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Regularity of the singular set of the free boundary for the obstacle problem in codimension one

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I am thankful to the organizers for giving me the opportunity of honoring a special friend whose work and mathematical taste have left an indelible mark on my own. My many collaborations with Ermanno have marked a significant period of my mathematical life and continue to be a source of inspiration. Happy Birthday Ermanno!

Abstract[1]. The study of the classical obstacle problem, initiated in the 60's with the pioneering works of G. Stampacchia, H. Lewy, J. L. Lions, has led over a period of three decades to beautiful and deep developments in calculus of variations and geometric partial differential equations. The crowning achievement has been the development, due to L. Caffarelli, of the theory of free boundaries. In this talk I will survey some new results on the lower-dimensional obstacle problem, or *Signorini problem*.

1. The thin obstacle problem

This lecture is a survey of recent works on the obstacle problem in codimension one. It is inspired in large part to the presentation in the recent joint work [9] with Arshak Petrosyan on the regularity of the singular part of the free boundary in the thin obstacle problem.

Let Ω be a domain in $\mathbb{R}^n$ and $\mathcal{M}$ a smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$ that divides Ω into two parts: Ω_+ and Ω_-. For given functions $\varphi : \mathcal{M} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ satisfying $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$, consider the problem of minimizing the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \ : \ u = g \text{ on } \partial\Omega, u \geq \varphi \text{ on } \mathcal{M} \cap \Omega\} \ .$$

This problem is known as the lower dimensional, or *thin obstacle problem*. The *thin obstacle* is the function φ. When u is constrained to stay above an obstacle φ which is assigned in the whole domain Ω, i.e. when $\mathcal{M} = \Omega$, then we obtain the *classical obstacle problem*. Whereas the latter is by now well-understood, the thin

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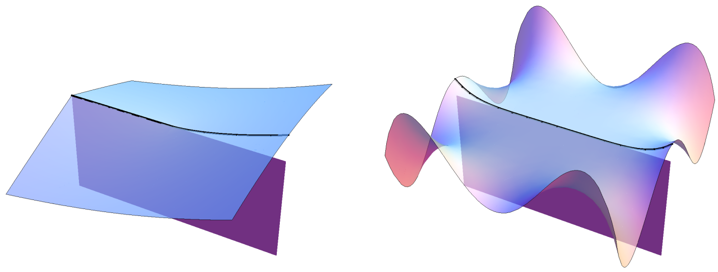

FIGURE 1. Graphs of $\mathrm{Re}\,(x_1 + i\,x_2)^{3/2}$ and $\mathrm{Re}\,(x_1 + i\,x_2)^6$

obstacle problem still presents considerable challenges and only recently there has been some significant progress on it.

Fig.1 gives an illustration of possible solutions to a thin obstacle problem. The profile of the shaded vertical region represents the obstacle.

The thin obstacle problem arises in a variety of situations of interest for the applied sciences:

- It presents itself in elasticity, when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.

- It also arises in financial mathematics in situations in which the random variation of an underlying asset changes discontinuously.

- It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction.

When $\mathcal{M}$ and φ are smooth, it has been proved by Caffarelli in 1979 [5] that the minimizer u in the thin obstacle problem is of class $C^{1,\alpha}_{\text{loc}}(\Omega_\pm \cup \mathcal{M})$. Since one can make free perturbations away from $\mathcal{M}$, it is easy to see that u satisfies

$$\Delta u = 0 \quad \text{in}\, \Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_- \ ,$$

but in general u does not need to be harmonic across $\mathcal{M}$. Instead, on $\mathcal{M}$, one has the following complementary conditions

$$u - \varphi \geq 0\ , \quad \partial_{\nu^+} u + \partial_{\nu^-} u \geq 0\ , \quad (u-\varphi)(\partial_{\nu^+} u + \partial_{\nu^-} u) = 0\ ,$$

where $\nu^\pm$ are the outer unit normals to $\Omega_\pm$ on $\mathcal{M}$.

One of the main objects of study in this problem is the so-called *coincidence set*

$$\Lambda(u) := \{x \in \mathcal{M} : u = \varphi\}$$

and its boundary (in the relative topology of $\mathcal{M}$)

$$\Gamma(u) := \partial_{\mathcal{M}} \Lambda(u)\ ,$$

known as the *free boundary*.

When the lower-dimensional obstacle φ is identically zero our constructions and proofs are most transparent. While most of our results hold also for smooth nonzero obstacles, the technicalities of the proofs are overwhelming and may easily distract from the main ideas. For this reason I will almost exclusively discuss the situation of a zero thin obstacle φ, and at the end quickly refer to some of the results for nonzero obstacles.

I will also assume that the manifold $\mathcal{M}$ is a flat portion of the boundary of the relevant domain. In this case the thin obstacle problem is known as the *Signorini problem.*

2. The Signorini problem: normalization

Consider then a solution u of the Signorini problem with *zero obstacle* on a *flat boundary.* Since we are interested in properties of minimizers near free boundary points, after a possible translation, rotation and scaling we will obtain a function u defined in the upper half-ball $B_1^+ \subset \mathbb{R}^{n-1} \times \{0\}$.

By a flat boundary we mean that $\mathcal{M}$ is the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ and $\varphi \equiv 0$ on $\mathcal{M}$.

We also let

$$B_1^+ := B_1 \cap \mathbb{R}^n_+ \ , \quad B_1' := B_1 \cap (\mathbb{R}^{n-1} \times \{0\}) \ .$$

In 1979 L. Caffarelli proved that the minimizer u in the thin obstacle problem is of class $C^{1,\alpha}_{\text{loc}}(B_1^+ \cup B_1')$.

Since we are interested in properties of minimizers near free boundary points, after a possible translation, rotation and scaling we will obtain a function

$$\Delta u = 0 \qquad \text{in } B_1^+ \tag{1}$$

$$u \geq 0 \ , \quad -\partial_{x_n} u \geq 0 \ , \quad u\, \partial_{x_n} u = 0 \quad \text{on } B_1' \tag{2}$$

$$0 \in \Gamma(u) = \partial\Lambda(u) := \partial\{(x',0) \in B_1' \mid u(x',0) = 0\} \ , \tag{3}$$

where $\Lambda(u)$ is the coincidence set and the boundary is in the relative topology of B_1'.

Definition 2.1. We denote by $\mathfrak{S}$ the class of solutions of the normalized Signorini problem (1)–(3).

Note that we may actually extend $u \in \mathfrak{S}$ by even symmetry to B_1

$$u(x', -x_n) := u(x', x_n) \ . \tag{4}$$

Then the resulting function will satisfy

$$\begin{aligned} \Delta u \leq 0 &\quad \text{in } B_1 \\ \Delta u = 0 &\quad \text{in } B_1 \setminus \Lambda(u) \\ u\, \Delta u = 0 &\quad \text{in } B_1 \ . \end{aligned}$$

It is useful to note the following relation between Δu and $\partial_{x_n} u$:

$$\Delta u = 2(\partial_{x_n} u)\, \mathcal{H}^{n-1}\big|_{\Lambda(u)} \qquad \text{in } \mathcal{D}'(B_1) \ .$$

There has been a recent surge of activity in the lower-dimensional obstacle problem.

In 2004 Athanasopoulos and Caffarelli [3] proved the *optimal* $C^{1,1/2}$ *interior regularity* for the solutions of the Signorini problem with flat $\mathcal{M}$ and $\varphi = 0$. This regularity is best possible as one can see from the explicit example of a solution $\hat{u}_0$ given by

$$\hat{u}_0(x) = \mathrm{Re}\,(x_1 + i x_n)^{3/2}\ .$$

A different perspective was taken in the paper [4], by Athanasopoulos, Caffarelli and Salsa, where extensive use was made of the celebrated monotonicity of Almgren's *frequency function*,

$$N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}\ .$$

This functional, or modifications of it, was extensively studied in 1986-87 by Garofalo and Lin [7], [8], in connection with problems of unique continuation for elliptic operators in divergence form. The name frequency comes from the fact that when u is a harmonic function in B_1 *homogeneous of degree* κ, then

$$N(r,u) = \frac{r\int_{\partial B_r} u\, u_\nu}{\int_{\partial B_r} u^2} = \frac{\int_{\partial B_r} u(x\cdot\nabla u)}{\int_{\partial B_r} u^2} \equiv \kappa\ .$$

Similarly to the classical obstacle problem, in the lower dimensional obstacle problem the analysis of the free boundary revolves around the behavior of the so-called blowups. In the Signorini problem one considers the *rescalings*

$$u_r(x) := \frac{u(rx)}{\left((1/r^{n-1})\ \int_{\partial B_r} u^2\right)^{1/2}},$$

and one wants to study the limits as $r \to 0+$, known as the *blowups.* Note that one has for any $0 < r < 1$

$$\|u_r\|_{L^2(\partial B_1)} = 1.$$

Generally the blowups might be different over different subsequences $r = r_j \to 0+$. The following result provides a tool to control the rescalings.

Theorem 2.2 (Monotonicity of the frequency)**.** *Let* $u \in \mathfrak{S}$*, then the function*

$$N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is monotone increasing in r *for* $0 < r < 1$*. Moreover,* $N(r,u) \equiv \kappa$ *for* $0 < r < 1$ *iff* u *is homogeneous of order* κ *in* B_1*, i.e.*

$$x\cdot\nabla u - \kappa u = 0 \quad \text{in } B_1\ .$$

When u is a harmonic function this is a classical result of Almgren [1] (1979), subsequently generalized to solutions of divergence form elliptic operators with Lipschitz coefficients by Garofalo and Lin (1986-87), [7], [8]. For the thin obstacle problem this formula has been first used by Athanasopoulos, Caffarelli and Salsa (2007) [4].

2.1. **The blowups.** A key property of the frequency is that it is invariant under the rescalings. By this we mean that

$$N(1, u_r) = N(r, u) \ .$$

One thus obtains for $r \leq 1$

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(1, u) \ ,$$

where in the last inequality we have used the monotonicity of the frequency $N(r, u)$ claimed in Theorem 2.2. The above inequality, and the $C^{1,\alpha}_{loc}$ estimates of Caffarelli, imply that there exists a nonzero function $u_0 \in W^{1,2}(B_1)$, called a *blowup* of u at the origin, such that for a subsequence $r = r_j \to 0+$

$$\begin{aligned} u_{r_j} &\to u_0 \qquad \text{in } W^{1,2}(B_1) \\ u_{r_j} &\to u_0 \qquad \text{in } L^2(\partial B_1) \\ u_{r_j} &\to u_0 \qquad \text{in } C^1_{\text{loc}}(B'_1 \cup B^{\pm}_1) \end{aligned}$$

The monotonicity of the frequency easily implies the following

Proposition 2.3 (Homogeneity of blowups)**.** *Let $u \in \mathfrak{S}$ and denote by u_0 any blowup of u as described above. Then, $u_0 \in \mathfrak{S}$ and it is a homogeneous function of degree $\kappa = N(0+, u)$.*

The following result was proved in part by L. Silvestre in his Ph.D. Dissertation (2006) [11], and in part by Caffarelli, Salsa and Silvestre (2008) [6].

Lemma 2.4 (Minimal homogeneity)**.** *Let $u \in \mathfrak{S}$. Then*

$$N(0+, u) \geq 2 - \frac{1}{2} \ .$$

Moreover, one has either

$$N(0+, u) = 2 - \frac{1}{2} \quad \text{or} \quad N(0+, u) \geq 2 \ .$$

Definition 2.5. Given $u \in \mathfrak{S}$, for $\kappa \geq 2 - (1/2)$ we define

$$\Gamma_\kappa(u) := \{x_0 \in \Gamma(u) : N^{x_0}(0+, u) = \kappa\} \ ,$$

where with

$$N^{x_0}(r, u) = \frac{r \int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2} \ ,$$

we have let $N^{x_0}(0+, u) = \lim_{r \to 0^+} N^{x_0}(r, u)$.

Notice that the sets $\Gamma_\kappa(u)$ may be nonempty only for κ in a certain set of values. For instance, the previous Lemma implies that

$$\Gamma_\kappa(u) = \varnothing \quad \text{whenever } 2 - \frac{1}{2} < \kappa < 2 \ .$$

On the other hand, for the functions

$$u_\kappa(x) = \operatorname{Re}(x_1 + i\, x_n)^\kappa \ , \quad \text{for } \kappa = 2m - \frac{1}{2} \ , \quad \kappa = 2m \ , \quad m \in \mathbb{N} \ ,$$

one has $0 \in \Gamma_\kappa(u_\kappa)$.

In fact, for dimension $n = 2$, a simple analysis of homogeneous harmonic functions in a halfplane shows that $\kappa = 2m - (1/2)$, $2m$, $m \in \mathbb{N}$, *are the only possible values of frequencies in the thin obstacle problem.*

Basic open problem: It is plausible that a similar result should be true when $n \geq 3$. In other words: Is it true that, at any free boundary point x_0 the only possible frequencies in the thin obstacle problem with zero obstacle on a flat boundary are

$$\kappa = N^{x_0}(0^+, u) \in \bigcup_{m\in\mathbb{N}} \left\{2m - \frac{1}{2}\right\} \cup \{2m\} \ ?$$

This is not known.

3. Regularity of the free boundary at regular points

Of special interest is the case of the *smallest possible value of the frequency*, i.e. $\kappa = 2 - (1/2)$.

Definition 3.1 (Regular free boundary points)**.** For $u \in \mathfrak{S}$ we say that $x_0 \in \Gamma(u)$ is *regular* if $N^{x_0}(0+, u) = 2 - (1/2)$, i.e., if $x_0 \in \Gamma_{2-(1/2)}(u)$.

The following result was proved by Athanasopoulos, Caffarelli and Salsa in 2007, see [4].

Theorem 3.2 (Smoothness of the free boundary at regular points)**.** *Let $u \in \mathfrak{S}$, then that part of the free boundary composed of regular points $\Gamma_{2-(1/2)}(u)$ is locally a $C^{1,\alpha}$ regular $(n-2)$-dimensional surface.*

3.1. **Singular free boundary points.** The previous theorem settles the question of what happens at regular free boundary points. The free boundary, however, might be formed (in fact, exclusively, see Fig.2) of points which are not regular according to the previous definition. One type of free boundary points which presents itself is given in the following definition.

Definition 3.3 (Singular points)**.** Let $u \in \mathfrak{S}$. We say that x_0 is a *singular point* of the free boundary $\Gamma(u)$, if the coincidence set $\Lambda(u)$ has vanishing $(n-1)$-dimensional density at x_0, i.e.

$$\lim_{r\to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(u) \cap B'_r(x_0))}{\mathcal{H}^{n-1}(B'_r(x_0))} = 0 \ .$$

We denote by $\Sigma(u)$ the subset of singular points of $\Gamma(u)$.

As we will see, the singular portion $\Sigma(u)$ of the free boundary is precisely composed by points at which the frequency equals

$$\kappa = 2m \qquad m \in \mathbb{N} \ .$$

Note that if, for instance, $0 \in \Sigma(u)$, then in terms of the rescalings the condition $0 \in \Sigma(u)$ is equivalent to

$$\lim_{r\to 0+} \mathcal{H}^{n-1}(\Lambda(u_r) \cap B'_1) = 0 \ .$$

We denote by

$$\Sigma_\kappa(u) := \Sigma(u) \cap \Gamma_\kappa(u) \ .$$

the set of those singular free boundary points at which the frequency is equal to κ. It is important to observe that the singular set $\Sigma(u)$ is not necessarily a small set

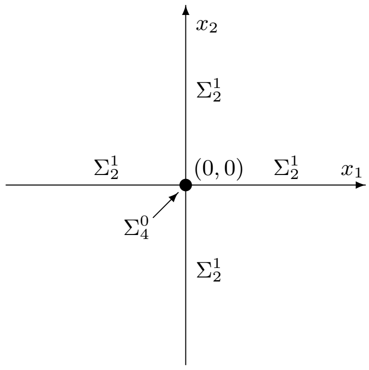

FIGURE 2. Free boundary for $u(x) = x_1^2 x_2^2 - \left(x_1^2 + x_2^2\right) x_3^2 + (1/3)\, x_3^4$ in $\mathbb{R}^3$ with zero thin obstacle on $\mathbb{R}^2 \times \{0\}$.

in any sense. In fact, it can fill up the whole free boundary. An illustration of this is given in Fig.2.

3.2. **Central question: uniqueness of the blowups at singular points?** Similarly to the classical obstacle problem, the central goal now is to prove the *uniqueness of the blowups at singular free boundary points.* This is one of the main difficulties in the analysis.

Proving such uniqueness is equivalent to showing that at any $x_0 \in \Sigma(u)$ one has a Taylor expansion

$$u(x', x_n) = p_\kappa^{x_0}(x - x_0) + o(|x - x_0|^\kappa) \ ,$$

where $p = p_\kappa^{x_0}$ is a nondegenerate homogeneous polynomial of a certain order κ, satisfying

$$\Delta p = 0 \ , \quad x \cdot \nabla p - \kappa p = 0 \ , \quad p(x', 0) \geq 0 \ , \quad p(x', -x_n) = p(x', x_n) \ .$$

The value of κ *must be an even integer*, and it is obtained from the above mentioned generalization of Almgren's frequency formula to the thin obstacle problem.

4. Historical development: the Alt-Caffarelli-Friedman, Weiss' and Monneau's monotonicity formulas

In the classical obstacle problem uniqueness of the blowups of the rescalings $u_r(x) = u(rx)/r^2$ was proved by Caffarelli using the following deep result:

4.1. **Alt-Caffarelli-Friedman monotonicity formula (1984)** [2]**.** *If u is a minimizer with $u(0) = 0$, then*

$$r \ \rightarrow \ \frac{1}{r^4} \int_{B_r} \frac{|\nabla u^+|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla u^-|^2}{|x|^{n-2}}$$

is monotone increasing.

4.2. **Weiss' monotonicity formula(1999)** [12]**.** Fifteen years later, George Weiss discovered a simpler monotonicity formula and showed that with such formula one can still prove the uniqueness of the blowups. *If u is a minimizer, then*

$$r \;\to\; W(r) := \frac{1}{r^{n+2}} \int_{B_r} (|\nabla u|^2 + 2u) - \frac{2}{r^{n+3}} \int_{\partial B_r} u^2$$

is monotone increasing. One has in fact

$$\frac{d}{dr} W(r) = \frac{2}{r^{n+4}} \int_{\partial B_r} (x \cdot \nabla u - 2\,u)^2 \ .$$

Note: Unlike Almgren's monotonicity formula, in Weiss' formula $W'(r) \equiv 0$ if and only if u is **homogeneous of degree $\kappa = 2$!**

4.3. **Monneau's monotonicity formula at singular points, (2003)** [10]**.** More recently, Regis Monneau has used Weiss' result to prove yet another monotonicity formula which is tailor made for the study of the blowups at singular free boundary points in the classical obstacle problem. A free boundary point is called *singular* if the coincidence set has vanishing n-dimensional density at that point. For instance, for the global solution (blowup) $u(x) = (1/2)\,(x_1^+)^2$ of the classical obstacle problem, the free boundary is $\{0\} \times \mathbb{R}^{n-1}$, and so all free boundary points are singular!

Let u be a minimizer and assume that 0 is a singular free boundary point for u. Then, for an arbitrary harmonic polynomial $p \geq 0$ homogeneous of degree 2 the function

$$r \;\to\; \frac{1}{r^{n+3}} \int_{\partial B_r} (u-p)^2$$

is monotone increasing.

Now, in the classical obstacle problem the only value of the frequency that appears is $\kappa = 2$. And in fact the above mentioned monotonicity formulas of Alt-Caffarelli-Friedman, Weiss and Monneau are only suitable for $\kappa = 2$.

In the thin obstacle problem, instead, one the main complications is that at a singular point the frequency κ may be an arbitrary even integer $2m$, $m \in \mathbb{N}$.

With this observation in mind, and with the objective of studying singular points, our original desire was to construct an analogue of Monneau's formula based on Almgren's frequency formula, rather than that of Weiss'.

This was suggested by the fact that, at least in principle, Almgren's frequency formula does not display the limitation of the specific value $\kappa = 2$.

In the process, however, we have discovered a new one-parameter family of monotonicity formulas $\{W_\kappa\}$ of Weiss type which is tailor made for studying the thin obstacle problem, and that, remarkably, is inextricably connected to Almgren's frequency formula. In the sense that Almgren's monotonicity formula is equivalent to the one-parameter family of monotonicity formulas of Weiss type.

5. Weiss type monotonicity formulas

Theorem 5.1 (Weiss type monotonicity formula)**.** *Given $u \in \mathfrak{S}$, for any $\kappa \geq 0$ we introduce the functional*

$$W_\kappa(r,u) := \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} u^2 \ .$$

Then

$$\frac{d}{dr}W_\kappa(r,u) = \frac{2}{r^{n+2\kappa}}\int_{\partial B_r}(x\cdot\nabla u - \kappa\, u)^2 \ .$$

As a consequence, $r \to W_\kappa(r,u)$ is monotone increasing.

Note that for $\kappa = 2$ we obtain Weiss' monotonicity formula for the classical obstacle problem (except that Weiss' energy has the additional term $2u$ to account for the fact that his equation is $\Delta u = \chi_{\{u>0\}}$). But in the thin obstacle problem one needs the full range of $\kappa \geq 0$, or at least $\kappa = 2m - (1/2)$, or $\kappa = 2m$, with $m \in \mathbb{N}$.

With these new formulas in hand, inspired by Monneau, we have discovered another one-parameter family $\{M_\kappa\}$ of monotonicity formulas which are ad hoc for studying singular free boundary points with frequency $\kappa = 2m$, $m \in \mathbb{N}$.

6. Monneau type monotonicity formulas

Our result is a κ-homogeneous analogue of Monneau's formula, and it is tailor made for the study of singular points in the thin obstacle problem.

For $\kappa = 2m$, $m \in \mathbb{N}$, we denote by $\mathfrak{P}_\kappa$ be the family of harmonic homogeneous polynomial p_κ of degree κ, positive on $x_n = 0$; i.e.

$$\mathfrak{P}_\kappa = \{p_\kappa(x) : \Delta p_\kappa = 0,\ x\cdot\nabla p_\kappa - \kappa p_\kappa = 0,\ p_\kappa(x',0) \geq 0\} \ .$$

Theorem 6.1 (Monneau type monotonicity formula)**.** *Let $u \in \mathfrak{S}$ with $0 \in \Sigma_\kappa(u)$, $\kappa = 2m$, $m \in \mathbb{N}$. Then for arbitrary $p_\kappa \in \mathfrak{P}_\kappa$ the functional*

$$M_\kappa(r,u,p_\kappa) := \frac{1}{r^{n-1+2\kappa}}\int_{\partial B_r}(u-p_\kappa)^2$$

is monotone increasing in r for $0 < r < 1$.

Note that when $m = 1$ and then $\kappa = 2$, the above result is the monotonicity formula of Monneau for the classical obstacle problem.

6.1. The main results: uniqueness of the blowups and structure of the singular set.

Using the monotonicity of the family $\{M_\kappa\}$ we can prove one of our main results: the desired **Taylor expansion for the blowups** and therefore **the uniqueness of the blowups**.

Furthermore, the monotonicity formulas $\{M_\kappa\}$ allow to establish the nondegeneracy and the continuous dependence on the singular free boundary point x_0 with frequency κ of the polynomial $p_\kappa^{x_0}$.

Theorem 6.2 (κ-differentiability at singular points)**.** *Let $u \in \mathfrak{S}$ and $0 \in \Sigma_\kappa(u)$ with $\kappa = 2m$, $m \in \mathbb{N}$. Then there exists a* nonzero *$p_\kappa \in \mathfrak{P}_\kappa$ such that*

$$u(x) = p_\kappa(x) + o(|x|^\kappa) \ .$$

Moreover, if for $x_0 \in \Sigma_\kappa(u)$ the polynomial $p_\kappa^{x_0} \in \mathfrak{P}_\kappa$ is such that we have the Taylor expansion

$$u(x) = p_\kappa^{x_0}(x - x_0) + o(|x-x_0|^\kappa) \ ,$$

then $p_\kappa^{x_0}$ depends continuously on $x_0 \in \Sigma_\kappa(u)$.

To state our second main result we need the following

Definition 6.3 (Dimension at the singular point)**.** For a singular point $x_0 \in \Sigma_\kappa(u)$ we denote

$$d_\kappa^{x_0} := \dim\{\xi \in \mathbb{R}^{n-1} : \xi \cdot \nabla_{x'} p_\kappa^{x_0} \equiv 0\} ,$$

the degree of degeneracy of the polynomial $p_\kappa^{x_0}$, which we call the dimension of $\Sigma_\kappa(u)$ at x_0. Note that since $p_\kappa^{x_0} \not\equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$ one has

$$0 \leq d_\kappa^{x_0} \leq n-2 .$$

For $d = 0, 1, \ldots, n-2$ we define

$$\Sigma_\kappa^d(u) := \{x_0 \in \Sigma_\kappa(u) : d_\kappa^{x_0} = d\} .$$

Using the Taylor expansion

$$u(x) = p_\kappa^{x_0}(x - x_0) + o(|x - x_0|^\kappa) ,$$

in combination with Whitney extension theorem and the implicit function theorem we can prove our second main theorem.

Theorem 6.4 (Structure of the singular set)**.** *Let $u \in \mathfrak{S}$. Then, $\Sigma_\kappa(u) = \Gamma_\kappa(u)$ for $\kappa = 2m$, $m \in \mathbb{N}$, and every set $\Sigma_\kappa^d(u)$, $d = 0, 1, \ldots, n-2$ is contained in a countable union of d-dimensional C^1 manifolds.*

7. The case of a nonzero obstacle

By allowing nonzero obstacles one sacrifices Almgren's frequency formula in its purest form. However, a modified version of it does hold. In the case $k = 2$ such result has first been established by Caffarelli-Salsa-Silvestre in 2008. For their purposes it was sufficient to consider the class $\mathfrak{S}_2$ as it allows to capture the slowest growth rate of the solution at a regular free boundary point and thus establish the optimal regularity.

For the analysis of the free boundary at singular points we need instead to consider the full range of values of k.

Assume now that the lower dimensional obstacle is given by $\phi \in C^{k,1}(B_1')$. Let $Q_k(x')$ be the Taylor polynomial of degree k of ϕ at the origin , i.e.,

$$\phi(x') = Q_k(x') + O(|x'|^{k+1}) .$$

Moreover, we will also have

$$\Delta_{x'}\phi(x') = \Delta_{x'}Q_k(x') + O(|x'|^{k-1}) .$$

By a technical lemma we can find a harmonic extension $\tilde{Q}_k$ of Q_k into $\mathbb{R}^n$. For the solution v of the Signorini problem with thin obstacle ϕ we consider the difference

$$u(x', x_n) := v(x) - \tilde{Q}_k(x', x_n) - (\phi(x') - Q_k(x')) .$$

It is easy to see that u satisfies

$$|\Delta u| = |\Delta_{x'}(\phi - Q_k)| \leq M|x'|^{k-1} \quad \text{in } B_1^+ \tag{5}$$

$$u \geq 0 , \quad -\partial_{x_n} u \geq 0 , \quad u\,\partial_{x_n} v = 0 \quad \text{on } B_1\prime \tag{6}$$

$$0 \in \Gamma(u) := \partial\{u(\cdot, 0) > 0\} . \tag{7}$$

Definition 7.1. We say that $u \in C^{1,\alpha}(B_1^+ \cup B_1')$ belongs to the class $\mathfrak{S}_k(M)$ if it satisfies (5)–(7) and moreover

$$\|u\|_{C^1(B_1)} \leq M .$$

We write $\mathfrak{S}_k$ when the constant M is not important.

7.1. **The generalized frequency formula.**

Theorem 7.2 (Generalized frequency formula)**.** *Let* $u \in \mathfrak{S}_k(M)$*. With* $H(r) = \int_{\partial B_r} u^2$*, there exists* $C_M > 0$ *and* $r_M > 0$ *such that the functional*

$$\Phi_k(r,u) := (r + C_M r^2)\frac{d}{dr}\log\max\left\{H(r), r^{n-1+2k}\right\} \ ,$$

is nondecreasing in r *for* $0 < r < r_M$*.*

The proof of this theorem is technically quite involved! We refer the reader to [9].

7.2. **The generalized Weiss type monotonicity formula.**

Theorem 7.3 (Weiss type monotonicity formula)**.** *Let* $u \in \mathfrak{S}_k(M)$ *and* $\kappa \le k$*. Then there exist* C_M *and* $r_M > 0$ *such that*

$$\begin{aligned} W_\kappa(r,u) &:= \frac{1}{r^{n-2+2\kappa}}\int_{B_r}|\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}}\int_{\partial B_r} u^2 = \\ &= \frac{1}{r^{n-2+2\kappa}}\, D(r) - \frac{\kappa}{r^{n-1+2\kappa}}\, H(r) \ . \end{aligned}$$

satisfies

$$\frac{d}{dr}W_\kappa(r) \ge -C_M \quad \text{for } 0 < r < r_M \ .$$

7.3. **The generalized Monneau type monotonicity formula.**

Theorem 7.4 (Monneau type monotonicity formula)**.** *Let* $u \in \mathfrak{S}_k(M)$ *and suppose that* $0 \in \Sigma_\kappa(u)$ *with* $\kappa = 2m < k$*,* $m \in \mathbb{N}$*. Then for any* $p_\kappa \in \mathfrak{P}_\kappa$ *there exist* C_M *and* $r_M > 0$ *such that*

$$M_\kappa(r,u,p_\kappa) = \frac{1}{r^{n-1+2\kappa}}\int_{\partial B_r}(u - p_\kappa)^2$$

satisfies

$$\frac{d}{dr}M_\kappa(r,u,p_\kappa) \ge -C_M\left(1 + \|p_\kappa\|_{L^2(B_1)}\right) \quad \text{for } 0 < r < r_M \ .$$

7.4. **The singular set.**

Theorem 7.5 (κ-differentiability at singular points)**.** *Let* $u \in \mathfrak{S}_k$ *and* $0 \in \Sigma_\kappa(u)$ *for* $\kappa = 2m < k$*,* $m \in \mathbb{N}$*. Then there exist nonzero* $p_\kappa \in \mathfrak{P}_\kappa$ *such that*

$$u(x) = p_\kappa(x) + o(|x|^\kappa) \ .$$

Moreover, if $v \in \mathfrak{S}^\phi$ *with* $\phi \in C^{k,1}(B_1')$*,* $x_0 \in \Sigma_\kappa(v)$ *and* $u_k^{x_0}$ *is obtained by translating to* x_0*, then in the Taylor expansion*

$$u_k^{x_0}(x) = p_\kappa^{x_0}(x) + o(|x|^\kappa)$$

the mapping $x_0 \mapsto p_\kappa^{x_0}$ *from* $\Sigma_\kappa(v)$ *to* $\mathfrak{P}_\kappa$ *is continuous.*

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