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Atomic decompositions of holomorphic Hardy spaces in $\mathbb{S}^1$ and applications

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Dedicated to Ermanno Lanconelli on the occasion of his 65^{th} *birthday*

Abstract[1]. We present a proof of the atomic decomposition theorem for boundary values of holomorphic functions in the unit disc of $\mathbb{C}$ whose means over the circle $|z| = r$ are uniformly bounded in L^p, $p \leq 1$. As an application, we show the continuity of the Hilbert transform on the subspace of distributions that are boundary values of such holomorphic functions and describe the linear functionals in H^p.

1. Introduction

The general theory of Hardy spaces H^p originated in the extraordinary discoveries made seventy or eighty years ago by G. Hardy, V. Krylov, J. E. Littlewood, I. I. Privalov, F. and M. Riesz ([10], [13], [16], [15]) to cite just the main contributions. Chapters 7 and 14 of Zygmund's treatise about Trigonometric Series [20], exhibit in an unified way the H^p spaces in the context of holomorphic functions of one complex variable. In the end of the fifties, the development of the real variable methods carried in previous years by the Calderón-Zygmund school em Chicago —that furnished new proofs of classical results without resorting to holomorphic functions theory, among them the L^p continuity of the Hilbert's transform — opened the door to a purely real maximal theory of Hardy spaces in several variables. This was initiated by E. Stein and G. Weiss [19] (departing from ideas of Burkholder, Gundy and Silverstein [1]) with their maximal characterization and was complemented by the duality theorem between H^1 and BMO due to C. Fefferman and E. Stein ([5], [6], [18]) and by the atomic characterization formulated and proved by R. Coifman [2] in one dimension and by R. H. Latter [14] in higher dimensions, although present in an implicit and primitive way in the duality result. Today the H^p spaces and their local versions H^p_{loc} based on the localizable spaces or Goldberg [9] are important functional spaces in which framework it is possible to carry out

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the study of many questions in analysis beyond the threshold $p = 1$ that limits the use of the Lebesgue spaces L^p.

Since atomic decompositions are essentially a real tool, they are usually carried out within the framework of functional spaces that do not involve holomorphic functions and the case of spaces holomorphic functions is obtained as a byproduct of the atomic decompositions of a bigger space that contains them. An exception to this rule is the proof of the atomic decomposition in $H^1(\Delta)$ by Koosis ([12]) where a specific ingredient of holomorphic functions, namely Cauchy's formula, plays a key role. Here we extend this construction for $0 < p < 1$. Although the result is not new, only the case $p = 1$ seems to have been written down. Two applications are given: i) the continuity of Hilbert transform in $h^p(\mathbb{T})$ (see Definition 6.1); and ii) a description of linear bounded functionals in H^p.

This paper is organized as follows: In section 2 we give the basic definitions and results to be used later and we state the atomic decomposition theorem, Theorem 2.2. Section 3 is devoted to study boundary values (in the sense of distributions) in $H^p(\Delta)$, in particular we show that functions in $H^p(\Delta)$ possesses boundary values and that they admit the Poisson representation in the sense of distributions. In Section 4 we prove some preliminaries results. Section 5 is devoted to prove the atomic decomposition theorem. In Section 6 we prove the continuity of Hilbert transform, Corollary 6.2. In Section 7 we describe the set of linear functionals in H^p and show the relevance of atomic decompositions for the case $0 < p \leq 1$ in the proof of Theorem 7.2.

2. Definitions

Let Δ the unit disc in $\mathbb{C}$, i.e.

$$\Delta = \{z \in \mathbb{C} \ : \ |z| < 1\} .$$

The boundary of Δ, $\partial\Delta$ we will denote by $\mathbb{T}$. The Poisson kernel in Δ is given by

$$P_r(\theta) = \frac{1+r}{1+r^2-2r\cos\theta}$$

and the conjugate Poisson kernel is given by

$$Q_r(\theta) = \frac{2r\sin\theta}{1+r^2-2r\cos\theta} .$$

If $f \in L^1(\mathbb{T})$ the integral of Poisson of f is the harmonic function $P(f)$ defined by the convolution

$$P(f)(re^{i\theta}) = (P_r * f)(\theta) = \int_{\mathbb{T}} \frac{1+r}{1+r^2-2r\cos(\theta-t)}\, f(t)\, dt .$$

The set of holomorphic functions in Δ will be denoted by $H(\Delta)$.

Definition 2.1. For $0 < p \leq \infty$, we define the Hardy spaces in the unit disc Δ, $H^p(\Delta)$, by

$$H^p(\Delta) = \{F \in H(\Delta) \ : \ \|F\|_{H^p} \equiv \sup_{0 \leq r < 1} M_p(r,F) < \infty\} ,$$

where

$$M_p^p(r,F) = \frac{1}{2\pi} \int_{\mathbb{T}} |F(re^{i\theta})|^p \, d\theta .$$

Remark 2.1. If $0 < p < q < \infty$, then $H^\infty(\Delta) \subset H^q(\Delta) \subset H^p(\Delta)$.

Remark 2.2. By the Hardy's convexity theorem, see [10], we have that

$$\|F\|_{H^p} = \sup_{0\le r<1} M_p(r,F) = \lim_{r\to 1} M_p(r,F) \ .$$

We will introduce now two additional equivalent definitions of the Hardy spaces $H^p(\Delta)$, $0 < p \le \infty$, by means of two different maximal functions that involve comparable norms (a more detailed discussion could be found in [11]).

The maximal radial function. Given a measurable function in Δ, G, we define the *maximal radial function* $G^{\perp}$, of G, by

$$G^{\perp}(\theta) = \sup_{r<1} \left|G\left(re^{i\theta}\right)\right| \ .$$

The non-tangential maximal function. Given a measurable function in Δ, G, we define the *non-tangential maximal function* G^*, of G, by

$$G^*_c(\theta) = \sup_{re^{i\rho}\in\Gamma^c_\theta} \left|G\left(re^{i\rho}\right)\right| \ , \tag{2.1}$$

where Γ^c_θ is Privalov's ice-cream cone given by (see [12], page 59 for an illustration):

$$\Gamma^c_\theta = \Big\{z \ : \ |\arg(e^{i\theta}-z)| < c\pi \text{ and } 1/\sqrt{2} < |z| < 1\Big\} \cup B(0,1/\sqrt{2}) \ ,$$

for any $0 < c \le 1/4$. We will fix c and write only G^* for G^*_c and Γ_θ for Γ^c_θ. Observe that the function G^* is lower semi-continuous and non negative.

Theorem 2.1. *There exist constants A_p, B_p, depending only on p such that for each $F \in H^p(\Delta)$ we have $F^{\perp}(\theta)$, $F^*(\theta) \in L^p(\mathbb{T})$ and*

$$\|F^*\|_{L^p} \le A_p\|F^{\perp}\|_{L^p} \le B_p\|F\|_{H^p} \ .$$

It is not evident that the norm in L^p of F^*, $0 < p \le \infty$ given by maximal functions corresponding to different apertures are equivalent. However, if $0 < c_1, c_2 \le 1/4$, there are positive constants C and D, depending on c_1, c_2 and p, such that

$$C\|F^*_{c_1}\|_{L^p} \le \|F^*_{c_2}\|_{L^p} \le D\|F^*_{c_1}\|_{L^p} \ .$$

Clearly, $F^*_{c_1}(\theta) \le F^*_{c_2}(\theta)$ for every θ if $c_1 \le c_2$. One can adapt Stein's argument [17, p. 62] to prove

Proposition 2.1. *Let G a measurable function in Δ and $0 < c_2 < c_1 \le 1/4$, then there is a positive constant $K = K(c_1, c_2)$ such that*

$$\int_0^{2\pi} G^*_{c_1}(\theta)\,d\theta \le K\int_0^{2\pi} G^*_{c_2}(\theta)\,d\theta \ .$$

Applying Proposition 2.1 to F^p, we have the equivalence of the quantities $\|F^*_{c_1}\|_{L^p}$ and $\|F^*_{c_2}\|_{L^p}$ to any $0 < c_1, c_2 \le 1/4$ and $0 < p \le \infty$. Thus we will fix, from now on, $c = 1/4$.

If $X \subset \mathbb{T}$ is a measurable set, we will denote by $|X|$ its normalized Lebesgue measure, for instance, if X is the arc between $-\pi < a < b < \pi$, then $|X| = (b-a)/2\pi$.

Definition 2.2. Given a measurable function $a(\theta)$ defined in $\mathbb{T}$, we will say that $a(\theta)$ is a p-atom if:

(i) $\operatorname{supp}(a) \subset J$, where J is a arc in $\mathbb{T}$ (that could be all of $\mathbb{T}$);

(ii) $|a(\theta)| \leq |J|^{-1/p}$;

(iii) $\displaystyle\int_0^{2\pi} a(\theta)\theta^k \, d\theta = 0$, for $k \leq 1/p - 1$.

We still say that $a(\theta)$ is a p-atom if $a(\theta) \equiv c$, with constant $|c| \leq 1$. We will refer to the arc J as the carrier of a.

We are now ready to state the atomic decomposition theorem:

Theorem 2.2. *There exists a universal constant* $C > 0$ *such that for any* $F \in H^p(\Delta)$ *with boundary value, in the sense of distributions* $bF \doteq f_b$, *there exist*

(a) *a sequence* (a_j) *of* p*-atoms and*

(b) *a sequence* (λ_j) *of complex numbers satisfying*

$$\sum_j |\lambda_j|^p \leq C \int |F^*(\theta)|^p \, d\theta \ ,$$

so that $f_b = \sum_j \lambda_j a_j$ *in the topology of* $\mathcal{D}'(\mathbb{T})$.

Remark 2.3. The convergence of the series f_b follows from Theorem 3.1 and Corollary 3.1 below.

3. Boundary Values in H^p

According with Fatou's theorem (see [7], [8]), every function F in H^p possesses a non-tangential limit almost everywhere. So it defines a function in $\mathbb{T}$:

$$f(\theta) \doteq \lim_{z \xrightarrow{N.T.} e^{i\theta}} F(z) \ .$$

where the notation $z \xrightarrow{N.T.} e^{i\theta}$ means that $z = re^{it}$ converge to $e^{i\theta}$ in the region $\{re^{it} \ : \ |\theta - t| < c(1-r)\}$, for any $c > 0$.

One can show that (see [8], Theorems 3.1 and 3.6 for a proof) if $F \in H^p$ then:

(1) $f \in L^p(\mathbb{T})$;

(2) $\|F\|_{H^p} = \|f\|_{L^p}$;

(3) if $p \geq 1$, then we can recover F from f, that is, $F = P(f)$.

However, we can not expect to recover the function F from f when $p < 1$. For instance, let

$$F(re^{i\theta}) = P_r(\theta) + iQ_r(\theta) = \frac{1+z}{1-z} \ ,$$

then $F \in H^p$ for all $p < 1$, $\lim_{z \xrightarrow{N.T.} e^{i\theta}} F(z) = Q_1(\theta)$ a.e. but $F(re^{i\theta}) \neq P_r(Q_1)(\theta)$. On the other hand, we can recover F if we take the Poisson integral of its boundary value in the sense of distributions.

Theorem 3.1. *Let* $F(z) \in H(\Delta)$. *The following conditions are equivalent:*

(i) *For every* $\phi \in C^\infty(\mathbb{T})$ *there exists the limit*

$$\langle f, \phi \rangle \ \doteq \ \lim_{r \nearrow 1} \int_0^{2\pi} F(re^{i\theta})\phi(\theta) \, d\theta \ .$$

(ii) *There is a distribution* $f \in \mathcal{D}'(\mathbb{T})$ *so that* F *is the Poisson integral of* f

$$F(re^{i\theta}) = \frac{1}{2\pi} \langle f(t), P_r(\theta - t) \rangle \ . \tag{3.1}$$

(*iii*) *There are constants* $C > 0$, $\alpha \geq 0$, *such that*

$$\left|F(re^{i\theta})\right| \leq \frac{C}{(1-r)^{\alpha}}\ , \quad 0 \leq r < 1\ . \tag{3.2}$$

Proof. (i) $\Longrightarrow$ (ii). Consider a sequence $r_k \nearrow 1$ and write $f_k(\theta) = F(r_k e^{i\theta})$. Then, $f_k \in D'(\mathbb{T})$ and $\langle f_k, \phi\rangle \to \langle f, \phi\rangle$ for every $\phi \in C^\infty(\mathbb{T})$. By the uniform boundedness theorem in $C^\infty(\mathbb{T})$, there are $C > 0$, $N > 0$, so that

$$|\langle f_k, \phi\rangle| \leq C \max_{0\leq j\leq N} \|D^j\phi\|_{L^\infty}$$

and the same estimate holds for f in the place of f_k, that is,

$$|\langle f, \phi\rangle| \leq C \max_{0\leq j\leq N} \|D^j\phi\|_{L^\infty} \tag{3.3}$$

proving that f distribution of order N. The Poisson formula to $z \mapsto F(r_k z)$ that is in H^∞ can be written as

$$F(r\,r_k\,e^{i\theta}) = \frac{1}{2\pi}\ \left\langle F(r_k\,e^{it}), P_r(\theta - t)\right\rangle$$

and taking $k \to \infty$ we obtain (3.1).
(ii) $\Longrightarrow$ (iii). If (3.1) holds with $f \in D'(\mathbb{T})$, f satisfies (3.3), for convenient C and N. Applying this estimate to $\phi(t) = P_r(\theta - t)$ and keeping in mind that $\|D_t^j P_r(\theta - t)\|_{L^\infty} \leq C_j(1-r)^{-j-1}$ we obtain (3.2).
(iii) $\Longrightarrow$ (i). Suppose first that $0 \leq \alpha < 1$ in (3.2). Adding a constant to F the same estimate (with another C) still holds and there is no loss of generality in assuming that $F(0) = 0$. Write

$$F(re^{i\theta}) = \int_0^r \frac{\partial}{\partial s}\ F(se^{i\theta})\,ds$$

using the Cauchy-Riemann equations for F and integration by parts we have

$$\begin{aligned}
\left|\left\langle F(re^{it}) - F(r'e^{it}), \phi(t)\right\rangle\right| &= \left|\int_0^{2\pi}\int_{r'}^{r} \frac{\partial F}{\partial s}(se^{i\theta})\,\phi(\theta)\,ds\,d\theta\right| = \\
&= \left|\int_0^{2\pi}\int_{r'}^{r} \frac{i\partial F}{s\partial\theta}(se^{i\theta})\,\phi(\theta)\,ds\,d\theta\right| = \\
&= \left|\int_0^{2\pi}\int_{r'}^{r} F(se^{i\theta})\ \frac{\partial\phi}{s\partial\theta}(\theta)\,ds\,d\theta\right| \leq \\
&\leq C(\phi)\int_{r'}^{r} \frac{1}{(1-s)^{\alpha}}\,ds\ .
\end{aligned}$$

The right hand side converges to 0 when $r, r' \to 1$ since $\alpha < 1$, showing that the limit $\lim_{r\nearrow 1}\int_0^{2\pi} F(re^{i\theta})\phi(\theta)\,d\theta$ exists in this case. If $1 \leq \alpha < 2$, assuming that $F(0) = F'(0) = 0$, we can write

$$F(re^{i\theta}) = \int_0^r ds \int_0^s \frac{\partial^2 F}{\partial\sigma^2}(\sigma e^{i\theta})\,d\sigma$$

and proceed as before. Taking a convenient number of derivatives, we can adapt the argument for any $\alpha > 0$.

□

If a function $F \in H(\Delta)$ satisfies the equivalent properties in Theorem 3.1 we say that F is of tempered growth at the boundary.

Corollary 3.1. *The functions in $H^p(\Delta)$, $0 < p \le \infty$, are of tempered growth at the boundary. In particular, they admit the Poisson representation in the sense of distributions.*

Proof. Let $F \in H^p$, we will prove that F satisfies (3.2). It is enough to prove the case $p < \infty$. Fix $z \in \Delta$ and consider the mean value inequality for the sub-harmonic function $|F|^p$

$$|F(z)|^p \le \frac{1}{2\pi} \int_0^{2\pi} \left|F(z + se^{it})\right|^p \, dt \ , \quad 0 < s < \rho \doteq \frac{1 - |z|}{2} \ .$$

Integrating this inequality with respect to $s\, ds$ we obtain

$$|F(z)|^p \le \frac{1}{\rho^2 \pi} \int\int_{D_\rho(z)} |F(\zeta)|^p \, dA(\zeta) \ ,$$

where $D_\rho(z)$ is the disc centered in z with radius ρ and dA is the area element. We can use polar coordinates $re^{i\theta}$ centered at the origin and include $D_\rho(z)$ in the annulus $|z| - \rho < r < |z| + \rho$. Integrating first in θ, using that $F \in H^p$ and that $2\rho = 1 - |z|$, we get

$$|F(z)|^p \le C \, \frac{\|F\|_{H^p}^p}{1 - |z|}$$

proving (3.2) with $\alpha = 1/p$.

$\square$

Remark 3.4. The space of holomorphic functions with tempered growth is strictly bigger than $\bigcup_{p>0} H^p(\Delta)$, actually for any $\alpha > 0$ there is a holomorphic function on Δ that satisfies (3.2) while its radial limit $\lim_{r\to 1} F(re^{i\theta})$ does not exist for almost every θ, in particular, $F \notin H^p(\Delta)$ for any $p > 0$. The construction of such a function can be found in [3, Ch. 5].

Definition 3.1. The space of functions which are boundary values of functions in $H^p(\Delta)$ in the sense of Fatou's theorem (pointwise radial limits) will be denoted by $\mathcal{H}^p(\mathbb{T})$. The space of distributions which are boundary values of functions in $H^p(\Delta)$ in the weak sense will be denoted by $\mathcal{H}^p_b(\mathbb{T})$.

4. Atomic decompositions in H^p

In this section we shall prove the atomic decomposition theorem. We recall that atoms where defined in Definition 2.2. While the non constant atoms have media zero, the constant atoms satisfying (i) and (ii) but not (iii) unless the constant is zero. Observe that the two first properties in Definition 2.2 imply that $\|a\|_{L^2} \le |J|^{-1/p}$ and that the real and imaginary part of a p-atom still is a p-atom. Conversely, if the real and imaginary part of a function f are p-atoms with the same carrier, then $f/2$ is a p-atom.

Definition 4.1 (Hilbert Transform)**.** Let a be a p-atom or a function in $L^1(\mathbb{T})$. Let $u(re^{i\theta})$ be its Poisson integral and let $v(re^{i\theta})$ be the harmonic conjugate of u uniquely determined by the condition $v(0) = 0$. Then we define the Hilbert transform of a to be the function

$$\tilde{a}(\theta) = \lim_{r\to 1} v(re^{i\theta}) \ .$$

It is well known (see, e.g., [8, Theorem 5.14]) that

$$\tilde{a}(\theta) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\epsilon < |\theta - t| \leq \pi} \frac{1}{2\tan((\theta - t)/2)}\, a(t)\, dt \,.$$

If a is a real p-atom, we see that $\tilde{a} \in L^2(\mathbb{T})$ (this is the trivial case of the Marcel Riesz inequality, for $p = 2$) therefore $\tilde{a} \in L^1(\mathbb{T})$. Then we conclude[2] that $a + i\tilde{a}$ is the boundary value of the holomorphic function $F = u + iv$, where $u = Pa$ and $v = P\tilde{a}$. Since $a + i\tilde{a} \in L^1(\mathbb{T})$ we see that $F \in H^1 \subset H^p$. This shows that $a + i\tilde{a} \in \mathcal{H}^1 \subset \mathcal{H}^p$. The crucial point is that the L^p norm of $a + i\tilde{a}$ is bounded by a constant independent of a for any real p-atom a. This fact is still valid for complex atoms if we extend the Hilbert transform $f \mapsto \tilde{f}$ to complex functions as a complex linear operator. In fact,

Proposition 4.1. *There exist a universal constant $C > 0$ such that if a is a p-atom then $\|a + i\tilde{a}\|_{L^p} \leq C$.*

Proof. Is enough to consider a non constant real p-atom. Follows from (i), (ii) and the definition of p-atom that $\|a\|_{L^p} \leq 1$ so will be enough to show that $\|\tilde{a}\|_{L^p} \leq C$. If J is the arc appearing in (i) and (ii) and $|J| \geq 1/2$, we obtain a estimate to $\|a\|_{L^2}$ that implies in the boundedness of $\|\tilde{a}\|_{L^2}$ moreover, by Holder's inequality, $\|\tilde{a}\|_{L^p}$ is bounded, and we can restrict ourselves to the case where $|J| < 1/2$. In this case, denote by J^* the concentric arc with J and with twice of its length, that is, $|J^*| = 2|J|$. We can assume, without loss of generality, that the center of J is $\theta = 0$ and write $J = (-\delta, \delta)$, $J^* = (-2\delta, 2\delta)$. Therefore,

$$\int_{-\pi}^{\pi} |\tilde{a}(\theta)|^p\, d\theta = \int_{-2\delta}^{2\delta} |\tilde{a}(\theta)|^p\, d\theta + \int_{2\delta < |\theta| < \pi} |\tilde{a}(\theta)|^p\, d\theta = \mathrm{I}_1 + \mathrm{I}_2 \,.$$

Using the Holder's and Marcel Riesz's inequality, we obtain

$$\mathrm{I}_1 \leq (4\delta)^{(2-p)/2} \|\tilde{a}\|_{L^2}^p \leq C_1 \delta^{(2-p)/2} \|a\|_{L^2}^p \leq C_2 \delta^{(2-p)/2} \left(\delta^{(-2+p)/p}\right)^{p/2} = C \,.$$

To estimate I_2 we write ([11, Thm. 1.4.8])

$$\tilde{a}(\theta) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\epsilon < |\theta - t| \leq \pi} \frac{1}{2\tan((\theta - t)/2)}\, a(t)\, dt \,.$$

Considering that $|\theta| \geq 2\delta$ in I_2 and $|t| \leq \delta$ in the support of a we see that the denominator does not vanish in the integral that defines $\tilde{a}$ and we have

$$\tilde{a}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\tan((\theta - t)/2)}\, a(t)\, dt \,, \quad |\theta| \geq 2\delta \,. \tag{4.1}$$

By expanding the function

$$t \to \frac{1}{\tan((\theta - t)/2)} \,,$$

in its Taylor expansion of order $N_p = [1/p - 1]$ around[3] $t = 0$ we get

$$\frac{1}{\tan((\theta - t)/2)} = \sum_{k \leq N_p} \left(\frac{1}{\tan((\theta - t)/2)} \right)^{(k)}_{t=0} \frac{t^k}{k!} + R_{N_p+1}(t, \theta) \,.$$

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[2] The operator $f \mapsto \tilde{f}$ is of weak type $(1,1)$, then the Hilbert transform is bounded from L^1 to L^p, $0 < p < 1$. As a consequence if $f \in L^1$ is such that $\tilde{f} \in L^1$ then the harmonic conjugate of $u = P(f)$ is given by $P(\tilde{f})$, see [8].

[3] $[s]$ stands for the largest integer less than or equal to s.

It is not hard to see that over the conditions: $|\theta| \geq 2\delta$ and $|t| \leq \delta$, we have

$$|R_{N_p+1}(t,\theta)| \leq C|\theta|^{-(N_p+2)}|t|^{N_p+1} \ . \tag{4.2}$$

Since a has vanishing moments up to order N_p, we have, using (4.2)

$$|\tilde{a}(\theta)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} R_{N_p+1}(t,\theta)a(t)\,dt\right| \leq C|\theta|^{-(N_p+2)}|t|^{N_p+1}\|a\|_{L^1} \ .$$

Taking the p-th power in both sides of the above inequality we get

$$\begin{aligned}|\tilde{a}(\theta)|^p &\leq C|\theta|^{-p(N_p+2)}|t|^{p(N_p+1)}\|a\|_{L^1}^p \leq \\ &\leq C|\theta|^{-p(N_p+2)}\delta^{p(N_p+1)}\delta^{p-1} = \\ &= C|\theta|^{-p(N_p+2)}\delta^{p(N_p+2)-1} \ .\end{aligned}$$

Then

$$\begin{aligned}\mathrm{I}_2 &\leq C\delta^{p(N_p+2)-1}\int_{2\delta<|\theta|<\pi}|\theta|^{-p(N_p+2)}\,d\theta \leq \\ &\leq C\delta^{p(N_p+2)-1}\int_{\delta}^{\infty}|\theta|^{-p(N_p+2)}\,d\theta \leq \\ &\leq C\delta^{p(N_p+2)-1}\delta^{-p(N_p+2)+1} = C\end{aligned}$$

concluding the proof of proposition.

□

Let $F \in H^p$ not be identically zero and consider

$$\tilde{\lambda} = \inf_{\theta\in\mathbb{T}} F^*(\theta) \ .$$

Since F^* is lower semi-continuous its infimum is reached in some $\theta_0 \in \mathbb{T}$. But, if $F^*(\theta_0) = 0$, F should vanish in an open set, forcing F to vanish identically. We conclude that $\tilde{\lambda} > 0$. Define, for each $\lambda > 0$, $\mathcal{O}_\lambda = \{\theta \in \mathbb{T} \ : \ F^*(x) > \lambda\}$. Observe that $\mathcal{O}_\lambda$ is an open subset of $\mathbb{T}$ and it is a proper subset only if $\lambda > \tilde{\lambda}$.

From now on, unless stated otherwise, we will consider boundary values of $F \in H^p$ in the sense of distributions and we will denote it by $bF \doteq f_b$. Since $\mathcal{H}_b^p \cap L^1$ is dense in $\mathcal{H}_b^p$ there is no loss of generality if we assume that $f_b \in L^1$ and we will do so.

Proposition 4.2. *With the notations above, let* $\lambda > \tilde{\lambda}$ *and* $k \in \mathbb{N}$*. Then*

$$f_b = g_\lambda(\theta) + b_\lambda(\theta) \ ,$$

with $|g_\lambda(\theta)| \leq C\lambda$ *and* b_λ *satisfies*

(1) b_λ *is supported in* $\mathcal{O}_\lambda$ *and*

(2) $\displaystyle\int_0^\beta t^\ell b_\lambda(t)\,dt = 0 \ , \quad 0 \leq \ell \leq k.$

Proof. We start by stating and proving a lemma that is a key step in the proof of the proposition.

Lemma 4.1. *Let* $J \subset \mathbb{T}$ *a component of* $\mathcal{O}_\lambda$*,* $\lambda > 0$*,* c *its center and* $k \in \mathbb{N}$*, then*

$$\left|\int_J (\theta - c)^k f(\theta)\,d\theta\right| \leq C|J|^{k+1}\lambda \ . \tag{4.3}$$

Proof. Let $J = (c-\delta, c+\delta)$ and consider the curvilinear triangle, or trapezoid T constructed as follows (see [12] page 60):

(1) Triangle: if the opening of J is less than or equal to 90°, then

$$T = \{z \; : \; z \in \Gamma_\theta \text{ for all } \theta \in I\} \; .$$

(2) Trapezoid: if the opening of J is greater than 90°, then

$$T = \{z \; : \; z \in \Gamma_\theta \text{ for all } \theta \in I\} \cap \{1/\sqrt{2} < |z| < 1\} \; .$$

By Cauchy's theorem, (at this point that we use the assumption $f \in L^1(\mathbb{T})$),

$$\int_{\partial T} F(rz) \, \frac{dz}{z} = 0 \; ,$$

for any $r < 1$, where ∂T denotes the oriented boundary of T. Observe that $|F(rz)| \leq \lambda$ for all $z \in \partial T \setminus I$. Taking $r \to 1$ and observing that

$$|\partial T \setminus J| \leq \sqrt{2}|J| \quad \text{and that } |z| \geq \frac{1}{\sqrt{2}}, \; z \in \partial T \; .$$

we have the inequality (4.3) for $k = 0$.

A similar reasoning with the function $(z-c)F(z)$ in the place of $F(z)$ yields (4.3) for $k = 1$ where we have to use that $\sup_{z \in T} |z - c| \leq C|J|$. This proves the Lemma for $k = 1$ (which implies the case $k = 0$) and similar arguments to prove (4.3) for any $k \geq 2$ are already present.

□

We now continue with the proof of Proposition 4.2. We will only prove the case $k = 2$ because the general case is quite similar and will be left for the reader. Let $\mathcal{O}_\lambda = \bigcup_l J_l$ be the decomposition into connected components and write $J_l = (c_l - \delta_l, c_l + \delta_l)$. Define

$$g_\lambda(t) = \begin{cases} \alpha_l(t - c_l) + \beta_l \; , & \text{for } t \in J_l \; , \\ f(t) \; , & \text{for } t \in \mathbb{T} \setminus \mathcal{O}_\lambda \; . \end{cases}$$

Here α_l, β_l are constants that we determine by the conditions

$$\begin{aligned} m_0^l \doteq & \int_{J_l} f(t)\, dt = \int_{J_l} (\alpha_j(x - c_j) + \beta_j)\, dt \; , \\ m_1^l \doteq & \int_{J_l} f(t)(t - c_l)\, dt = \int_{J_l} (\alpha_j(t - c_l)^2 + \beta_j(t - c_l))\, dx \; . \end{aligned}$$

Since $\int_{J_l} (t - c_l)\, dt = 0$, we obtain

$$\alpha_l = \frac{m_1^l}{\rho_2^l} \; , \qquad \beta_l = \frac{m_0^l}{\rho_0^l} \; ,$$

with

$$\rho_i^l = \int_{J_l} (t - c_l)^i \, dt \; , \quad i = 0, 2 \; .$$

In view of (4.3) $|m_0^l| \leq C\lambda\delta_l$ and $|m_1^l| \leq C\lambda\delta_l^2$, while $\rho_0^l = 2\delta_l$ and $\rho_2^l = (2/3)\delta_l^3$. Thus, $|\alpha_l| \leq C\,\lambda/\delta_l$ and $|\beta_l| \leq C\lambda$ which implies that $|g_\lambda(t)| \leq C\lambda$ on J_l. Since $|f(t)| \leq F^*(t) \leq \lambda$ off $\mathcal{O}_\lambda$ we conclude that $|g_\lambda(t)| \leq C\lambda$ on $\mathbb{T}$. Set now

$$b_\lambda(t) = \begin{cases} f(t) - (\alpha_l(t - c_l) + \beta_l) \; , & \text{for } t \in J_l \; , \\ 0 \; , & \text{for } t \in \mathbb{T} \setminus \mathcal{O}_\lambda \end{cases}$$

It is clear that $f(t) = g_\lambda(t) + b_\lambda(t)$. Concerning the properties that $b_\lambda(t)$ must fulfill, (1) holds by the very definition of $b_\lambda(t)$ while, to show (2), it is enough to check that $\int_{J_l} b_\lambda(t)\,dt = \int_{J_l} tb_\lambda(t)\,dt = 0$ for any l and this follows from the choice of α_l and β_l.

$\square$

5. Proof of Theorem 2.2

Choose $n_0 \in \mathbb{Z}$ such that $2^{n_0} \leq \tilde{\lambda} < 2^{n_0+1}$. For all $n > n_0$, let

$$f(\theta) = g_{2^n}(\theta) + b_{2^n}(\theta)$$

the decomposition obtained from the Proposition 4.2. We still need to define $g_{2^{n_0}}$ e $b_{2^{n_0}}$. Observe that

$$\int_0^{2\pi} \{F(e^{i\theta}) - F(0)\}\,d\theta = 0 \ ,$$

then, if we define

$$g_{2^{n_0}}(\theta) = F(0) \ , \quad b_{2^{n_0}}(\theta) = F(e^{i\theta}) - F(0) \ ,$$

we will have

$$\int_0^{2\pi} b_{2^{n_0}}(\theta)\,d\theta = 0 \ ,$$

and

$$|g_{2^{n_0}}(\theta)| = |F(0)| \leq \inf_\theta F^*(\theta) \leq \tilde{\lambda} < 22^{n_0} \ .$$

Now we are going to make simultaneous use of all these decompositions. Observe that $g_{2^n}(\theta) \to f(\theta)$ a.e. when $n \to \infty$, because the function $f(\theta) - g_{2^n}(\theta)$ is supported in the set $\{\theta \ : \ F^*(\theta) > 2^n\}$ (if $n > n_0$) which decrease to set of measure zero when $n \to \infty$. Moreover, from the construction of g_{2^n}, is easy to see that $|g_{2^n}(\theta)| \leq CF^*(\theta)$. These facts alow us to write f as a telescopic series converging a.e., to know

$$\begin{aligned} f(\theta) \quad &= g_{2^{n_0}}(\theta) + \sum_{n=n_0}^{\infty} (g_{2^{n+1}}(\theta) - g_{2^n}(\theta)) = \\ &= F(0) + \sum_{n=n_0}^{\infty} (b_{2^n}(\theta) - b_{2^{n+1}}(\theta)) \ . \end{aligned}$$

The term $F(0)$ is a multiple of the constant p-atom $c = 1$, $F(0) = \lambda_0 c$, where $|\lambda_0| = |F(0)| \leq C\|F^*\|_{L^p}$. We will expand $b_{2^n}(\theta) - b_{2^{n+1}}(\theta)$, $n \geq n_0$, as a series of p-atoms with coefficients in ℓ^p. For each n write $\mathcal{O}_{2^n}$ as a disjoint union of intervals

$$\mathcal{O}_{2^n} = \bigcup_i J_n(i) \ .$$

Fix n, all $J_{n+1}(j)$ are contained in the disjoint union of $J_n(i)$. For every ordered pair (n, i), let

$$\psi_{n,i}(\theta) \quad = \begin{cases} b_{2^n}(\theta) - b_{2^{n+1}}(\theta) \ , & \theta \in J_n(i) \ ; \\ 0 \ , & \theta \in \mathbb{T} \setminus J_n(i) \ . \end{cases}$$

Then, for $0 \leq l \leq k = N_p$,

$$\begin{aligned}\int_{J_n(i)} \psi_{n,i}(\theta)\,\theta^l d\theta &= \int_{J_n(i)} b_{2^n}(\theta)\,\theta^l d\theta - \int_{J_n(i)} b_{2^{n+1}}(\theta)\,\theta^l d\theta = \\ &= \int_{J_n(i)} b_{2^n}(\theta)\,\theta^l d\theta - \sum_{j\in S(n,i)} \int_{J_{n+1}(j)} b_{2^{n+1}}(\theta)\,\theta^l d\theta = 0\ ,\end{aligned}$$

where $S(n,i)$ is the set of all j such that $J_{n+1}(j) \subset J_n(j)$. Define

$$\phi_{n,i}(\theta) = \frac{1}{2^n 3C}\ |J_n(i)|^{-1/p}\psi_{n,i}(\theta)\ , \qquad \lambda_{n,i} = 3C2^n|J_n(i)|^{1/p}\ .$$

With this definition, all $\phi_{n,i}$ are p-atoms and

$$\sum_{n=n_0+1}^{\infty} (b_{2^n}(\theta) - b_{2^{n+1}}(\theta)) = \sum_{n=n_0+1}^{\infty}\sum_{k=0}^{\infty} \lambda_{n,i}\,\phi_{n,i}(\theta)\ .$$

However,

$$\begin{aligned}\sum_{n=n_0+1}^{\infty}\sum_{i=1}^{\infty}|\lambda_{n,i}|^p &= (3C)^p \sum_{n=n_0+1}^{\infty}\sum_{i=1}^{\infty} 2^{np}|J_n(i)| = \\ &= C\sum_{n=n_0+1}^{\infty} 2^{n(p-1)}2^n|\{\theta\ :\ F^*(\theta) > 2^n\}| \leq \\ &\leq C\int_0^{\infty} \lambda^{p-1}|\{\theta\ :\ F^*(\theta) > \lambda\}|\,d\lambda = \\ &= C\int_{-\pi}^{\pi} (F^*(\theta))^p\,d\theta < \infty\ .\end{aligned}$$

Therefore, the series $\sum_{n=n_0+1}^{\infty}\sum_{i=1}^{\infty} \lambda_{n,i}\phi_{n,i}(\theta)$ converge a.e. and in L^p to $f(\theta)$, because $\|\phi_{n,k}\|_{L^p} \leq C$. Rearranging the terms of the series, we can write

$$f(\theta) = \sum_j \gamma_j a_j(\theta)\ ,$$

where the a_j are p-atoms and $(\gamma_j) \in \ell_p$, concluding the proof for the case $f_b \in \mathcal{H}_b^p \cap L^1$. The general case follows by a density argument.

□

6. Applications

In this section we will make use of atomic decompositions in H^p to prove the continuity of the Hilbert transform from $h^p \subset \mathcal{H}_b^p$ to itself (see Definition 6.1).

Let $f \in L^1$, we know that its Hilbert transform Hf is given by

$$H(f)(\theta) = \lim_{\epsilon\to 0}\ \frac{1}{\pi}\ \int_{\epsilon\leq|t|\leq\pi} \frac{f(\theta - t)}{2\tan t/2}\ dt\ .$$

For $1 < p < \infty$, H applies $L^p(\mathbb{T})$ continuously into itself and, by Marcel Riesz inequality, $\|Hf\|_{L^p} \leq C_p\|f\|_{L^p}$, $f \in L^p$, but the constants C_p tend to infinity when either $p \to 1$ or $p \to \infty$ and the continuity fails in the extremes of the interval $1 < p < \infty$. However, if we consider $\mathcal{H}_b^p$, $0 < p \leq 1$ instead of L^p we have the following:

For $f \in \mathcal{D}'(\mathbb{T})$ denote by $\mathcal{A}f$ the function

$$\mathcal{A}f(re^{i\theta}) = ((P+iQ)f)(re^{i\theta}) = \langle (P_r + iQ_r)(\theta), f(\theta)\rangle \ .$$

Theorem 6.1. *If $f \in \mathcal{H}^p_b$, that is, $f = bF$ for some $F \in H^p$, then $\mathcal{A}f \in H^p$, that is, $b(\mathcal{A}f) \in \mathcal{H}^p_b$.*

Proof. By the atomic decomposition of H^p, is enough to show that

$$\|\mathcal{A}a\|_{H^p} = \lim_{r\to 1} \|(P_r + iQ_r)a\|_{L^p} = \|a + i\tilde{a}\|_{L^p} \leq C$$

for all p-atom a, and this follows from Proposition 4.1.

□

The theorem has the following extension.

Definition 6.1. Denote $\Re\mathcal{H}^p_b$ the space of distributions that are boundary values of real part of some $F \in H^p$ and $h^p(\mathbb{T}) = \mathbb{C} \otimes \Re\mathcal{H}^p_b$, that is, f is in $h^p(\mathbb{T})$ if and only if $f = u + iv$ and $u, v \in \Re\mathcal{H}^p_b$.

The space $h^p(\mathbb{T})$ was defined using certain holomorphic functions in Δ, however the following theorem allows us to give an intrinsic characterization of this space in the sense that it does not make use of the fact that $\mathbb{T}$ is the boundary of Δ.

Theorem 6.2. *Let $f \in \mathcal{D}'(\mathbb{T})$. The following are conditions are equivalent:*

(i) *$f \in h^p(\mathbb{T})$;*

(ii) *$\mathcal{A}f \in H^p(\Delta)$;*

(iii) *there exists a sequence of p-atoms (a_j) and a sequence $(\lambda_j) \in \ell^p$ such that*

$$f = \sum_{j=0}^{\infty} \lambda_j a_j \quad (\textit{convergence in } \mathcal{D}') \ . \tag{6.1}$$

Proof. (i) $\Longrightarrow$ (iii) follows by an application of Theorem 2.2. The uniform bound $\|Ha_j\|_{L^p} \leq C$ (see proof of Theorem 6.1) shows the implication (iii) $\Longrightarrow$ (ii). Since condition (ii) implies that the boundary value of $(P+iQ)f$ exists and, moreover, $f = b(\Re(P+iQ)f) \in h^p(\mathbb{T})$, we conclude that (ii) $\Longrightarrow$ (i) and the proof is finished.

□

Given $f \in h^p(\mathbb{T})$ we can consider the function $af \in L^p(\mathbb{T})$ given by the non-tangential limit of the function $\mathcal{A}f \in H^p(\Delta)$.

Corollary 6.1. *The space $h^p(\mathbb{T})$ with "norm" given by*

$$\|f\|_{h^p} \doteq \|af\|_{L^p}$$

is a complete metric space (Banach space if $p = 1$).

Proof. Let (f_k) is a Cauchy sequence in $h^p(\mathbb{T})$, we can write $f_k = u_k + iv_k$, where u_k and v_k are the boundary values (in the sense of distributions) of real part of functions in H^p, that is

$$u_k = b(\Re F_k) \ , \quad \text{and} \quad v_k = b(\Re G_k) \ , \quad F_k \ , G_k \in H^p(\Delta) \ .$$

The sequence (af_k) is a Cauchy sequence in L^p and therefore converges in L^p, $af_k \to g \in L^p$. We want to show that there exists $f \in h^p$ such that the sequence f_k converge to f in h^p. Since (af_k) is a Cauchy sequence in L^p, (F_k) and (G_k) are

Cauchy sequences in H^p, therefore, they converge to F and G in H^p respectively. Let $f = b(\Re F) + ib(\Re G)$, then it follows that: $af = g$ and $f_k \to f$ in h^p.

□

Remark 6.5. Observe that for $p = 1$ the norm $\|f\|_{h^1}$ is equivalent to the norm $\|f\|_{L^1} + \|Hf\|_{L^1}$.

Now, we can improve Theorem 6.2.

Corollary 6.2. *The Hilbert transform applies $h^p(\mathbb{T})$ continuously to $h^p(\mathbb{T})$.*

Proof. An operator A is bounded in $h^p(\mathbb{T})$ if $\|a(Af)\|_{L^p} \leq C\|f\|_{h^p}$. But

$$\|a(Af)\|_{L^p} = \|\mathcal{A}(Af)\|_{H^p} \ .$$

In the case $A = H$, we have

$$\begin{aligned} \|\mathcal{A}(Af)\|_{H^p} &\simeq \| \lim_{r\to 1} \langle (P_r + iQ_r), Hf\rangle \|_{L^p} \leq \\ &\leq \| \lim_{r\to 1} -i\langle (P_r + iQ_r), f\rangle \|_{L^p} = \\ &= \|af\|_{L^p} \end{aligned}$$

where we have used the identity $H^2 = -I$. In fact, if the Fourier series of $f \in h^p(\mathbb{T})$ is $\sum_{-\infty}^{\infty} a_k e^{ik\theta}$ then the Fourier series of Hf is

$$-\sum_{k=-\infty}^{\infty} i \,\mathrm{sgn}(k)\, a_k\, e^{ik\theta} \ ,$$

and iterating this fact we see that the Fourier series of $H^2 f$ will be $-\sum_{-\infty}^{\infty} a_k e^{ik\theta}$, showing that $H^2 f = -f$.

□

Corollary 6.3. *The series* (6.1) *converges in metric of $h^p(\mathbb{T})$.*

Proof. It is enough to show that the partial sum of the series is a Cauchy sequence in h^p, and it follows from the fact that $(\lambda_j) \in \ell^p$ and that $\|a\|_{L^p} + \|Ha\|_{L^p} \leq C$ for every p-atom.

□

Now we can use the atomic decomposition to give another intrinsic norm in $h^p(\mathbb{T})$. Given $f \in h^p(\mathbb{T})$ write

$$\|f\|_{\mathrm{at}}^p = \inf \sum_{j=0}^{\infty} |\lambda_j|^p$$

where the infimum is taken over all atomic decompositions $f = \sum_{j=0}^{\infty} \lambda_j\, a_j$. If $f = \sum_{j=0}^{\infty} \lambda_j\, a_j$ is any atomic decomposition, from the triangle inequality and from the fact that $\|a_j\|_{h^p} \leq C$ we see that $\|f\|_{h^p}^p \leq C \sum_{j=0}^{\infty} |\lambda_j|^p$, showing that $\|f\|_{h^p} \leq C\|f\|_{\mathrm{at}}$. Conversely, given $f \in h^p$ exist a decomposition (Theorems 6.2 and 2.2) $f = \sum_j a_j$ with

$$\sum_j |\lambda_j|^p \leq C\|f\|_{h^p}^p \ .$$

Summing up, we have obtained

Corollary 6.4. *The norms $\|f\|_{h^p}$ and $\|f\|_{\mathrm{at}}$ are equivalent in $h^p(\mathbb{T})$, in particular, $(h^p(\mathbb{T}), \| \ \|_{\mathrm{at}})$ is a complete metric space.*

7. DUALITY

In this section we will describe all continuous linear functionals in $H^p(\Delta)$. The case $p > 1$ was known for a long time, the case $p < 1$ was proved by Duren, Romberg and Shields ([4]) and, finally, the case $p = 1$ is closely related to the famous duality theorem of Fefferman and Stein [6].

Theorem 7.1 (Theorem 7.3 in [3]). *For* $1 \leq p < \infty$, $(H^p(\Delta))^*$ *is isometrically isomorphic to* $L^q(\partial\Delta)/H^q(\Delta)$, *where* $1/p + 1/q = 1$. *Moreover, if* $1 < p < \infty$, *every* $\phi \in (H^p(\Delta))^*$ *is represented in the form*

$$\langle F, G\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(e^{it})G(e^{-it})\, dt\,. \tag{7.1}$$

for a unique $G \in H^q(\Delta)$ *and the norm of the functional* ϕ *in* $(H^p(\Delta))^*$ *is comparable with the norm of* G *in* H^q. *If* $\phi \in (H^1(\Delta))^*$, *the functional can be represented in the form* (7.1) *for some* $G \in L^\infty$ *but* $G(e^{i\theta})$ *might not be the boundary value of a holomorphic function defined on* Δ.

We will describe how the atomic decompositions could be used to study de dual of $H^p(\Delta)$ when $0 < p \leq 1$. Notice that in the case $p = 1$ theorem 7.1 gives a representation for bounded linear functionals in $H^1(\Delta)$ which is less satisfactory than the situation for $p > 1$ because the function used to represent the linear functional is not the boundary value of a holomorphic function. Suppose we are only interested in considering functions that are boundary values $G(e^{i\theta})$ of holomorphic functions. We may reason as follows. Let $\phi \in (H^p(\Delta))^*$ for some $0 < p \leq 1$. If $F \in H^{p'}(\Delta)$, $1 < p' < \infty$, we have

$$|\langle \phi, F\rangle| \leq \|\phi\|_{(H^p)^*}\|F\|_{H^p} \leq C_p\|\phi\|_{(H^p)^*}\|F\|_{H^{p'}}$$

showing that the restriction of ϕ to $H^p(\Delta)$ is a bounded functional that, by Theorem 7.1, can be represented as (7.1) by a function $G_q \in H^q(\Delta)$ with $\|bG_q\|_{L^q} \simeq \|G_q\|_{H^q} \simeq \|\phi\|_{(H^1)^*}$ where p' and q are conjugated exponents. Taking $F(z) = z^k$, $k = 0, 1, \ldots$, we conclude that $G_q(z) = G_{\tilde{q}}(z)$, $1 < q, \tilde{q} < \infty$, that is, there exist a fixed function $G(z) \in \bigcap_{1<q<\infty} H^q(\Delta)$ so that (7.1) is valid when F belongs, say, to $H^2(\Delta)$. In particular we obtain

$$\|bG\|_{L^1} \leq C\|\phi\|_{(H^p)^*} \tag{7.2}$$

Let us see which additional properties we can learn about the boundary function $g(\theta) \doteq bG(-\theta) = G(e^{-i\theta})$. Fix an arc $J \subset \mathbb{T}$ and consider a function $a(\theta)$ satisfying

(*i*) $\operatorname{supp}(a) \subset J$;

(*ii*) $\|a\|_{L^2} \leq |J|^{1/2-1/p}$;

(*iii*) $\displaystyle\int_0^{2\pi} \theta^k a(\theta)\, d\theta = 0,\ 0 \leq k \leq N_p$.

A function that satisfies properties (i), (ii) and (iii) above is called a generalized by p-atom. Notice the similarity with Definition 2.2: there is only a modification in the requirement (ii) with the role of the L^∞ norm played now by the L^2 norm. It is elementary to check that every p-atom is a generalized p-atom. A simple variation of Proposition 4.1 allows us to conclude that there exist a universal constant $C > 0$ so that

$$\|a\|_{h^p} = \|a\|_{L^p} + \|Ha\|_{L^p} \leq C$$

for all generalized p-atom.

Case $\mathbf{p = 1}$. We will denote by $L^2_0(J)$ the space of the functions in $L^2(\mathbb{T})$ supported in J and that are orthogonal to the function 1, i.e., if $f \in L^2_0(J)$, then

$$\int_{\mathbb{T}} f(\theta)\, d\theta = \int_J f(\theta)\, d\theta = 0\ .$$

If $f \in L^2_0(J)$, $f/(\|f\|_{L^2}|J|^{1/2})$ is a generalized 1-atom and we see that $\|f\|_{h^1} \leq C|J|^{1/2}\|f\|_{L^2}$. Given any real function $f \in L^2_0(J)$ we can find $F \in H^1$ with $\Re\, bF = f$ and obtain

$$\begin{aligned}\left|\frac{1}{2\pi}\int_0^{2\pi} bF(t)g(t)\, dt\right| &= |\langle \phi, F\rangle| \leq \|\phi\|_{(H^1)^*}\|F\|_{H^1} \leq \\ &\leq C\|\phi\|_{(H^1)^*}|J|^{1/2}\|f\|_{L^2}\ ,\end{aligned}$$

In particular,

$$\left|\frac{1}{|J|^{1/2}}\,\Re\int_J f(t)g(t)\, dt\right| \leq C\|\phi\|_{(H^1)^*}\|f\|_{L^2}\ .$$

Writing $g = \alpha - i\tilde{\alpha}$, α real, we observe that

$$\Re\,\langle f, g\rangle = \langle f, \alpha\rangle + \langle \tilde{f}, \tilde{\alpha}\rangle = 2\langle f, \alpha\rangle$$

since $\langle Hf, H\alpha\rangle = -\langle H^2 f, \alpha\rangle = \langle f, \alpha\rangle$. We have

$$\left|\frac{1}{|J|^{1/2}}\int_J f(t)\alpha(t)\, dt\right| \leq C\|\phi\|_{(H^1)^*}\|f\|_{L^2}\ . \tag{7.3}$$

Denote by

$$\alpha_J = \frac{1}{|J|}\int_J \alpha(t)\, dt\ .$$

the average of α over J. We see that

$$\begin{aligned}\int_J (\alpha(t) - \alpha_J)^2\, dt &= \int_J \alpha(t)(\alpha(t) - \alpha_J)\, dt - \alpha_J \int_J (\alpha(t) - \alpha_J)\, dt = \\ &= \int_J \alpha(t)(\alpha(t) - \alpha_J)\, dt\end{aligned}$$

and applying (7.3) with $f(t) = (\alpha(t) - \alpha_J)\chi(t)$, where χ denote the characteristic function of the arc J, we obtain

$$\frac{1}{|J|^{1/2}}\int_J (\alpha(t) - \alpha_J)^2\, dt \leq C\|\phi\|_{(H^1)^*}\left(\int_J (\alpha(t) - \alpha_J)^2\, dt\right)^{1/2}$$

or, equivalently,

$$\left(\frac{1}{|J|}\int_J (\alpha(t) - \alpha_J)^2\, dt\right)^{1/2} \leq C\|\phi\|_{(H^1)^*}\ .$$

An analogous estimate holds for $\tilde{\alpha}$ that is the real part of ig, that correspond to the functional $i\phi$. Then we obtain

$$\left(\frac{1}{|J|}\int_J |g(t) - g_J|\, dt\right)^{1/2} \leq C\|\phi\|_{(H^1)^*}\ . \tag{7.4}$$

Definition 7.1. We say that $g \in L^1(\mathbb{T})$ belongs to $\mathrm{bmo}\,(\mathbb{T})$ if

$$N(g) = \sup_J \left(\frac{1}{|J|} \int_J |g(t) - g_J|^2 \, dt \right)^{1/2} < \infty$$

and we will denote by

$$\|g\|_{\mathrm{bmo}} = N(g) + \|g\|_{L^1} \ .$$

Equations (7.2) and (7.4) show that $g \in \mathrm{bmo}\,(\mathbb{T})$ and $\|g\|_{\mathrm{bmo}} \le C\|\phi\|_{(H^1)^*}$. Then we conclude that every linear functional in H^1 can be represented uniquely in the form (7.1) where $G(z)$ is holomorphic and its boundary values $bG(\theta) = G(e^{i\theta}) \in \mathrm{bmo}\,(\mathbb{T})$. To complete this result we have to show that, conversely, if G is holomorphic and $bG \in \mathrm{bmo}\,(\mathbb{T})$, the form

$$F \mapsto \langle F, G \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it}) G(e^{-it}) \, dt \ , \quad F \in H^\infty \ , \tag{7.5}$$

is bounded in H^1 and therefore can be continuously extended to H^1 by the density of H^∞ in H^1. In this way we produce a linear functional $\phi \in (H^1)^*$ and we must further show that $\|\phi\|_{(H^1)^*} \simeq \|g\|_{\mathrm{bmo}}$.

Consider an atomic decomposition of $F \in H^\infty \subset H^1$, $F = \sum_{j=0}^\infty \lambda_j a_j$, with $\sum_j |\lambda_j| \simeq \|F\|_{H^1}$. If $a_0 \equiv 1$ is the constant atom, we have

$$\frac{1}{2\pi} \left| \int_0^{2\pi} a_0(t) g(t) \, dt \right| \le \|g\|_{L^1} \le \|g\|_{\mathrm{bmo}} \ .$$

For the atoms a_j, $j \ge 1$, with average zero and carrier J, we have

$$\begin{aligned} \frac{1}{2\pi} \left| \int_0^{2\pi} a_j(t) g(t) \, dt \right| &= \frac{1}{2\pi} \left| \int_J a_j(t)(g(t) - g_J) \, dt \right| \le \\ &\le \frac{1}{2\pi} \, \|a_j\|_{L^2} \left| \int_J |g(t) - g_J|^2 \, dt \right|^{1/2} \le \\ &\le \frac{1}{2\pi} \, |J|^{-1/2} |J|^{1/2} \|g\|_{\mathrm{bmo}} \le \\ &\le C\|g\|_{\mathrm{bmo}} \ . \end{aligned}$$

Then, we obtain

$$|\langle F, G \rangle| \le \frac{1}{2\pi} \sum_j |\lambda_j| \left| \int_0^{2\pi} a_j(t) g(t) \, dt \right| \le C\|F\|_{H^1} \|g\|_{\mathrm{bmo}}$$

with C independent of $F \in H^\infty$. Moreover $F \mapsto \langle F, G \rangle$ is bounded in the norm H^1 and extend uniquely to a functional ϕ defined in H^1 with norm $\|\phi\|_{(H^1)^*} \le C\|g\|_{\mathrm{bmo}}$. Observe that we already have showed that $\|\phi\|_{(H^1)^*} \le C\|g\|_{\mathrm{bmo}}$, therefore $\|\phi\|_{(H^1)^*} \simeq \|g\|_{\mathrm{bmo}}$.

Summarizing, we have proved the case $p = 1$ of the following theorem.

Theorem 7.2. *Every* $\phi \in H^p(\Delta)^*$ *may be represented in the form* (7.5) *by an unique* $G \in \bigcap_{1 \le q < \infty} H^q$.

(1) *If* $p = 1$, $bG \in \mathrm{bmo}\,(\mathbb{T})$ *and* $\|\phi\|_{(H^1)^*} \simeq \|g\|_{\mathrm{bmo}}$.

(2) *If* $p < 1$, $bG \in \Lambda^{1/p-1}\,(\mathbb{T})$ *and* $\|\phi\|_{(H^p)^*} \simeq \|g\|_{\Lambda^{1/p-1}}$.

To explain the meaning of (2) in Theorem 7.2 we introduce the following functional space.

Definition 7.2. We say that $g \in L^1(\mathbb{T})$ belongs to $\Lambda^s(\mathbb{T})$ if

$$|g|_s = \sup_J \left(\inf_{P \in \mathcal{P}_{[s]}} \frac{1}{|J|^{2s+1}} \int_J |g(t) - \mathcal{P}|^2 \, dt \right)^{1/2} < \infty \, ,$$

where $\mathcal{P}_{[s]}$ is the set of all polynomials $P(t)$, $t \in \mathbb{R}$, of degree less than or equal to $[s]$. We also set

$$\|g\|_{\Lambda^s} = |g|_s + \|g\|_{L^1} \, .$$

Case $\mathbf{0 < p < 1}$. We now conclude the proof of the theorem.

Proof. Assume $0 < p < 1$. For n a nonnegative integer, let $L^2_n(J)$ the space of the functions in $L^2(\mathbb{T})$ supported in J and that are orthogonal to the functions $1, \theta, \theta^2, \ldots, \theta^n$, i.e.,

$$\int_{\mathbb{T}} \theta^k f(\theta) \, d\theta = \int_J \theta^k f(\theta) \, d\theta = 0 \, , \quad 0 \le k \le n \, .$$

If $f \in L^2_{N_p}(J)$, $f/(\|f\|_{L^2}|J|^{1/p-1/2})$ is a generalized p-atom and we see that $\|f\|_{h^p} \le C|J|^{1/p-1/2}\|f\|_{L^2}$. Given any real function $f \in L^2_{N_p}(J)$ we can find $F \in H^1$ with $\Re \, bF = f$ and obtain

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^{2\pi} bF(t) g(t) \, dt \right| &= |\langle \phi, F \rangle| \le \|\phi\|_{(H^p)^*} \, \|F\|_{H^p} \le \\ &\le C \, \|\phi\|_{(H^p)^*} \, |J|^{1/p-1/2} \, \|f\|_{L^2} \le \\ &\le C \, \|\phi\|_{(H^p)^*} \, |J|^{1/p-1} \, |J|^{1/2} \, \|f\|_{L^2} \, . \end{aligned}$$

Then, reasoning as before, we obtain

$$\left(\frac{1}{|J|} \int_J |g(t) - g_J|^2 \, dt \right)^{1/2} \le C \, |J|^{1/p-1} \, \|\phi\|_{(H^p)^*} \, . \tag{7.6}$$

Equations (7.2) and (7.6) show that g is in $\Lambda^{1/p-1}(\mathbb{T})$ with $\|g\|_{\Lambda^{1/p-1}} \le C\|\phi\|_{(H^p)^*}$. Then we conclude that every linear functional in H^p can be represented uniquely in the form (7.1) where $G(z)$ is holomorphic and its boundary values $bG(\theta) = G(e^{i\theta}) \in \Lambda^{1/p-1}(\mathbb{T})$. To complete this result we have to show that, conversely, if G is holomorphic and $bG \in \Lambda^{1/p-1}(\mathbb{T})$, the form

$$F \mapsto \langle F, G \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it}) \, G(e^{-it}) \, dt \, , \quad F \in H^\infty \, , \tag{7.7}$$

is bounded in H^p - then can be extended (by density) continuously to H^p getting $\phi \in (H^p)^*$ - yet $\|\phi\|_{(H^p)^*} \simeq \|g\|_{\Lambda^{1/p-1}}$. But this can be done exactly as we did for the case $p = 1$.

□

Remark 7.6. The spaces $\Lambda^s(\mathbb{T})$ introduced in Definition 7.2 are Hölder spaces (also called Lipschitz spaces) in disguise. For instance, if $0 < s < 1$, $f(e^{it}) \in \Lambda^s(\mathbb{T})$ if and only if $f(t)$, $t \in \mathbb{R}$, is continuous and satisfies

$$\sup_{t_1, t_2 \in \mathbb{R}} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|} < \infty \, .$$

We refer the reader to section 5 of [8, Ch. 3] where the equivalence is proved for functions defined on $\mathbb{R}^n$.

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