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A uniqueness theorem for degenerate elliptic equations

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Dedicated to Ermanno Lanconelli on his 65th birthday

Abstract[1]. In this paper, we consider the uniqueness of a viscosity solution of the degenerate elliptic equation

$$F(x, u(x), \nabla u(x), D^2 u(x)) = 0$$

or its normalized counterpart

$$F(x, u(x), \widehat{\nabla u}(x), D^2 u(x)) = 0$$

subject to continuous boundary data. A useful strict comparison principle is obtained with very weak requirements, precisely that F is proper, degenerate elliptic, and locally uniformly continuous in x, on the equation. Under a further structural condition on F, namely

$$F(x, (1+t)r, (1+t)q, (1+t)X) \geq (1+\sigma(t))F(x, r, q, X)$$

or

$$F(x, (1+t)r, q, (1+t)X) \geq (1+\sigma(t))F(x, r, q, X)$$

for the normalized equation, where σ is a modulus of continuity, the comparison principle and whence the uniqueness of a viscosity solution of the inhomogeneous equation

$$F(x, u(x), \nabla u(x), D^2 u(x)) = f(x)$$

or its normalized counterpart

$$F(x, u(x), \widehat{\nabla u}(x), D^2 u(x)) = f(x) ,$$

where f is a positive or negative continuous function, are secured from the strict comparison principle. The inhomogeneous p-Laplace equation, the inhomogeneous infinity Laplace equation, and its normalized counterpart are the prototypes of the degenerate equations treated in this work.

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INTRODUCTION

This paper may be regarded as a further development of the theory of viscosity solutions presented in the work of Crandall-Ishii-Lions, [13], and also as summary and generalization of the authors' uniqueness res ults in [LW1,2]. This work is by no means a summary of the developments in the theory of degenerate elliptic equations up to date. To find the recent developments in the theory of degenerate elliptic equations, the reader may read the paper by Bardi and Da Lio, [5], and the references therein.

In this paper, we consider the partial differential equations of the form

$$F(x, u(x), \nabla u(x), D^2u(x)) = 0 \tag{0.1}$$

or

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) = 0 \ , \tag{0.2}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \to \mathbb{R}$, Ω is a bounded domain in $\mathbb{R}^n$, $\mathcal{S}_{n\times n}$ is the set of $n \times n$ symmetric matrices, and $\widehat{\nabla u} = \nabla u/|\nabla u|$. We say F is *degenerate elliptic* if

$$F(x, r, q, X) \le F(x, r, q, Y)$$

whenever $X \le Y$ in the sense of matrices and for any $x \in \Omega$, $r \in \mathbb{R}$, and $q \in \mathbb{R}^n$; and we say F is *proper* if

$$F(x, r, q, X) \ge F(x, s, q, X)$$

whenever $r \le s$ and for any $x \in \Omega$, $q \in \mathbb{R}^n$, and $X \in \mathcal{S}_{n\times n}$.

Examples of the proper degenerate elliptic operator F are the well-known Laplacian $\Delta(x, r, q, X) = \mathrm{Tr}\,(X)$, the p-Laplacian $\Delta_p(x, r, q, X) = (p-2)|q|^{p-4}\langle Xq, q\rangle + |q|^{p-2}\mathrm{Tr}\,(X)$, the infinity Laplacian $\Delta_\infty(x, r, q, X) = \langle Xq, q\rangle$, and the normalized infinity Laplacian $\Delta^N_\infty(x, r, q, X) = \langle X\hat{q}, \hat{q}\rangle$, where $\mathrm{Tr}\,(X)$ denotes the trace of the matrix X. The inhomogeneous nonlinear operators such as the inhomogeneous infinity Laplacian $(x, r, q, X) \mapsto \langle Xq, q\rangle - f(x, q)$, where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, are also important examples of the proper degenerate elliptic operators. Here $\langle x, y\rangle$ and $\hat{x} = x/|x|$, for x and $y \in \mathbb{R}^n$, denote the inner product of x and y, and the normalized vector of x.

In the manuscript [13], Crandall, Ishii and Lions dealt with the uniqueness problem of the degenerate elliptic proper partial differential equation

$$F(x, u(x), \nabla u(x), D^2u(x)) = 0 \ . \tag{0.3}$$

Under the structural conditions

(a) that there exists $\gamma > 0$ such that

$$\gamma(r - s) \le F(x, s, q, X) - F(x, r, q, X) \quad \text{for } r \ge s, \text{ and } (x, q, X) \in \overline{\Omega} \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \ ,$$

and

(b) that there exists a modulus of continuity $\omega : [0, \infty] \to [0, \infty]$ with $\omega(0+) = 0$ such that

$$F(x, r, \alpha(x - y), X) - F(y, r, \alpha(x - y), Y) \le \omega(\alpha|x - y|^2 + |x - y|) \ ,$$

where x, $y \in \Omega$, $r \in \mathbb{R}$, X, $Y \in \mathcal{S}_{n\times n}$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

holds for all sufficiently large α,

they proved there exists at most one viscosity solution of the equation (0.3), which is stated in the Theorem 3.3 in [13].

The study of infinity Laplacian was initiated in the works of Arronson [A1-3] in conjunction with Lipschitz extension. Jensen proved in his work [15] the uniqueness of a viscosity solution of the Dirichelt problem for the homogeneous infinity Laplace equation

$$\Delta_\infty u(x) = \langle D^2u(x)\nabla u(x), \nabla u(x)\rangle = 0$$

in a bounded domain in $\mathbb{R}^n$, using a truncation method according to the gradient of the solution and a delicate transform of the viscosity solution. Barles and Busca later gave a second proof of the uniqueness of an absolute minimizer, namely the viscosity solution of the infinity Laplace equation, in [6]. Their proof of the uniqueness theorem applies to many degenerate elliptic equations without zeroth-order term. On the other hand, Crandall, Gunnarsson and Wang proved a uniqueness theorem for a viscosity solution of the infinity Laplace equation in an unbounded domain subject to sub-linear growth in their work [12].

A partial list of related work on the degenerate equations, especially on the infinity Laplacian, can be found in the works [7], [9], [10], [11], [16], [17], [19] to name just a few. We also refer to [4] for a systematic treatment of the subject.

Though much work has been done for homogeneous infinity Laplacian, not so much has been investigated on inhomogeneous infinity Laplcian. In the manuscript [20], the authors considered the well-posedness for the inhomogeneous infinity Laplace equation $\Delta_\infty u(x) = \langle D^2u(x)\nabla u(x), \nabla u(x)\rangle = f(x)$. Under the assumption $\inf_\Omega f > 0$ (this assumption can be weakened to $f > 0$ as pointed out in [21]), the uniqueness of a viscosity solution subject to continuous boundary data is proved(Theorem 4, [20]). The normalized inhomogeneous infinity Laplace equation

$$\Delta_\infty^N u(x) = \langle D^2u(x)\widehat{\nabla u}(x), \widehat{\nabla u}(x)\rangle = f(x)$$

was studied by Peres, Schramm, Sheffield and Wilson in [22] using differential game theory. Existence and uniqueness were established in [22] for such inhomogeneous normalized infinity Laplacian (for related works, see also [18], and [8], [14], etc.). In [21], we investigated the inhomogeneous normalized infinity Laplacian from the perspective of PDE theory and thus reproved those results in [22] using a quite different method. A comparison principle and hence the uniqueness theorem of the similar type but done with more effort are also proved under the assumption $f > 0$ or $f < 0$ (Theorems 3.3 and 3.4, [21]). Furthermore, a stability result for the Dirichlet problem was derived in [21] when the inhomogeneous term and the boundary data were perturbed.

In this paper, we prove a strict comparison principle for the equations (0.1) and (0.2) under the rather mild requirement that F is locally uniformly continuous in x. With an additional structural condition ((1.3) for equation (0.1) and (2.14) for equation (0.2)), we can prove the comparison principle and hence the uniqueness of a viscosity solution for either equation(Theorems 1.3, 1.4, 2.4 and 2.5). Our strict comparison result holds for the typical degenerate elliptic equations such as the p-Laplace equation, the infinity Laplace equation and its normalized counterpart, their inhomogeneous versions, and with viscosity terms(and, of course, the linear equations and many other nonlinear equations). The comparison principle and the uniqueness theorem of the inhomogeneous equation include the uniqueness theorem for the inhomogeneous p-Laplace equation, the inhomogeneous infinity Laplace equation, or its normalized version. This fact indicates possible further applications of the results proved in the study of degenerate elliptic equations.

Throughout this paper, Ω denotes a bounded domain in $\mathbb{R}^n$. The notation $u \prec_{x_0} \varphi$ means that $u(x) \le \varphi(x)$ for all x near x_0 and $u(x_0) = \varphi(x_0)$.

A continuous function u defined in Ω is called a *viscosity sub-solution*,or simply a *sub-solution*, of a degenerate elliptic partial differential equation

$$F(x, u(x), \nabla u(x), D^2u(x)) = f(x) \tag{0.4}$$

in Ω, if

$$F(x_0, \varphi(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \ge f(x_0) ,$$

whenever $u \prec_{x_0} \varphi$ for any $x_0 \in \Omega$ and any C^2 test function φ. We sometimes write

$$F(x, u(x), \nabla u(x), D^2u(x)) \ge f(x)$$

in the viscosity sense to indicate that u is a viscosity sub-solution of the equation

$$F(x, u(x), \nabla u(x), D^2u(x)) = f(x) .$$

Similarly, u is called a *viscosity super-solution*, or simply a *super-solution*, of the partial differential equation

$$F(x, u(x), \nabla u(x), D^2u(x)) = f(x)$$

in Ω, if

$$F(x_0, \varphi(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \le f(x_0) ,$$

whenever $\varphi \prec_{x_0} u$ for any $x_0 \in \Omega$ and any C^2 test function φ.

A *viscosity solution*, or simply a *solution*, of the partial differential equation

$$F(x, u(x), \nabla u(x), D^2u(x)) = f(x)$$

in Ω is both a viscosity sub-solution and super-solution of the equation.

The definition of a viscosity solution of the normalized equation

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) = f(x) \tag{0.5}$$

need to be made clear at the points where $\nabla u(x)$ is 0.

For $u \in C(\Omega)$, $x_0 \in \Omega$, and $r > 0$ with $\overline{B}_r(x_0) \subset \Omega$, we define $g(r) = \max_{|x-x_0|=r} u(x)$ and $h(r) = \min_{|x-x_0|=r} u(x)$. In addition, x_r^+ denotes any point with $|x_r^+ - x_0| = r$ such that $u(x_r^+) = g(r)$, while x_r^- denotes any point with $|x_r^- - x_0| = r$ such that $u(x_r^-) = h(r)$.

If $x_0 \in \Omega$ and $u \in C(\Omega)$ such that u is twice differentiable at x_0, we define the *set of maximum directions* of u at x_0 to be the set

$$E_u^+(x_0) = \left\{ e \in S^1 \ : \ e = \lim_k \frac{x_r^+ - x_0}{r_k} \quad \text{for some sequence } r_k \downarrow 0 \right\}$$

and the *set of minimum directions* of u at x_0 to be the set

$$E_u^-(x_0) = \left\{ e \in S^1 \ : \ e = \lim_k \frac{x_r^- - x_0}{r_k} \quad \text{for some sequence } r_k \downarrow 0 \right\} .$$

Here S^1 denotes the unit sphere of $\mathbb{R}^n$. If u is twice differentiable at x_0 and $\nabla u(x_0) \ne 0$, it can be shown that $E^+(x_0) = \{\widehat{\nabla u}(x_0)\}$ and $E^-(x_0) = \{-\widehat{\nabla u}(x_0)\}$.

A function $u \in \Omega$ is called a *viscosity sub-solution* of the normalized partial differential equation

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) = f(x)$$

in Ω, if

$$F(x_0, \varphi(x_0), \widehat{\nabla \varphi}(x_0), D^2\varphi(x_0)) \geq f(x_0)\ , \quad \text{if } \nabla\varphi(x_0) \neq 0\ , \quad \text{or}$$

$$F(x_0, \varphi(x_0), e, D^2\varphi(x_0)) \geq f(x_0)\ , \quad \text{for every } e \in E^+_\varphi(x_0)\ , \quad \text{if } \nabla\varphi(x_0) = 0$$

whenever $u \prec_{x_0} \varphi$ for any $x_0 \in \Omega$ and any C^2 test function φ.

Similarly, u is a *viscosity super-solution* of the partial differential equation

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) = f(x)$$

in Ω, if

$$F(x_0, \varphi(x_0), \widehat{\nabla \varphi}(x_0), D^2\varphi(x_0)) \leq f(x_0)\ , \quad \text{if } \nabla\varphi(x_0) \neq 0\ , \quad \text{or}$$

$$F(x_0, \varphi(x_0), e, D^2\varphi(x_0)) \geq f(x_0)\ , \quad \text{for every } e \in E^-_\varphi(x_0)\ , \quad \text{if } \nabla\varphi(x_0) = 0$$

whenever $\varphi \prec_{x_0} u$ for any $x_0 \in \Omega$ and any C^2 test function φ.

A *viscosity solution*, or simply a *solution*, of the partial differential equation

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) = f(x)$$

in Ω is both a viscosity sub-solution and super-solution of the equation.

We will need the concepts of superjets and subjets in our approach.

Definition 0.1. Suppose $u \in C(\Omega)$. The second-order *superjet* of u at x_0 is defined to be the set

$$J^{2,+}_\Omega u(x_0) = \big\{(D\varphi(x_0), D^2\varphi(x_0)) \ : \ \varphi \text{ is } C^2 \text{ and } u \prec_{x_0} \varphi\big\}\ ,$$

whose closure is defined to be the set

$$\begin{aligned}\overline{J}^{2,+}_\Omega u(x_0) = \big\{&(q, X) \in \mathbb{R}^n \times \mathcal{S}_{n\times n} \ : \ \exists\ (x_n, q_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \text{ such that} \\ &(q_n, X_n) \in J^{2,+}_\Omega u(x_n) \text{ and } (x_n, u(x_n), q_n, X_n) \to (x_0, u(x_0), q, X)\big\}\ .\end{aligned}$$

The second-order *subjet* of u at x_0 is defined to be the set

$$J^{2,-}_\Omega u(x_0) = \big\{(D\varphi(x_0), D^2\varphi(x_0)) \ : \ \varphi \text{ is } C^2 \text{ and } \varphi \prec_{x_0} u\big\}\ ,$$

whose closure is defined to be the set

$$\begin{aligned}\overline{J}^{2,-}_\Omega u(x_0) = \big\{&(q, X) \in \mathbb{R}^n \times \mathcal{S}_{n\times n} \ : \ \exists\ (x_n, q_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \text{ such that} \\ &(q_n, X_n) \in J^{2,-}_\Omega u(x_n) \text{ and } (x_n, u(x_n), q_n, X_n) \to (x_0, u(x_0), q, X)\big\}\ .\end{aligned}$$

1. A STRICT COMPARISON PRINCIPLE

We prove the uniqueness theorem of the equation (0.4) in this section. The key result in this section is the following strict comparison principle.

Theorem 1.1. *Suppose* $F : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}_{n\times n} \to \mathbb{R}$ *is degenerate elliptic and proper. We further assume that* F *is locally uniformly continuous in* x*, i.e. for any* $\Omega' \subset\subset \Omega$*, there exists a modulus of continuity* $\rho_{\Omega'}$ *such that* $|F(x, r, q, X) - F(y, r, q, X)| \leq \rho_{\Omega'}(|x - y|)$ *for any* x*,* $y \in \Omega'$*,* $r \in \mathbb{R}$*,* $q \in \mathbb{R}^n$*, and* $X \in \mathcal{S}_{n\times n}$*. Assume* u*,* $v \in C(\Omega)$ *satisfy in the viscosity sense*

$$F(x, u(x), \nabla u(x), D^2u(x)) \geq f_1(x) > f_2(x) \geq F(x, v(x), \nabla v(x), D^2v(x))$$

in Ω*, where* $f_j \in C(\Omega)$ *for* $j = 1, 2$*. Then*

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} (u - v)$$

holds.

Proof. As $F_C(x, r, q, X) := F(x, r-C, q, X)$ is also degenerate elliptic and proper for any constant $C \in \mathbb{R}$, we may without the loss of generality assume $u \leq v$ on $\partial\Omega$ and intend to prove $u \leq v$ in Ω. Furthermore, for any small $\delta > 0$, let $u_\delta = u - \delta$. Then $u_\delta < u$ on $\partial\Omega$ and, as F is proper,

$$F(x, u_\delta(x), \nabla u_\delta(x), D^2u_\delta(x)) = F(x, u(x) - \delta, \nabla u(x), D^2u(x)) \geq$$

$$\geq F(x, u(x), \nabla u(x), D^2u(x)) \geq f_1(x)$$

holds in the viscosity sense. If we can show that $u_\delta \leq v$ in Ω for every small $\delta > 0$, then it follows that $u \leq v$ in Ω. So we may additionally assume $u < v$ on $\partial\Omega$ in the following proof.

Suppose $\sup_\Omega (u - v) = \max_{\overline{\Omega}} (u - v) > 0$. For every small $\varepsilon > 0$, we define

$$w_\varepsilon(x, y) = u(x) - v(y) - \frac{1}{2\varepsilon} |x - y|^2 \quad \text{for all } (x, y) \in \overline{\Omega} \times \overline{\Omega} .$$

We also define $M_0 = \max_{\overline{\Omega}} (u - v)$ and

$$M_\varepsilon = \max_{\overline{\Omega}\times\overline{\Omega}} w_\varepsilon = u(x_\varepsilon) - v(y_\varepsilon) - \frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 ,$$

for some $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$.

By Lemma 3.1 of [13],

$$\lim_{\varepsilon\downarrow 0} M_\varepsilon = M_0 ,$$

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 = 0$$

and

$$\lim_{\varepsilon\downarrow 0}(u(x_\varepsilon) - v(y_\varepsilon)) = M_0 .$$

The inequality

$$|x_\varepsilon - y_\varepsilon| \leq \frac{\varepsilon}{2} + \frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2$$

implies that $\lim_{\varepsilon\downarrow 0} |x_\varepsilon - y_\varepsilon| = 0$.

As $M_\varepsilon \to M_0 > \max_{\partial\Omega} (u - v)$, we know x_ε, $y_\varepsilon \in \widetilde{\Omega}$ for all small enough ε and some sub-domain $\widetilde{\Omega} \subset\subset \Omega$.

Now we apply Theorem 3.2 of [13]. There exist X, $Y \in \mathcal{S}_{n\times n}$ such that $((x_\varepsilon - y_\varepsilon)/\varepsilon, X) \in \overline{J}^{2,+}_{\Omega} u(x_\varepsilon)$ and $((x_\varepsilon - y_\varepsilon)/\varepsilon, Y) \in \overline{J}^{2,-}_{\Omega} v(y_\varepsilon)$ and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} .$$

In particular, $X \le Y$.

The definition of the superjet $\overline{J}^{2,+}_{\Omega} u(x_\varepsilon)$ and of the subjet $\overline{J}^{2,-}_{\Omega} v(y_\varepsilon)$ implies that

$$F\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \ge f_1(x_\varepsilon) \tag{1.1}$$

and

$$F\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right) \le f_2(y_\varepsilon) \ . \tag{1.2}$$

Let $\delta = \inf_{\widetilde{\Omega}}(f_1 - f_2) > 0$. As a result of the facts that $X \le Y$, $u(x_\varepsilon) \ge v(y_\varepsilon)$, and the assumption that F is proper, the above inequalities (1.1) and (1.2)) imply

$$\begin{aligned} f_1(x_\varepsilon) \quad &\le F\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right) \le F\left(x_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right) \le \\ &\le F\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right) + \rho_{\widetilde{\Omega}}(|x_\varepsilon - y_\varepsilon|) \le \\ &\le f_2(y_\varepsilon) + \rho_{\widetilde{\Omega}}(|x_\varepsilon - y_\varepsilon|) \le f_1(y_\varepsilon) - \delta + \rho_{\widetilde{\Omega}}(|x_\varepsilon - y_\varepsilon|) < \\ &< f_1(x_\varepsilon) \ , \quad \text{if } \varepsilon \text{ is sufficiently small} \ . \end{aligned}$$

The last inequality holds as $\lim_{\varepsilon\downarrow 0}|x_\varepsilon - y_\varepsilon| = 0$ and x_ε, $y_\varepsilon \in \widetilde{\Omega}$.

□

It is clear that either f_1 or f_2 can be absorbed in the function F so that one may assume either f_1 or f_2 is identically zero. Furthermore, a careful examination of the preceding proof show that f may depend on the gradient of u as well as on x. We leave the work to the reader and state only the theorem below.

Theorem 1.2. *Suppose $F : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}_{n\times n} \to \mathbb{R}$ is degenerate elliptic and proper. We further assume that F is locally uniformly continuous in x, i.e. for any $\Omega' \subset\subset \Omega$, there exists a modulus of continuity $\rho_{\Omega'}$ such that $|F(x,r,q,X) - F(y,r,q,X)| \le \rho_{\Omega'}(|x-y|)$ for any x, $y \in \Omega'$, $r \in \mathbb{R}$, $q \in \mathbb{R}^n$, and $X \in \mathcal{S}_{n\times n}$. Assume u, $v \in C(\Omega)$ satisfy in the viscosity sense*

$$\begin{aligned} F(x, u(x), \nabla u(x), D^2u(x)) \ge f_1(x, \nabla u(x)) > f_2(x, \nabla v(x)) \ge \\ \ge F(x, v(x), \nabla v(x), D^2v(x)) \end{aligned}$$

in Ω, where $f_j \in C(\Omega\times\mathbb{R}^n)$ for $j = 1, 2$. Then

$$\sup_{\Omega}\,(u - v) \le \max_{\partial\Omega}\,(u - v)$$

holds.

In order to prove the standard comparison principle, we need to impose one more structural condition on F. We a ssume that there is a modulus of continuity $\sigma : (0,\infty) \to (0,\infty)$ such that

$$F(x, (1+t)r, (1+t)q, (1+t)X) \ge (1+\sigma(t))F(x,r,q,X) \tag{1.3}$$

holds for any $t > 0$, $x \in \Omega$, $q \in \mathbb{R}^n$, and $X \in \mathcal{S}_{n\times n}$. Under this condition and the assumption that F is degenerate elliptic, proper, and locally uniformly continuous in x, the following theorem is true.

Theorem 1.3. *Suppose u, $v \in C(\overline{\Omega})$ satisfy*

$$F(x, u(x), \nabla u(x), D^2u(x)) \geq f(x)$$

and

$$F(x, v(x), \nabla v(x), D^2v(x)) \leq f(x)$$

in the viscosity sense in the domain Ω, where f is a continuous positive function defined on Ω. Then

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)\ . \tag{1.4}$$

Proof. Without the loss of generality, we may assume that $u \leq v$ on $\partial\Omega$ and intend to prove $u \leq v$ in Ω. For every small $\delta > 0$, we take

$$u_\delta(x) = (1 + \delta)u(x) - \delta \|u\|_{L^\infty(\partial\Omega)}\ .$$

Then $u_\delta \leq u \leq v$ on $\partial\Omega$, and it is easy to see that

$$\begin{aligned} F(x, u_\delta(x), \nabla u_\delta(x), D^2u_\delta(x)) &\geq (1 + \sigma(\delta))F(x, u(x), \nabla u(x), D^2u(x)) \geq \\ &\geq (1 + \sigma(\delta))f(x) > f(x) \geq \\ &\geq F(x, v(x), \nabla v(x), D^2v(x)) \end{aligned}$$

in Ω in the viscosity sense.

The preceding strict comparison, Theorem 1.1, is readily applied to u_δ and v. We have thus shown that $u_\delta \leq v$ in Ω for any small $\delta > 0$. Sending δ to 0, we have obtained $u \leq v$ in Ω as desired.

□

It is obvious that the above theorem 1.3 is true if the condition $f > 0$ in Ω is replaced by the condition $f < 0$ in Ω, as the function $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \to \mathbb{R}$ defined by $G(x, r, q, X) = -F(x, -r, -q, -X)$ is degenerate elliptic and proper, and verifies the structural condition and the local uniform continuity in x.

The uniqueness theorem for equation (0.4) follows as a direct corollary of the preceding comparison principle.

Theorem 1.4. *Suppose $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \to \mathbb{R}$ is the same as in the preceding theorem, and that u and $v \in C(\overline{\Omega})$ are both viscosity solutions of the equation* (0.4)

$$F(x, w(x), \nabla w(x), D^2w(x)) = f(x)$$

in Ω, where f is a continuous function defined on Ω such that either $f > 0$ in Ω or $f < 0$ in Ω holds. If, in addition, $u = v$ on $\partial\Omega$, then $u = v$ in Ω.

The condition that the right-hand-side f does not change sign in Ω is indispensable, as a counter-example for the infinity Laplacian

$$F(x, w(x), \nabla w(x), D^2w(x)) = \langle D^2w(x)\nabla w(x), \nabla w(x)\rangle$$

provided in [20] shows the uniqueness of a viscosity solution subject to given boundary data fails without such a condition.

2. THE NORMALIZED VERSION OF UNIQUENESS

In this section, we consider the normalized equation (0.5)

$$F(x, u(x), \widehat{\nabla u}(x), D^2 u(x)) = f(x) \ .$$

We also assume that $F(x, r, q, 0_{n\times n}) = 0$, for any $x \in \Omega$, $r \in \mathbb{R}$, and $q \in S^1$, in this section besides the degenerate ellipticity and properness, and that F is locally uniformly continuous in x and continuous in all other arguments. Here $0_{n\times n}$ is the $n \times n$ zero matrix.

We start with a maximum principle in the special case in which a sub-solution u is semi-convex.

Lemma 2.1. *Suppose $u \in C(\overline{\Omega})$ satisfies the normalized differential inequality*

$$F(x, u(x), \widehat{\nabla u}(x), D^2 u(x)) \geq f(x) > 0$$

in the viscosity sense in Ω, where $f \in C(\Omega)$. In addition, u is semi-convex in $\mathbb{R}^n$. Then u attains its maximum over $\overline{\Omega}$ only on $\partial\Omega$.

Proof. Assume $u(x_0) \geq \max_{\partial\Omega} u$ for some $x_0 \in \Omega$. For $\delta > 0$, we define $u_\delta(x) = u(x) - \delta|x - x_0|^2$. As F is continuous, $F(x, u_\delta(x), \widehat{\nabla u}(x), D^2 u_\delta(x)) \geq \tilde{f}(x) > 0$ for a new continuous function $\tilde{f}$ if δ is small enough. So we may without the loss of generality assume that $u(x_0) > \max_{\partial\Omega} u$ in the first place. Then there exists $\varepsilon > 0$ such that

$$\sup_{\Omega} (u(x) + \langle p, x\rangle) > \max_{\partial\Omega} (u(x) + \langle p, x\rangle)$$

for any $p \in \mathbb{R}^n$ and $|p| \leq \varepsilon$. For every $p \in \mathbb{R}^n$ with $|p| \leq \varepsilon$, the function $x \mapsto u(x) + \langle p, x\rangle$ is semi-convex and thus is differentiable at every point in the set

$$\operatorname{Argmax}(x \mapsto u(x) + \langle p, x\rangle) := \{z \in \overline{\Omega} \ : \ u(z) + \langle p, z\rangle = \max_{\overline{\Omega}} (u(x) + \langle p, x\rangle)\} \ .$$

According to a theorem of Alexandroff, a convex function is twice differentiable almost everywhere. So is the semi-convex function u.

As u is locally Lipschitz continuous, the set

$$\bigcup_{0<|p|\leq\varepsilon} \operatorname{Argmax}(x \mapsto u(x) + \langle p, x\rangle)$$

has positive measure. So there exists a point $x_1 \in \cup_{0<|p|\leq\varepsilon} \operatorname{Argmax}(x \mapsto u(x) + \langle p, x\rangle)$ at which u is twice differentiable. At x_1, $\nabla u(x_1) = p \neq 0$ and $D^2 u(x_1) \leq 0$ for some $p \in \mathbb{R}^n$ with $|p| \leq \varepsilon$. Therefore, the following inequalities hold.

$$0 < F(x_1, u(x_1), \widehat{\nabla u}(x_1), D^2 u(x_1)) \leq$$

$$\leq F(x_1, u(x_1), \hat{p}, 0_{n\times n}) = 0 \ , \quad \text{as } F \text{ is degenerate elliptic} \ .$$

We end up with a contradiction.

□

We now prove the following strict comparison principle for the normalized equation (0.5).

Theorem 2.2. *We further assume that F is locally uniformly continuous in x, i.e. for any $\Omega' \subset\subset \Omega$, there exists a modulus of continuity $\rho_{\Omega'} : (0, \infty) \to (0, \infty)$ with $\rho_{\Omega'}(0+) = 0$ such that*

$$|F(x, r, q, X) - F(y, r, q, X)| \leq \rho_{\Omega'}(|x - y|)$$

for any $x,\ y \in \Omega'$, $r \in \mathbb{R}$, $q \in S^1$, *and* $X \in \mathcal{S}_{n\times n}$. *Suppose*

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) \geq f_1(x)$$

and

$$F(x, v(x), \widehat{\nabla v}(x), D^2v(x)) \leq f_2(x)$$

in Ω *in the viscosity sense explained in the beginning section even at the points where the gradient of a test function is zero, where* $u,\ v \in C(\overline{\Omega})$. *In addition,* $f_1 > f_2$, *and* $f_j \in C(\Omega)$ *for* $j = 1, 2$. *Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)$$

holds.

Proof. Without the loss of generality, we assume $u \leq v$ on $\partial\Omega$ and prove that $u \leq v$ in Ω. Furthermore, for any small $\delta > 0$, let $u_\delta = u - \delta$. Then $u_\delta < v$ on $\partial\Omega$ and the inequalities

$$F(x, u_\delta(x), \widehat{\nabla u_\delta}(x), D^2u_\delta(x)) = F(x, u(x) - \delta, \widehat{\nabla u}(x), D^2u(x)) \geq$$
$$\geq F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) \geq f_1(x)$$

hold in the viscosity sense. The first inequality is due to the fact F is proper.

If it can be shown that $u_\delta \leq v$ in Ω for every small δ, then it follows that $u \leq v$ in Ω. So we may assume $u < v$ on $\partial\Omega$ in the following proof.

If $\sup_\Omega(u - v) > \max_{\partial\Omega}(u - v)$, then there exists a sub-domain $\Omega' \subset\subset \Omega$ such that $\sup_{\Omega'}(u - v) > \max_{\partial\Omega}(u - v)$.

Take any $A \geq \max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\}$. For any sufficiently small real number $\varepsilon > 0$, we take $\delta = 3\sqrt{A\varepsilon}$ and $\Omega_\delta = \{x \in \Omega\ :\ \operatorname{dist}(x, \partial\Omega) > \delta\}$. We define, on $\mathbb{R}^n$,

$$u^\varepsilon(x) = \sup_{y\in\Omega}\left(u(y) - \frac{1}{2\varepsilon}\,|x - y|^2\right) \tag{2.1}$$

and

$$v_\varepsilon(x) = \inf_{y\in\Omega}\left(v(y) + \frac{1}{2\varepsilon}\,|x - y|^2\right) \tag{2.2}$$

as the sup- and inf-convolutions of u and v respectively.

For any $y \in \Omega$ such that $|y - x| \geq 2\sqrt{A\varepsilon}$, $u(y) - (1/2\varepsilon)|x - y|^2 \leq u(x)$ holds.

So, in Ω_δ,

$$u^\varepsilon(x) = \sup_{y\in\Omega, |x-y|\leq 2\sqrt{A\varepsilon}}\left(u(y) - \frac{1}{2\varepsilon}\,|x - y|^2\right) = \tag{2.3}$$
$$= \sup_{|z|\leq 2\sqrt{A\varepsilon}}\left(u(x + z) - \frac{1}{2\varepsilon}|z|^2\right)\ ,$$

as $x + z \in \Omega$ for any $x \in \Omega_\delta$ and $|z| \leq 2\sqrt{A\varepsilon}$. Similarly, for $x \in \Omega_\delta$,

$$v_\varepsilon(x) = \inf_{y\in\Omega, |x-y|\leq 2\sqrt{A\varepsilon}}\left(v(y) + \frac{1}{2\varepsilon}\,|x - y|^2\right) = \tag{2.4}$$
$$= \inf_{|z|\leq 2\sqrt{A\varepsilon}}\left(v(x + z) + \frac{1}{2\varepsilon}|z|^2\right)\ .$$

Let

$$f_{1,\varepsilon}(x) = \inf_{x+z\in\Omega, |z|\le 2\sqrt{A\varepsilon}} f_1(x+z) = \inf_{|z|\le 2\sqrt{A\varepsilon}} f_1(x+z) \tag{2.5}$$

and

$$f_2^{\varepsilon}(x) = \sup_{x+z\in\Omega, |z|\le 2\sqrt{A\varepsilon}} f_2(x+z) = \sup_{|z|\le 2\sqrt{A\varepsilon}} f_2(x+z)\ , \tag{2.6}$$

for $x \in \Omega_\delta$. Clearly, $f_{1,\varepsilon}$ is lower semi-continuous. It is continuous due to the equicontinuity of the one parameter family of the functions $x \mapsto f_1(x+z)$ in any compact subset of Ω. f_2^{ε} is continuous for a similar reason.

For every z with $|z| \le 2\sqrt{A\varepsilon}$ and $x \in \Omega_\delta$, we define $u_z(x) = u(x+z) - (1/2\varepsilon)\,|z|^2$ and $v_z(x) = v(x+z) + (1/2\varepsilon)\,|z|^2$. Then the following inequalities hold in the viscosity sense.

$$\begin{aligned}
&F(x, u_z(x), \widehat{\nabla u_z}(x), D^2u_z(x)) = \\
&= F(x, u(x+z) - \frac{1}{2\varepsilon}\,|z|^2, \widehat{\nabla u}(x+z), D^2u(x+z)) \ge \\
&\ge F(x, u(x+z), \widehat{\nabla u}(x+z), D^2u(x+z)) \ge \quad \text{as } F \text{ is proper ;} \\
&\ge f_1(x+z) - (F(x+z, u(x+z), \widehat{\nabla u}(x+z), D^2u(x+z)) - \\
&-F(x, u(x+z), \widehat{\nabla u}(x+z), D^2u(x+z))) \ge \\
&\ge f_1(x+z) - \rho_{\Omega'}(|z|) \ge \\
&\ge f_{1,\varepsilon}(x) - \rho_{\Omega'}(2\sqrt{A\varepsilon}) > \\
&> f_2^{\varepsilon}(x) + \rho_{\Omega'}(2\sqrt{A\varepsilon}) \ge \quad \text{in } \Omega_\delta \subset\subset \Omega\ , \quad \text{if } \varepsilon \text{ is small enough .} \\
&\ge f_2^{\varepsilon}(x) + \rho_{\Omega'}(|z|) \ge \\
&\ge f_2(x+z) + \Big(F(x, v(x+z), \widehat{\nabla v}(x+z), D^2v(x+z)) - \\
&-F(x+z, v(x+z), \widehat{\nabla v}(x+z), D^2v(x+z))\Big) \ge \\
&\ge F(x, v(x+z), \widehat{\nabla v}(x+z), D^2v(x+z)) \ge \\
&\ge F\left(x, v(x+z) + \frac{1}{2\varepsilon}\,|z|^2, \widehat{\nabla v}(x+z), D^2v(x+z)\right) = \quad \text{as } F \text{ is proper} \\
&= F(x, v_z(x), \widehat{\nabla v_z}(x), D^2v_z(x))\ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&F(x, u^{\varepsilon}, \widehat{\nabla u}^{\varepsilon}, D^2u^{\varepsilon}) \ge \tilde{f}_1(x) := f_{1,\varepsilon} - \rho_{\Omega'}(2\sqrt{A\varepsilon}) > \\
&> \tilde{f}_2(x) := f_2^{\varepsilon} + \rho_{\Omega'}(2\sqrt{A\varepsilon}) \ge F(x, v_{\varepsilon}, \widehat{\nabla v}_{\varepsilon}, D^2v_{\varepsilon})
\end{aligned}$$

hold in Ω_δ in the viscosity sense.

The rest properties of u^{ε} and v_{ε} are summarized in the following proposition, the proof of which is well known (see, for example, [4] Proposition 6.4).

Proposition 2.3. *u^ε and $-v_\varepsilon$ are semi-convex in $\mathbb{R}^n$. $u^\varepsilon \geq u$ and $v_\varepsilon \leq v$ in Ω. u^ε and v_ε converge locally uniformly to u and v respectively in Ω, as $\varepsilon \to 0$. u^ε and v_ε are both differentiable at the maximum points of $u^\varepsilon - v_\varepsilon$.*

As a result, if we take the value of ε smaller if necessary, then $v_\varepsilon > u^\varepsilon$ on $\partial\Omega_\delta$, $F(x, u^\varepsilon(x), \widehat{\nabla u}^\varepsilon(x), D^2u^\varepsilon(x)) \geq f_{1,\varepsilon}(x)$ and $F(x, v_\varepsilon(x), \widehat{\nabla} v_\varepsilon(x), D^2 v_\varepsilon(x)) \leq f_2^\varepsilon(x)$ in Ω_δ, and $f_{1,\varepsilon} > f_2^\varepsilon$ in Ω_δ.

If we can prove $u^\varepsilon \leq v_\varepsilon$ in Ω_δ for any small $\varepsilon > 0$ and $\delta = 3\sqrt{A\varepsilon}$, then $u \leq v$ in Ω holds. So we may without the loss of generality assume that u and $-v$ are semi-convex in $\mathbb{R}^n$. In particular, u and v are locally Lipschitz continuous in Ω.

Suppose $u(x_0) > v(x_0)$ for some $x_0 \in \Omega$. Without the loss of generality, we may assume that $u(x_0) - v(x_0) = \max_{\overline{\Omega}}(u - v)$. Then there exists $\delta > 0$ such that for any $h \in \mathbb{R}^n$ with $|h| < \delta$, we have $v(x_0) < u(x_0 + h)$, while $u(\cdot + h) < v(\cdot)$ holds in $\Omega \backslash \Omega_\delta$, an d $f_1(x + h) > f_2(x)$, $\forall\ x \in \Omega_\delta$. For any small positive number ε and $h \in \mathbb{R}^n$ with $|h| < \delta$, we define

$$w_{\varepsilon,h}(x, y) = u(x + h) - v(y) - \frac{1}{2\varepsilon}|x - y|^2\ , \tag{2.7}$$

$\forall\ (x, y) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$.

Let

$$M_0 = \max_{\overline{\Omega}}(u - v)\ , \tag{2.8}$$

$$M_h = \max_{\overline{\Omega}_\delta}(u(\cdot + h) - v(\cdot))\ , \tag{2.9}$$

and

$$M_{\varepsilon,h} = \max_{\overline{\Omega}_\delta \times \overline{\Omega}_\delta} w_{\varepsilon,h} = u(x_{\varepsilon,h} + h) - v(y_{\varepsilon,h}) - \frac{1}{2\varepsilon}|x_{\varepsilon,h} - y_{\varepsilon,h}|^2 \tag{2.10}$$

for some $(x_{\varepsilon,h}, y_{\varepsilon,h}) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$. Our assumption implies $M_h > 0$ for all $h \in \mathbb{R}^n$ with $0 \leq |h| < \delta$, and clearly $\lim_{h\to 0} M_h = M_0$.

We notice that M_h is Lipschitz continuous in $h \in \mathbb{R}^n$ with $|h| < \delta$, if δ is taken smaller, since both u and $-v$ are semi-convex and whence locally Lipschitz continuous.

By Lemma 3.1 of [13], we know that

$$\lim_{\varepsilon\downarrow 0} M_{\varepsilon,h} = M_h\ , \tag{2.11}$$

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}|x_{\varepsilon,h} - y_{\varepsilon,h}|^2\ , \tag{2.12}$$

and

$$\lim_{\varepsilon\downarrow 0}(u(x_{\varepsilon,h} + h) - v(y_{\varepsilon,h})) = M_h\ . \tag{2.13}$$

As a result of the second equality, $\lim_{\varepsilon\downarrow 0}|x_{\varepsilon,h} - y_{\varepsilon,h}| = 0$.

As $M_h > 0 \geq \max_{\partial\Omega_\delta}(u(\cdot + h) - v(\cdot))$, we know $x_{\varepsilon,h},\ y_{\varepsilon,h} \in \Omega_1$ for some $\Omega_1 \subset\subset \Omega_\delta$ and all small $\varepsilon > 0$.

Theorem 3.2 of [13] implies that there exist $X = X_{\varepsilon,h}$, $Y = Y_{\varepsilon,h} \in \mathcal{S}_{n\times n}$ such that $((x_{\varepsilon,h} - y_{\varepsilon,h})/\varepsilon, X) \in \overline{J}_{\Omega}^{2,+} u(x_{\varepsilon,h} + h)$, $((x_{\varepsilon,h} - y_{\varepsilon,h})/\varepsilon, Y) \in \overline{J}_{\Omega}^{2,-} v(y_{\varepsilon,h})$ and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

In particular, $X \le Y$.

Next, we solve the problem via a dichotomy.

Case 1. Suppose that there exists $h \in \mathbb{R}^n$ with $|h| < \delta$, and a sequence $\varepsilon_k \to 0$ such that $x_{\varepsilon_k,h} \ne y_{\varepsilon_k,h}$ for each k.

In this case, the following inequalities hold.

$$\begin{aligned}
& f_1(x_{\varepsilon_k,h} + h) \le \\
& \le F(x_{\varepsilon_k,h} + h, u(x_{\varepsilon_k,h} + h), \frac{\widehat{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}}{\varepsilon_k}, X_{\varepsilon_k,h}) \le \\
& \le F(x_{\varepsilon_k,h} + h, v(y_{\varepsilon_k,h}), \frac{\widehat{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}}{\varepsilon_k}, X_{\varepsilon_k,h}) \le \quad \text{as } F \text{ is proper .} \\
& \le F(x_{\varepsilon_k,h} + h, v(y_{\varepsilon_k,h}), \frac{\widehat{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}}{\varepsilon_k}, Y_{\varepsilon_k,h}) = \quad \text{as } F \text{ is degenerate elliptic .} \\
& = F(y_{\varepsilon_k,h}, v(y_{\varepsilon_k,h}), \frac{\widehat{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}}{\varepsilon_k}, Y_{\varepsilon_k,h}) + \\
& + \Bigg\{ F(x_{\varepsilon_k,h} + h, v(y_{\varepsilon_k,h}), \frac{\widehat{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}}{\varepsilon_k}, Y_{\varepsilon_k,h}) - \\
& - F(y_{\varepsilon_k,h}, v(y_{\varepsilon_k,h}), \frac{\widehat{x_{\varepsilon_k,h} - y_{\varepsilon_k,h}}}{\varepsilon_k}, Y_{\varepsilon_k,h}) \Bigg\} \le \\
& \le f_2(y_{\varepsilon_k,h}) + \sigma_{\Omega_1}(|x_{\varepsilon_k,h} - y_{\varepsilon_k,h} + h|) \ .
\end{aligned}$$

For a subsequence of $\{\varepsilon_k\}$, $x_{\varepsilon_k,h} \to x_h$ and $y_{\varepsilon_k,h} \to y_h$. As $\lim_{\varepsilon\downarrow 0} |x_{\varepsilon_k,h} - y_{\varepsilon_k,h}| = 0$, we know that $x_h = y_h$ and $\sigma_{\Omega_1}(|x_{\varepsilon_k,h} - y_{\varepsilon_k,h} + h|) < (1/2)\inf_{\Omega_1}(f_1 - f_2)$ for h small enough, which leads to a contradiction with the assumption $f_1(x_h) > f_2(x_h)$.

Case 2. For every $h \in \mathbb{R}^n$ with $|h| < \delta$, $x_{\varepsilon,h} = y_{\varepsilon,h}$ holds for every small $\varepsilon > 0$ and one pair $(x_{\varepsilon,h}, y_{\varepsilon,h})$.

Then $M_{\varepsilon,h} = u(x_{\varepsilon,h} + h) - v(y_{\varepsilon,h}) = M_h$. We simply write $x_{\varepsilon,h} = y_{\varepsilon,h} = x_h$.

The semi-convexity of $u(\cdot + h)$ and $-v(\cdot)$ implies that the two functions are differentiable at the maximum point x_h of their sum. The definition of x_h shows that

$$u(x_h + h) - v(x_h) \ge u(y + h) - v(x_h) - \frac{1}{2\varepsilon}\,|x_h - y|^2 \ ,$$

which in turn implies

$$u(x_h + h) \ge u(y + h) - \frac{1}{2\varepsilon}\,|x_h - y|^2 \ ,$$

for small $\varepsilon > 0$. So $\nabla u(x_h + h) = \nabla v(x_h) = 0$.

For small $h,\ k \in \mathbb{R}^n$,

$$\begin{aligned} M_h \quad &= u(x_h + h) - v(x_h) \geq \\ &\geq u(x_k + h) - v(x_k) = \\ &= M_k + u(x_k + h) - u(x_k + k) \geq \\ &\geq M_k - \circ(|h - k|) \ , \quad \text{as } \nabla u(x_k + k) = 0 \ . \end{aligned}$$

So $DM_h = 0$ a. e. as M_h is Lipschitz continuous, which implies $M_h = M_0$ for all small $h \in \mathbb{R}^n$.

At x_0, either $f_1(x_0) > 0$ or $f_2(x_0) < 0$ holds due to the fact $f_1 > f_2$. Without the loss of generality, we assume that $f_1(x_0) > 0$. The proof for the case $f_2(x_0) < 0$ is parallel. So $F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) \geq f_1(x) > 0$ in the viscosity sense in a neighborhood of x_0.

For any h with $|h| < \delta$,

$$u(x_0 + h) - v(x_0) \leq u(x_h + h) - v(x_h) = u(x_0) - v(x_0) \ .$$

So $u(x_0)$ is a local maximum of u, which is a contradiction to the preceding lemma 2.1.

□

In order to prove the standard comparison principle for the normalized equation (0.5), we again impose one additional structural condition on F. We assume that there is a modulus of continuity $\sigma : (0, \infty) \to (0, \infty)$ with $\sigma(0+) = 0$ such that

$$F(x, (1 + t)r, q, (1 + t)X) \geq (1 + \sigma(t))F(x, r, q, X) \tag{2.14}$$

holds for any $t > 0$, $x \in \Omega$, $q \in S^1$, and $X \in \mathcal{S}_{n \times n}$. Under this condition and the assumption that F is degenerate elliptic, proper, locally uniformly continuous in x and continuous in all other arguments, the following theorem is true.

Theorem 2.4. *Suppose $u,\ v \in C(\overline{\Omega})$ satisfy*

$$F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) \geq f(x)$$

and

$$F(x, v(x), \widehat{\nabla v}(x), D^2v(x)) \leq f(x)$$

in the viscosity sense in the domain Ω, where f is a continuous positive function defined on Ω. Then

$$\sup_{\Omega} (u - v) \leq \max_{\partial\Omega} (u - v) \ .$$

Proof. Without the loss of generality, we may assume that $u \leq v$ on $\partial\Omega$ and intend to prove $u \leq v$ in Ω.

For every small $\delta > 0$, we take

$$u_\delta(x) = (1 + \delta)u(x) - \delta\|u\|_{L^\infty(\partial\Omega)} \ .$$

Then $u_\delta \leq u \leq v$ on $\partial\Omega$, and it is easy to see that

$$\begin{aligned} F(x, u_\delta(x), \widehat{\nabla u_\delta}(x), D^2u_\delta(x)) \quad &\geq (1 + \sigma(\delta))\, F(x, u(x), \widehat{\nabla u}(x), D^2u(x)) \geq \\ &\geq (1 + \sigma(\delta))\, f(x) > f(x) \geq \\ &\geq F(x, v(x), \widehat{\nabla v}(x), D^2v(x)) \end{aligned}$$

in Ω in the viscosity sense.

The preceding strict comparison theorem 2.2 is readily applied to u_δ and v. We have shown that $u_\delta \leq v$ in Ω for any small $\delta > 0$. Sending δ to 0, we have $u \leq v$ in Ω as desired.

□

The uniqueness theorem follows from the preceding comparison principle.

Theorem 2.5. *Suppose $F : \Omega \times \mathbb{R} \times S^1 \times \mathcal{S}_{n\times n}$ is the same as in the preceding theorem, and that u and $v \in C(\overline{\Omega})$ are both viscosity solutions of the normalized equation* (0.5)

$$F(x, w(x), \widehat{\nabla w}(x), D^2 w(x)) = f(x)$$

in Ω, where f is a continuous function defined in Ω such that either $f > 0$ or $f < 0$ in Ω holds. If, in addition, $u = v$ on $\partial\Omega$, then $u = v$ in Ω.

The condition that f does not change sign in Ω is necessary. A counter-example for the normalized infinity Laplacian

$$F(x, w(x), \widehat{\nabla w}(x), D^2 w(x)) = \langle D^2 w(x) \widehat{\nabla w}(x), \widehat{\nabla w}(x) \rangle$$

is provided in [22] to show that a uniqueness theorem fails without such a condition.

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