives $Xf \in C^\alpha_{\mathrm{loc}}(\Omega), \partial_y f \in L^p_{\mathrm{loc}}(\Omega)$ with $p > 1/\alpha$, then for all $\xi \in \Omega$ the Lie-derivatives $Xf(\xi)$ exist and coincide with the weak ones.*

We are now ready to prove the result concerning the foliation.

Proof of Corollary 1.3. The equation $\gamma' = X(\gamma)$ has an unique solution, of the form

$$\gamma(x) = (x, y(x)) ,$$

where $y'(x) = u(x, y(x))$. In view of the regularity of u and of the previous proposition then $y''(x) = Xu(x, y(x))$, and $y'''(x) = X^2u(x, y(x)) = 0$. This shows that γ is a polynomial of order 2 and concludes the proof.

□

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