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The near field refractor

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To Ermanno Lanconelli in occasion of his 65th birthday

Abstract[1]. Given an $n-1$ dimensional target screen we prove the existence of an interface optical surface in $\mathbb{R}^n$ that separates two media with different index of refraction, say glass and air, such that all rays emanating from one point and with given intensity illuminate the screen with a prescribed intensity.

1. INTRODUCTION

Geometric optics is one of the oldest fields of mathematical research, and according to Sommerfeld [7], Leonardo Da Vinci called optics "the paradise of mathematicians". The equality of the angles of incidence and reflection was known to Euclid (c. 325 BC-265 BC). Claudius Ptolemæus (85-165AD) published an optics treatise in which he describes experiments measuring the angles of incidence and refraction, without, however, discovering the law. Kepler also made experiments, but he was equally unsuccessful in finding the law. The real discoverer of the law of refraction was Snell, around 1626, and it was first published by Descartes in 1637, who according to historians knew Snell's research. This was the solid ground for subsequent huge amounts of research and applications to innumerable instruments and optics devices used in everyday life. This continued for centuries until today, with many great mathematical minds of all times working in this area: Gauss, Huygens, Newton, Lagrange, Moëbius, Bessel, Hamilton, just to name a few. A beautiful presentation on the origin and historical development of general concepts of optics can be found in the book by Mach [6]. In recent decades important connections between geometric optics and nonlinear partial differential equations were found, and it is the purpose of this paper to consider a problem in this area. In order to describe our problem we then recall the law of refraction.

Suppose Γ is a surface in $\mathbb{R}^n$ that separates two media I and II that are homogeneous and isotropic. Let v_1 and v_2 be the velocities of propagation of light in the media I and II respectively. The index of refraction of medium I is $n_1 = c/v_1$, where c is the speed of propagation of light in the vacuum, and similarly $n_2 = c/v_2$.

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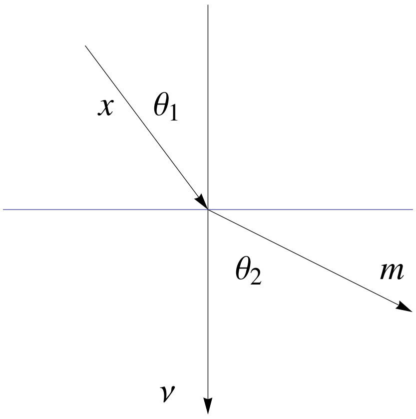

FIGURE 1. $x - \kappa m$ parallel to ν

If a ray of light[2] having direction $x \in S^{n-1}$ and traveling through medium I hits Γ at the point P, and ν is the unit normal to Γ at P going towards medium II, then this ray is refracted in the direction $m \in S^{n-1}$ through medium II according with the Snell-Descartes law in vector form: the vectors x, ν and m are all coplanar, and the vector $n_2 m - n_1 x$ is parallel to the normal vector ν, that is, setting $\kappa = n_2/n_1$, we have

$$x - \kappa\, m = \lambda \nu\ , \tag{1.1}$$

for some $\lambda \in \mathbb{R}$, see Figure 1. Making the vector product of this equation with the normal ν we obtain the well known form of the Snell law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where θ_1 is the angle between x and ν (the angle of incidence), θ_2 the angle between m and ν (the angle of refraction).

It can be easily seen that in (1.1), $\lambda = \cos\theta_1 - \kappa\cos\theta_2$, $\cos\theta_1 = x\cdot\nu > 0$, and $\cos\theta_2 = m\cdot\nu = \sqrt{1-\kappa^{-2}[1-(x\cdot\nu)^2]}$.

To fix ideas, we will concentrate throughout the paper in case $\kappa < 1$, or equivalently when $v_1 < v_2$, that is, waves propagate in medium II faster than in medium I, or equivalently, medium I is denser than medium II. In this case the refracted rays tend to bent away from the normal, that is the case for example, when medium I is glass and medium II is air. For this reason, the maximum angle of refraction is $\pi/2$ achieved when $\sin\theta_1 = n_2/n_1 = \kappa$. So there cannot be refraction when the incidence angle θ_1 is beyond this critical value, that is, we must have $0 \le \theta_1 \le \theta_c = \arcsin\,\kappa$. If $\theta_1 > \theta_c$, then the important phenomenon of total internal reflection occurs, see Figure 2, which explains the behavior of fiber optics, see Figure 3. It is easy to verify that

$$\theta_2 - \theta_1 = \arcsin(\kappa^{-1}\sin\theta_1) - \theta_1 \tag{1.2}$$

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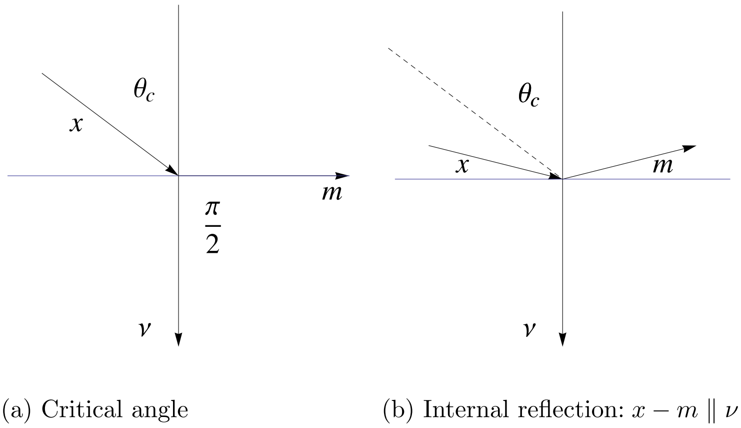
[2] Since the refraction angle depends on the frequency of the radiation, we assume our light ray is monochromatic, see [7, Chapter 1].



(a) Critical angle (b) Internal reflection: $x - m \parallel \nu$

FIGURE 2. Internal reflection

FIGURE 3. Fiber optics

is strictly increasing for $\theta_1 \in [0,\ \theta_c]$, and therefore $0 \leq \theta_2 - \theta_1 \leq (\pi/2) - \theta_c$. We are then lead to the following physical constraint:

$$\begin{aligned} &\textit{if } \kappa = n_2/n_1 < 1 \textit{ and a ray of direction } x \textit{ through medium I} \\ &\textit{is refracted into medium II in the direction } m, \textit{ then } m \cdot x \geq \kappa\,. \end{aligned} \tag{1.3}$$

Conversely, given $x, m \in S^{n-1}$ with $x \cdot m \geq \kappa$ and $\kappa < 1$, it follows from (1.2) that there exists a hyperplane refracting any ray through medium I with direction x into a ray of direction m in medium II. Therefore condition (1.3) is necessary and sufficient for refraction when $\kappa < 1$.

To illustrate we show the following table

Media I	n_1	κ with $n_2 = 1$ (air)	**Critical angle**
water	1.33	0.751	48.67°
crown glass	1.52	0.657	41.07°
diamond	2.41	0.414	24.45°

The problem considered in this paper is the following. Suppose we have a domain Ω in the sphere S^{n-1} and a domain $D \subset \mathbb{R}^n$ contained in an $n-1$ dimensional surface; D is referred as the target domain or screen to be illuminated. Let I and II be two homogeneous and isotropic media, and suppose that from a point O surrounded by medium I, light emanates with intensity $f(x)$ for $x \in \Omega$, and D is surrounded by media II. We seek an optical surface $\mathcal{R}$ parameterized by $\mathcal{R} = \{\rho(x)x\ :\ x \in \overline{\Omega}\}$, interface between media I and II, such that all rays refracted by $\mathcal{R}$ into medium II illuminate the object D, and the prescribed illumination intensity received at each point $P \in D$ is $g(P)$. Of course some conditions on the relative position of D and

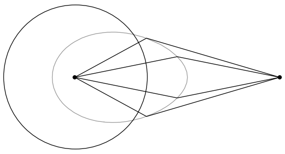


FIGURE 4. The lens outside the sphere and inside the oval refracts all rays into the same point

the set of directions in Ω are need to illuminate D. Assuming no loss of energy in this process, we have the conservation of energy equation $\int_\Omega f(x)\,dx = \int_D g(P)\,dP$.

The purpose of this paper is to show the existence of the interface surface $\mathcal{R}$ under general conditions on Ω and D and also for the case that g is a Radon measure in D. This implies that one can design a lens refracting light beams so that the screen D is illuminated in a prescribed way. The lens is bounded by two optical surfaces, the "outer" surface is $\mathcal{R}$ and the "inner" one is a sphere with center at the point from where the radiation emanates. These ideal lenses do not have spherical aberration, i.e., they focus all incoming rays into exactly one point. This is called a near field problem as opposed to the far field problem which consists in sending rays from O into a prescribed set of directions.

These type of geometric optics problems -far and near field- recently received attention because both of its applications and their mathematical interest and difficulty. We enumerate some related results. The problem of the far field reflector has been considered by several authors, both mathematicians and engineers. For example, existence and uniqueness up to dilations of solutions for the far field reflector problem were proved by Caffarelli and Oliker in [2] and X-J. Wang in [8]. C^1 regularity of solutions for the far field reflector was established in [1]. The near field reflector problem was considered by Kochengin and Oliker in [4], and in a recent work by Karakhanyan and X-J. Wang [5]. The far field refractor problem was considered for the first time in [3] where existence and uniqueness up to dilations of solutions are established. A difference between near and far field problems is that the latter can be cast in the frame of optimal transportation. Instead, near field problems are in general not optimal transportation problems, which makes their mathematical treatment more difficult.

In this note we describe how we solve our problem in case $\kappa < 1$. To do that, we use surfaces that refract all light rays emanating from a point O into a fixed point P; these surfaces are Cartesian ovals and are essential to our approach, see Figure 5. With them and energy conservation, we then formulate the concept of weak solution. Existence is first established when the right hand side is a sum of delta functions and in the general case we proceed by approximation and passing to the limit. Our method is constructive and perhaps can be implemented numerically.

2. CARTESIAN OVALS

To resolve our problem it is important to solve first the following simpler problem: given a point O inside medium I and a point P inside medium II, find an interface surface $\mathcal{S}$ between media I and II that refracts all rays emanating from the point

O into the point P. Suppose we are in dimension two, assume $O = (0,0)$ and $P = (a,0)$, and $\mathcal{S}$ has equation $r(t) = (x(t), y(t))$. By the Snell law of refraction the tangent vector $r'(t)$ satisfies

$$r'(t) \cdot \left(\frac{r(t)}{|r(t)|} - \kappa \frac{(a,0)-(x,y)}{\sqrt{(a-x)^2+y^2}} \right) = 0 \; .$$

That is,

$$\frac{xx'+yy'}{\sqrt{x^2+y^2}} + \kappa \frac{(x-a)(x-a)'+yy'}{\sqrt{(a-x)^2+y^2}} = 0 \; ,$$

or equivalently

$$\left((x^2+y^2)^{1/2} \right)' + \kappa \left(((x-a)^2+y^2)^{1/2} \right)' = 0 \; .$$

Therefore $\mathcal{S}$ is the Cartesian oval

$$|X| + \kappa |X-P| = C \; . \tag{2.1}$$

Since $f(X) = |X| + \kappa|X-P|$ is a convex function, the oval is a convex set.

In our treatment of the problem, we need to find and analyze the polar equation of the oval. Write $X = \rho(x)x$ with $x \in S^{n-1}$. Then writing $\kappa|\rho(x)x - P| = C - \rho(x)$, squaring this quantity and solving the quadratic equation yields

$$\rho(x) = \frac{(C-\kappa^2 x\cdot P) \pm \sqrt{(C-\kappa^2 x\cdot P)^2 - (1-\kappa^2)(C^2-\kappa^2|P|^2)}}{1-\kappa^2} \tag{2.2}$$

Suppose $0 < \kappa < 1$. Set

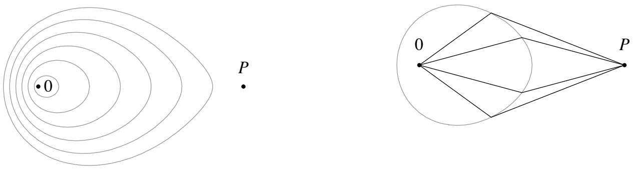
$$\Delta(t) = (C-\kappa^2 t)^2 - (1-\kappa^2)(C^2-\kappa^2|P|^2) \; .$$

We have

$$\Delta(x\cdot P) \geq \kappa^2 (x\cdot P - C)^2 \tag{2.3}$$

If the oval is non empty we have that $\kappa|P| \leq C$. In case $\kappa|P| = C$, the oval reduces to the point 0. Also from the equation of the oval we get that $\rho(x) \leq C$. So we now decide which values $\pm$ we should take in the definition of $\rho(x)$. Let ρ_+ and ρ_- the corresponding ρ's. We claim that $\rho_+(x) \geq C$ and $\rho_-(x) \leq C$. We have

$$\begin{aligned}
\rho_+(x) \quad &= \frac{(C-\kappa^2 x\cdot P) + \sqrt{\Delta(x\cdot P)}}{1-\kappa^2} \geq \\
&\geq \frac{(C-\kappa^2 x\cdot P) + \kappa|C - x\cdot P|}{1-\kappa^2} = \\
&= \frac{1}{1-\kappa^2} \begin{cases} (C-\kappa^2 x\cdot P) + k(C - x\cdot P) & \text{for } x\cdot P \leq C \\ (C-\kappa^2 x\cdot P) - k(C - x\cdot P) & \text{for } x\cdot P \geq C \end{cases} \\
&\geq C \; .
\end{aligned}$$



(a) $|X| + 2/3|X - P| = 1.4 - 1.9,\ P = (2,0)$ (b) $|X| + 2/3|X - P| = 1.7,\ P = (2,0)$

Figure 5. Cartesian ovals $\kappa < 1$, e.g., glass to air

Also

$$\begin{aligned}\rho_-(x) &= \frac{(C - \kappa^2 x \cdot P) - \sqrt{\Delta(x \cdot P)}}{1-\kappa^2} \leq \\ &\leq \frac{(C - \kappa^2 x \cdot P) - \kappa|C - x \cdot P|}{1-\kappa^2} = \\ &= \frac{1}{1-\kappa^2}\begin{cases}(C - \kappa^2 x \cdot P) - k(C - x \cdot P) & \text{for } x \cdot P \leq C \\ (C - \kappa^2 x \cdot P) + k(C - x \cdot P) & \text{for } x \cdot P \geq C\end{cases} \\ &\leq C.\end{aligned}$$

So the claim is proved. Therefore the polar equation of the oval is then given by

$$\rho(x) = \rho_-(x) = \frac{(C - \kappa^2 x \cdot P) - \sqrt{\Delta(x \cdot P)}}{1-\kappa^2}\,.$$

From the physical constraint for refraction we must have

$$x \cdot \left(\frac{P - \rho(x)x}{|P - \rho(x)x|}\right) \geq \kappa\,,$$

and from the equation of the oval we then get that to have refraction we need

$$x \cdot P \geq C\,.$$

The estimates of ρ are contained in the following lemma.

Lemma 2.1. *Let $0 < \kappa < 1$ and assume that $\kappa|P| < C < |P|$. Then we have*

$$\min_{x \in S^{n-1}} \rho(x) = \frac{C - \kappa|P|}{1+\kappa} \quad \text{and} \quad \max_{x \in S^{n-1}} \rho(x) = \frac{C - \kappa|P|}{1-\kappa}\,. \tag{2.4}$$

Also

$$\min_{x \in S^{n-1}} |P - \rho(x)x| = \frac{|p| - C}{1-\kappa} = \min_{\{x\,:\,x \cdot P \geq C\}} |P - \rho(x)x| = \left|P - \rho\left(\frac{P}{|P|}\right)\frac{P}{|P|}\right|\,. \tag{2.5}$$

Proof. We write

$$\rho(x) = \frac{C^2 - \kappa^2|P|^2}{(C - \kappa^2 x \cdot P) + \sqrt{\Delta(x \cdot P)}}$$

and let $g(t) = (C - \kappa^2 t) + \sqrt{\Delta(t)}$. We have g is decreasing for $-|P| \leq t \leq |P|$, and so $g(-|P|) \geq g(x \cdot P) \geq g(|P|)$. Hence

$$\frac{C^2 - \kappa^2|P|^2}{g(-|P|)} \leq \rho(x) \leq \frac{C^2 - \kappa^2|P|^2}{g(|P|)}$$

and calculating $g(-|P|)$ and $g(|P|)$ the estimate (2.4) follows.

To prove (2.5), since $\kappa|P - \rho(x)x| = C - \rho$, the first equality follows from the right identity in (2.4). To show the second identity in (2.5), notice that since the oval is convex and symmetric with respect to the line joining 0 and P we have that $\min_{x \in S^{n-1}} |P - \rho(x)x|$ is attained in when $x = P/|P|$. In particular, this gives the explicit value of the distance from P to the oval.

□

Remark 2.2. If $|P| \to \infty$, then the oval converges to an ellipsoid which is the surface having the uniform refraction property in the far field case, see [3]. In fact, if $m = P/|P|$ and $C = \kappa|P| + b$ with b positive constant we have

$$\rho(x) = \frac{C^2 - \kappa^2|P|^2}{C - \kappa^2 x \cdot P + \sqrt{\Delta(x \cdot P)}} =$$

$$= \frac{b(2\kappa|P| + b)}{(\kappa|P| - \kappa^2 x \cdot m|P| + b) + \sqrt{(\kappa|P| - \kappa^2 x \cdot m|P| + b)^2 - (1 - \kappa^2)b(2\kappa|P| + b)}} \to$$

$$\to \frac{2\kappa b}{(\kappa - \kappa^2 x \cdot m) + \sqrt{(\kappa - \kappa^2 x \cdot m)^2}} = \frac{b}{1 - \kappa x \cdot m}$$

as $|P| \to \infty$.

3. Near field refractor map and weak solutions

Let $\Omega \subset S^{n-1}$ be a domain with $|\partial\Omega| = 0$ (measure in the sphere). Let $D \subset \mathbb{R}^n$ be a "target" domain that we want to illuminate and suppose it is contained in an $n-1$ dimensional hyper surface, and assume $\overline{D}$ is compact, and $0 \notin \overline{D}$. Points in the sphere will be denoted with lower case letters and points in $\mathbb{R}^n$ by capitals.

We make the following assumptions on Ω and D:

(H1) There exists τ with $0 < \tau < 1 - \kappa$ such that $x \cdot P \geq (\kappa + \tau)|P|$ for all $x \in \overline{\Omega}$ and all $P \in \overline{D}$.

(H2) Let $0 < r_0 \leq (\tau/(1 + \kappa))\, \mathrm{dist}\,(0, \overline{D})$ and consider the cone in $\mathbb{R}^n$

$$Q_{r_0} = \{tx \;:\; x \in \overline{\Omega},\ 0 \leq t \leq r_0\}\,.$$

For each $m \in S^{n-1}$ and for each $X \in Q_{r_0}$ we assume that $\overline{D} \cap \{X + tm \;:\; t \geq 0\}$ contains at most one point. That is, for each $X \in Q_{r_0}$ each ray emanating from X intersects $\overline{D}$ at most in one point.

Given $P \in \mathbb{R}^n$ and $\kappa|P| < b < |P|$, a refracting oval is the set

$$O_b(P) = \{h(x, P, b)\, x \;:\; x \in S^{n-1},\ x \cdot P \geq b\}$$

where

$$h(x, P, b) = \frac{(b - \kappa^2 x \cdot P) - \sqrt{(b - \kappa^2 x \cdot P)^2 - (1 - \kappa^2)(b^2 - \kappa^2|P|^2)}}{1 - \kappa^2}\,.$$

Definition 3.1. Let $\mathcal{S} = \{x\rho(x) \ : \ x \in \overline{\Omega}\} \subset Q_{r_0}$ be a surface. We say that $\mathcal{S}$ is a *near field refractor* if for any point $y\rho(y) \in \mathcal{S}$ there exist $P \in \overline{D}$ and $b > 0$ such that the refracting oval $O_b(P)$ supports $\mathcal{S}$ at $y\rho(y)$, i.e. $\rho(x) \leq h(x, P, b)$ for all $x \in \overline{\Omega}$ with equality at $x = y$.

The near field refractor map of $\mathcal{S}$ is defined by

$$\mathcal{R}_{\mathcal{S}}(x) = \{P \in \overline{D} \ : \ \text{there exists a supporting oval } O_b(P)to\mathcal{S} \text{ at } \rho(x)x\} \ .$$

Given $P \in \overline{D}$, the near field tracing mapping of $\mathcal{S}$ is defined by

$$\mathcal{T}_{\mathcal{S}}(P) = \mathcal{R}_{\mathcal{S}}^{-1}(P) = \{x \in \overline{\Omega} \ : \ P \in \mathcal{R}_{\mathcal{S}}(x)\} \ .$$

Remark 3.2. Since $\mathcal{S}$ is a near field refractor, then $\mathcal{R}_{\mathcal{S}}(\overline{\Omega}) = \overline{D}$. Indeed, let $P \in D$ and $b_0 = (\kappa + \tau)|P|$. Then from the left identity in (2.4) and the assumption on r_0 in (H2) we get that

$$r_0 \leq \frac{b_0 - \kappa|P|}{1+\kappa} \leq h(x, P, b_0) \quad \text{for } x \in \overline{\Omega} \ .$$

Also from (H1), we have $x \cdot P \geq b_0$. Hence $\mathcal{S} \subset Q_{r_0} \subset O(P, b_0)$. Let

$$b_1 = \inf\{b \ : \ \rho(x) \leq h(x, P, b), \ x \cdot P \geq b \quad \forall x \in \overline{\Omega}\} \ .$$

Thus, the oval $O_{b_1}(P)$ supports $\mathcal{S}$ for some $y \in \overline{\Omega}$.

Remark 3.3. If $\mathcal{S}$ is a near field refractor defined with $\rho(x)$ satisfying $C_1 \leq \rho(x) \leq C_2$ for all $x \in \overline{\Omega}$ for some positive constants C_1, C_2, then ρ is Lipschitz in $\overline{\Omega}$ with a Lipschitz constant depending only on C_1 and C_2. Indeed, given $x_0 \in \overline{\Omega}$, $\mathcal{S}$ has a supporting oval $h(x, P, b)$ at $\rho(x_0)x_0$ with $P \in \overline{D}$. Then

$$\rho(x) - \rho(x_0) \leq h(x, P, b) - h(x_0, P, b) =$$

$$= \frac{1}{1-\kappa^2}\left((b - \kappa^2 x \cdot P) - (b - \kappa^2 x_0 \cdot P) + \sqrt{\Delta(x_0 \cdot P)} - \sqrt{\Delta(x \cdot P)}\right) =$$

$$= \frac{1}{1-\kappa^2}\left(I + II\right) \ .$$

We have $|I| \leq C|x - x_0|$ and

$$II = \frac{\Delta(x_0 \cdot P) - \Delta(x \cdot P)}{\sqrt{\Delta(x_0 \cdot P)} + \sqrt{\Delta(x \cdot P)}} \ .$$

We have from (2.3) and (2.4) that $\sqrt{\Delta(x_0 \cdot P)} \geq \kappa|b - x_0 \cdot P| \geq \kappa(1-\kappa)\, h(x_0, P, b) \geq \kappa(1-\kappa)C_1$, and $\sqrt{\Delta(x \cdot P)} \geq \kappa|b - x \cdot P| \geq \kappa(1-\kappa)\, h(x, P, b) \geq \kappa(1-\kappa)C_1$. Also from (2.4), $b \leq (1+\kappa)C_2 + \kappa|P|$ and so $|\Delta(x_0 \cdot P) - \Delta(x \cdot P)| \leq C\,|x - x_0|$. This completes the proof of the remark.

Lemma 3.4. *The class* $\mathcal{C} = \{F \subset \overline{D} \ : \ \mathcal{T}_{\mathcal{S}}(F)$ *is Lebesgue measurable*$\}$ *is a* σ*-algebra containing all Borel sets in* $\overline{D}$.

Proof. Obviously, $\mathcal{T}_{\mathcal{S}}(\emptyset) = \emptyset$ and $\mathcal{T}_{\mathcal{S}}(\overline{D}) = \overline{\Omega}$. Since $\mathcal{T}_{\mathcal{S}}(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \mathcal{T}_{\mathcal{S}}(F_i)$, $\mathcal{C}$ is closed under countable unions. Clearly for $F \subset \overline{D}$

$$\mathcal{T}_{\mathcal{S}}(F^c) = \{x \in \overline{\Omega} \ : \ \mathcal{R}_{\mathcal{S}}(x) \cap F^c \neq \emptyset\} =$$

$$= \{x \in \overline{\Omega} \ : \ \mathcal{R}_{\mathcal{S}}(x) \cap F = \emptyset\} \cup \{x \in \overline{\Omega} \ : \ \mathcal{R}_{\mathcal{S}}(x) \cap F^c \neq \emptyset \ , \ \mathcal{R}_{\mathcal{S}}(x) \cap F \neq \emptyset\} =$$

$$= [\mathcal{T}_{\mathcal{S}}(F)]^c \cup [\mathcal{T}_{\mathcal{S}}(F^c) \cap \mathcal{T}_{\mathcal{S}}(F)] \ . \tag{3.1}$$

If $x \in \mathcal{T}_{\mathcal{S}}(F^c) \cap \mathcal{T}_{\mathcal{S}}(F) \cap \Omega$, then $\mathcal{S}$ parameterized by ρ has two distinct supporting ovals $O_{c_1}(P_1)$ and $O_{c_2}(P_2)$ at $\rho(x)x$ with $P_1 \neq P_2$. We claim that $\rho(x)x$ is a singular point of $\mathcal{S}$. Otherwise, if $\mathcal{S}$ has tangent hyperplane Π at $\rho(x)x$, then Π must coincide both with the tangent hyperplane of $O_{c_1}(P_1)$ and that of $O_{c_2}(P_2)$ at $\rho(x)x$. From the Snell law we get that

$$\frac{P_1 - \rho(x)x}{|P_1 - \rho(x)x|} = \frac{P_2 - \rho(x)x}{|P_2 - \rho(x)x|} := m \ ,$$

and so the ray through $X = \rho(x)x$ with direction m contains P_1, P_2 and therefore $P_1 = P_2$ from assumption (H2), a contradiction. Since the graph of $\mathcal{S}$ is Lipschitz, the set of singular points of $\mathcal{S}$ has measure zero and therefore the surface area measure of $\mathcal{T}_{\mathcal{S}}(F^c) \cap T_{\mathcal{S}}(F)$ is zero. So $\mathcal{C}$ is closed under complements, and therefore a σ-algebra.

To prove that $\mathcal{C}$ contains all Borel subsets, it suffices to show that $\mathcal{T}_{\mathcal{S}}(K)$ is compact for $K \subset \overline{D}$ compact. Let $x_i \in \mathcal{T}_{\mathcal{S}}(K)$ for $i \geq 1$, and $P_i \in \mathcal{R}_{\mathcal{S}}(x_i) \cap K$. Let $O_{C_i}(P_i)$ be a supporting oval to $\mathcal{S}$ at $\rho(x_i)x_i$. Then

$$\rho(x) + \kappa|P_i - \rho(x)x| \leq C_i \qquad \text{for } x \in \overline{\Omega} \ , \tag{3.2}$$

with equality at $x = x_i$ and $x \cdot P_i \geq C_i$ for all $x \in \overline{\Omega}$. Assume that $a_1 \leq \rho(x) \leq a_2$ on $\overline{\Omega}$ for some constants $a_2 \geq a_1 > 0$ (from (H2) one can take $a_2 = r_0$). From (2.4) we get $a_1(1-\kappa) + \kappa|P_i| \leq C_i \leq \kappa|P_i| + a_2(1+\kappa)$. Therefore selecting a subsequence we can assume that $x_i \longrightarrow x_0$, $P_i \longrightarrow P_0 \in K$, and $C_i \longrightarrow C_0$, as $i \longrightarrow \infty$. By taking limit in (3.2), one obtains that the oval $\mathcal{O}(P_0, C_0)$ supports $\mathcal{S}$ at $\rho(x_0)x_0$, $x \cdot P_0 \geq C_0$, and $x_0 \in \mathcal{T}_{\mathcal{S}}(P_0)$. This proves $\mathcal{T}_{\mathcal{S}}(K)$ is compact.

□

Lemma 3.5. *Given a nonnegative $f \in L^1(\overline{\Omega})$, the set function*

$$\mathcal{M}_{\mathcal{S},f}(F) = \int_{\mathcal{T}_{\mathcal{S}}(F)} f \, dx$$

is a finite Borel measure defined on $\mathcal{C}$ and is called the near field refractor measure associated with f and the surface $\mathcal{S}$.

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{C}$. Let $H_1 = \mathcal{T}_{\mathcal{S}}(F_1)$, and $H_k = \mathcal{T}_{\mathcal{S}}(F_k) \setminus \cup_{i=1}^{k-1} \mathcal{T}_{\mathcal{S}}(F_i)$, for $k \geq 2$. Since $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\cup_{k=1}^{\infty} H_k = \cup_{k=1}^{\infty} \mathcal{T}_{\mathcal{S}}(F_k)$, it is easy to get

$$\mathcal{M}_{\mathcal{S},f}(\cup_{k=1}^{\infty} F_k) = \int_{\cup_{k=1}^{\infty} H_k} f \, dx = \sum_{k=1}^{\infty} \int_{H_k} f \, dx \ .$$

Observe that $\mathcal{T}_{\mathcal{S}}(F_k) \setminus H_k = \mathcal{T}_{\mathcal{S}}(F_k) \cap (\cup_{i=1}^{k-1} \mathcal{T}_{\mathcal{R}}(F_i))$ is a subset of the singular set of $\mathcal{S}$ and has surface area measure zero for $k \geq 2$. Therefore, $\int_{H_k} f \, dx = \mathcal{M}_{\mathcal{S},f}(F_k)$ and the σ-additivity of $\mathcal{M}_{\mathcal{S},f}$ follows.

□

Lemma 3.6. *Let $\mathcal{S}_j = \{\rho_j(x) \ : \ x \in \overline{\Omega}\}$ be a sequence of near-field refractors. Suppose that $a \leq \rho_j(x) \leq r_0$ for all $x \in \overline{\Omega}$ and $\rho_j \to \rho$ uniformly in $\overline{\Omega}$. Then $\mathcal{S} = \{\rho(x)x \ : \ x \in \overline{\Omega}\}$ is a near-field refractor and moreover*

1. $\limsup_{j\to\infty} \mathcal{T}_{\mathcal{S}_j}(K) \subset \mathcal{T}_{\mathcal{S}}(K)$, *for each $K \subset \overline{D}$;*
2. $\mathcal{T}_{\mathcal{S}}(G) \setminus E_0 \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{S}_j}(G)$, *for each $G \subset \overline{D}$ open, where E_0 is the singular set of $\mathcal{S}$.*

Proof. To show the first part, let $x_0 \in \mathcal{T}_{\mathcal{S}_{k_j}}(K)$ for $j = 1, \cdots$. Then there exists $p_{k_j} \in K$ such that $\rho_{k_j}(x) \leq h(x, p_{k_j}, b_{k_j})$ for all $x \in \overline{\Omega}$ with equality at $x = x_0$ and with $x \cdot p_{k_j} \geq b_{k_j}$. Since $\kappa|p_{k_j}| + a(1-\kappa) \leq b_{k_j} \leq (\kappa + \tau)|p_{k_j}|$, passing through a subsequence we get (1).

For the second part, we have from (3.1) and the first part that

$$\begin{aligned}\limsup_{j\to\infty} \left(\mathcal{T}_{\mathcal{S}_j}(G)\right)^c &\subset \limsup_{j\to\infty} \left(\left(\mathcal{T}_{\mathcal{S}_j}(G)\right)^c \cup \left(\mathcal{T}_{\mathcal{S}_j}(G) \cap \mathcal{T}_{\mathcal{S}_j}(G^c)\right)\right) = \\ &= \limsup_{j\to\infty} \mathcal{T}_{\mathcal{S}_j}(G^c) \subset \\ &\subset \mathcal{T}_{\mathcal{S}}(G^c) = \left(\mathcal{T}_{\mathcal{S}}(G)\right)^c \cup \left(\mathcal{T}_{\mathcal{S}}(G) \cap \mathcal{T}_{\mathcal{S}}(G^c)\right).\end{aligned}$$

Since $\mathcal{T}_{\mathcal{S}}(G) \cap \mathcal{T}_{\mathcal{S}}(G^c) \subset E_0$, part (2) follows taking complements.

□

The notion of weak solution is introduced through the conservation of energy.

Definition 3.7. A near field refractor $\mathcal{S}$ is a weak solution of the near field refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ on $\overline{\Omega}$ and prescribed refracted illumination intensity μ on $\overline{D}$ if for any Borel set $F \subset \overline{D}$

$$\mathcal{M}_{\mathcal{S},f}(F) = \int_{\mathcal{T}_{\mathcal{S}}(F)} f\,dx = \mu(F). \tag{3.3}$$

4. Existence of solutions

4.1. **Existence for sum of Dirac measures.**

Theorem 4.1. *Suppose (H1) and (H2) hold. Let $P_1, \cdots, P_N$ be distinct points in $\overline{D}$, $g_1, \cdots, g_N$ are positive numbers, and $f \in L^1(\Omega)$ such that*

$$\int_{\Omega} f(x)\,dx = \sum_{i=1}^{N} g_i. \tag{4.1}$$

Then, for each b_1 with $\kappa|P_1| < b_1 \leq \kappa|P_1| + r_0(1-\kappa)$, there exist positive numbers $b_2, \cdots, b_N$ such that the poly-oval $\mathcal{S} = \{\rho(x)x : x \in \overline{\Omega}\}$ with

$$\rho(x) = \min_{1\leq i\leq N} h(x, P_i, b_i) \tag{4.2}$$

is a weak solution to the near field refractor problem. Moreover, $\mathcal{M}_{\mathcal{S},f}(\{P_i\}) = g_i$ for $1 \leq i \leq N$.

Proof. For each $b = (b_1, \cdots, b_N)$ with $b_i > 0$, let $\mathcal{S}_b$ be the poly-oval with ρ defined by (4.2). Notice that if $b_1 \in (\kappa|P_1|, \kappa|P_1| + r_0(1-\kappa)]$, then from the second identity in (2.4) we get $\rho(x) \leq r_0$ and so $\mathcal{S}_b \subset Q_{r_0}$. Also, if $b_k \leq (\kappa+\tau)|P_k|$, then from (H1) $\overline{\Omega} \subset \{x \in S^{n-1} : x \cdot P_k \geq b_k\}$.

Let

$$\mathcal{W} = \{(b_2, \cdots, b_N) : \kappa|P_i| < b_i \leq (\kappa+\tau)|P_i|;\ \mathcal{M}_{\mathcal{S}_b,f}(\{P_i\}) \leq g_i, \text{ for } 2 \leq i \leq N\}.$$

From (4.1), $\mathcal{M}_{\mathcal{S}_b,f}(\{P_1\}) \geq g_1 > 0$.

We first claim that $\mathcal{W} \neq \emptyset$. Let $b_i = (\kappa+\tau)|P_i|$ for $2 \leq i \leq N$. Then $S_b = \mathcal{O}(P_1, b_1)$ because

$$h(x, P_1, b_1) \leq \frac{b_1 - \kappa|P_1|}{1-\kappa} \leq r_0 \leq \frac{b_i - \kappa|P_i|}{1+\kappa} \leq h(x, P_i, b_i)$$

for $2 \leq i \leq N$ from (H2) and (2.4). Hence $\mathcal{M}_{\mathcal{S}_b,f}(\{P_i\}) = 0$ for $i \neq 1$.

For each $b' = (b_2, \cdots, b_N) \in \mathcal{W}$,

$$b_i \geq \kappa|P_i| + \frac{1-\kappa}{1+\kappa}\left(b_1 - \kappa|P_1|\right). \tag{4.3}$$

Because there exists $x_0 \in \Omega$ such that $\mathcal{O}(P_1, b_1)$ supports $\mathcal{S}_b$ at $\rho(x_0)x_0$. Hence from (2.4)

$$\frac{b_1 - \kappa|P_1|}{1+\kappa} \leq \rho(x_0) \leq h(x_0, P_i, b_i) \leq \frac{b_i - \kappa|P_i|}{1-\kappa}$$

for $i \neq 1$, and then (4.3) follows.

The set $\mathcal{W}$ is compact. By the definition of $\mathcal{W}$ it is enough to show that $\mathcal{M}_{\mathcal{S}_{(b_1,b')},f}(\{P_i\})$ is continuous as a function of b' for each $i \neq 1$. Let $b'_m = (b_2^m, \cdots, b_N^m) \in \mathcal{W}$ with $b'_m \to b'_0 = (b_2^0, \cdots, b_N^0)$ as $m \to \infty$. From (4.3) and (2.4),

$$\frac{1-\kappa}{(1+\kappa)^2}\left(b_1 - \kappa|P_1|\right) \leq \rho_m(x) := \min_{1 \leq i \leq N} h(x, P_i, b_i^m) \leq r_0$$

with $b_1^m = b_1$, and therefore $\rho_m(x) \to \rho_0(x) := \min_{1\leq i\leq N} h(x, P_i, b_i^0)$ uniformly in $\overline{\Omega}$ with $b_1^0 = b_1$. From Lemma 3.6 we then get (we set $b_m = (b_1, b'_m)$, and $\mathcal{S}_m = \mathcal{S}_{b_m}$)

$$\limsup_{m\to\infty} \mathcal{M}_{\mathcal{S}_m,f}(\{P_i\}) \leq \int_{\limsup_{m\to\infty} \mathcal{T}_{S_m}(P_i)} f(x)\,dx \leq \int_{\mathcal{T}_{S_0}(P_i)} f(x)\,dx\,,$$

and for each $G \subset \overline{D}$ open

$$\liminf_{m\to\infty} \mathcal{M}_{S_m,f}(G) \geq \int_{\liminf_{m\to\infty} \mathcal{T}_{S_m}(G)} f(x)\,dx \geq \int_{\mathcal{T}_{S_0}(G)} f(x)\,dx\,.$$

If for each i we choose G small open neighborhood of P_i such that $P_j \notin G$ for $j \neq i$, then we get that $\mathcal{M}_{\mathcal{S}_m,f}(G)$ is equal to $\mathcal{M}_{\mathcal{S}_m,f}(\{P_i\})$ because $\mathcal{T}_{S_m}(G \setminus \{P_i\})$ is contained in the singular set of S_m and we are done.

We prove now prove the existence of weak solutions. The function $w(b_2, \cdots, b_N) = b_2 + \cdots + b_N$ attains its minimum on $\mathcal{W}$ at some point $(a_2, \cdots, a_N)$. We claim that S_a with $a = (b_1, a_2, \cdots, a_N)$ is a weak solution of the near field refractor problem. Assume by contradiction that this is not true, then say $\mathcal{M}_{\mathcal{S}_a,f}(\{P_2\}) < g_2$. Let $\bar{a} = (b_1, a_2 - \varepsilon, a_3, \cdots, a_N)$ and the corresponding surface $\mathcal{S}_{\bar{a}}$. From the continuity $\mathcal{M}_{\mathcal{S}_{\bar{a}},f}(\{P_2\}) < g_2$ for all ε sufficiently small and since $\mathcal{T}_{S_{\bar{a}}}(P_i) \subset \mathcal{T}_{S_a}(P_i)$ a.e. for $i \neq 1, 2$ we get that $\bar{a} \in \mathcal{W}$, a contradiction.

□

Lemma 4.2. *Let $\mathcal{S}_b, \mathcal{S}_{b^*}$ be two solutions obtained in Theorem 4.1, with $b = (b_1, \cdots, b_N)$, and $b^* = (b_1^*, \cdots, b_N^*)$ satisfying $\kappa|P_1| < b_1, b_1^* \leq \kappa|P_1| + r_0(1-\kappa)$. Assume that $f > 0$ a.e. in Ω.*

If $b_1^* \leq b_1$, then $b_i^* \leq b_i$ for all $1 \leq i \leq N$. In particular, if $b_1^* = b_1$, then $b_i^* = b_i$ for all $1 \leq i \leq N$.

Proof. Let $J = \{j \,:\, b_j < b_j^*\}$ and $I = \{i \,:\, b_i \geq b_i^*\}$. Suppose by contradiction that $J \neq \emptyset$. We have $I \neq \emptyset$ since $1 \in I$. For each $j \in J$ we have $h(z, P_j, b_j) \leq h(z, P_j, b_j^*)$ for all $z \in \Omega$ since $b_j < b_j^*$. And also $h(z, P_i, b_i^*) \leq h(z, P_i, b_i)$ for all $i \in I$ and all $z \in \Omega$. Fix $j \in J$ and let $x \in \mathcal{T}_{\mathcal{S}_{b^*}}(P_j)$. We then have $P_j \in \mathcal{R}_{\mathcal{S}_{b^*}}(x)$. So there exists a supporting oval $h(z, P_j, b')$ to $\mathcal{S}_{b^*}$ at the point x. Since the function defining the poly-oval $\mathcal{S}_{b^*}$ is given by $\rho(z) = \min_i h(z, P_i, b_i^*)$, then $b' = b_j^*$ and

$\rho(x) = h(x, P_j, b_j^*)$, and consequently $h(x, P_j, b_j^*) \leq h(x, P_i, b_i^*)$ for all $1 \leq i \leq N$. Thus, we obtain

$$h(x, P_j, b_j) < h(x, P_j, b_j^*) \leq h(x, P_i, b_i^*) \leq h(x, P_i, b_i) \quad \forall i \in I \ .$$

Hence by continuity, there exists N_x a neighborhood of x such that

$$h(y, P_j, b_j) < h(y, P_i, b_i) \quad \forall i \in I \quad \forall y \in N_x \ .$$

Since the function defining S_b is $\rho(z) = \min_{1 \leq i \leq N} h(z, P_i, b_i)$, we get for $y \in N_x$ that $\rho(y) = \min_{j \in J} h(y, P_j, b_j)$, that is, $\rho(y) = h(y, P_j, b_j)$ for some $j \in J$ which means that $h(z, P_j, b_j)$ is a supporting oval to S_b at y. Therefore

$$N_x \subset \mathcal{T}_{\mathcal{S}_b}\left(\cup_{j \in J} P_j\right) \ .$$

We then have that every point $x \in \mathcal{T}_{\mathcal{S}_{b^*}}\left(\cup_{j \in J} P_j\right)$ has a neighborhood contained in $\mathcal{T}_{\mathcal{S}_b}\left(\cup_{j \in J} P_j\right)$, that is,

$$\mathcal{T}_{\mathcal{S}_{b^*}}\left(\cup_{j \in J} P_j\right) \subset \left(\mathcal{T}_{\mathcal{S}_b}\left(\cup_{j \in J} P_j\right)\right)^{\circ} \neq \overline{\Omega} \ .$$

Since $\overline{\Omega}$ is connected and $\mathcal{T}_{\mathcal{S}_{b^*}}\left(\cup_{j \in J} P_j\right)$ is closed, we get that $\left(\mathcal{T}_{\mathcal{S}_b}\left(\cup_{j \in J} P_j\right)\right)^{\circ} \setminus \mathcal{T}_{\mathcal{S}_{b^*}}\left(\cup_{j \in J} P_j\right)$ is a non empty open set. This is a contradiction with the fact that

$$\int_{\mathcal{T}_{\mathcal{S}_b}\left(\cup_{j \in J} P_j\right)} f(x)\, dx = \sum_{j \in J} g_j = \int_{\mathcal{T}_{\mathcal{S}_{b^*}}\left(\cup_{j \in J} P_j\right)} f(x)\, dx \ ,$$

since $f > 0$ a.e.

□

4.2. **Existence in the general case.**

Theorem 4.3. *Assume conditions (H1) and (H2). Let μ be a Radon measure on $\overline{D}$, $f \in L^1(\Omega)$ with $f > 0$ a.e., and satisfying the energy conservation condition*

$$\int_{\Omega} f(x)\, dx = \mu(\overline{D}) \ .$$

Then given $X_0 \in Q_{r_0}$ with

$$0 < |X_0| \leq \left(\frac{1-\kappa}{1+\kappa}\right)^2 r_0 \ ,$$

there exists a weak solution of the near field refractor problem passing through X_0.

Proof. We assume first that $\mu = \sum_{i=1}^{N} g_i \delta_{P_i}$, with $g_i > 0$ and P_i pairwise distinct points in D. From Theorem 4.1 and Lemma 4.2, given $b_1 \in (\kappa|P_1|, \kappa|P_1| + (1-\kappa) r_0]$ there exists a unique $(b_2, \cdots, b_N)$ such that $\mathcal{S}_{(b_1, \cdots, b_N)}$ is a weak solution to the near field refractor problem defined by the radial function $\rho(x, b_1) = \min_i h(x, P_i, b_i)$. From the proof of Theorem 4.1 and Lemma 4.2, the function $\rho(x, b_1)$ is continuous for $(x, b_1) \in \overline{\Omega} \times (\kappa|P_1|, \kappa|P_1| + (1-\kappa) r_0]$. We shall first prove that

$$\rho(x, \kappa|P_1| + (1-\kappa) r_0) \geq \left(\frac{1-\kappa}{1+\kappa}\right)^2 r_0 \ , \quad \forall x \in \overline{\Omega} \ . \tag{4.4}$$

We have $\rho(x, \kappa|P_1| + (1-\kappa) r_0) = \min_{1 \leq i \leq N} h(x, P_i, b_i)$ with $b_1 = \kappa|P_1| + (1-\kappa) r_0$ and some $b_2, \cdots, b_N$. From (2.4)

$$\frac{b_i - \kappa|P_i|}{1-\kappa} \geq h(x, P_i, b_i) \geq \rho(x, \kappa|P_1| + (1-\kappa) r_0) \ , \quad i = 1, \cdots, N \ ; \quad \forall x \in \overline{\Omega} \ . \tag{4.5}$$

Also there exists $x_1 \in \overline{\Omega}$ such that $\rho(x_1, \kappa|P_1| + (1-\kappa)r_0) = h(x_1, P_1, \kappa|P_1| + (1-\kappa)r_0)$ and then again by (2.4),

$$\rho\left(x_1, \kappa|P_1| + (1-\kappa)r_0\right) \geq \frac{1-\kappa}{1+\kappa}\, r_0 \,.$$

Hence from (4.5) we get

$$\frac{b_i - \kappa|P_i|}{1-\kappa} \geq \frac{1-\kappa}{1+\kappa}\, r_0 \,, \quad i = 1, \cdots, N \,. \tag{4.6}$$

Once again by (2.4),

$$h(x, P_i, b_i) \geq \frac{b_i - \kappa|P_i|}{1+\kappa} \,,$$

which combined with (4.6) yields

$$h(x, P_i, b_i) \geq \left(\frac{1-\kappa}{1+\kappa}\right)^2 r_0 \,, \quad i = 1, \cdots, N \,,$$

and hence (4.4) follows. On the other hand,

$$\rho(x, b_1) \leq h(x, P_1, b_1) \leq \frac{b_1 - \kappa|P_1|}{1-\kappa}$$

by (2.4) for all $(x, b_1) \in \overline{\Omega} \times (\kappa|P_1|, \kappa|P_1| + (1-\kappa)r_0]$ and hence given $\delta > 0$, we get $\rho(x, b_1) < \delta$ for all $x \in \overline{\Omega}$ as long as b_1 is sufficiently close to $\kappa|P_1|$. Suppose now that $X_0 \in Q_{r_0}$ with

$$0 < |X_0| \leq \left(\frac{1-\kappa}{1+\kappa}\right)^2 r_0$$

and with $x_0 = X_0/|X_0| \in \overline{\Omega}$. Hence from (4.4) and the continuity of $\rho(x_0, \cdot)$, we obtain that there exists $b_1 \in (\kappa|P_1|, \kappa|P_1| + (1-\kappa)r_0]$ such that $\rho(x_0, b_1) = |X_0|$.

For the general case of a Radon measure μ in D, we choose a sequence of measures μ_ℓ such that each one is a finite combination of Dirac measures and $\mu_\ell \to \mu$ weakly with $\mu_\ell(\overline{D}) = \mu(\overline{D})$. From the first step, let $\mathcal{S}_\ell$ be the near field refractor corresponding to the measure μ_ℓ and parameterized by $\rho_\ell(x)x$ and passing through the point X_0. We claim that

$$\frac{1-\kappa}{1+\kappa}\, |X_0| \leq \rho_\ell(x) \leq \frac{1+\kappa}{1-\kappa}\, |X_0| \,, \quad \forall \ell \,, \quad \forall x \in \overline{\Omega} \,. \tag{4.7}$$

Indeed, $x_0 = X_0/|X_0| \in \overline{\Omega}$, $\rho_\ell(x_0) = |X_0| = h(x_0, P_0, b_0)$ for some $P_0 \in \overline{D}$ and b_0. Hence from (2.4),

$$\frac{b_0 - \kappa|P_0|}{1+\kappa} \leq |X_0| \,.$$

So for $x \in \overline{\Omega}$,

$$\rho_\ell(x) \leq h(x, P_0, b_0) \leq \frac{b_0 - \kappa|P_0|}{1-\kappa} \leq \frac{1+\kappa}{1-\kappa}\, |X_0| \,.$$

On the other hand, for each $z \in \overline{\Omega}$ there exists P_1 and b_1 such that $\rho_\ell(z) = h(z, P_1, b_1)$ and from (2.4),

$$h(z, P_1, b_1) \geq \frac{b_1 - \kappa|P_1|}{1+\kappa} \geq \frac{1-\kappa}{1+\kappa}\, h(x_0, P_1, b_1) \geq \frac{1-\kappa}{1+\kappa}\, \rho_\ell(x_0) \geq \frac{1-\kappa}{1+\kappa}\, |X_0| \,.$$

Thus, (4.7) is proved.

From (4.7) and Remark 3.3, ρ_ℓ is uniformly Lipschitz in $\overline{\Omega}$ and therefore there exists a subsequence, denoted also ρ_ℓ, that converges uniformly in $\overline{\Omega}$ to some function ρ. From Lemma 3.6 the surface $\mathcal{S}$ defined by ρ is a near field refractor, and $\limsup_{\ell\to\infty} \mathcal{M}_{\mathcal{S}_\ell,f}(K) \leq \mathcal{M}_{\mathcal{S},f}(K)$ for each K compact, and $\liminf_{\ell\to\infty} \mathcal{M}_{\mathcal{S}_\ell,f}(G) \geq \mathcal{M}_{\mathcal{S},f}(G)$ for each G open. That is, $\mathcal{M}_{\mathcal{S}_\ell,f}$ converges weakly to $\mathcal{M}_{\mathcal{S},f}$ and since $\mathcal{M}_{\mathcal{S}_\ell,f} = \mu_\ell$ we obtain $\mathcal{M}_{\mathcal{S},f} = \mu$. □

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