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Cauchy-Riemann geometry and subelliptic theory

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Dedicated to Ermanno Lanconelli on the occasion of his 65th birthday

Abstract¹. CR structures are a bundle theoretic recast of the tangential Cauchy-Riemann equations $\bar{\partial}_b u = 0$ and may be thought of as a geometric framework for their study. Any strictly pseudoconvex CR manifold M endowed with a contact form θ carries a natural second order subelliptic (of order $\varepsilon = 1/2$) operator, the *sublaplacian* Δ_b of (M, θ) . Thus the equations $\Delta_b \phi^i + \sum_{a=1}^{2n} (\Gamma_{jk}^i \circ \phi) X_a(\phi^j) X_a(\phi^k) = 0$ describing all S^1 -invariant harmonic maps $\Phi = \phi \circ \pi : C(M) \rightarrow N$ from the total space $C(M)$ of the canonical circle bundle $S^1 \rightarrow C(M) \xrightarrow{\pi} M$ endowed with the Fefferman metric F_θ form a nonlinear subelliptic system of variational origin. The present paper is devoted mainly to the description of the impact of subelliptic theory on the study of geometric objects appearing on a strictly pseudoconvex CR manifold, such as subelliptic harmonic maps.

1. INTRODUCTION

CR analysis and geometry grew during the last forty years into a quite popular argument cf. e.g. A. Boggess, [9], and M.S. Baouendi & P. Ebenfelt & L.P. Rothschild, [3], for the analysis of CR functions and maps, and G. Tomassini et al., [29], for the differential geometric side of the subject. Subelliptic operators on the other hand appear in function theory in several complex variables related to the analysis of the $\bar{\partial}_b$ -complex and the corresponding Kohn-Rossi operator $\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ on a CR submanifold of \mathbb{C}^n (the principal part of $-\square_b$ is a subelliptic operator, cf. J.J. Kohn, [53]). Cf. also D. Jerison & A. Sánchez-Calle, [49]. Though the monograph [29] contains already rudiments of subelliptic theory (leading to the solution of the CR Yamabe problem, cf. D. Jerison & J.M. Lee, [50], N. Gamara & R. Yacoub, [37]) a systematic application of subelliptic theory to CR and pseudohermitian geometry (at the level elliptic theory applies to Riemannian geometry) is still missing from the present day mathematical literature. The present paper is thus an attempt to fill in the gap between CR geometry and analysis of subelliptic operators, on the line of thought by E. Lanconelli, [56], and of course influenced by J. Jost & C-J. Xu's program (cf. e.g. [52]) of recovering results on nonlinear

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elliptic equations of variational origin to the at least hypoelliptic setting. Cf. also E. Barletta et al., [4], L. Capogna & N. Garofalo, [15], and C. Wang, [74].

2. THE ENERGY INTEGRAL

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n (cf. e.g. Definition 1.7 in [29], p. 6). Let θ be a contact form on M such that the Levi form G_θ is positive definite and let $dv = \theta \wedge (d\theta)^n$ be the corresponding volume form on M . Let $S^1 \rightarrow C(M) \xrightarrow{\pi} M$ be the canonical circle bundle over M (cf. Definition 2.9 in [29], p. 119) and F_θ the Fefferman metric associated to θ (cf. Definition 2.15 in [29], p. 128). F_θ is a Lorentz metric on $C(M)$ and its restricted conformal class $\{e^{u\circ\pi}F_\theta : u \in C^\infty(M)\}$ is a CR invariant (cf. J.M. Lee, [60]). Let (N, h) be a ν -dimensional Riemannian manifold (with the Riemannian metric h). Let us assume that M is compact. Hence $C(M)$ is compact as well. We consider the ordinary Dirichlet energy functional

$$(1) \quad \mathbb{E}(\Phi) = \frac{1}{2} \int_{C(M)} \text{trace}_{F_\theta} (\Phi^*h) \, d\text{vol}(F_\theta), \quad \Phi \in C^\infty(C(M), N),$$

where $d\text{vol}(F_\theta)$ is the canonical volume form of the Lorentzian manifold $(C(M), F_\theta)$. A C^∞ map $\Phi : C(M) \rightarrow N$ is *harmonic* if it is a critical point of \mathbb{E} i.e.

$$\frac{d}{dt} \{\mathbb{E}(\Phi_t)\}_{t=0} = 0$$

for any smooth 1-parameter variation $\{\Phi_t\}_{|t|<\varepsilon}$ of Φ . Cf. B. Fuglede, [36]. Let $\phi : M \rightarrow N$ be a C^∞ map and let $\Phi = \phi \circ \pi : C(M) \rightarrow N$ be its vertical lift. Then integration along the fibre in (1) shows that

$$(2) \quad \mathbb{E}(\phi \circ \pi) = 2\pi E(\phi)$$

where²

$$(3) \quad E(\phi) = \frac{1}{2} \int_M \text{trace}_{G_\theta} (\Pi_H \phi^*h) \, dv.$$

Here $\Pi_H \phi^*h$ denotes the restriction of the bilinear form ϕ^*h to $H(M) \otimes H(M)$ and $H(M)$ is the Levi, or maximally complex, distribution on M (cf. e.g. [29], p. 4). A C^∞ map $\phi : M \rightarrow N$ is a critical point of E if $\{dE(\phi_t)/dt\}_{t=0} = 0$ for any smooth 1-parameter variation $\{\phi_t\}_{|t|<\varepsilon}$ of ϕ . Critical points of E are called *subelliptic harmonic maps*. Cf. E. Barletta et al., [4] (where critical points of E are referred to as *pseudoharmonic maps*). Let $x \in M$ be a point and let $\{X_a : 1 \leq a \leq 2n\}$ be a local G_θ -orthonormal frame of $H(M)$ defined on an open neighborhood $U \subseteq M$ of x . Then $\text{trace}_{G_\theta} (\Pi_H \phi^*h) : M \rightarrow [0, +\infty)$ is given by

$$\text{trace}_{G_\theta} (\Pi_H \phi^*h)(x) = \sum_{a=1}^{2n} (\phi^*h)(X_a, X_a)_x.$$

To each C^∞ map $\phi : M \rightarrow N$ we associate its second fundamental form

$$\beta(\phi)(X, Y) = \nabla_X^\phi \phi_* Y - \phi_* \nabla_X Y, \quad X, Y \in \mathfrak{X}^\infty(M),$$

cf. R. Petit, [66]. Here ∇ is the Tanaka-Webster connection of (M, θ) (cf. Definition 1.25 in [29], p. 26). It is the unique linear connection ∇ on M such that i) $H(M)$ and its complex structure J are parallel with respect to ∇ , ii) $\nabla g_\theta = 0$ where g_θ is the

²The symbol π in the right hand side of (2) is the irrational number $\pi \in \mathbb{R} \setminus \mathbb{Q}$.

Webster metric of (M, θ) (a Riemannian metric on M , cf. Definition 1.10, in [29], p. 9) and iii) the torsion tensor field T_∇ of ∇ is pure (cf. Definition 1.24 in [29], p. 25). Also $\nabla^\phi \in \mathcal{C}(\phi^{-1}TN, h^\phi)$ is the connection in the pullback bundle $\phi^{-1}TN \rightarrow M$ naturally induced by the Levi-Civita connection ∇^h of (N, h) . Here h^ϕ is the Riemannian bundle metric on $\phi^{-1}TN \rightarrow M$ induced by h and $\mathcal{C}(\phi^{-1}TN, h^\phi)$ is the affine space consisting of all $D \in \mathcal{C}(\phi^{-1}TN)$ such that $Dh^\phi = 0$ ($\mathcal{C}(\phi^{-1}TN, h^\phi)$ is an affine subspace of the affine space $\mathcal{C}(\phi^{-1}TN)$ of all connections in $\phi^{-1}TN \rightarrow M$).

The tension field $\tau(\phi)$ of $\phi : M \rightarrow N$ is given by

$$\tau(\phi) = \text{trace}_{G_\theta} (\Pi_H \beta(\phi)) \in \Gamma^\infty(\phi^{-1}TN) .$$

By a result in [4] $\phi \in C^\infty(M, N)$ is a subelliptic harmonic map if and only if $\tau(\phi) = 0$ (the Euler-Lagrange equations of the variational principle associated to $E : C^\infty(M, N) \rightarrow \mathbb{R}$). The sublaplacian Δ_b of (M, θ) is the formally self-adjoint second order differential operator

$$\Delta_b u = -\text{div} (\nabla^H u) , \quad u \in C^2(M) .$$

Cf. Definition 2.1 in [29], p. 111. The divergence is computed with respect to the volume form dv i.e.

$$\mathcal{L}_X dv = \text{div}(X) dv$$

for any tangent vector field X of class C^1 on M . Here \mathcal{L}_X is the Lie derivative. Moreover $\nabla^H u$ is the horizontal gradient of u i.e. $\nabla^H u = \Pi_H \nabla u$ and $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. Also $\Pi_H : T(M) \rightarrow H(M)$ is the projection associated to the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ and T is the characteristic direction of $d\theta$ (cf. Proposition 1.2 in [29], p. 8). Let (V, y^i) be a local coordinate system on N such that $\phi^{-1}(V) = U$ and let us set $Y_i = (\partial/\partial y^i) \circ \phi$. Then $\{Y_i : 1 \leq i \leq \nu\}$ is a local frame in $\phi^{-1}TN \rightarrow M$ defined on U . Let us set $\tau(\phi) = \tau(\phi)^i Y_i$ on U . By a result in [4]

$$\tau(\phi)^i = \Delta_b \phi^i + \sum_{a=1}^{2n} X_a(\phi^j) X_a(\phi^k) (\Gamma_{jk}^i \circ \phi) , \quad 1 \leq i \leq \nu ,$$

where $\phi^i = y^i \circ \phi$ and Γ_{jk}^i are the Christoffel symbols of the second kind of the Riemannian metric h . A hint to the study of systems $\tau(\phi)^i = 0$ where Δ_b is replaced by an arbitrary second order linear X -elliptic operator (in the sense of [57]) is given in Appendix A to this paper.

The energy (3) is but a particular case of σ -integral. Precisely let $\sigma(\xi_1, \dots, \xi_{2n})$ be an arbitrary symmetric function of $2n$ variables. Let $B \in \Gamma^\infty(H(M)^* \otimes H(M)^*)$ be a smooth symmetric bilinear form on $H(M)$. Let $x \in M$ be a point and $\{X_a : 1 \leq a \leq 2n\}$ a local G_θ -orthonormal frame of $H(M)$ defined on a neighborhood $U \subseteq M$ of x . Let $\{\lambda_1(x), \dots, \lambda_{2n}(x)\}$ be the eigenvalues of B at x relative to the Levi form G_θ i.e. the $2n$ real roots of the equation

$$\det [\delta_{ab} \lambda - B_{ab}(x)] = 0$$

where $B_{ab} = B(X_a, X_b)$. Let us consider the function $\sigma(B) : M \rightarrow \mathbb{R}$ given by $\sigma(B)(x) = \sigma(\lambda_1(x), \dots, \lambda_{2n}(x))$ for any $x \in M$. The σ -integral of B is the number

$$I_\sigma(B) = \int_M \sigma(B) dv .$$

Let $\sigma_p(\xi_1, \dots, \xi_{2n})$ denote the p -th elementary symmetric function. In particular

$$\sigma_1(B) = \text{trace}_{G_\theta}(B) , \quad \sigma_{2n}(B)(x) = \det [B_{ab}(x)] , \quad x \in U .$$

As well as in the Riemannian category (cf. e.g. J. Eells & J.H. Sampson, [31], p. 5) the σ_p -integrals $I_{\sigma_p}(B)$ lead to a wealth of variational problems of interest within pseudohermitian geometry. For instance let $B = \Pi_H \text{Ric}$ where

$$\text{Ric}(X, Y) = \text{trace} \{ Z \in \mathfrak{X}^\infty(M) \rightarrow R^\nabla(Z, Y)X \}$$

and R^∇ is the curvature tensor field of the Tanaka-Webster connection ∇ of (M, θ) . Let us set $R_{ab} = \text{Ric}(X_a, X_b)$. By a result in [29], p. 58, $[R_{ab}]$ is symmetric so that

$$\det [\delta_{ab}\xi - R_{ab}(x)] = 0$$

has $2n$ real roots $\{\xi_a(x) : 1 \leq a \leq 2n\}$ (a pseudohermitian analog to the principal curvatures of a surface in the Euclidean space, cf. e.g. [40], p. 210). Then

$$(4) \quad I_{\sigma_1}(\Pi_H \text{Ric}) = \int_M \text{trace}_{G_\theta}(\Pi_H \text{Ric}) dv = 2 \int_M \rho dv$$

where ρ is the pseudohermitian scalar curvature of (M, θ) (cf. Definition 1.30 in [29], p. 50). The functional (4) was studied by S.S. Chern & R.S. Hamilton, [18]. In spite of the fact that a theory of CR and pseudohermitian immersions is already available (cf. e.g. [28]) the pseudohermitian analog to the σ_m -integral $\int_M \sigma_m(\alpha) d\text{vol}(g)$ (where $\alpha \otimes \nu$ is the second fundamental form in the normal direction ν of a given isometric immersion of the m -dimensional Riemannian manifold (M, g) , hence $\sigma_m(\alpha)$ is the Gauss-Kroneker curvature in the normal direction ν) has not been studied.

Let $\phi : M \rightarrow N$ be a C^∞ map. The σ -energy of ϕ is the number

$$E_\sigma(\phi) = I_\sigma(\Pi_H \phi^* h) = \int_M \sigma(\Pi_H \phi^* h) dv .$$

Then the energy (3) of ϕ is its $(\sigma_1/2)$ -energy i.e. $2E(\phi) = E_{\sigma_1}(\phi)$.

3. THE SUB-RIEMANNIAN APPROACH

Let M be a C^∞ manifold and $S \subset T(M)$ a smooth distribution of real rank m on M . S is *bracket generating* if the C^∞ sections in S together with their commutators span $T_x(M)$ at each point $x \in M$. Given $v \in S_x$ let $X \in \Gamma^\infty(S)$ such that $X_x = v$. Let $S_x + [v, S_x] \subset T_x(M)$ be the subspace spanned by

$$S_x \cup \{[X, Y]_x : Y \in \Gamma^\infty(S)\} .$$

The definition of $S_x + [v, S_x]$ doesn't depend upon the choice of C^∞ extension X of v (because $[V, Y]_x \in S_x$ for any $V \in \Gamma^\infty(S)$ such that $V_x = 0$). Next we define inductively the spaces $\mathcal{D}_k(v) \subset T_x(M)$ by setting

$$\mathcal{D}_2(v) = S_x + [v, S_x] , \quad \mathcal{D}_k(v) = S_x + [\mathcal{D}_{k-1}(v), S_x] , \quad k \geq 3 .$$

A tangent vector $v \in S_x$ is a *k-step bracket generator* if $\mathcal{D}_k(v) = T_x(M)$. The distribution S satisfies the *strong bracket generating hypothesis* if for arbitrary $x \in M$ each $v \in S_x$ is a 2-step bracket generator.

Let S be a bracket generating distribution on M . A *sub-Riemannian metric* on S is a Riemannian bundle metric on S i.e. a smooth positive definite section $Q \in \Gamma^\infty(S^* \otimes S^*)$. Given a sub-Riemannian metric Q on S and a point $x \in M$ we consider the linear map $a(x) : T_x^*(M) \rightarrow S_x$ given by

$$Q_x(v, a(x)\xi) = \xi(v) , \quad \xi \in T_x^*(M) , \quad v \in S_x .$$

Then $\text{Ker}[a(x)] = S_x^\perp$ (where $S^\perp \subset T^*(M)$ is the conormal bundle associated to S). If (U, x^A) is a local coordinate system on M we set

$$a(x)(dx^A)_x = a^{AB}(x) \frac{\partial}{\partial x^B} \Big|_x, \quad x \in U, \quad 1 \leq A \leq m.$$

The functions $a^{AB} \in C^\infty(U)$ are the sub-Riemannian analogs to the inverse of the matrix consisting of the local components of the given metric in Riemannian geometry. The analogy may not be pushed further as the matrix $[a^{AB}]$ is never invertible.

Let M be a strictly pseudoconvex CR manifold, of CR dimension n . The Levi distribution $H(M)$ is bracket generating. Indeed let $x \in M$ and $\{T_\alpha : 1 \leq \alpha \leq n\}$ a local frame of $T_{1,0}(M)$ defined on $U \subseteq M$ such that $x \in U$. Let $\{X_a : 1 \leq a \leq 2n\}$ be the local frame of $H(M)$ given by

$$X_\alpha = \frac{1}{2} (T_\alpha + T_{\bar{\alpha}}), \quad X_{\alpha+n} = \frac{i}{2} (T_\alpha - T_{\bar{\alpha}}), \quad 1 \leq \alpha \leq n,$$

where $T_{\bar{\alpha}} = \bar{T}_\alpha$. We set as customary

$$g_{\alpha\bar{\beta}} = G_\theta(T_\alpha, T_{\bar{\beta}}), \quad \nabla_{T_B} T_C = \Gamma_{BC}^A T_A.$$

Here $A, B, C \in \{0, 1, \dots, n, \bar{1}, \dots, \bar{n}\}$ with the convention $T_0 = T$. As the torsion of the Tanaka-Webster connection ∇ is pure

$$\Gamma_{\beta\bar{\gamma}}^{\bar{\alpha}} T_{\bar{\alpha}} - \Gamma_{\bar{\gamma}\beta}^\alpha T_\alpha - [T_\beta, T_{\bar{\gamma}}] = 2ig_{\beta\bar{\gamma}} T.$$

Consequently $T_x(M) \otimes_{\mathbb{R}} \mathbb{C}$ is the span of $\{T_\alpha(x), T_{\bar{\alpha}}(x), [T_\alpha, T_{\bar{\beta}}]_x : 1 \leq \alpha, \beta \leq n\}$ and then $\{X_a(x), [X_a, X_b]_x : 1 \leq a, b \leq 2n\}$ spans $T_x(M)$.

Let $x \in M$ and $v \in H(M)_x$. For each $1 \leq a \leq 2n$ we may consider a local frame $\mathcal{R}_a = \{X_{ab} : 1 \leq b \leq 2n\}$ of $H(M)$ on $U \subseteq M$ such that $x \in U$ and $X_{aa}(x) = v$. If $X \in \mathcal{R}_a$ let \tilde{X} denote a C^∞ extension of X to M . Then

$$\begin{aligned} \mathcal{D}_2(v) &= H(M)_x + [v, H(M)_x] = \\ &= \bigcup_{a=1}^{2n} \text{Span}_{\mathbb{R}} \left[H(M)_x \cup \left\{ [\tilde{X}_{aa}, Y]_x : Y \in \Gamma^\infty(H(M)) \right\} \right] = \\ &= \bigcup_{a=1}^{2n} \text{Span}_{\mathbb{R}} \{X_{ab}(x), [X_{aa}, X_{ab}]_x : 1 \leq b \leq 2n\} = T_x(M) \end{aligned}$$

i.e. v is a 2-step bracket generator. Therefore $H(M)$ satisfies the strong bracket generating hypothesis and G_θ is a sub-Riemannian metric on $H(M)$. By a result in [5], p. 411, if $g(x) : T_x^*(M) \rightarrow H(M)_x$ is given by $G_{\theta,x}(v, g(x)\xi) = \xi(v)$ for any $\xi \in T_x^*(M)$ and any $v \in H(M)_x$ then

$$(5) \quad a^{AB} = g^{AB} - T^A T^B, \quad 1 \leq A, B \leq 2n+1,$$

where

$$[g^{AB}] = [g_{AB}]^{-1}, \quad g_{AB} = g_\theta(\partial_A, \partial_B), \quad T = T^A \partial_A,$$

and $\partial_A = \partial/\partial x^A$ for simplicity. For each $\varepsilon > 0$ let g_ε be the Riemannian metric on M locally given by

$$g_\varepsilon(\partial_A, \partial_B) = g_{AB}(\varepsilon), \quad [g_{AB}(\varepsilon)] = [a^{AB} + \varepsilon g^{AB}]^{-1}.$$

By a result in [7]

$$(6) \quad g_\varepsilon(X, Y) = \frac{1}{1 + \varepsilon} \left\{ g_\theta(X, Y) + \frac{1}{\varepsilon} \theta(X)\theta(Y) \right\}$$

for any $X, Y \in \mathfrak{X}^\infty(M)$. As a consequence of (6)

$$(7) \quad \nabla^\varepsilon u = \nabla^H u + \varepsilon \nabla u, \quad u \in C^1(M).$$

Here $\nabla^\varepsilon u$ is the gradient of u with respect to g_ε i.e. $g_\varepsilon(\nabla^\varepsilon u, X) = X(u)$ for any $X \in \mathfrak{X}^\infty(M)$. By (7) $\nabla^\varepsilon u$ approaches the horizontal gradient of u as $\varepsilon \rightarrow 0$.

Let $\phi : M \rightarrow N$ be a C^1 map and let us consider the ordinary energy density

$$\begin{aligned} e_\varepsilon(\phi) &= \frac{1}{2} \text{trace}_{g_\varepsilon}(\phi^* h) = \\ &= \frac{1}{2} (a^{AB} + \varepsilon g^{AB}) \frac{\partial \phi^i}{\partial x^A} \frac{\partial \phi^j}{\partial x^B} (h_{ij} \circ \phi). \end{aligned}$$

As well as J. Jost & C-J. Xu, [52], we dissociate the underlying integration measure from the domain metric. Then the energy of ϕ is

$$E_\varepsilon(\phi) = \int_M e_\varepsilon(\phi) dv$$

and

$$\lim_{\varepsilon \rightarrow 0^+} E_\varepsilon(\phi) = E(\phi).$$

Let us adopt for a moment the notations and conventions in [75], p. 113-116. Let $\sigma_2(\Delta_b) \in \text{Smb}_2(E, E)$ be the symbol of Δ_b where $E = M \times \mathbb{C}$ is the trivial line bundle over M . Let $T'(M) = T^*(M) \setminus (0)$ and $\pi : T'(M) \rightarrow M$ the projection. Let $\omega \in T'(M)$ with $\pi(\omega) = x$. Then $\sigma_2(\Delta_b)_\omega : (\pi^{-1}E)_\omega \rightarrow (\pi^{-1}E)_\omega$ is given by

$$\sigma_2(\Delta_b)_\omega v = [\|\omega\|^2 - \omega(T_x)^2] v, \quad v \in E_x = (\pi^{-1}E)_\omega,$$

hence ellipticity of Δ_b fails in the characteristic direction T of $d\theta$. The sublaplacian Δ_b is actually a degenerate elliptic operator (cf. J.M. Bony, [11]) i.e. locally

$$\Delta_b u = - \sum_{A, B=1}^{2n+1} \frac{\partial}{\partial x^A} \left(a^{AB} \frac{\partial u}{\partial x^B} \right) + \sum_{B=1}^{2n+1} c^B \frac{\partial u}{\partial x^B}$$

(where a^{AB} given by (5) is but positive semi-definite) and

$$\Delta_b = \sum_{a=1}^{2n} X_a^2 + Y$$

for some smooth real vector fields $\{X_a, Y : 1 \leq a \leq 2n\}$ generating a Lie algebra $\mathcal{L}(X_1, \dots, X_{2n}, Y)$ of rank $2n$. Also Δ_b is subelliptic of order $\varepsilon = 1/2$ i.e. for any point $x \in M$ there is an open neighborhood $U \subseteq M$ and a constant $C > 0$ such that

$$\|u\|_{1/2}^2 \leq C [(\Delta_b u, u) + \|u\|^2], \quad u \in C_0^\infty(U).$$

Here $\|\cdot\|_\varepsilon$ is the Sobolev norm of order ε . Also $(f, g) = \int_M f \bar{g} dv$ and $\|f\| = (f, f)^{1/2}$ for any $f, g \in L^2(M)$. Consequently (cf. L. Hörmander, [46]) Δ_b is hypoelliptic.

As a consequence of (6) one may relate the Levi-Civita connection ∇^ε of (M, g_ε) and the Tanaka-Webster connection of (M, θ) (cf. [7])

$$(8) \quad \nabla_X^\varepsilon Y = \nabla_X Y - \left\{ (d\theta)(X, Y) + \frac{\varepsilon}{1 + \varepsilon} g_\theta(\tau X, Y) \right\} T,$$

$$(9) \quad \nabla_X^\varepsilon T = \tau(X) + \left(1 + \frac{1}{\varepsilon}\right) JX ,$$

$$(10) \quad \nabla_T^\varepsilon T = 0 ,$$

for any $X, Y \in \mathfrak{X}^\infty(M)$. Here τ is the pseudohermitian torsion of ∇ (cf. [29], p. 26). Then (8)-(10) yield

$$(11) \quad \Delta_\varepsilon u = (1 + \varepsilon)\Delta_b u - \varepsilon T(T(u)) , \quad u \in C^2(M) ,$$

where Δ_ε is the Laplace-Beltrami operator of (M, g_ε) . By (11) the sublaplacian Δ_b is approximated by the second order elliptic operator Δ_ε (very much the way Hörmander's operator H is approximated by H_ε in [52], p. 4635).

4. SUBELLIPTIC HARMONIC MAPS INTO SPHERES

Let $S^\nu \subset \mathbb{R}^{\nu+1}$ be the standard sphere and $\Sigma = \{x = (x_1, \dots, x_{\nu+1}) \in S^\nu : x_1 = x_2 = 0\}$. A continuous map $\phi : M \rightarrow S^\nu$ *meets* Σ if $\phi(M) \cap \Sigma \neq \emptyset$. Let $\phi : M \rightarrow S^\nu$ be a continuous map that doesn't meet Σ . Then $\phi : M \rightarrow S^\nu$ *links* Σ if the map $\phi : M \rightarrow S^\nu \setminus \Sigma$ is not null-homotopic. By a result of Y. Kamishima et al., [30]

Theorem 1. *Any nonconstant subelliptic harmonic map $\phi : M \rightarrow S^\nu$ of a compact strictly pseudoconvex CR manifold M into a sphere S^ν either links or meets Σ .*

The Riemannian counterpart of Theorem 1 is due to B. Solomon, [70]. The proof of Theorem 1 relies on the following

Lemma 1. *Let $\phi : M \rightarrow N$ be a subelliptic harmonic map of a compact strictly pseudoconvex CR manifold into a Riemannian warped product $N = L \times_w \mathbb{R}$ with $w \in C^\infty(L)$. Then $\phi(M) \subset L \times \{t_\phi\}$ for some $t_\phi \in \mathbb{R}$.*

Let us set $F = p_1 \circ \phi$ and $u = p_2 \circ \phi$ where $p_1 : N \rightarrow L$ and $p_2 : N \rightarrow \mathbb{R}$ are the natural projections. Subelliptic theory comes into picture because (under the assumptions of Lemma 1) u may be shown to satisfy the subelliptic equation

$$(12) \quad \operatorname{div}((w \circ F)^2 \nabla^H u) = 0 .$$

As a consequence of (12) u is shown to be a \mathbb{R} -valued smooth solution to the tangential Cauchy-Riemann equations $\bar{\partial}_b u = 0$ hence (by the nondegeneracy of the CR structure) a constant. The key differential geometric feature is that $S^\nu \setminus \Sigma$ is isometric to the warped product $S_+^{\nu-1} \times_\nu S^1$ where

$$v : S_+^{\nu-1} \times S^1 \rightarrow (0, +\infty) , \quad v(y, z) = y_\nu , \quad y \in S_+^{\nu-1} , \quad z \in S^1 \subset \mathbb{C} ,$$

$$S_+^{\nu-1} = \{y = (y', y_\nu) \in \mathbb{R}^\nu : y \in S^\nu, y_\nu > 0\} .$$

Indeed let us assume that the subelliptic harmonic map $\phi : M \rightarrow S^\nu$ doesn't meet Σ and $\phi : M \rightarrow S^\nu \setminus \Sigma$ is null-homotopic. Then $\phi : M \rightarrow S_+^{\nu-1} \times_\nu S^1$ admits a natural lift $\tilde{\phi} : M \rightarrow S_+^{\nu-1} \times_w \mathbb{R}$ which is subelliptic harmonic too. Here $w \in C^\infty(S_+^{\nu-1})$ is given by $w(y) = y_\nu$ for any $y \in S_+^{\nu-1}$. By Lemma 1 such a map is actually $S_+^{\nu-1}$ -valued. Finally there are no subelliptic harmonic maps from a strictly pseudoconvex CR manifold into an open upper hemisphere but the constants as (by a version of the first variation formula got in the presence of an isometric immersion of the target manifold into an Euclidean space)

$$(13) \quad \Delta_b \Phi^A = |X\Phi|^2 \Phi^A , \quad 1 \leq A \leq \nu ,$$

where $\Phi = \iota \circ \phi$ and $\iota : S^{\nu-1} \rightarrow \mathbb{R}^\nu$ is the inclusion. Also $|X\Phi|^2 = \sum_{a,A} (X_a \Phi^A)^2$ on U with respect to a local G_θ -orthonormal frame $\{X_a : 1 \leq a \leq 2n\}$ on $U \subseteq M$. Once again it may be easily shown that smooth solutions to (13) are real valued CR functions (i.e. $\bar{\partial}_b \Phi^A = 0$) hence constants.

We recall that $S_+^{\nu-1} \times S^1 \simeq S^1$ (a homotopy equivalence). Therefore a continuous map $\phi : M \rightarrow S_+^{\nu-1} \times S^1$ is null homotopic if and only if $p_2 \circ \phi : M \rightarrow S^1$ is null-homotopic, where $p_2 : S_+^{\nu-1} \times S^1 \rightarrow S^1$ is the projection. The homotopy classes of continuous maps $M \rightarrow S^1$ form an abelian group $\pi^1(M)$ (the Bruschi group of M , cf. e.g. [48], p. 48). Also (by Theorem 7.1 in [48], p. 49) there is a natural isomorphism $\pi^1(M) \approx H^1(M, \mathbb{Z})$. Then we may state the following

Corollary 1. *Let M be a compact strictly pseudoconvex CR manifold with $H^1(M, \mathbb{Z}) = 0$. Then any nonconstant subelliptic harmonic map $\phi : M \rightarrow S^\nu$ meets Σ .*

We end this section by hinting to certain open problems regarding the regularity of energy minimizing maps $\phi : M \rightarrow S^\nu \setminus \Sigma$ from a strictly pseudoconvex CR manifold M . Let us briefly recall the state-of-the-art of the problem within Riemannian geometry. Let (M, g) be a Riemannian manifold such that $\dim(M) = m \geq 4$ and $\phi \in W_{\text{loc}}^{1,2}(M, N)$ (cf. e.g. [45], cf. Definition 1.4.1, p. 32). If $K \subset M$ is a compact subset the ordinary Dirichlet energy integral is

$$E_K(\phi) = \frac{1}{2} \int_K \|d\phi\|^2 \, d\text{vol}(g).$$

Then ϕ is *energy minimizing* if $E_K(\phi) \leq E_K(\psi)$ for any compact set $K \subset M$ and any $\psi \in W_{\text{loc}}^{1,2}(M, N)$ with $\psi = \phi$ a.e. in $M \setminus K$. By a classical result of R. Schoen & K. Uhlenbeck (cf. Theorem II in [68], p. 310) energy minimizing maps $\phi : M \rightarrow N$ are always smooth on $M \setminus \text{Sing}(\phi)$, where $\text{Sing}(\phi)$ is the singular set of ϕ , a closed subset of M of Hausdorff codimension at least 3. The result may be improved (cf. [69]) for energy minimizing maps into a sphere $N = S^\nu$. Indeed let

$$d(2) = 2, \quad d(3) = 3, \quad d(\nu) = \left[\min \left\{ \frac{\nu}{2} + 1, 6 \right\} \right], \quad \nu \geq 4.$$

Then (by Theorem 2.7 in [69], p. 96) i) if $m \leq d(\nu)$ then every minimizing map $\phi : M \rightarrow S^\nu$ is smooth, ii) if $m = d(\nu) + 1$ then $\phi : M \rightarrow S^\nu$ has at most isolated singularities, and in general iii) $\text{Sing}(\phi)$ is a closed set of Hausdorff dimension $\leq m - d(\nu) - 1$. It should be emphasized that the lower bound on the codimension of $\text{Sing}(\phi)$ may be improved from information on the target manifold N alone.

By a result of B. Solomon, [70]

Theorem 2. *Let M be a m -dimensional Riemannian manifold, $m \geq 4$. Any energy minimizing map $\phi : M \rightarrow S^\nu$ which omits a neighborhood of a codimension 2 totally geodesic submanifold $\Sigma \subset S^\nu$ is everywhere smooth.*

The proof of this result relies on Theorem IV in [68], p. 310, according to which the Hausdorff dimension of $\text{Sing}(\phi)$ is $\leq m - k - 3$ for any energy minimizing map $\phi : M \rightarrow N$ provided that N satisfies the property $P(N, k)$ for some $k \in \mathbb{Z}$, $1 \leq k \leq m - 3$. The predicate $P(N, k)$ is for any $d \in \{1, \dots, k\}$ each C^∞ harmonic map $S^{d+1} \rightarrow N$ is constant. This may be applied to $N = S^\nu \setminus \Sigma$ as $H^1(S^{d+1}, \mathbb{Z}) = 0$ for any $d \in \mathbb{Z}$, $d > 0$. Indeed (by the Riemannian counterpart of Corollary 1, cf. [70], p. 155-156) each harmonic map $S^{d+1} \rightarrow S^\nu \setminus \Sigma$ is constant i.e. $P(S^\nu \setminus \Sigma, k)$ holds true for any $1 \leq k \leq m - 3$.

The obstacle towards recovering Theorem 2 within CR geometry is then due to the lack of information on the Hausdorff dimension of $\text{Sing}(\phi)$ for a given energy minimizing (relative to the functional (3) above) map $\phi : M \rightarrow S^\nu$ from a strictly pseudoconvex CR manifold M . So far the known results contemplate solely subelliptic harmonic maps from a Carnot group, cf. C. Wang, [74], and L. Capogna & N. Garofalo, [15]. The next section is devoted to a brief survey of the known regularity results on weakly subelliptic harmonic maps and to a few remarks as to their relationship to CR geometry.

5. SUBELLIPTIC HARMONIC MAPS FROM CARNOT GROUPS

Let \mathbb{G} be a simply connected Lie group. \mathbb{G} is a Carnot group if its Lie algebra \mathfrak{g} is stratified and nilpotent i.e. there is $r \geq 1$ such that $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ and i) $[V_i, V_j] = V_{j+1}$ for any $1 \leq j \leq r-1$ (i.e. \mathfrak{g} is *stratified*) and ii) $[V_j, V_r] = (0)$ for any $1 \leq j \leq r$ (i.e. \mathfrak{g} is *r-nilpotent*). The smallest such r is the *step* of \mathbb{G} . The exponential map $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a diffeomorphism (cf. e.g. [73]). Let $m_j = \dim_{\mathbb{R}} V_j$ and $\{X_{j,i} : 1 \leq i \leq m_j\}$ a linear basis of V_j for any $1 \leq j \leq r$. The exponential coordinates $p_{ji} : \mathbb{G} \rightarrow \mathbb{R}$ are given by

$$x = \exp \left(\sum_{j=1}^r \sum_{i=1}^{m_j} p_{ji}(x) X_{j,i} \right), \quad x \in \mathbb{G}.$$

For each $\lambda > 0$ we consider the dilation $\delta_\lambda : \mathbb{G} \rightarrow \mathbb{G}$ given by

$$\delta_\lambda(x) = \exp \left(\sum_{j=1}^r \sum_{i=1}^{m_j} \lambda^j p_{ji}(x) X_{j,i} \right), \quad x \in \mathbb{G}.$$

We shall need the gauge norm

$$|x|^{2r!} = \sum_{j=1}^r \left(\sum_{i=1}^{m_j} |p_{ji}(x)|^2 \right)^{r!/j}, \quad x \in \mathbb{G},$$

and the gauge metric

$$d(x, y) = |x^{-1}y|, \quad x, y \in \mathbb{G}.$$

The gauge metric is a *pseudo-metric* i.e.

$$d(x, y) \geq 0, \quad d(x, y) = 0 \Leftrightarrow x = y, \quad d(x, y) = d(y, x),$$

$$d(x, y) \leq C_{\mathbb{G}} \{d(x, z) + d(z, y)\},$$

for some constant $C_{\mathbb{G}} > 0$ and any $x, y, z \in \mathbb{G}$. Also

$$(14) \quad d(zx, zy) = d(x, y), \quad d(\delta_\lambda(x), \delta_\lambda(y)) = d(x, y).$$

Next we set $B(x, r) = \{y \in \mathbb{G} : d(x, y) < r\}$ (the ball of center $x \in \mathbb{G}$ and radius $r > 0$ with respect to the gauge metric). Using (14) one may easily show that

$$(15) \quad \mu[B(x, r)] = r^Q \mu[B(e, 1)], \quad x \in \mathbb{G}, \quad r > 0.$$

Here $Q = \sum_{j=1}^r j m_j$ is the homogeneous dimension of \mathbb{G} (cf. e.g. [10], p. 126) and $e \in \mathbb{G}$ is the group identity. Also $\mu(A)$ is the Haar measure of $A \subset \mathbb{G}$ (μ is got by pushing forward the Lebesgue measure on \mathfrak{g} to \mathbb{G}). The growth (15) of the balls with respect to d accounts for the role as a critical dimension played by Q (in

the regularity results to be surveyed below). We shall make use of the Hausdorff measure with respect to the gauge metric

$$H^s(A) = \lim_{\delta \rightarrow 0^+} H_\delta^s(A), \quad A \subset \mathbb{G}, \quad 0 \leq s < \infty,$$

$$H_\delta^s(A) = \inf \left\{ \sum_{k=1}^{\infty} r_k^s : A \subset \bigcup_{k=1}^{\infty} B(x_k, r_k), \quad \sup_{k \geq 1} r_k < \delta \right\}.$$

Let \mathbb{G} be a Carnot group of step $r \geq 2$. We set $m = m_1$ and $X_a = X_{1,a}$ for $1 \leq a \leq m$ for simplicity (so that $X = \{X_1, \dots, X_m\}$ is a linear basis in V_1). Let $\Omega \subset \mathbb{G}$ be a bounded domain and $1 \leq p < \infty$. Let $W_X^{1,p}(\Omega)$ consist of all $u \in L^2(\Omega)$ admitting weak derivatives $X_a u \in L^p(\Omega)$ for $1 \leq a \leq m$. We set $Xu = (X_1 u, \dots, X_m u)$. Then $W_X^{1,p}(\Omega)$ is a Banach space with the norm

$$(16) \quad \|u\|_{W_X^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}.$$

We denote by $W_{X,0}^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm (16). For each $K \geq 1$ let $W_X^{1,p}(\Omega, \mathbb{R}^K)$ consist of all maps $\phi = (\phi^1, \dots, \phi^K) : \Omega \rightarrow \mathbb{R}^K$ such that $\phi^j \in W_X^{1,p}(\Omega)$ for any $1 \leq j \leq K$. We also need the Hölder like spaces $\Gamma^\alpha(\Omega)$, $0 < \alpha < 1$, consisting of all $u : \Omega \rightarrow \mathbb{R}$ such that $[u]_\alpha < \infty$. Here

$$[u]_\alpha = \sup \left\{ \frac{u(x) - u(y)}{d(x,y)^\alpha} : x, y \in \Omega, \quad x \neq y \right\}.$$

$\Gamma^\alpha(\Omega)$ is a Banach space with the norm

$$\|u\|_{\Gamma^\alpha(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_\alpha.$$

Let N be a closed (i.e. compact without boundary) Riemannian manifold isometrically immersed into an Euclidean space \mathbb{R}^K for some $K \geq 1$. Let $W_X^{1,p}(\Omega, N)$ consist of all $\phi \in W_X^{1,p}(\Omega, \mathbb{R}^K)$ such that $\phi(x) \in N$ for a.e. $x \in \Omega$. We consider the *horizontal* energy functional

$$(17) \quad E_\Omega(\phi) = \frac{1}{2} \int_\Omega |X\phi|^2 d\mu, \quad \phi \in W_X^{1,2}(\Omega, N).$$

Here $|X\phi|^2 = \sum_{a=1}^m \sum_{j=1}^K |X_a \phi^j|^2$. In this context a subelliptic harmonic map is a critical point $\phi \in W_X^{1,2}(\Omega, N)$ of E_Ω . We also need the Morrey spaces $\mathcal{M}^{p,q}(U)$ consisting of all $u : U \rightarrow \mathbb{R}$ such that

$$(18) \quad \|u\|_{\mathcal{M}^{p,q}(U)}^p \equiv \sup_{B(x,r) \subset U} \frac{1}{r^q} \int_{B(x,r)} |u(y)|^p d\mu(y) < \infty.$$

Here $1 \leq p < \infty$, $0 < q < \infty$ and $U \subset \mathbb{G}$ is an open subset. $\mathcal{M}^{p,q}(U)$ is a Banach space with the norm (18). Morrey spaces in the subelliptic setting are also employed³ by G. Citti & G. Di Fazio, [21]. Appendix B is devoted to a brief review

³The result in [21] is that weak solutions to $\sum_{a=1}^m X_a^2 u + V(x)u = 0$ (where $\{X_a : 1 \leq a \leq m\}$ is a Hörmander system of smooth vector fields on an open set $\Omega \subset \mathbb{R}^n$) are Hölder continuous provided that the potential $V(x)$ belongs to a Morrey type space i.e.

$$(19) \quad \sup_{r>0, x \in \Omega} r^{-q} \int_{\Sigma_r(x)} |V(y)| dy < \infty$$

for some $q > 1$ (compare (19) to (18) above). Here $\Sigma_r(x) = \{y \in \Omega : 1/\Gamma(x,y) < r\}$ (rather than a Carnot-Carathéodory ball) where $\Gamma(x,y)$ is a fundamental solution to $\sum_{a=1}^m X_a^2$. The geometry of the sets $\Sigma_r(x)$ and that of the Carnot-Carathéodory balls are however intimately related (as shown for instance in the Appendix to [20]).

of the results by M. Geisler, [38], hinting to the construction of Morrey spaces on a strictly pseudoconvex CR manifold. It may be shown that

$$\lim_{r \rightarrow 0} \|X\phi\|_{\mathcal{M}^{2, Q-2}(B(x, r))} = 0$$

for any $\phi \in W_X^{1,2}(\Omega, N) \cap C^\infty(\Omega, N)$. The work in [74] is mainly concerned with the converse of this statement for a given subelliptic harmonic map $\phi \in W_X^{1,2}(\Omega, N)$. To survey the results in [74] we need to recall the concept of regular point. Let $\phi \in W_X^{1,2}(\Omega, N)$ and $\varepsilon > 0$. A point $x_0 \in \Omega$ is ε -regular for ϕ if there is $r_0 = r_0(x_0, \varepsilon) > 0$ such that $|X\phi| \in \mathcal{M}^{2, Q-2}(B(x_0, r_0))$ and

$$\|X\phi\|_{\mathcal{M}^{2, Q-2}(B(x_0, r_0))} \leq \varepsilon.$$

Let $\text{Reg}_\varepsilon(\phi) \subset \Omega$ be the set of all ε -regular point for ϕ and let us set $\text{Sing}_\varepsilon(\phi) = \Omega \setminus \text{Reg}_\varepsilon(\phi)$. As an immediate consequence of definitions $\text{Reg}_\varepsilon(\phi)$ is an open set (eventually empty) and

$$0 < \varepsilon_1 < \varepsilon_2 \implies \text{Reg}_{\varepsilon_1}(\phi) \subset \text{Reg}_{\varepsilon_2}(\phi)$$

so that the limit

$$\text{Reg}(\phi) \equiv \bigcap_{\varepsilon > 0} \text{Reg}_\varepsilon(\phi)$$

exists. Also

$$\text{Reg}(\phi) \subset \left\{ x \in \Omega : \lim_{r \rightarrow 0} r^{2-Q} \int_{B(x, r) \cap \Omega} |X\phi|^2 d\mu = 0 \right\}.$$

Cf. (1.7) in [74], p. 99. A point $x_0 \in \Omega$ is a *smooth point* for $\phi \in W_X^{1,2}(\Omega, N)$ if there is $r_0 = r_0(x_0) > 0$ such that $\phi \in C^\infty(B(x_0, r_0), N)$. The main result of C. Wang (cf. Theorem A in [74], p. 99) is

Theorem 3. *Let \mathbb{G} be a Carnot group and $\Omega \subset \mathbb{G}$ a bounded domain. Let $\phi \in W^{1,2}(\Omega, N)$ be a subelliptic harmonic map. There is $\varepsilon_0 > 0$ such that any ε_0 -regular point for ϕ is a smooth point for ϕ . In particular $\text{Reg}(\phi) = \text{Reg}_{\varepsilon_0}(\phi)$ is open and $\phi \in C^\infty(\text{Reg}(\phi), N)$.*

A subelliptic harmonic map $\phi \in W_X^{1,2}(\Omega, N)$ is *stationary* if there exist two constants $K_0 > 0$ and $R_0 > 0$ depending on Ω and N such that

$$s^{2-Q} \int_{B(x, s)} |X\phi|^2 d\mu \leq K_0 r^{2-Q} \int_{B(x, r)} |X\phi|^2 d\mu$$

(the horizontal Dirichlet energy monotonicity inequality) for any $x \in \mathbb{G}$ and any $0 < s \leq r \leq \min\{R_0, d(x, \partial\Omega)\}$. In the elliptic setting the concept is due to P. Price, [67]. As an immediate consequence of definitions if $\phi \in W_X^{1,2}(\Omega, N)$ is a stationary subelliptic harmonic map there is a constant $C > 0$ such that

$$\text{Reg}_\varepsilon(\phi) = \left\{ x \in \Omega : \lim_{r \rightarrow 0} r^{2-Q} \int_{B(x, r)} |X\phi|^2 d\mu \leq C \varepsilon^2 \right\}.$$

Hence (cf. L. Evans & R. Gariepy, [32]) $H^{Q-2}(\text{Sing}_\varepsilon(\phi)) = 0$ for any $\varepsilon > 0$ so that Theorem 3 yields (cf. Corollary B in [74], p. 100)

Corollary 2. *Let \mathbb{G} be a Carnot group and $\Omega \subset \mathbb{G}$ a bounded domain. If $\phi \in W_X^{1,2}(\Omega, N)$ is a stationary subelliptic harmonic map then $H^{Q-2}(\Omega \setminus \text{Reg}(\phi)) = 0$ and $\phi \in C^\infty(\text{Reg}(\phi), N)$.*

It should be observed that if $\phi \in W_X^{1,Q}(\Omega, N)$ then $\Omega = \text{Reg}_\varepsilon(\phi)$ for any $\varepsilon > 0$. Indeed for each ball $B(x, r) \subset \Omega$ (by Hölder's inequality)

$$\begin{aligned} & \|X\phi\|_{\mathcal{M}^{2,Q-2}(B(x,r))} \leq \\ & \leq \sup \left\{ \left(\int_{B(y,s)} |X\phi|^Q d\mu \right)^{1/Q} : B(y,s) \subset B(x,r) \right\} \leq \\ & \leq \left(\int_{B(x,r)} |X\phi|^Q d\mu \right)^{1/Q}. \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0^+} \|X\phi\|_{\mathcal{M}^{2,Q-2}(B(x,r))} = 0, \quad x \in \Omega,$$

by the absolute continuity of $\int_{B(x,r)} |X\phi|^Q d\mu$. Therefore one may state (cf. Corollary C in [74], p. 100)

Corollary 3. *Let \mathbb{G} be a Carnot group and $\Omega \subset \mathbb{G}$ a bounded domain. If $\phi \in W_X^{1,Q}(\Omega, N)$ is a subelliptic harmonic map then $\phi \in C^\infty(\Omega, N)$.*

Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the natural coordinates (z^1, \dots, z^n, t) and $z^\alpha = y^\alpha + iy^\alpha$. We consider the left invariant tangent vector fields

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + 2y^\alpha T, \quad X_{\alpha+n} = \frac{\partial}{\partial y^\alpha} - 2x^\alpha T, \quad 1 \leq \alpha \leq n,$$

where $T = \partial/\partial t$. Then \mathbb{H}_n is a Carnot group of step $r = 2$ whose Lie algebra \mathfrak{h} may be decomposed as $\mathfrak{h} = V_1 \oplus V_2$ where V_1 is the span of $\{X_a : 1 \leq a \leq 2n\}$ while $V_2 = \mathbb{R}T$. The homogeneous dimension of \mathbb{H}_n as a Carnot group is $Q = 2n + 2$. \mathbb{H}_n is also a strictly pseudoconvex CR manifold of CR dimension n when equipped with the CR structure $T_{1,0}(\mathbb{H}_n)$ spanned by

$$T_\alpha = \frac{\partial}{\partial z^\alpha} + i\bar{z}^\alpha T, \quad 1 \leq \alpha \leq n.$$

Let $\Omega \subset \mathbb{H}_n$ be a bounded domain. In this context a map $\phi \in W_X^{1,2}(\Omega, N)$ is *horizontal energy minimizing* if $E_\Omega(\phi) \leq E_\Omega(\psi)$ for any $\psi \in W_X^{1,2}(\Omega, N)$ with $\psi - \phi \in W_{X,0}^{1,2}(\Omega, \mathbb{R}^K)$. It is an open problem whether one may exploit C. Wang's result (cf. Theorem 3 above) to establish a CR analog to Theorem 2 for subelliptic harmonic maps $\phi \in W_X^{1,2}(\Omega, S^\nu)$. Unfortunately it is readily clear (cf. [74], p. 101) that Theorem 3 doesn't include the possible partial regularity of minimizing subelliptic harmonic maps. The key to the relationship among subelliptic harmonic maps from a CR manifold and from a Carnot group is perhaps J. Mitchell's result (cf. Theorem 1 in [62], p. 36) reformulated as it applies to our situation

Theorem 4. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that the Levi form G_θ is positive definite. Let d_θ be the Carnot-Carathéodory metric of the sub-Riemannian manifold $(M, H(M), G_\theta)$. Then the tangent cone*

$$(20) \quad \mathcal{C}_x(M) = \lim_{\lambda \rightarrow \infty} ((M, x), \lambda \cdot d_\theta), \quad x \in M,$$

is isometric to a nilpotent Lie group endowed with a Carnot-Carathéodory metric.

The limit in (20) is the Hausdorff-Gromov limit of the sequence of (pointed) metric spaces $((M, x), \lambda \cdot d_\theta)$. The explicit calculation of the tangent cone $\mathcal{C}_x(M)$ is also performed in [62].

6. FURTHER REGULARITY RESULTS

Let \mathbb{G} be a Carnot group of step $r = 2$ with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ and let $X = \{X_1, \dots, X_m\}$ be a fixed linear basis in the first layer V_1 of \mathfrak{g} . Let $\Omega \subset \mathbb{G}$ be a bounded domain. Partial regularity of local minimizers of functionals of the form

$$(21) \quad \mathcal{F}_\Omega(\phi) = \int_\Omega F(X\phi) d\mu$$

is dealt with by L. Capogna & N. Garofalo, [15]. Here $\phi : \Omega \rightarrow \mathbb{R}^K$ and $X\phi$ is the matrix $[X_a\phi^j]$. Also $F : \mathbb{R}^{mK} \rightarrow [0, +\infty)$ is a C^2 function such that

$$F(\xi) \leq C(1 + |\xi|^q) ,$$

$$\sum_{i,j=1}^K \sum_{a,b=1}^m \frac{\partial^2 F}{\partial \xi_a^i \partial \xi_b^j}(\xi) \eta_a^i \eta_b^j \geq \lambda(1 + |\xi|)^{q-2} |\eta|^2 > 0 ,$$

for some $2 \leq q < \infty$ and any $\xi, \eta \in \mathbb{R}^{mK}$. A typical example of a functional of the form (21) is the regularized horizontal p -energy

$$E_p^\varepsilon(\phi, \Omega) = \int_\Omega (\varepsilon + |Xu|^2)^{p/2} d\mu$$

with $2 \leq p < \infty$. The finding in [15] is

Theorem 5. *Let \mathbb{G} be a Carnot group of step 2 and $\Omega \subset \mathbb{G}$ a bounded domain. If ϕ is a local minimizer of \mathcal{F}_Ω then there is a closed set $S \subset \Omega$ of measure zero such that $X\phi$ is Hölder continuous in $\Omega \setminus S$. Moreover if $q = 2$ then $H^{Q-2+\sigma}(S) = 0$ for any $\sigma > 0$.*

The paper [15] is also concerned with the partial regularity of weak solutions to quasi-linear systems of the form

$$(22) \quad - \sum_{j=1}^K \sum_{a,b=1}^m X_a (A_{ij}^{ab}(x, \phi) X_b \phi^j) = f_i(x, \phi, X\phi) , \quad 1 \leq i \leq K ,$$

where A_{ij}^{ab} are Hölder continuous functions which together with f_i obey to the following requirements. For any $L > 0$ i) there is a constant $\lambda(L) > 0$ such that

$$(23) \quad \sum_{i,j=1}^K \sum_{a,b=1}^m A_{ij}^{ab}(x, \phi) \xi_a^i \xi_b^j \geq \lambda(L) |\xi|^2 ,$$

and ii) $f = (f_1, \dots, f_K)$ is at most of quadratic growth in $X\phi$ i.e. there exist constants $a(L) > 0$ and $b(L) > 0$ such that

$$(24) \quad |f(x, \phi, \xi)| \leq a(L) |\xi|^2 + b(L) ,$$

for any $x \in \Omega$ and $|\phi| \leq L$. When $\{X_a : 1 \leq a \leq m\}$ is a Hörmander system of vector fields on an open set $\Omega \subset \mathbb{R}^n$ quasi-linear systems of the form

$$(25) \quad \sum_{a,b=1}^m X_b^* (B^{ab}(x, \phi) X_a \phi^i) = f_i(x, \phi, X\phi)$$

whose right hand side satisfies (24) have been considered by C-J. Xu & C. Zuily, [76]. Their result is that continuous solutions $\phi^i \in W_X^{1,2}(\Omega)$ to (25) are C^∞ provided that B^{ab} and f^i are C^∞ (cf. Theorem 1.1 in [76], p. 323). As left translations on \mathbb{G} are isometries of $L^2(\mathbb{G})$ the formal adjoint of X_a is $X_a^* = -X_a$ for any $1 \leq a \leq m$ (cf. e.g. [35]). Thus the systems (25) (restated on a Carnot group \mathbb{G}) fit into the larger class (22) (for $A_{ij}^{ab} = \delta^{ij} B^{ab}$). A prototype for both (22) and (25) is the subelliptic harmonic map system $\tau(\phi)^i = 0$. Precisely the class of systems (25) includes the subelliptic harmonic map system (4.6) in [52], p. 4639, while (22) includes the Euler-Lagrange equations of the variational principle associated to the horizontal energy functional (17). The finding in [15] is

Theorem 6. *Let \mathbb{G} be a Carnot group of step 2 and $\Omega \subset \mathbb{G}$ a bounded domain. Let ϕ be a weak solution to (22) with the assumptions (23)-(24). If $a(L)L \leq \lambda(L)/2$ there is a closed set $S \subset \Omega$ such that $X\phi$ is Hölder continuous in $\Omega \setminus S$ and $H^{Q-2+\sigma}(S) = 0$ for each $\sigma > 0$.*

Cf. Theorem 1.2 in [15], p. 5. The main technical novelty in [15] are the $W^{2,2}$ -estimates for constant coefficient systems of the form

$$-\sum_{j=1}^K \sum_{a,b=1}^m X_a (A_{ij}^{ab} X_b \phi^j) = f_i, \quad 1 \leq i \leq K.$$

Aside from regarding a smaller class of systems the setting in [76] is of course more general (working with arbitrary Hörmander systems of vector fields, *versus* systems of left invariant vector fields $\{X_1, \dots, X_m\} \subset V_1 \subset \mathfrak{g}$ in [74] and [15]). It should be mentioned however that the approach in [76] relies on G.B. Folland & E.M. Stein's idea of approximating by a nilpotent Lie group (cf. [34]) later developed by L.P. Rothschild & E.M. Stein (cf. [72]) into a lifting technique exhibiting the approximate behavior near the origin - of the vector fields X_a belonging to the given Hörmander system - as left invariant vector fields on a neighborhood of the identity of a connected simply connected nilpotent Lie group. Precisely let $\{X_1, \dots, X_m\}$ be a system of C^∞ vector fields defined in a neighborhood of the origin in \mathbb{R}^n . For a multi-index $\alpha = (a_1, \dots, a_d) \in \mathbb{Z}^d$, $1 \leq a_j \leq m$, we set $|\alpha| = d$ and

$$X_\alpha = [X_{a_d}, [X_{a_{d-1}}, \dots, [X_{a_2}, X_{a_1}] \dots]]$$

(a *commutator* of order d of the vector fields X_a). Let us assume that Hörmander's condition is satisfied at the origin i.e. $\{X_\alpha(0) : |\alpha| \leq r\}$ span $T_0(\mathbb{R}^n)$ for some $r \in \mathbb{Z}$, $r \geq 1$. Let $\mathfrak{g}(m, r)$ be the free nilpotent Lie algebra of step r with m generators i.e. the quotient of the free Lie algebra with m generators by the ideal generated by the commutators of order $\geq r+1$. The system of vector fields $\{X_1, \dots, X_m\}$ is *free* up to order r at 0 if $\dim_{\mathbb{R}} \mathfrak{g}(m, r) = n$ (as a \mathbb{R} -linear space). The lifting technique mentioned above is described by (cf. [72])

Theorem 7. *There is a neighborhood of the origin $U \subset \mathbb{R}^n \times \mathbb{R}^\ell$ and a system of vector fields $\{\tilde{X}_a : 1 \leq a \leq m\} \subset \mathfrak{X}^\infty(U)$ of the form*

$$\tilde{X}_a(x, y) = X_a(x) + \sum_{i=1}^{\ell} f_a^i(x, y^1, \dots, y^{i-1}) \frac{\partial}{\partial y^i}, \quad y = (y^1, \dots, y^\ell),$$

such that i) $\{\tilde{X}_\alpha(0, 0) : |\alpha| \leq r\}$ span $T_{(0,0)}(\mathbb{R}^n \times \mathbb{R}^\ell)$ and ii) the system $\{\tilde{X}_a : 1 \leq a \leq m\}$ is free up to order r at $(0, 0)$.

A simple proof of Theorem 7 is provided by L. Hörmander & A. Melin, [47]. On the other hand J. Mitchell's explicit calculation (cf. [62]) of the tangent cone (20) in Theorem 4 relies on the work by L.P. Rothschild & E.M. Stein, [72], R. Goodman, [39], and G. Metivier, [61], raising the expectation that the study of subelliptic harmonic maps from a strictly pseudoconvex CR manifold may be reduced to that of subelliptic harmonic maps from a Carnot group. Precisely the proof of Theorem 4 is based on

Theorem 8. *Let M be a strictly pseudoconvex CR manifold of CR dimension n and $\{X_a : 1 \leq a \leq 2n\}$ a local frame of $H(M)$ defined on the open set $U \subset M$. For each point $x_0 \in U$ there exist neighborhoods $U_1 \subset \subset U_0 \subset \subset U$ of x_0 , a neighborhood of the origin $\Omega_0 \subset \mathbb{R}^{2n+1}$, and a C^∞ map $F : \bar{U}_1 \times U_0 \rightarrow \mathbb{R}^{2n+1}$ such that*

- i) *For each $x \in \bar{U}_1$ the map $F_x(y) = F(x, y)$ is a C^∞ diffeomorphism of U_0 onto $F_x(U_0) = \Omega_0$ and $F_x(x) = 0$;*
- ii) *For each $x \in \bar{U}_1$ the vector fields $X_a^x \in \mathfrak{X}^\infty(\Omega_0)$ given by*

$$X_a^x(y) = (d_y F_x)X_{a,y}, \quad y \in U_0,$$

are of degree ≤ 1 at 0;

- iii) *If \hat{X}_a^x is the homogeneous part of degree one of X_a^x then the vector fields \hat{X}_a^x generate a nilpotent Lie algebra of dimension $2n + 1$. Moreover for each $\xi \in \Omega_0$ let $\hat{V}_d(\xi)$ denote the subspace of $T_\xi(\mathbb{R}^{2n+1})$ spanned by all commutators $\{\hat{X}_\alpha(\xi) : |\alpha| \leq d\}$. Then*

$$\dim_{\mathbb{R}} \hat{V}_1(\xi) = 2n, \quad \dim_{\mathbb{R}} \hat{V}_d(\xi) = 2n + 1, \quad d \geq 2, \quad \xi \in \Omega_0;$$

- iv) *The vector fields X_a^x and \hat{X}_a^x depend smoothly on $x \in U_1$.*

Theorem 8 is taken from [61] (cf. also [62], p. 38) and restated as it applies to the CR category.

7. SUB-RIEMANNIAN GEOMETRY AND CONNECTION THEORY

Manifest in the work by R.S. Strichartz, [71], is the absence of any notion of co-variant derivative and curvature (associated to a given sub-Riemannian structure). Riemannian curvature measures the higher order deviation of the given manifold from its Euclidean model, yet (as argued in [71], p. 223) sub-Riemannian manifolds exhibit no approximate Euclidean behavior. However if the given sub-Riemannian structure is associated to a pair $(T_{1,0}(M), \theta)$ consisting of a strictly pseudoconvex CR structure $T_{1,0}(M)$ and a contact form θ (as in Section 2 of this paper) then the Tanaka-Webster connection ∇ of (M, θ) turns out to be a useful tool (cf. [5]). Indeed if (M, θ) is a Sasakian manifold (i.e. $\tau = 0$) then every lengthy geodesic of ∇ is a sub-Riemannian geodesic (cf. Corolary 1 in [5], p. 413). Nevertheless the question remains whether R.S. Strichartz's theory (cf. *op. cit.*) may be geometrized i.e. whether there is a connection whose auto-parallel curves are the sub-Riemannian geodesics of the given sub-Riemannian structure. An answer to this question was recently provided by B. Langerock, [58], though by embedding sub-Riemannian geometry into a generalized connection theory, dealing with generalized connections on $T^*(M)$ over a vector bundle morphism $a : T^*(M) \rightarrow T(M)$ (these are never ordinary connections as $\text{Ker}(a) \neq (0)$). We briefly review B. Langerock's theory (cf. also F. Cantrijn & B. Langerock, [16]) stripped of the physics considerations there and hint to its impact on CR geometry.

Let M be a C^∞ manifold and $\pi : E \rightarrow M$ a vector bundle over M . Let $f : E \rightarrow T(M)$ be a vector bundle morphism such that $\Pi \circ f = \pi$ where $\Pi : T(M) \rightarrow M$ is the projection. Let $\rho : W \rightarrow M$ be a bundle with arbitrary standard fibre. We shall need the pullback bundle $\rho^{-1}E \rightarrow W$. The various bundles considered so far are summarized in the following commutative diagram

$$\begin{array}{ccccc} T(M) & \xleftarrow{f} & E & \xleftarrow{\tilde{\rho}} & \rho^{-1}E \\ \text{id} \downarrow & & \pi \downarrow & & \tilde{\pi} \downarrow \\ T(M) & \xrightarrow{\Pi} & M & \xrightarrow{\rho} & W . \end{array}$$

Here $\rho^{-1}E$ is a submanifold of the product manifold $W \times E$ and the maps $\tilde{\pi}$ and $\tilde{\rho}$ are the restrictions to $\rho^{-1}E$ of the natural projections $W \times E \rightarrow W$ and $W \times E \rightarrow E$. We adopt the following definition (cf. B. Langerock, [58]). A *generalized connection* on W over the vector bundle morphism f (a *f-connection* on W) is a vector bundle morphism $h : \rho^{-1}E \rightarrow T(W)$ such that the following diagram is commutative

$$\begin{array}{ccc} \rho^{-1}E & \xrightarrow{h} & T(W) \\ \tilde{\rho} \downarrow & & \downarrow d\rho \\ E & \xrightarrow{f} & T(M) . \end{array}$$

Let us look at the particular case where $E = T(M)$ (hence $\pi = \Pi$) and f is the identical transformation

$$\begin{array}{ccccc} W & \xleftarrow{\tilde{\Pi}} & p^{-1}T(M) & \xrightarrow{h} & T(W) \\ \rho \downarrow & & \downarrow \tilde{\rho} & & \downarrow d\rho \\ M & \xleftarrow{\Pi} & T(M) & \xrightarrow{\text{id}} & T(M) . \end{array}$$

If this is the case we set

$$N_w = h_w(p^{-1}T(M))_w = h_w(\{w\} \times T_{p(w)}(M)) , \quad w \in W .$$

Let $V_w = \text{Ker}(d_w\rho) \subset T_w(W)$ and let $X \in N_w \cap V_w$. Then $X = h_w(w, v)$ for some $v \in T_{\rho(w)}(M)$ and (by the commutativity of the diagram above)

$$0 = (d_w\rho)X = (d_w\rho)h_w(w, v) = \tilde{\rho}(w, v) = v$$

hence $X = 0$. Therefore

$$(26) \quad N_w \cap V_w = (0) , \quad w \in W .$$

Each $X \in \mathfrak{X}(M)$ gives rise to a *natural lift* $\hat{Y} \in \Gamma^\infty(\rho^{-1}T(M))$ given by

$$\hat{Y}(w) = (w, X_{\rho(w)}) , \quad w \in W .$$

Let (U, x^i) be a local coordinate system on M and let us denote by $X_i \in \Gamma^\infty(\rho^{-1}(U), \rho^{-1}T(M))$ the natural lift of $\partial/\partial x^i$. Then $\{X_i : 1 \leq i \leq \dim(M)\}$ is a local frame in $\rho^{-1}T(M) \rightarrow W$ defined on the open set $\rho^{-1}(U) \subset W$. Next we set

$$\delta_i = \frac{\delta}{\delta x^i} = h(X_i) .$$

We claim that $\{\delta_i : 1 \leq i \leq \dim(M)\}$ is a local frame of N defined on $\rho^{-1}(U)$ hence $N : w \in W \mapsto N_w$ is a C^∞ distribution on W of rank $\dim(M)$. To prove the claim it suffices to check that $h : \rho^{-1}T(M) \rightarrow T(W)$ is a vector bundle monomorphism.

Indeed if $Y = (w, v) \in \text{Ker}(h_w)$ with $v \in T_{\rho(w)}(M)$ then $h_w(Y) = 0$ yields $0 = (d_w \rho)h_w(Y) = \tilde{\rho}(Y) = v$ hence $Y = 0$. Then

$$(27) \quad T_w(W) = N_w \oplus V_w, \quad w \in W.$$

In particular $d_w \rho : N_w \rightarrow T_{\rho(w)}(M)$ is a \mathbb{R} -linear isomorphism whose inverse is $h_w : (\rho^{-1}T(M))_w \rightarrow N_w$ for any $w \in W$. Gathering the information in (26)-(27) N is nothing but a *nonlinear connection* on W (in the sense of W. Barthel, [8]) and the bundle isomorphism $h : \rho^{-1}T(M) \rightarrow N$ is the corresponding horizontal lift. It should be observed that in general, for an arbitrary f -connection $h : \rho^{-1}E \rightarrow T(W)$ on W , the sum

$$h_w(\rho^{-1}E)_w + \text{Ker}(d_w \rho) \subset T_w(W), \quad w \in W,$$

may fail to be direct. Indeed it may be easily shown that

Proposition 1. *Let $h : \rho^{-1}E \rightarrow T(W)$ be a f -connection on W . Let $w \in W$ and $v \in E_{\rho(w)}$. Then*

$$h_w(w, v) \in N_w \cap V_w \iff v \in \text{Ker}(f_{\rho(w)})$$

where $N_w = h_w(\rho^{-1}E)_w$ and $V_w = \text{Ker}(d_w \rho)$.

Let $\pi_P : P \rightarrow M$ be a principal G -bundle. A *principal f -connection* on P is a f -connection $h : \pi_P^{-1}E \rightarrow T(P)$ on P such that $N = h(\pi_P^{-1}E)$ is G -invariant i.e.

$$(d_p R_g)N_p = N_{pg}, \quad p \in P, \quad g \in G.$$

Here $R_g : P \rightarrow P$, given by $R_g(p) = \alpha(p, g) = pg$ for any $p \in P$, is the right translation by $g \in G$ and $\alpha : P \times G \rightarrow P$ is the given action of G on P . Cf. again [58], p. 210. Let F be a manifold on which G acts to the left and let $\beta : G \times F \rightarrow F$ be the given action of G on F . There is a natural action of G on the product manifold $P \times F$

$$\gamma : (P \times F) \times G \rightarrow P \times F,$$

$$\gamma((p, f), g) = (p, f) \cdot g = (pg, g^{-1}f), \quad p \in P, \quad f \in F, \quad g \in G.$$

For each $L_{(p,f)} : G \rightarrow P \times F$ be given

$$L_{(p,f)}(g) = (p, f) \cdot g, \quad g \in G.$$

Let $W = P \times_G F$ be the quotient space

$$W = P \times_G F = (P \times F)/G = \{L_{(p,f)}(G) : (p, f) \in P \times F\}.$$

Moreover let us consider the map

$$\sigma : P \times F \rightarrow W, \quad \sigma(p, f) = L_{(p,f)}(G) \in W, \quad (p, f) \in P \times F.$$

The projection $\rho : W \rightarrow M$ is then given by

$$\rho(w) = \pi_P(p), \quad w = \sigma(p, f) \in W, \quad (p, f) \in P \times F.$$

The definition of $\rho(w)$ doesn't depend upon the choice of representative (p, f) of the G -orbit $w \in W$. A *verbatim* repetition of the arguments in [22] leads to

Proposition 2. *Let M be a manifold and G a Lie group. Let $P(M, \pi_P, G)$ be a principal G -bundle and let $W(M, \rho, F, G)$ be the associated bundle of standard fibre F . Then any principal f -connection $h : \pi_P^{-1}E \rightarrow T(W)$ on P induces a f -connection $h^* : \rho^{-1}E \rightarrow T(W)$ on W .*

Proof. Given $f \in F$ let $\sigma_f : P \rightarrow W$ be the map given by

$$\sigma_f(p) = \sigma(p, f) \in W, \quad p \in P.$$

Next we consider the space

$$N_w^* = (d_p \sigma_f) N_p \subset T_w(W), \quad w = \sigma(p, f) \in W,$$

where $N_p = h_p(\pi_P^{-1}E)_p \subset T_p(P)$. The definition of N_w^* doesn't depend upon the choice of representative (p, f) of w . Then

$$N_w^* = (d_p \sigma_f) h_p E_{\pi_P(p)} = (d_p \sigma_f) h_p E_{\rho(w)} = (d_p \sigma_f) h_p (\rho^{-1}E)_w.$$

At this point we may consider

$$h_w^* : (\rho^{-1}E)_w \rightarrow T_w(W), \quad w \in W,$$

$$h_w^* = (d_p \sigma_f) \circ h_p, \quad w = \sigma(p, f).$$

Then $h^* : \rho^{-1}E \rightarrow T(W)$ is a f -connection on W . Q.e.d.

By Proposition 1 if $f : E \rightarrow T(M)$ is a vector bundle monomorphism then N^* is a connection in the associated bundle $W(M, \rho, G, F)$. If $h^* : \rho^{-1}E \rightarrow T(W)$ is but a generalized connection on W over f one may nevertheless follow the arguments in [22] to produce a covariant differentiation operator ∇ naturally associated to N^* and acting on sections in W . Cf. also [27]. The explicit construction of ∇ (though confined to the case where the standard fibre F is a linear space) is given in [16].

Let us assume that $\rho : W \rightarrow M$ is a vector bundle. A generalized connection $h : \rho^{-1}E \rightarrow T(W)$ on W over f is a *linear f -connection* (according to the terminology in [58]) if

$$[(d_w \phi_t) \circ h_w](w, v) = h_{\phi_t(w)}(\phi_t(w), v), \quad (w, v) \in (\rho^{-1}E)_w, \quad t \in \mathbb{R},$$

where $\phi_t : W \rightarrow W$ is multiplication by $t \in \mathbb{R}$ i.e. $\phi_t(w) = tw$ for any $w \in W$. The corresponding covariant differentiation operator $\nabla : \Gamma^\infty(E) \times \Gamma^\infty(W) \rightarrow \Gamma^\infty(W)$ possesses (cf. [16] or [58], p. 210) the following properties i) ∇ is \mathbb{R} -bilinear, ii) ∇ is $C^\infty(M)$ -homogeneous in the first argument, iii) $\nabla_s(u\sigma) = u\nabla_s\sigma + (f \circ s)(u)\sigma$ for any $s \in \Gamma^\infty(E)$, $u \in C^\infty(M)$ and $\sigma \in \Gamma^\infty(W)$.

A smooth curve $\tilde{\gamma} : [a, b] \rightarrow E$ is *f -admissible* if $\dot{\gamma}(t) = f(\tilde{\gamma}(t))$ where $\gamma(t) = \pi(\tilde{\gamma}(t)) \in M$, $a \leq t \leq b$. A *base curve* is the projection on M of a f -admissible curve in W . As well as in classical connection theory a linear f -connection on $W \rightarrow M$ gives rise to a covariant differentiation operator $\nabla_{\tilde{\gamma}}$ along any f -admissible curve $\tilde{\gamma} : [a, b] \rightarrow E$. Precisely let us set

$$k = \dim_{\mathbb{R}} E_x, \quad \ell = \dim_{\mathbb{R}} W_x, \quad x \in M,$$

and let us consider local frames

$$\{s_\alpha : 1 \leq \alpha \leq k\} \subset \Gamma^\infty(U, E), \quad \{\sigma_A : 1 \leq A \leq \ell\} \subset \Gamma^\infty(U, W),$$

in E and W respectively, defined on the open set $U \subseteq M$. Let $\Gamma_{\alpha B}^A \in C^\infty(U)$ be defined by

$$\nabla_{s_\alpha} \sigma_B = \Gamma_{\alpha B}^A s_\alpha \sigma_B, \quad 1 \leq \alpha \leq k, \quad 1 \leq B \leq \ell.$$

Let $\gamma^{-1}W \rightarrow [a, b]$ be the pullback of $W \rightarrow M$ via $\gamma : [a, b] \rightarrow M$. We consider the map

$$\begin{aligned} \nabla_{\tilde{\gamma}} : \Gamma^\infty(\gamma^{-1}W) &\rightarrow \Gamma^\infty(\gamma^{-1}W), \\ (\nabla_{\tilde{\gamma}}\sigma)(t) &= \left\{ \frac{d\mu^A}{dt}(t) + \Gamma_{\alpha B}^A(\gamma(t))\lambda^\alpha(t)\mu^B(t) \right\} \sigma_B(\gamma(t)), \end{aligned}$$

$$\begin{aligned} \sigma &\in \Gamma^\infty(\gamma^{-1}W), \quad a \leq t \leq b, \\ \tilde{\gamma}(t) &= \lambda^\alpha(t)s_\alpha(\gamma(t)), \quad \sigma(t) = \mu^A(t)\sigma_A(\gamma(t)). \end{aligned}$$

A section $\sigma \in \Gamma^\infty(\gamma^{-1}W)$ is *parallel* along the base curve $\gamma = \pi \circ \tilde{\gamma}$ (with respect to the linear f -connection ∇) if $\nabla_{\tilde{\gamma}}\sigma = 0$.

Let (M, S, Q) be a sub-Riemannian manifold and $a : T^*(M) \rightarrow S \subset T(M)$ the corresponding bundle morphism. We shall work with linear a -connections $h : \rho^{-1}T^*(M) \rightarrow T(T^*(M))$ over the bundle map $a : T^*(M) \rightarrow T(M)$. Here $E = W = T^*(M)$, $f = a$ and $\rho = \pi = \Pi^* : T^*(M) \rightarrow M$ is the projection. The relevant commutative diagram is

$$\begin{array}{ccc} \rho^{-1}T^*(M) & \xrightarrow{h} & T(T^*(M)) \\ \tilde{\rho} \downarrow & & \downarrow d\rho \\ T^*(M) & \xrightarrow{a} & T(M). \end{array}$$

To proceed we need a few additional notions of sub-Riemannian geometry. A piecewise C^1 curve $\gamma : [a, b] \rightarrow M$ is *lengthy* if $\dot{\gamma}(t) \in S_{\gamma(t)}$ for each value of the parameter t such that $\dot{\gamma}(t)$ exists. The *length* of a lengthy curve $\gamma : [a, b] \rightarrow M$ is given by $L(\gamma) = \int_a^b Q_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt$. Let us assume that M is pathwise connected. As S is bracket generating a well known result of W.L. Chow, [19], implies that any two points in M can be joined by a lengthy curve. Therefore it makes sense to consider the problem of finding length minimizing curves among all curves connecting two given points. Precisely if $x, y \in M$ and $\Omega_S(x, y)$ is the set of all lengthy curves joining x and y (where $\Omega_S(x, y) \neq \emptyset$ by Chow's theorem, as argued above) a curve $\gamma \in \Omega_S(x, y)$ is *length minimizing* if $L(\gamma) \leq L(c)$ for any $c \in \Omega_S(x, y)$.

Let $\alpha : [a, b] \rightarrow T^*(M)$ be a piecewise C^1 curve and let us set $\gamma(t) = \Pi^*(\alpha(t))$ for any $a \leq t \leq b$. The curve $\alpha : [a, b] \rightarrow T^*(M)$ is a -admissible if $a(\alpha(t)) = \dot{\gamma}(t)$ whenever $\dot{\gamma}(t)$ makes sense (and then γ is a base curve in M). A lengthy curve $\gamma : [0, 1] \rightarrow M$ is *parametrized by constants times arc-length* if there is a constant $C > 0$ such that $Q_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = C$ for each t . The following *Pontryagin maximum principle* holds (cf. Theorem 4 in [58], p. 208-209, restated as it applies to CR geometry)

Theorem 9. *Let M be a connected strictly pseudoconvex CR manifold and θ a contact form on M with G_θ positive definite. Let $\gamma : [0, 1] \rightarrow M$ be a length minimizing curve, parametrized by constants times arc-length, on the semi-Riemannian manifold $(M, H(M), G_\theta)$. There is a piecewise C^1 curve $\beta : [0, 1] \rightarrow T^*(M)$ such that $\Pi^*(\beta(t)) = \gamma(t)$ and $\beta(t) \neq 0$ for any $0 \leq t \leq 1$ and one of the following conditions is satisfied*

i) $\beta : [0, 1] \rightarrow T^*(M)$ is an integral curve of the Hamiltonian vector field X_H associated to the Hamiltonian $H : T^*(M) \rightarrow \mathbb{R}$ given by

$$(28) \quad H(\omega) = \frac{1}{2} Q_x^*(\omega, \omega), \quad \omega \in T_x^*(M), \quad x \in M.$$

Here Q^* is given by $Q_x^*(\lambda, \mu) = \langle a_x(\lambda), \mu \rangle = \mu[a_x(\lambda)]$ for any $\lambda, \mu \in T_x^*(M)$ and $x \in M$. In particular both β and γ are C^∞ .

ii) $\beta(t) \in H(M)_{\gamma(t)}^\perp$ for any $0 \leq t \leq 1$ and for any a -admissible curve $\alpha : [0, 1] \rightarrow T^*(M)$ with base curve $\gamma : [0, 1] \rightarrow M$

$$(29) \quad \frac{d}{dt} \{ \langle \beta(t), X_{\gamma(t)} \rangle \}_{t=0} = \langle \beta(t), [a_{\gamma(t)}(\alpha(t)), X] \rangle$$

for any $X \in \mathfrak{X}^\infty(M)$ and any $0 \leq t \leq 1$ such that $\dot{\gamma}(t)$ is defined.

As to the notation in (29) for any $\eta \in \Omega^1(M) = \Gamma^\infty(T^*(M))$ and $X, Y \in \mathfrak{X}^\infty(M)$ with $\eta_x \in H(M)_x^\perp$ and $X_x \in H(M)_x$ we set

$$\langle \eta_x, [X_x, Y_x] \rangle = \langle \eta, [X, Y] \rangle(x) .$$

A lengthy curve γ in M is a *normal extremal* (respectively an *abnormal extremal*) if there is a piecewise C^1 curve β in $T^*(M)$ such that $\Pi^* \circ \beta = \gamma$ and $\beta(t) \neq 0$ for all t and the condition (i) (respectively the condition (ii)) in Theorem 9 above is satisfied.

Let M be a strictly pseudoconvex CR manifold of CR dimension n and θ a contact form on M . Let $\nabla : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)$ be a linear a -connection on $T^*(M) \rightarrow M$. Let $I \subset \mathbb{R}$ be an interval and let $\alpha : I \rightarrow T^*(M)$ be an a -admissible curve with base curve $\gamma = \Pi^* \circ \alpha$. Let

$$\nabla_\alpha : \Gamma^\infty(\gamma^{-1}T^*(M)) \rightarrow \Gamma^\infty(\gamma^{-1}T^*(M))$$

be the covariant differential operator along γ . Note that

$$\alpha(t) \in T_{\gamma(t)}^*(M) = (\gamma^{-1}T^*(M))_t$$

hence $\alpha \in \Gamma^\infty(\gamma^{-1}T^*(M))$. The a -admissible curve $\alpha : I \rightarrow T^*(M)$ is an *auto-parallel curve* of ∇ if $\nabla_\alpha \alpha = 0$. The base curve γ corresponding to an auto-parallel curve α is a *geodesic* of ∇ . If (U, x^i) is a local coordinate system on M then

$$\alpha(t) = \xi_i(t) (dx^i)_{\gamma(t)} , \quad |t| < \varepsilon .$$

Here $\varepsilon > 0$ is small enough such that $\gamma(t) \in U$ for any $|t| < \varepsilon$. Let us also set $\gamma^i(t) = x^i(\gamma(t))$, $|t| < \varepsilon$, for any $1 \leq i \leq m$ where $m = 2n + 1$. The a -admissibility condition $\dot{\gamma}(t) = a(\alpha(t))$ together with $\nabla_\alpha \alpha = 0$ may be locally written

$$(30) \quad \frac{d\gamma^i}{dt}(t) = a^{ij}(\gamma(t)) \xi_j(t) ,$$

$$(31) \quad \frac{d\xi_i}{dt}(t) + \Gamma_i^{jk}(\gamma(t)) \xi_j(t) \xi_k(t) = 0 .$$

The functions $\Gamma_i^{jk} \in C^\infty(U)$ are of course given by $\nabla_{dx^j} dx^k = \Gamma_i^{jk} dx^i$. The equations (30)-(31) should be compared to the equations (H-J) in [71], p. 233 (describing the "geodesics" of a sub-Riemannian manifold). Also $Q^*(dx^i, dx^j) = a^{ij}$ hence the Hamiltonian functions in [71], p. 233, and in our (28) coincide. In view of [5] it is interesting to look for applications of B. Langerock's theory (cf. [58]) of linear a -connections to pseudohermitian geometry. The *symmetric product* (associated to the linear a -connection ∇ , cf. [58], p. 213) is

$$\langle \cdot : \cdot \rangle_\nabla : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M) ,$$

$$\langle \alpha : \beta \rangle_\nabla = \nabla_\alpha \beta + \nabla_\beta \alpha , \quad \alpha, \beta \in \Omega^1(M) .$$

The *symmetric bracket* associated to the sub-Riemannian structure $(H(M), G_\theta)$ is the mapping (cf. Definition 13 in [58], p. 214)

$$\{ \cdot, \cdot \} : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M) ,$$

$$\{ \alpha, \beta \} = \mathcal{L}_{a(\alpha)} \beta + \mathcal{L}_{a(\beta)} \alpha - d(Q^*(\alpha, \beta)) , \quad \alpha, \beta \in \Omega^1(M) .$$

A linear a -connection ∇ on $T^*(M)$ is *normal* if

$$\langle \alpha : \beta \rangle_\nabla = \{ \alpha, \beta \} , \quad \alpha, \beta \in \Omega^1(M) .$$

Cf. Definition 15 in [58], p. 214. It may be shown that the local coefficients Γ_i^{jk} of a normal linear a -connection satisfy

$$\Gamma_i^{jk} + \Gamma_i^{kj} = \frac{\partial a^{jk}}{\partial x^i}.$$

If this is the case the equations (30)-(31) become

$$\frac{d\gamma^i}{dt}(t) = a^{ij}(\gamma(t))\xi_j(t), \quad \frac{d\xi_i}{dt}(t) = -\frac{1}{2} \frac{\partial a^{jk}}{\partial x^i}(\gamma(t))\xi_j(t)\xi_k(t),$$

(which coincide with (H-J) in [71], p. 233). By a result of B. Langerock (cf. Theorem 17 in [58], p. 215) a linear a -connection ∇ is normal if and only if every geodesic of ∇ is a normal extremal and conversely.

8. SUBELLIPTIC p -HARMONIC MAPS

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\{X_a : 1 \leq a \leq m\}$ a Hörmander system of vector fields (cf. e.g. [29], p. 233) defined on an open neighborhood U of $\bar{\Omega}$. Let us consider the energy functional

$$E_p(\phi) = \int_{\Omega} |X\phi(x)|^2 dx, \quad \phi \in C^\infty(\Omega, S^\nu).$$

A critical point $\phi = (\phi_1, \dots, \phi_{\nu+1}) : \Omega \rightarrow S^\nu \subset \mathbb{R}^{\nu+1}$ of E_p satisfies the Euler-Lagrange equations

$$(32) \quad -\sum_{a=1}^m X_a^* (|X\phi|^{p-2} X_a \phi) = |X\phi|^p \phi$$

and the constraint $\sum_{j=1}^{\nu+1} \phi_j^2 = 1$. Such maps are referred to as *subelliptic p -harmonic* maps (cf. [42]). Here X_a^* is the formal adjoint of X_a

$$\int_U (X_a^* u) \varphi dx = - \int_U u X_a \varphi dx, \quad u, \varphi \in C_0^\infty(U).$$

We need the Sobolev type spaces $W_X^{1,p}(\Omega, S^\nu)$ consisting of all maps $\phi : \Omega \rightarrow S^\nu$ such that $\phi_j \in W_X^{1,p}(\Omega)$, $1 \leq j \leq \nu + 1$. Also $W_X^{1,p}(\Omega)$ consists of the functions $u \in L^p(\Omega)$ such that $X_a u \in L^p(\Omega)$ for any $1 \leq a \leq m$. Of course $X_a u$ are intended as weak derivatives i.e.

$$\int_U (X_a u) \varphi dx = - \int_U u X_a^* \varphi dx, \quad u \in L_{loc}^1(U), \quad \varphi \in C_0^\infty(U).$$

A map $\phi \in W_X^{1,p}(\Omega, S^\nu)$ is a *weak* subelliptic p -harmonic map if

$$\int_{\Omega} \left\{ \sum_{a=1}^m |X\phi|^{p-2} (X_a \phi) \cdot (X_a \phi) - |X\phi|^p \phi \cdot \phi \right\} dx = 0$$

for any $\varphi \in C_0^\infty(\Omega, \mathbb{R}^{\nu+1})$. The main finding of P. Hájlasz & P. Strzelecki, [42], is

Theorem 10. *Each weak subelliptic Q -harmonic map $\phi \in W_X^{1,Q}(\Omega, S^\nu)$ is locally Hölder continuous.*

Here Q is a homogeneous dimension relative to Ω . We recall that given a bounded open set $\Omega \subset \mathbb{R}^n$ a positive integer Q is a homogeneous dimension relative to Ω if for some positive constant $C > 0$

$$\frac{\mu(B_X(x, r))}{\mu(B_X(x_0, r_0))} \geq C \left(\frac{r}{r_0} \right)^Q$$

where $B_X(x_0, r_0)$ is an arbitrary Carnot-Carathéodory ball centered at a point $x_0 \in \Omega$ of radius $0 < r_0 \leq \text{diam}(\Omega)$ and $B_X(x, r)$ is another Carnot-Carathéodory ball such that $x \in B_X(x_0, r_0)$ and $0 < r \leq r_0$. The Carnot-Carathéodory metric on Ω is of course associated to the Hörmander system $X = \{X_1, \dots, X_m\}$ (cf. e.g. Definition 4.20 in [29], p. 258). Also $\mu(A)$ is the Lebesgue measure of $A \subset \mathbb{R}^n$. Cf. [6] for a generalization of Theorem 10 to critical points of functionals of the form $\int_{\Omega} F(|X\phi|) dx$. The notion of subelliptic p -harmonic map carries over easily to CR geometry. A smooth map $\phi : M \rightarrow N$ of a compact strictly pseudoconvex CR manifold M into a Riemannian manifold (N, h) is *subelliptic p -harmonic* if ϕ is a critical point of

$$E_p(\phi) = \int_M [\text{trace}_{G_\theta} (\Pi_H \phi^* h)]^p dv .$$

Appendix C to this paper is devoted to a description of the main known results on subelliptic p -harmonic functions ($N = \mathbb{R}$). It is an open problem whether a smooth subelliptic p -harmonic map $\phi : M \rightarrow S^\nu \setminus \Sigma$ links Σ . Also if $\phi \in W_X^{1,p}(M, S^\nu \setminus U)$ is a map minimizing E_p (where $U \subseteq S^\nu$ is an open neighborhood of Σ) then it is an open question whether $\phi : M \rightarrow S^\nu$ is smooth. To solve the above problems one would of course need analogs to R. Schoen & K. Uhlenbeck's regularity results (cf. [68]) adapted to weak subelliptic p -harmonic maps minimizing E_p . At the present state of research only analogs of R. Schoen & K. Uhlenbeck's results adapted to p -harmonic maps among Riemannian manifold are actually known. For instance, by a result of R. Hardt & F.H. Lin, [43], any energy minimizing p -harmonic map $\phi : M \rightarrow N$ of class $W^{1,p}$ from a m -dimensional Riemannian manifold is of class C^1 on $M \setminus \text{Sing}(\phi)$ and the Hausdorff dimension of $\text{Sing}(\phi)$ is $\leq m - [p] - 1$ where $[p]$ is the integer part of p . The estimate on the Hausdorff dimension of $\text{Sing}(\phi)$ was improved by N. Nakauchi, [64], when $N = S^\nu$. Precisely

Theorem 11. i) *Let M be a m -dimensional Riemannian manifold. Let $\nu \geq p + 2$. Then any minimizing weak p -harmonic map $\phi \in W^{1,p}(M, S^\nu)$ is of class C^1 on $M \setminus \text{Sing}(\phi)$ and the Hausdorff dimension of $\text{Sing}(\phi)$ is $\leq m - [p] - 2$.*

ii) *If $\nu \geq p + 4$ then the Hausdorff dimension of $\text{Sing}(\phi)$ is $\leq m - [p] - 3$.*

Keeping in mind B. Solomon's proof to Theorem 2 the following result (due again to [64]) is relevant

Theorem 12. *Let N be a Riemannian manifold. Let us assume that there is an integer $k \geq [p] + 1$ such that any energy minimizing map*

$$f \in C^1(S^{j-1}, N), \quad [p] + 1 \leq j \leq k,$$

is a constant. Then the Hausdorff dimension of $\text{Sing}(\phi)$ is $\leq m - k - 1$ for any energy minimizing p -harmonic map $\phi \in W^{1,p}(M, N)$. If $m < k + 1$ then $\text{Sing}(\phi) = \emptyset$.

In view of Theorem 12 it is reasonable to conjecture that B. Solomon's results (cf. [70]), and in particular Theorem 2 above, carry over to p -harmonic maps from a Riemannian manifold into the sphere S^ν . The subelliptic case depends of course on

whether Theorems 8 and 12 may be generalized (to maps $\phi : \mathbb{G} \rightarrow S^\nu$ from a Carnot group \mathbb{G} , or to maps $\phi : \Omega \rightarrow S^\nu$ from a domain $\Omega \subset \mathbb{R}^n$ carrying a Hörmander system of vector fields, or to maps $\phi : M \rightarrow S^\nu$ from a strictly pseudoconvex CR manifold M).

APPENDIX A. A THEOREM OF E. LANCONELLI

In his seminal 1982 paper⁴ E. Lanconelli considered (cf. [55]) second order differential operators

$$(33) \quad Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(a^{ij}(x) \frac{\partial u}{\partial x^j} \right) + c(x)u$$

such that $a^{ij}, c \in L^\infty(\mathbb{R}^n)$ and

$$(34) \quad C_0 \sum_{j=1}^n \lambda^j(x)^2 \xi_j^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \leq C_1 \sum_{j=1}^n \lambda^j(x)^2 \xi_j^2$$

for some constants $C_0 > 0$ and $C_1 > 0$ and some continuous function $\lambda = (\lambda^1, \dots, \lambda^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the following assumptions i) $\lambda^j(x) \geq 0$ for any $x \in \mathbb{R}^n$, ii) $\lambda^j(x) = \prod_{k=1}^n \lambda_{(k)}^j(x^k)$ for any $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ where $\lambda_{(j)}^j \in \text{Lip}(\mathbb{R})$ and

$$j \neq k \implies \lambda_{(k)}^j \in C^1(\mathbb{R} \setminus \{0\}) \quad \text{and} \quad \lambda_{(k)}^j(-t) = \lambda_{(k)}^j(t), \quad t \in \mathbb{R}.$$

Finally one requires that iii) there exist constants $C_k^j > 0$ such that

$$j \neq k \implies 0 \leq x^k \frac{\partial \lambda^j}{\partial x^k}(x) \leq C_k^j \lambda^j(x), \quad x \in \mathbb{R}^n.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. Let $W_\lambda(\Omega)$ (respectively $\overset{\circ}{W}_\lambda(\Omega)$) be the completion of $\{u \in C^\infty(\Omega) : \|u\|_{W_\lambda(\Omega)} < \infty\}$ (respectively of $C_0^\infty(\Omega)$) in the norm

$$\|u\|_{W_\lambda(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \left\| \lambda^j \frac{\partial u}{\partial x^j} \right\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Next let us consider the bilinear form

$$(35) \quad \mathcal{A}_\lambda : [W_\lambda(\Omega) \cap C^\infty(\Omega)] \times [W_\lambda(\Omega) \cap C^\infty(\Omega)] \rightarrow \mathbb{R},$$

$$\mathcal{A}_\lambda(u, v) = \int_\Omega \left(\sum_{i,j=1}^n a^{ij}(x) \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} - c(x)uv \right) dx,$$

for any $u, v \in W_\lambda(\Omega) \cap C^\infty(\Omega)$. As a consequence of (34) and $c \in L^\infty(\mathbb{R}^n)$

$$|\mathcal{A}_\lambda(u, v)| \leq C \|u\|_{W_\lambda(\Omega)} \|v\|_{W_\lambda(\Omega)}$$

hence one may extend (35) by continuity to a bilinear form

$$\mathcal{A}_\lambda : W_\lambda(\Omega) \times W_\lambda(\Omega) \rightarrow \mathbb{R}.$$

⁴Our presentation doesn't aim at historical accuracy. It should be nevertheless mentioned that E. Lanconelli's paper [55] was published in 1983 (while the fundamental work by A. Nagel & E.M. Stein & S. Wainger, [63], appeared in 1985).

Let $W_\lambda^{\text{loc}}(\Omega)$ consist of all $u \in L_{\text{loc}}^2(\Omega)$ such that $\varphi u \in W_\lambda(\Omega)$ for each $\varphi \in C_0^\infty(\Omega)$. Let $f \in L_{\text{loc}}^2(\Omega)$. A natural notion of weak solution to $Lu = f$ may be introduced by using the bilinear form

$$\mathcal{L}_\lambda : W_\lambda^{\text{loc}}(\Omega) \times C_0^\infty(\Omega) \rightarrow \mathbb{R},$$

$$\mathcal{L}_\lambda(u, \varphi) = \mathcal{A}_\lambda(\psi u, \varphi), \quad u \in W_\lambda^{\text{loc}}(\Omega), \quad \varphi \in C_0^\infty(\Omega),$$

where $\psi \in C_0^\infty(\Omega)$ is a bump function such that $\psi(x) = 1$ for any $x \in \text{Supp}(\varphi)$. The definition of $\mathcal{L}_\lambda(u, \varphi)$ doesn't depend upon the choice of ψ . Then $u \in W_\lambda^{\text{loc}}(\Omega)$ is a *weak solution* to $Lu = f$ in Ω if

$$\mathcal{L}_\lambda(u, \varphi) = - \int_\Omega f(x) \varphi(x) dx, \quad \varphi \in C_0^\infty(\Omega).$$

Let us consider the tangent vector fields

$$X_j = \lambda^j \frac{\partial}{\partial x^j}, \quad 1 \leq j \leq n.$$

A continuous curve $\gamma : [0, \ell] \rightarrow \mathbb{R}^n$ is *λ -admissible* if γ is the juxtaposition of a finite number of curves $\{\gamma_a : 1 \leq a \leq N\}$ such that each γ_a is an integral curve of some $X \in \{\pm X_j : 1 \leq j \leq n\}$. Let $C_\lambda([0, \ell], \mathbb{R}^n)$ be the set of all λ -admissible curves $\gamma : [0, \ell] \rightarrow \mathbb{R}^n$. An open set $\Omega \subseteq \mathbb{R}^n$ is *locally λ -connected* if for each point $x \in \Omega$ and each open neighborhood $x \in V \subseteq \mathbb{R}^n$ there is an open neighborhood $x \in W \subseteq V$ such that each $y \in W$ may be joined to x by a λ -admissible curve $\gamma \in C_\lambda([0, \ell], \mathbb{R}^n)$ lying in V i.e. $\gamma(t) \in V$ for any $0 \leq t \leq \ell$. The main finding in [55] is

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ be a locally λ -connected open subset. Then each weak solution $u \in W_\lambda^{\text{loc}}(\Omega)$ to $Lu = 0$ is locally Hölder continuous in Ω .*

Cf. Theorem 2.1 in [55], p. IX-6. An important qualitative ingredient in the proof of Theorem 13 is the following Harnack type inequality: if $u > 0$ is a weak solution to $Lu = 0$ in a locally λ -connected open set $\Omega \subset \mathbb{R}^n$ then for any compact subset $K \subset \Omega$ there is a constant $C(K) > 0$ such that $\sup_{x \in K} u(x) \leq C(K) \inf_{x \in K} u(x)$ (cf. Theorem 2.2 in [55], p. IX-6). The technical approach is i) to study the properties of the distance function

$$d_\lambda(x, y) = \inf \{ \ell > 0 : \exists \gamma \in C_\lambda([0, \ell], \mathbb{R}^n) \text{ with } \gamma(0) = x, \gamma(\ell) = y \}$$

(which turns out to be intimately related to the operator L , very much the way the Euclidean distance is related to the ordinary Laplacian), ii) to obtain an embedding $\overset{\circ}{W}_\lambda(\Omega) \hookrightarrow L^q(\Omega)$ for some $q > 2$, and iii) to prove a result on functions of bounded mean oscillation on spheres relative to the distance d_λ (cf. F. John & L. Nirenberg, [51], for the Euclidean counterpart of the result in [55]).

To motivate the choice of d_λ let $\lambda \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\lambda^j(x) > 0$ for any $x \in \mathbb{R}^n$ and $1 \leq j \leq n$. Then \mathbb{R}^n may be endowed with the Riemannian metric g_λ given by

$$g_\lambda(X, Y)_x = \sum_{j=1}^n \lambda^j(x)^{-2} X^j(x) Y^j(x), \quad x \in \mathbb{R}^n,$$

for any $X, Y \in \mathfrak{X}^\infty(\mathbb{R}^n)$ represented as $X = X^j(\partial/\partial x^j)$ and $Y = Y^j(\partial/\partial x^j)$. Let $\nabla^\lambda u$ be the gradient of $u \in C^1(\mathbb{R}^n)$ with respect to g_λ i.e. $g_\lambda(\nabla^\lambda u, X) = X(u)$ for

any $X \in \mathfrak{X}^\infty(\mathbb{R}^n)$. As $g^{ij} = (\lambda^i)^2 \delta^{ij}$

$$\nabla^\lambda u = \sum_{j=1}^n (\lambda^j)^2 \frac{\partial u}{\partial x^j} \frac{\partial}{\partial x^j} = \sum_{j=1}^n X_j(u) X_j .$$

The length of a C^1 curve $\gamma : [0, \ell] \rightarrow \mathbb{R}^n$ is

$$L(\gamma) = \int_0^\ell g_\lambda(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt = \int_0^\ell \left[\sum_{i=1}^n \lambda_i(\gamma(t))^{-2} \left(\frac{d\gamma^i}{dt}(t) \right)^2 \right]^{1/2} dt .$$

Consequently when γ is an integral curve of $X_j = \lambda^j(\partial/\partial x^j) \in \mathfrak{X}^\infty(\mathbb{R}^n)$ then $L(\gamma) = \ell$. Let $d_{g_\lambda} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ be the distance function associated to the Riemannian metric g_λ i.e. $d_{g_\lambda}(x, y)$ is the infimum of all lengths of all piecewise C^1 curves joining x and y . It may be shown that there exist constants $a > 0$ and $A > 0$ such that

$$a d_\lambda(x, y) \leq d_{g_\lambda}(x, y) \leq A d_\lambda(x, y) , \quad x, y \in \mathbb{R}^n ,$$

i.e. the distance functions d_λ and d_{g_λ} are equivalent. The constants a and A depend only on n and on the Lipschitz constants of $\lambda_{(j)}^j$, $1 \leq j \leq n$. Of course d_λ makes sense in the more general case where one assumes but $\lambda^j(x) \geq 0$. However the set

$$\{\ell > 0 : \exists \gamma \in C_\lambda([0, \ell], \mathbb{R}^n) \text{ with } \gamma(0) = x, \gamma(\ell) = y\}$$

is nonempty only provided that x and y may be joined by a λ -admissible curve. Hence d_λ is a distance function on each connected locally λ -connected open subset $\Omega \subset \mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and N a ν -dimensional Riemannian manifold covered by a single coordinate chart $\chi = (y^1, \dots, y^\nu)$. Let $W_\lambda(\Omega, N)$ consist of all maps $\phi : \Omega \rightarrow N$ such that $\phi^\alpha = y^\alpha \circ \phi \in W_\lambda(\Omega)$ for any $1 \leq \alpha \leq \nu$. Let us assume from now on that $\lambda^j(x) > 0$ for any $x \in \Omega$, $1 \leq j \leq n$. Let us consider the energy functional

$$E(\phi) = \frac{1}{2} \int_\Omega a^{ij}(x) \frac{\partial \phi^\alpha}{\partial x^i}(x) \frac{\partial \phi^\beta}{\partial x^j}(x) h_{\alpha\beta}(\phi(x)) dx ,$$

defined *a priori* on $W_\lambda(\Omega, N) \cap C^\infty(\Omega, N)$ where $a^{ij} = a^{ji} \in L^\infty(\mathbb{R}^n)$ are assumed to satisfy (34). The Euler-Lagrange equations of the variational principle associated to E are

$$(36) \quad \tau_L(\phi)^\alpha \equiv L\phi^\alpha + a^{ij} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} (\Gamma_{\beta\gamma}^\alpha \circ \phi) = 0 , \quad 1 \leq \alpha \leq \nu ,$$

where L is given by (33) with $c(x) = 0$. It is an open problem to study the existence and regularity of weak solutions to (36). The solution to this problem would yield yet another generalization of the theory of harmonic maps (among Riemannian manifolds) to the subelliptic setting and corresponding in the scalar case (i.e. $N = \mathbb{R}$) to the theory of X -elliptic operators, cf. e.g. C.E. Gutiérrez & E. Lanconelli, [41], and G. Di Fazio et al., [25]. Precisely let $X = \{X_a : 1 \leq a \leq m\}$ be a family of vector fields $X_a = b_a^i(x) \partial/\partial x^i$ on \mathbb{R}^n whose coefficients $b_a^i(x)$ are locally Lipschitz continuous on \mathbb{R}^n . Let $a^{ij}(x) = a^{ji}(x)$, $a^i(x)$, $b^i(x)$ and $c(x)$ be measurable functions. Let $\Omega \subset \mathbb{R}^n$ be an open set. A linear second order differential

operator

$$(37) \quad Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(a^{ij}(x) \frac{\partial u}{\partial x^j} + a^i(x)u \right) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x^i} + c(x)u$$

is *X-elliptic* in Ω if the following conditions are satisfied. 1) There is a constant $C_0 > 0$ such that

$$C_0 \sum_{a=1}^m \langle X_a(x), \xi \rangle^2 \leq \langle g(x)\xi, \xi \rangle ,$$

$$\xi = \xi_j (dx^j)_x , \quad (\xi_1, \dots, \xi_n) \in \mathbb{R}^n , \quad x \in \Omega ,$$

where $\langle \cdot, \cdot \rangle : T_x(\mathbb{R}^n) \times T_x^*(\mathbb{R}^n) \rightarrow \mathbb{R}$ is the natural pairing. Also

$$g(x) (dx^i)_x = a^{ij}(x) \frac{\partial}{\partial x^j} \Big|_x , \quad \langle g(x)\xi, \xi \rangle = a^{ij}(x) \xi_i \xi_j .$$

Moreover 2) there is a function $\gamma(x) \geq 0$ such that

$$[a(x)\xi]^2 + [b(x)\xi]^2 \leq \gamma(x)^2 \sum_{a=1}^n \langle X_a(x), \xi \rangle^2 ,$$

$$\xi = \xi_j (dx^j)_x , \quad (\xi_1, \dots, \xi_n) \in \mathbb{R}^n , \quad x \in \Omega ,$$

$$a(x) (dx^i)_x = a^i(x) , \quad b(x) (dx^i)_x = b^i(x) .$$

A *X-elliptic* operator (37) is *uniformly X-elliptic* in Ω if there is a constant $C_1 > 0$ such that

$$\langle g(x)\xi, \xi \rangle \leq C_1 \sum_{a=1}^m \langle X_a(x), \xi \rangle^2 , \quad \xi \in T_x^*(\mathbb{R}^n) , \quad x \in \Omega .$$

The condition of *X-ellipticity* is explicitly stated by E. Lanconelli & A.E. Kogoj, [57] (for operators (37) with $a^i(x) = b^i(x) = c(x) = 0$). Nevertheless (34) is of course the condition that the differential operator (33) is uniformly *X-elliptic* (relative to the system of vector fields $X = \{\lambda^a \partial / \partial x^a : 1 \leq a \leq n\}$). Let N be a Riemannian manifold of sectional curvature $\text{Sect}(N) \leq \kappa^2$ and let

$$0 < \mu < \min \left\{ \frac{\pi}{2\kappa} , i(x) \right\} ,$$

where $i(x)$ is the injectivity radius of $x \in N$. Let $f \in C(\bar{\Omega}, N) \cap W_X^{1,2}(\Omega, N)$ such that $f(\bar{\Omega}) \subset B(x, \mu)$ (a *regular* ball of N). Let us assume that $\lambda^j(x) > 0$ for any $x \in \mathbb{R}^n$, $1 \leq j \leq n$. The Dirichlet problem for the system (36) is

$$(38) \quad \tau_L(\phi) = 0 , \quad \phi \in W_X^{1,2}(\Omega, N) \cap L^\infty(\Omega, N) ,$$

$$(39) \quad \phi^\alpha - f^\alpha \in W_{X,0}^{1,2}(\Omega) , \quad 1 \leq \alpha \leq \nu .$$

When (38) is the subelliptic harmonic map system, existence and regularity of solutions to (38)-(39) were studied by J. Jost & C-J. Xu, [52]. In view of A. Parmeggiani & C-J. Xu's result (dealing with the Dirichlet problem (38)-(39) when $N = \mathbb{R}$ and $f \in C^\infty(\partial\Omega)$, cf. [65]) the key to the solution (to the regularity part) of the problem (38)-(39) is perhaps the comparison principle of the operator (33) with the Hörmander operator $\sum_a X_a^* X_a$.

APPENDIX B. MORREY-CAMPANATO SPACES ON MANIFOLDS

Following M. Geisler, [38], we present the construction and main properties of Morrey and Campanato spaces on a compact orientable Riemannian manifold and hint to the development of the theory within the CR category.

Morrey-Campanato spaces are commonly considered on bounded domains in \mathbb{R}^n (cf. e.g. S. Campanato, [12]). Precisely if $\Omega \subset \mathbb{R}^n$ is a bounded domain and $0 < \rho \leq \infty$, $0 < \delta \leq \infty$, $0 \leq \lambda < \infty$, $1 \leq p < \infty$, $x_0 \in \mathbb{R}^n$ one sets

$$\begin{aligned} \Omega_\rho(x_0) &= \Omega \cap B(x_0, \rho) , \\ f_{x_0, \rho} &= \frac{1}{|\Omega_\rho(x_0)|} \int_{\Omega_\rho(x_0)} f(x) dx , \\ [f]_{p, \lambda, \Omega, \delta} &= \left(\sup_{\substack{0 < \rho \leq \delta \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(x) - f_{x_0, \rho}|^p dx \right)^{1/p} . \end{aligned}$$

The Morrey space $\mathcal{M}^{p, \lambda}(\Omega)$ consists of all $f \in L^p(\Omega)$ such that

$$(40) \quad \|f\|_{\mathcal{M}^{p, \lambda}(\Omega)} = \left(\sup_{\substack{\rho > 0 \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(x)|^p dx \right)^{1/p} < \infty .$$

The Campanato space $\mathcal{L}^{p, \lambda}(\Omega)$ consists of all $f \in L^p(\Omega)$ such that

$$(41) \quad \|f\|_{\mathcal{L}^{p, \lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p, \lambda, \Omega} < \infty$$

where $[f]_{p, \lambda, \Omega} = [f]_{p, \lambda, \Omega, \infty}$ (we omit δ when $\delta = \infty$). We recall

Theorem 14. i) *The Morrey and Campanato spaces $\mathcal{M}^{p, \lambda}(\Omega)$, $\mathcal{L}^{p, \lambda}(\Omega)$ are Banach spaces with respect to the norms (40) and (41) respectively. Moreover the following natural embeddings are continuous*

$$\begin{aligned} \mathcal{M}^{p, \lambda}(\Omega) &\hookrightarrow \mathcal{L}^{p, \lambda}(\Omega) \hookrightarrow L^p(\Omega) , \\ \mathcal{L}^{q, \nu}(\Omega) &\hookrightarrow \mathcal{L}^{p, \lambda}(\Omega) , \quad 1 \leq p \leq q < \infty , \quad \frac{\lambda - n}{p} < \frac{\nu - n}{q} . \end{aligned}$$

ii) *If $0 \leq \lambda < n$ and the domain $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary then $\mathcal{M}^{p, \lambda}(\Omega)$ and $\mathcal{L}^{p, \lambda}(\Omega)$ are isomorphic.*

iii) *$\mathcal{M}^{p, n}(\Omega)$ and $L^\infty(\Omega)$ are isomorphic. There exist domains $\Omega \subset \mathbb{R}^n$ such that $\mathcal{M}^{p, n}(\Omega) \subsetneq \mathcal{L}^{p, n}(\Omega)$ (strict inclusion).*

iv) *Let us assume that $n < \lambda \leq n + p$ and the boundary $\partial\Omega$ is Lipschitz. Let us set $\alpha = (\lambda - n)/p$. Then $\mathcal{L}^{p, \lambda}(\Omega)$ is isomorphic to the Hölder space $\Gamma^\alpha(\bar{\Omega})$ with the norm*

$$\|f\|_{\Gamma^\alpha(\bar{\Omega})} = \sup_{x \in \bar{\Omega}} |f(x)| + \sup_{\substack{x, y \in \bar{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} .$$

For a proof of Theorem 14 one may see A. Kufner et al., [54]. Morrey and Campanato spaces are useful in describing local properties. For instance (cf. Proposition 1 in [38], p. 311)

Proposition 3. Let $\Omega \subset \mathbb{R}^n$ and $D_i \subset \mathbb{R}^n$, $1 \leq i \leq m$, be bounded domains and let us set $\Omega_i = \Omega \cap D_i$. Let $\delta > 0$. Let us assume that $\overline{\Omega} \subset \bigcup_{i=1}^m D_i$. Then

$$\|f\|_{L^p(\Omega)} + [f]_{p,\lambda,\Omega,\delta}, \quad \sum_{i=1}^m \|f\|_{\mathcal{L}^{p,\lambda}(\Omega_i)}, \quad f \in \mathcal{L}^{p,\lambda}(\Omega),$$

are norms on the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$. Both are equivalent to the original norm (41).

Proof. Obviously $[f]_{p,\lambda,\Omega,\delta} \leq [f]_{p,\lambda,\Omega}$. Next (by Hölder's inequality with $1/p + 1/q = 1$)

$$\begin{aligned} \|f_{x_0,\rho}\|_{L^p(\Omega_\rho(x_0))} &= \left(\int_{\Omega_\rho(x_0)} |f_{x_0,\rho}|^p dx \right)^{1/p} = |f_{x_0,\rho}| |\Omega_\rho(x_0)|^{1/p} = \\ &= |\Omega_\rho(x_0)|^{-1+1/p} \left| \int_{\Omega_\rho(x_0)} f(x) dx \right| \leq \\ &\leq |\Omega_\rho(x_0)|^{-1+1/p} \|f\|_{L^2(\Omega_\rho(x_0))} |\Omega_\rho(x_0)|^{1/q} \leq \|f\|_{L^p(\Omega)}. \end{aligned}$$

Thus (by Minkowski's inequality and $\rho > \delta$)

$$\begin{aligned} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(x) - f_{x_0,\rho}|^p dx &= \rho^{-\lambda} \|f - f_{x_0,\rho}\|_{L^p(\Omega_\rho(x_0))}^p \leq \\ &\leq \delta^{-\lambda} (\|f\|_{L^p(\Omega_\rho(x_0))} + \|f_{x_0,\rho}\|_{L^p(\Omega_\rho(x_0))})^p \leq 2^p \delta^{-\lambda} \|f\|_{L^p(\Omega)}^p. \end{aligned}$$

Finally

$$\begin{aligned} [f]_{p,\lambda,\Omega} &\leq [f]_{p,\lambda,\Omega,\delta} + \\ &+ \left(\sup_{\substack{\rho > \delta \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(x) - f_{x_0,\rho}|^p dx \right)^{1/p} \leq \\ &\leq [f]_{p,\lambda,\Omega,\delta} + 2\delta^{-\lambda/p} \|f\|_{L^p(\Omega)} \end{aligned}$$

hence $\|f\|_{L^p(\Omega)} + [f]_{p,\lambda,\Omega,\delta}$ and $\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}$ are equivalent norms on the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$. To deal with the norm $\sum_{i=1}^m \|f\|_{\mathcal{L}^{p,\lambda}(\Omega_i)}$ we need the following

Lemma 2. There is $\delta > 0$ such that for any $x \in \Omega$ and $0 < \rho < \delta$ there is $1 \leq j \leq m$ such that $\Omega_\rho(x) \subset \Omega_j$.

Proof. The proof is by contradiction. Let us assume that for any $\delta > 0$ there exist $x_\delta \in \Omega$ and $\rho_\delta \in (0, \delta)$ such that $\Omega_{\rho_\delta}(x_\delta)$ is not contained in any of the domains $\{\Omega_j : 1 \leq j \leq m\}$. Let $\delta = 1/k$ with $k \in \mathbb{Z}$, $k \geq 1$, arbitrary and let us set $y_k = x_{1/k} \in \Omega$ and $r_k = \rho_{1/k} \in (0, 1/k)$. Then $\Omega_{r_k}(y_k)$ is not contained in any of the domains $\{\Omega_j : 1 \leq j \leq m\}$ and $r_k \rightarrow 0$ as $k \rightarrow \infty$. As $y_k \in \Omega \subset \overline{\Omega}$ and $\overline{\Omega}$ is a compact set, hence sequentially compact. Hence there is a subsequence $\{x_\nu\}_{\nu \geq 1} = \{y_{k_\nu}\}_{\nu \geq 1}$ such that $x_\nu \rightarrow x_0$ as $\nu \rightarrow \infty$ for some $x_0 \in \overline{\Omega}$. By hypothesis $\overline{\Omega} \subset \bigcup_{i=1}^m D_i$ so that $x_0 \in D_i$ for some $1 \leq i \leq m$. Consequently there is $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset D_i$. Also there is $\nu_\varepsilon \geq 1$ such that $x_\nu \in B(x_0, \varepsilon)$ for any $\nu \geq \nu_\varepsilon$. Let us set $\rho_\nu = r_{k_\nu}$ for any $\nu \geq 1$. Let $x \in \Omega_{\rho_\nu}(x_\nu)$. Then

$$|x - x_0| \leq |x - x_\nu| + |x_\nu - x_0| < \rho_\nu + \varepsilon$$

hence (as $\rho_\nu \rightarrow 0$ for $\nu \rightarrow \infty$) $x \in B(x_0, \varepsilon)$. We have shown that $\Omega_{\rho_\nu}(x_\nu) \subset B(x_0, \varepsilon)$ so that

$$\Omega_{\rho_\nu}(x_\nu) \subset B(x_0, \varepsilon) \cap \Omega \subset D_i \cap \Omega = \Omega_i ,$$

a contradiction. Lemma 2 is proved.

Let us go back to the proof of Proposition 3. Note that

$$\Omega_i \subset \Omega \implies [f]_{p,\lambda,\Omega_i} \leq [f]_{p,\lambda,\Omega} \implies \sum_{i=1}^m \|f\|_{\mathcal{L}^{p,\lambda}(\Omega_i)} \leq \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} .$$

On the other hand (by Lemma 2)

$$[f]_{p,\lambda,\Omega,\delta} \leq \max \{ [f]_{p,\lambda,\Omega_i,\delta} : 1 \leq i \leq m \}$$

hence the norms $\sum_{i=1}^m \|f\|_{\mathcal{L}^{p,\lambda}(\Omega_i)}$ and $\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}$ are equivalent. Q.e.d.

A statement on the Morrey spaces $\mathcal{M}^{p,\lambda}(\Omega)$ similar to Proposition 3 may be easily obtained. Morrey-Campanato spaces are invariant under diffeomorphisms. Indeed (cf. Proposition 2 in [38], p. 311-312)

Proposition 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\phi : \Omega \rightarrow D = \phi(\Omega)$ be a C^1 diffeomorphism such that the functions $\partial\phi^j/\partial x^k$ and $\partial(\phi^{-1})^j/\partial y^k$ are bounded respectively on Ω and D for any $1 \leq j, k \leq n$. Then the map*

$$\phi^* : \mathcal{L}^{p,\lambda}(D) \rightarrow \mathcal{L}^{p,\lambda}(\Omega) , \quad \phi^*(f) = f \circ \phi , \quad f \in \mathcal{L}^{p,\lambda}(D) ,$$

is a continuous linear isomorphism.

Proof. Let $f \in \mathcal{L}^{p,\lambda}(D)$. Then

$$\begin{aligned} \|\phi^*(f)\|_{L^p(\Omega)}^p &= \int_{\Omega} |f(\phi(x))|^p dx = \\ &= \int_D |f(y)|^p |\det J(\phi^{-1})(y)| dy \leq C \|f\|_{L^p(D)}^p \end{aligned}$$

with $C = \sup_{y \in D} |\det J(\phi^{-1})(y)|$ ($0 < C < \infty$ as $\det J(\phi^{-1})$ is bounded on D). Thus

$$\|\phi^*(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p(D)} .$$

Next we apply the well known inequality (cf. e.g. [54])

$$(42) \quad [g]_{p,\lambda,\Omega} \leq 2 \left[\sup_{\substack{\rho>0 \\ x_0 \in \Omega}} \rho^{-\lambda} \left(\inf_{s \in \mathbb{R}} \int_{\Omega_\rho(x_0)} |g(x) - s|^p dx \right) \right]^{1/p}$$

to the function $g = \phi^*(f)$ so that to obtain

$$[\phi^*(f)]_{p,\lambda,\Omega} \leq 2 \left[\sup_{\substack{\rho>0 \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(\phi(x)) - s|^p dx \right]^{1/p}$$

for any $s \in \mathbb{R}$ (a relevant choice of s will be chosen later on. As $\partial\phi^j/\partial x^k$ are bounded functions the mean value formula yields

$$\phi[\Omega_\rho(x_0)] \subset D_{a\rho}[\phi(x_0)]$$

for some constant $a > 0$ independent of x_0 and ρ . Here $D_r(x) = D \cap B(x, r)$. Therefore

$$\begin{aligned} \int_{\Omega_\rho(x_0)} |f(\phi(x)) - s|^p dx &= \int_{\phi[\Omega_\rho(x_0)]} |f(y) - s| |\det J(\phi^{-1})(y)| dy \leq \\ &\leq C \int_{D_{a\rho}(x_0)} |f(y) - s| dy \end{aligned}$$

and then (with $y_0 = \phi(x_0)$)

$$\begin{aligned} &\sup_{\substack{\rho > 0 \\ x_0 \in \Omega}} \rho^{-\lambda} \int_{\Omega_\rho(x_0)} |f(\phi(x)) - s|^p dx \leq \\ &\leq C \sup_{\substack{\rho > 0 \\ y_0 \in D}} \rho^{-\lambda} \int_{D_{a\rho}(y_0)} |f(y) - s|^p dy = \\ &= Ca^\lambda \sup_{\substack{r > 0 \\ y_0 \in D}} r^{-\lambda} \int_{D_r(y_0)} |f(y) - s|^p dy \end{aligned}$$

so that

$$[\phi^*(f)]_{p,\lambda,\Omega} \leq 2C^{1/p} a^{\lambda/p} \left[\sup_{\substack{r > 0 \\ y_0 \in D}} r^{-\lambda} \int_{D_r(y_0)} |f(y) - s|^p dy \right]^{1/p}.$$

In particular for $s = f_{r,y_0}$

$$[\phi^*(f)]_{p,\lambda,\Omega} \leq A [f]_{p,\lambda,D}$$

where $A = 2C^{1/p} a^{\lambda/p}$. The same arguments apply to ϕ^{-1} . Proposition 4 is proved.

Propositions 3 and 4 lead to the construction of spaces $\mathcal{M}^{p,\lambda}(M)$ and $\mathcal{L}^{p,\lambda}(M)$ on a compact n -dimensional manifold M without boundary. Let $\mathcal{A} = \{(U_i, \varphi_i) : 1 \leq i \leq m\}$ be a finite collection of local charts covering M . Given a function $f : M \rightarrow \mathbb{R}$ we denote by $\varphi_i^* f = f \circ \varphi_i^{-1}$ the local expression of f with respect to (U_i, φ_i) . Let $\mathcal{L}^{p,\lambda}(M)$ be the space consisting of all functions $f : M \rightarrow \mathbb{R}$ such that

$$\varphi_i^* f \in \mathcal{L}^{p,\lambda}(\varphi_i(U_i)), \quad 1 \leq i \leq m.$$

Moreover we endow $\mathcal{L}^{p,\lambda}(M)$ with the norm

$$(43) \quad \|f\|_{\mathcal{L}^{p,\lambda}(M)} = \sum_{i=1}^m \|\varphi_i^* f\|_{\mathcal{L}^{p,\lambda}(\Omega_i)}, \quad \Omega_i = \varphi_i(U_i).$$

Note that the construction of $\mathcal{L}^{p,\lambda}(M)$ and of its norm depends solely on the differential structure of M . Moreover it may be easily shown that the definition of $\mathcal{L}^{p,\lambda}(M)$ doesn't depend on the choice of \mathcal{A} and that different choices of \mathcal{A} yield equivalent norms. A similar construction gives the Morrey spaces $\mathcal{M}^{p,\lambda}(M)$. Theorem 14 readily implies (cf. Proposition 3 in [38], p. 313)

Proposition 5. *Let M be a compact n -dimensional smooth manifold. Then i) $\mathcal{M}^{p,\lambda}(M)$ and $\mathcal{L}^{p,\lambda}(M)$ are Banach spaces and the natural embeddings*

$$\mathcal{M}^{p,\lambda}(M) \hookrightarrow \mathcal{L}^{p,\lambda}(M) \hookrightarrow L^p(M),$$

$$\mathcal{L}^{q,\nu}(M) \hookrightarrow \mathcal{L}^{p,\lambda}(M), \quad 1 \leq p \leq q < \infty, \quad \frac{\lambda - n}{p} < \frac{\nu - n}{q},$$

are continuous.

ii) If $0 \leq \lambda < n$ then $\mathcal{M}^{p,\lambda}(M)$ and $\mathcal{L}^{p,\lambda}(M)$ are isomorphic.

iii) $\mathcal{M}^{p,n}(M)$ and $L^\infty(M)$ are isomorphic.

The relationship of $\mathcal{L}^{p,\lambda}(M)$ to Hölder spaces is more subtle. Let (M, g) be a compact orientable connected n -dimensional Riemannian manifold and $d_g : M \times M \rightarrow [0, +\infty)$ the distance function on M associated to the Riemannian metric g . Let $0 < \alpha \leq 1$. The Hölder space $\Gamma^\alpha(M)$ consists of all continuous functions $f : M \rightarrow \mathbb{R}$ such that

$$(44) \quad \|f\|_{\Gamma^\alpha(M)} = \sup_{x \in M} |f(x)| + \sup_{\substack{x, y \in M \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

$\Gamma^\alpha(M)$ is a Banach space with the norm (44).

Proposition 6. *Let $\mathcal{A} = \{(V_i, \varphi_i) : 1 \leq i \leq m\}$ be a finite collection of local charts covering M and such that i) $\Omega_i = \varphi(V_i)$ is a convex subset of \mathbb{R}^n and ii) for each $1 \leq i \leq m$ there is a local chart $(U_i, \tilde{\varphi}_i)$ such that $\bar{V}_i \subset U_i$ and $\varphi_i : V_i \rightarrow \mathbb{R}^n$ is the restriction to V_i of $\tilde{\varphi}_i : U_i \rightarrow \mathbb{R}^n$. Then $f \in \Gamma^\alpha(M)$ if and only if $\varphi_i^*(f) \in \Gamma^\alpha(\Omega_i)$ for each $1 \leq i \leq m$.*

Cf. Proposition 4 in [38], p. 314. The proof of Proposition 6 depends heavily on the Riemannian structure on M (in particular it relies on the classical Hopf-Rinow theorem that any two points $x, y \in M$ can be joined by a minimizing geodesic of length $d(x, y)$). Stated loosely the contents of Proposition 6 is that a function $f : M \rightarrow \mathbb{R}$ belongs to $\Gamma^\alpha(M)$ if and only if the local expression of f with respect to an arbitrary local chart (U, φ) belongs to $\Gamma^\alpha(\varphi(U))$. By constructing a finite collection \mathcal{A} satisfying the requirements in Proposition 6 one may establish

Proposition 7. *Let us consider a compact orientable connected n -dimensional Riemannian manifold M . If $n < \lambda \leq n + p$ and $\alpha = (\lambda - n)/p$ then $\mathcal{L}^{p,\lambda}(M)$ is isomorphic to $\Gamma^\alpha(M)$.*

A description of the Morrey and Campanato spaces $\mathcal{M}^{p,\lambda}(M)$ and $\mathcal{L}^{p,\lambda}(M)$ closer to the spirit of the Euclidean case (40)-(41) requires additional knowledge of Riemannian geometry. Let (M, g) be a complete Riemannian manifold. Let $x \in M$ and $v \in T_x(M)$. Let $\gamma : \mathbb{R} \rightarrow M$ be the geodesic of initial data $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. The exponential map $\exp_x v = \gamma(1)$ is well defined for sufficiently small v . The injectivity radius of M is

$$r_M = \inf_{x \in M} \sup\{r > 0 : \exp_x \text{ is injective on } N_x(r)\},$$

$$N_x(r) = \{v \in T_x(M) : g_x(v, v) < r^2\}.$$

When M is compact $r_M > 0$ and each ball $B(x, \rho) = \{y \in M : d(x, y) < \rho\}$ of radius $0 < \rho \leq r_M$ is given by $B(x, \rho) = \exp_x N_x(\rho)$ i.e. $B(x, \rho) = \{\exp_x v :$

$g_x(v, v) < \rho^2\}$. Let us set

$$f_{x_0, \rho} = \frac{1}{\mu(B(x_0, \rho))} \int_{B(x_0, \rho)} f(x) \, d\text{vol}(g)(x),$$

$$[f]_{p, \lambda, M, \delta} = \left(\sup_{\substack{0 < \rho \leq \delta \\ x_0 \in M}} \rho^{-\lambda} \int_{B(x_0, \rho)} |f(x) - f_{x_0, \rho}|^p \, d\text{vol}(g)(x) \right)^{1/p}.$$

We may state (cf. Proposition 6 in [38], p. 315)

Theorem 15. *Let M be a compact orientable connected Riemannian manifold. Then $\mathcal{M}^{p, \lambda}(M)$ consists of all $f \in L^p(M)$ such that*

$$\|f\|'_{\mathcal{M}^{p, \lambda}(M)} = \left(\sup_{\substack{\rho > 0 \\ x \in M}} \rho^{-\lambda} \int_{B(x, \rho)} |f|^p \, d\text{vol}(g) \right)^{1/p} < \infty.$$

Also $\mathcal{L}^{p, \lambda}(M)$ consists of all $f \in L^p(M)$ such that

$$\|f\|'_{\mathcal{L}^{p, \lambda}(M)} = \|f\|_{L^p(M)} + [f]_{p, \lambda, M, \delta} < \infty.$$

Moreover the norms $\|\cdot\|_{\mathcal{M}^{p, \lambda}(M)}$ and $\|\cdot\|'_{\mathcal{M}^{p, \lambda}(M)}$ (respectively $\|\cdot\|_{\mathcal{L}^{p, \lambda}(M)}$ and $\|\cdot\|'_{\mathcal{L}^{p, \lambda}(M)}$) are equivalent.

Proof. Let $0 < \delta < r_M$. As M is compact we may assume that

$$M = \bigcup_{i=1}^m B(x_i, \rho), \quad 0 < \rho < \frac{r_M}{2}, \quad x_i \in M, \quad 1 \leq i \leq m.$$

For each $x_i \in M$ let $\{v_j^{(i)} : 1 \leq j \leq n\} \subset T_{x_i}(M)$ be an orthonormal basis i.e. $g_x(v_j^{(i)}, v_k^{(i)}) = \delta_{jk}$. Let $u_i : \mathbb{R}^n \rightarrow T_{x_i}(M)$ be the \mathbb{R} -linear isomorphism given by $u_i(e_j) = v_j^{(i)}$ for any $1 \leq j \leq n$, where $\{e_1, \dots, e_n\} \subset \mathbb{R}^n$ is the canonical linear basis. Let us consider the local chart

$$\varphi_i : U_i \rightarrow \mathbb{R}^n, \quad U_i = B(x_i, \rho),$$

$$\varphi_i(x) = (\exp_{x_i} \circ u_i)^{-1}(x), \quad x \in U_i.$$

Let $\mathcal{A} = \{(U_i, \varphi_i) : 1 \leq i \leq m\}$. Let $f \in L^p(M)$ such that $\|f\|'_{\mathcal{L}^{p, \lambda}(M)} < \infty$ and let $f_i = \varphi_i^*(f)$ its local expression with respect to (U_i, φ_i) . We claim that $f_i \in \mathcal{L}^{p, \lambda}(B(0, \rho))$. Indeed

$$\begin{aligned} \|f_i\|_{L^p(B(0, \rho))}^p &= \int_{B(0, \rho)} |f_i(\xi)|^p \, d\xi \leq \\ &\leq \sup_{\xi \in B(0, \rho)} G(\xi)^{-1/2} \int_{B(0, \rho)} |f_i(\xi)|^p G(\xi)^{1/2} \, d\xi \leq C \|f\|_{L^p(M)}^p \end{aligned}$$

where $G = \det[g_{jk}] \circ \varphi_i^{-1}$. Next we apply the inequality (42) for $g = f_i$ and $\Omega = B(0, \rho)$ so that for any $a \in \mathbb{R}$

$$\frac{1}{2} [f_i]_{p, \lambda, B(0, \rho), \delta} \leq$$

$$\begin{aligned} & \leq \left(\sup_{\substack{0 < s < \delta \\ \xi \in B(0, \rho)}} s^{-\lambda} \int_{B(0, \rho) \cap B(\xi, s)} |f_i(\eta) - a|^p d\eta \right)^{1/p} \leq \\ & \leq \left(\sup_{\substack{0 < s < \delta \\ \xi \in B(0, \rho)}} C s^{-\lambda} \int_{B(0, \rho) \cap B(\xi, s)} |f_i(\eta) - a|^p G(\eta)^{1/2} d\eta \right)^{1/p}. \end{aligned}$$

Let us set $\xi = \varphi_i^{-1}(x)$ and $\eta = \varphi_i^{-1}(y)$. Then

$$\begin{aligned} \beta |\xi - \eta| & \leq d(x, y) \leq \alpha |x - y|, \\ \alpha & = \sup \{ \mu \in \text{Spec}([g_{jk}(\varphi_i^{-1}(\xi))]) : \xi \in B(0, \rho) \}, \\ \beta & = \inf \{ \mu \in \text{Spec}([g_{jk}(\varphi_i^{-1}(\xi))]) : \xi \in B(0, \rho) \}. \end{aligned}$$

Consequently

$$\eta \in B(0, \rho) \cap B(\xi, s) \implies y \in B(x, \alpha s)$$

and then (by choosing $s = \sigma/\alpha$ and $a = f_{x, \sigma}$)

$$\begin{aligned} & [f_i]_{p, \lambda, B(0, \rho), \delta} \leq \\ & \leq C' \left(\sup_{\substack{0 < \sigma < \delta/\alpha \\ x \in M}} \sigma^{-\lambda} \int_{B(x, \sigma)} |f(y) - f_{x, \sigma}|^p d\text{vol}(g)(y) \right)^{1/p} \leq \\ & \leq C' [f]_{p, \lambda, M}. \end{aligned}$$

Viceversa let $f \in L^2(M)$ such that $\|f\|_{\mathcal{L}^{p, \lambda}(M)} < \infty$. The proof is by contradiction. If there exist $x_k \in M$ and $\rho_k \rightarrow 0$ as $k \rightarrow \infty$ such that

$$\left(\rho_k^{-\lambda} \int_{B(x_k, \rho_k)} |f - f_{x_k, \rho_k}|^p d\text{vol}(g) \right)^{1/p} > k.$$

As M is compact we may pick up a subsequence, denoted again by x_k , such that $x_k \rightarrow x$ as $k \rightarrow \infty$ (convergence is meant in (M, d) as a metric space). For sufficiently large k one has $B(x_k, \rho_k) \subset B(x, \delta)$. Let $u : \mathbb{R}^n \rightarrow T : x(M)$ be a \mathbb{R} -linear isomorphism and $\varphi = (\exp_x \circ u)^{-1}$. Then (by an argument similar to that in the proof of Proposition 4) for any $a \in \mathbb{R}$

$$\left(\rho_k^{-\lambda} \int_{\varphi(B(x_k, \rho_k))} |\varphi^*(f)(\xi) - a|^p G(\xi)^{1/2} d\xi \right)^{1/p} \geq \frac{k}{2}$$

and then

$$\left(\rho_k^{-\lambda} \int_{B(\varphi(x_k), \rho_k/\beta)} |\varphi^*(f)(\xi) - a| d\xi \right)^{1/p} \geq C \frac{k}{2}.$$

By choosing $a = \varphi^*(f_{\rho_k/\beta, B(\varphi(x_k), \rho_k/\beta)})$ the previous inequality is in contradiction with $f \in \mathcal{L}^{p, \lambda}(M)$. Q.e.d.

If M is a compact strictly pseudoconvex CR manifold of CR dimension n then $d_H(x, y)$ denotes the Carnot-Carathéodory distance among $x, y \in M$ i.e. the

greatest lower bound of the lengths of lengthy curves joining x and y . Carnot-Carathéodory balls $B_H(x_0, \rho) = \{x \in M : d_H(x_0, x) < \rho\}$ are open in the manifold topology and measurable (with respect to the Borel measure $\mu(A) = \int_M \chi_A(x) dv(x)$ with $A \subset M$ and $dv = \theta \wedge (d\theta)^n$). We set

$$f_{x_0, \rho} = \frac{1}{\mu(B_H(x_0, \rho))} \int_{B_H(x_0, \rho)} f(x) dv(x),$$

$$[f]_{p, \lambda, M, \delta} = \left(\sup_{\substack{0 < \rho \leq \delta \\ x_0 \in M}} \rho^{-\lambda} \int_{B_H(x_0, \rho)} |f(x) - f_{x_0, \rho}|^p dv(x) \right)^{1/p}.$$

It is reasonable to expect that for producing a pseudohermitian analog to the Morrey and Campanato spaces (on a strictly pseudoconvex CR manifold) one must generalize Propositions 3 and 4 to the case of a bounded domain $\Omega \subset \mathbb{H}_n$ where \mathbb{H}_n is the Heisenberg group (by replacing the Euclidean norm by the Heisenberg norm). As a pseudohermitian counterpart of local normal coordinates on a Riemannian manifold one should use the (normal) Folland-Stein coordinates (cf. [35]).

APPENDIX C. SUBELLIPTIC p -LAPLACIANS

Given a Hörmander system $X = \{X_1, \dots, X_m\}$ of C^∞ vector fields defined on an open set $\Omega \subset \mathbb{R}^n$ the *subelliptic p -Laplacian* is the operator

$$\mathcal{L}_p u = \sum_{a=1}^m X_a^* (|Xu|^{p-2} X_a u)$$

cf. L. Capogna et al., [14], p. 1766. Here $Xu = (X_1 u, \dots, X_m u)$. We denote by $W_X^{1,p}(\Omega)$ the completion of $C^1(\Omega)$ in the norm

$$(45) \quad \|u\|_{W_X^{1,p}(\Omega)} = \left(\int_{\Omega} (|Xu|^p + |u|^p) dx \right)^{1/p}.$$

Let us also consider

$$W_{X, \text{loc}}^{1,p}(\Omega) = \{u \in L_{\text{loc}}^p(\Omega) : \varphi u \in W_X^{1,p}(\Omega), \quad \forall \varphi \in C_0^1(\Omega)\}.$$

Next let $\overset{\circ}{W}_X^{1,p}(\Omega)$ be the completion of $C_0^1(\Omega)$ in the norm (45). The equation $\mathcal{L}_p u = 0$ enters the class of equations of the form

$$(46) \quad \sum_{a=1}^m X_a^* [A_a(x, u, Xu)] = f(x, u, Xu).$$

Weak solutions to (46) may be defined in a natural way. Precisely a function $u \in W_{X, \text{loc}}^{1,p}(\Omega)$ is a weak solution to (46) if

$$\sum_{a=1}^m \int_{\Omega} A_a(x, u, Xu) X_a \varphi dx = \int_U f(x, u, Xu) \varphi dx$$

for any $\varphi \in \overset{\circ}{W}_X^{1,p}(\Omega)$. A weak solution $u \in W_{X, \text{loc}}^{1,p}(\Omega)$ to $\mathcal{L}_p u = 0$ is a (weak) *subelliptic p -harmonic* function. By a result in [14] the following Harnack type inequality holds

Theorem 16. *Let $u \in W_{X,loc}^{1,p}(\Omega)$ be a nonnegative subelliptic p -harmonic function. Then there exist $C > 0$, $R_0 > 0$ such that for any ball $B(x, R)$ with $B(x, 4R) \subset \Omega$ and $0 < R \leq R_0$*

$$(47) \quad \operatorname{ess\,sup}_{y \in B(x,R)} u(y) \leq C \operatorname{ess\,inf}_{y \in B(x,R)} u(y) .$$

This is Theorem 3.1 in [14], p. 1778 (restated for nonnegative subelliptic p -harmonic functions). Balls $B(x, R)$ are meant with respect to the Carnot-Carathéodory metric associated to X . The following structural conditions on the measurable functions $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ appearing in (46) are assumed to hold (cf. [14]). There exist $p \in (1, +\infty)$ and $c_1 \geq 0$ and there exist measurable functions $f_1, f_2, f_3, g_2, g_3, h_3$ on \mathbb{R}^n such that for a.e. $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^m$

$$(48) \quad |A(x, u, \xi)| \leq c_1 |\xi|^{p-1} + g_2(x) |u|^{p-1} + g_3(x) ,$$

$$(49) \quad |f(x, u, \xi)| \leq f_1(x) |\xi|^{p-1} + f_2(x) |u|^{p-1} + f_3(x) ,$$

$$(50) \quad A(x, u, \xi) \cdot \xi \geq |\xi|^p - f_2(x) |u|^p - h_3(x) .$$

Cf. [14], p. 1767, where the structural conditions (48)-(50) are referred to as the conditions (S). The functions f_i, g_i, h_i in the conditions (S) are required to satisfy the following integrability properties

$$(51) \quad g_2, g_3 \in L_{loc}^r(\Omega) , \quad \text{with} \quad \begin{cases} r = \frac{Q}{p-1} & \text{if } p < Q \\ r > \frac{Q}{Q-1} & \text{if } p = Q , \end{cases}$$

$$(52) \quad f_2, f_3, h_3 \in L_{loc}^s(\Omega) \quad \text{with} \quad s > \frac{Q}{p} ,$$

$$(53) \quad f_1 \in L_{loc}^t(\Omega) , \quad \text{with } t > Q .$$

Here Q is a homogeneous dimension relative to Ω . It should be observed that we may write

$$s = \frac{Q}{p-\varepsilon} , \quad t = \frac{Q}{1-\varepsilon} ,$$

for some $0 < \varepsilon < 1$. By Theorem 3.1 in [14] under the assumptions (48)-(50) and (51)-(53) the nonnegative solutions to (46) satisfy the Harnack inequality

$$(54) \quad \operatorname{ess\,sup}_{y \in B_R} u(y) \leq C \left(\operatorname{ess\,inf}_{y \in B_R} u(y) + K(R) \right)$$

where $B_R = B(x, R)$ and

$$K(R) = \left(|B_R|^{\varepsilon/Q} \|f_3\|_{L^s(B_R)} + \|g_3\|_{L^r(B_R)} \right)^{1/(p-1)} + \left(|B_R|^{\varepsilon/Q} \|h_3\|_{L^s(B_R)} \right)^{1/p} ,$$

with r, s as in (51)-(52). The proof of Theorem 3.1 in [14] relies essentially on the existence of subelliptic cut-off functions (as established by G. Citti et al., [20]). Cf. [14], p. 1785-1791. In the special case of the equation $\mathcal{L}_p u = 0$ one has

$$A_a(x, u, \xi) = A_a(\xi) = |\xi|^{p-2} \xi_a, \quad f = 0, \quad 1 \leq a \leq m.$$

Therefore

$$|A(\xi)| \leq |\xi|^{p-1}, \quad A(\xi) \cdot \xi = |\xi|^p,$$

so that the structural conditions (S) are satisfied with the choice

$$c_1 = 1, \quad g_2 = g_3 = 0, \quad f_1 = f_2 = f_3 = 0, \quad h_3 = 0.$$

In particular $K(R) = 0$ and (54) implies (47).

A Harnack inequality for the function $\tilde{u} \in L^p(\Omega, \gamma)$ (the weighted Lebesgue space, with a suitable weight function γ) associated (in the sense of S. Chanillo & R.L. Wheeden, [17]) to a nonnegative weak subelliptic p -harmonic function u was established by F. Ferrari, [33].

As an application of Theorem 16 one may prove the following regularity result (cf. Theorem 3.35 in [14], p. 1791)

Theorem 17. *Let $u \in W_{X, \text{loc}}^{1,p}(\Omega)$ be a weak solution to $\mathcal{L}_p u = 0$ such that $\text{ess sup}_\Omega |u| = A < \infty$. Then there exist $C > 0$ and $0 < \alpha < 1$ depending on Ω and A such that*

$$\text{ess sup}_{x,y \in \Omega} \frac{|u(x) - u(y)|}{d(x,y)^\alpha} \leq C.$$

Further results on subelliptic p -harmonic functions are available on Carnot groups, cf. Z.M. Balogh & I. Holopainen & J.T. Tyson, [1], and Z.M. Balogh & J.J. Manfredi & J.T. Tyson, [2]. Let $\Omega \subset \mathbb{G}$ be a domain in a Carnot group of homogeneous dimension Q . The basic nonlinear potential theory of subelliptic p -harmonic functions is developed by J. Heinonen & I. Holopainen, [44], including the Harnack inequality

$$(55) \quad \sup_{B(x,r)} u \leq C(\mathbb{G}, p) \inf_{B(x,r)} u$$

holding for positive continuous weak subelliptic p -harmonic functions

$u \in W_{X, \text{loc}}^{1,p}(B(x, 2r))$ with $x \in \mathbb{G}$ and $r > 0$. Therefore subelliptic p -harmonic functions have locally Hölder continuous representatives and there are no positive subelliptic p -harmonic functions $u : \mathbb{G} \rightarrow \mathbb{R}$ but the constant functions. Another consequence of (55) is the *strong maximum principle*: a continuous nonconstant subelliptic p -harmonic function on Ω cannot reach its supremum or infimum. Also continuous subelliptic p -harmonic functions satisfy the following *comparison principle*: if u and v are two continuous subelliptic p -harmonic functions on a domain $\Omega \subset \subset \mathbb{G}$ such that $u \geq v$ on $\partial\Omega$ (in an appropriate sense) then $u \geq v$ on Ω . Moreover by a result of L. Capogna, [13], if a subelliptic p -harmonic function u satisfies

$$0 < \frac{1}{C} \leq |(Xu)(x)| \leq C < \infty \quad \text{for a.e. } x \in \mathbb{G},$$

then u is smooth. Let $(E, F; \Omega)$ be a *condenser* in \mathbb{G} i.e. E and F are closed sets such that $E \cup F \subset \bar{\Omega}$. The p -capacity of $(E, F; \Omega)$ is

$$\text{Cap}_p(E, F; \Omega) = \inf \left\{ \int_\Omega |Xu|^p d\mu : u \in \mathcal{F} \right\},$$

$$\mathcal{F} = \{u \in C^\infty(\Omega) : \lim_{x \rightarrow y} u(x) = 0, \quad \lim_{x \rightarrow z} u(x) = 1, \quad \forall y \in E, z \in F\}.$$

Cf. [1], p. 165. The Q -capacity $\text{Cap}_Q(E, F; \Omega)$ is referred to as the *conformal capacity* of $(E, F; \Omega)$. Note that $\text{Cap}_p(E, F; \Omega)$ doesn't change if one replaces $C^\infty(\Omega)$ in the definition of \mathcal{F} by $C^0(\Omega) \cap W_{X, \text{loc}}^{1,p}(\Omega)$.

A domain $R \subset \mathbb{G}$ is a *ring domain* if its complement $\mathbb{G} \setminus R$ has exactly two components. If $R \subset \mathbb{G}$ is a ring domain with boundary components E and F then one writes $\text{Cap}_p(R) = \text{Cap}_p(E, F; R)$ and $\text{Cap}(R) = \text{Cap}_Q(R)$. Also a function $u : R \rightarrow \mathbb{R}$ is a p -potential of R if u is subelliptic p -harmonic in R , continuous in \bar{R} , and has boundary values $u(y) = 0$ for $y \in E$ and $u(z) = 1$ for $z \in F$. If a p -potential u of R then it is unique and

$$\text{Cap}_p(R) = \int_R |Xu|^p d\mu.$$

Let $u : \mathbb{G} \setminus \{0\} \rightarrow \mathbb{R}$ be a continuous subelliptic Q -harmonic function such that

$$\lim_{x \rightarrow 0} u(x) = \infty, \quad \lim_{|x| \rightarrow \infty} u(x) = -\infty,$$

$$(56) \quad \text{Cap}\{x \in \mathbb{G} \setminus \{0\} : \beta < u(x) < \alpha\} = (\alpha - \beta)^{1-Q}.$$

Such u is a *Green's function* for the subelliptic Q -Laplacian with pole at 0. To see that Green's functions exist let v be a subelliptic Q -harmonic ($Q \geq 2$) function on $\Omega = \mathbb{G} \setminus \{0\}$ such that $\lim_{x \rightarrow 0} v(x) = \infty$ and $\lim_{|x| \rightarrow \infty} v(x) = -\infty$ (such v exists by Proposition 4.16 in [44]). Next let us set

$$k = (\text{Cap}\{x \in \Omega : 0 < v(x) < 1\})^{1/(1-Q)}, \quad u = kv.$$

Then (by Lemma 2.11 in [1], p. 167) for any $-\infty < \beta < \alpha < \infty$ the relation (56) is satisfied. The finding in [1] is

Theorem 18. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q . Let u and v be Green's functions for the subelliptic Q -Laplacian with pole at the neutral element 0. If $u(x_0) = v(x_0)$ for some $x_0 \neq 0$ then $u = v$.*

Cf. Theorem 1.2 in [1], p. 160. A *homogeneous norm* on \mathbb{G} is a continuous function $N : \mathbb{G} \setminus \{0\} \rightarrow (0, +\infty)$ such that

$$N(x^{-1}) = N(x), \quad N(\delta_\lambda(x)) = \lambda N(x), \quad x \in \mathbb{G}, \quad \lambda > 0.$$

As a consequence of Theorem 18 (cf. Theorem 1.3 in [1], p. 160)

Corollary 4. *Let u be a Green's function for the subelliptic Q -Laplacian with pole at 0. Then there is a constant $\gamma(\mathbb{G}) > 0$ such that $N = e^{-\gamma(\mathbb{G})u}$ is a homogeneous norm on \mathbb{G} .*

Let $S = \{x \in \mathbb{G} \setminus \{0\} : N(x) = 1\}$ where N is the homogeneous norm in Corollary 4. The following version of the integration in polar coordinates holds (by a result in [35]): there is a Radon measure on S such that for any $f \in L^1(\mathbb{G})$

$$(57) \quad \int_{\mathbb{G}} f(x) d\mu(x) = \int_0^\infty d\lambda \int_S f(\delta_\lambda(u)) \lambda^{Q-1} d\sigma(u).$$

The Green's function $u = u_Q$ in Corollary 4 may be written as

$$(58) \quad u_Q(x) = (1/\gamma(\mathbb{G})) \log(1/N(x)).$$

It may be easily shown that composition of a smooth function with an element of $W_{X,\text{loc}}^{1,Q}(\mathbb{G} \setminus \{0\})$ is again an element of $W_{X,\text{loc}}^{1,Q}(\mathbb{G} \setminus \{0\})$ and that the chain rule holds almost everywhere. Then (by (58))

$$|X(u_Q)(x)| = \frac{1}{\gamma(\mathbb{G})} \frac{|X(N)(x)|}{N(x)} \quad \text{for a.e. } x \in \mathbb{G}.$$

Using (57) we may integrate in polar coordinates in

$$\int_{B(0,2) \setminus B(0,1)} |X(u_Q)(x)|^Q d\mu(x) < \infty$$

to conclude that $X(N)|_S \in L^Q(S, d\sigma)$. This is a crucial ingredient in the proof of (cf. Theorem 3.1 in [2], p. 41)

Theorem 19. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q and u_Q the Green's function of the subelliptic Q -Laplacian given by (58). Then up to a constant multiple u_Q is the fundamental solution to the subelliptic Q -Laplacian i.e. $\mathcal{L}_Q u_Q = b_Q \delta$ for some $b_Q \in \mathbb{R}$.*

We end Appendix C by formulating an open problem as to the relationship among the subelliptic p -Laplacian and the duality mapping on the space $W_X^{1,p}(\Omega)$ corresponding to the normalization function $\varphi(t) = t^{p-1}$ (with $1 < p < \infty$).

Let X be a real Banach space with the norm $\|\cdot\|$ and let X^* be its dual. The range of a multi-valued operator $A : X \rightarrow \mathcal{P}(X^*)$ is given by

$$\mathcal{R}(A) = \bigcup_{x \in \mathcal{D}(A)} A(x), \quad \mathcal{D}(A) = \{x \in X : A(x) \neq \emptyset\}.$$

The operator $A : X \rightarrow \mathcal{P}(X^*)$ is *monotone* if

$$\langle \Lambda_1 - \Lambda_2, x_1 - x_2 \rangle \geq 0$$

for any $x_1, x_2 \in \mathcal{D}(A)$ and any $\Lambda_i \in A(x_i)$, $i \in \{1, 2\}$. A continuous function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a *normalization function* if i) φ is strictly increasing, ii) $\varphi(0) = 0$, and iii) $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$. The *duality mapping* corresponding to the normalization function φ is the multi-valued operator $J_\varphi : X \rightarrow \mathcal{P}(X^*)$ given by

$$J_\varphi(x) = \{\Lambda \in X^* : \Lambda(x) = \varphi(\|x\|)\|x\|, \|\Lambda\| = \varphi(\|x\|)\}$$

for any $x \in X$. As a consequence of the Hahn-Banach theorem $\mathcal{D}(J_\varphi) = X$. Let us also recall (cf. e.g. J.L. Lions, [59])

Theorem 20. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a normalization function. Then*

- i) $J_\varphi(x) \subset X^*$ is a bounded, closed and convex subset for each $x \in X$.
- ii) J_φ is monotone.
- iii) For any $x \in X$

$$J_\varphi(x) = (\partial\Phi)(x), \quad \Phi(x) = \int_0^{\|x\|} \varphi(t) dt.$$

Here $\partial\Phi : X \rightarrow \mathcal{P}(X^*)$ is the subdifferential of Φ i.e.

$$(\partial\Phi)(x) = \{\Lambda \in X^* : \Phi(y) - \Phi(x) \geq \Lambda(y - x), \quad \forall y \in X\},$$

for any $x \in X$. A functional $f : X \rightarrow \mathbb{R}$ is G -differentiable at $x \in X$ if there is $\Lambda \in X^*$ such that $\lim_{t \rightarrow 0} t^{-1}[f(x+th) - f(x)] = \Lambda(h)$ for any $h \in X$. If $f : X \rightarrow \mathbb{R}$ is

G -differentiable at $x \in X$ we adopt the customary notation $\Lambda = f'(x)$. If $f : X \rightarrow \mathbb{R}$ is a convex function and G -differentiable at $x \in X$ then $(\partial f)(x) = \{f'(x)\}$.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $X = \{X_1, \dots, X_m\}$ is a Hörmander system of smooth vector fields defined on a neighborhood of $\bar{\Omega}$. What is the relationship among the subelliptic p -Laplacian \mathcal{L}_p and the duality mapping $J_\varphi : W_X^{1,p}(\Omega) \rightarrow W_X^{1,p}(\Omega)^*$ associated to the normalization function $\varphi(t) = t^{p-1}$? As an application of Theorem 20 it may be shown (cf. J. Crînganu, [23]) that the duality mapping $J_\varphi : H^{1,p}(\Omega) \rightarrow H^{1,p}(\Omega)^*$ corresponding to $\varphi(t) = t^{p-1}$ is the single-valued map $J_\varphi(u) = -\Delta_p u + |u|^{p-2}u$ for each $u \in H^{1,p}(\Omega)$. Here $H^{1,p}(\Omega)$ and Δ_p are the ordinary Sobolev space and p -Laplacian. See also G. Dinçă et al., [26], and J. Crînganu, [24].

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