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**On a class of geometric PDE's: the prescribed
Levi curvature equations**

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Dedicated to Ermanno Lanconelli, on the occasion of his 65th birthday

Abstract¹. In this paper we are concerned with the following problems: the problem of characterizing real hypersurfaces in \mathbb{C}^{n+1} with some constant Levi curvatures, the Dirichlet problem for the prescribed Levi curvature equation with sharp conditions on the boundary data and the problem of local regularity of viscosity solutions of the prescribed Levi curvature equations.

1. INTRODUCTION

In this paper we give a survey about some notions of curvatures associated with pseudoconvexity and the Levi form the way the classical Gauss and Mean curvatures are related to the convexity and to the Hessian matrix. We shall first show that these curvature equations contain information about the geometric feature of a closed hypersurface. Then, we shall show that the curvature operators lead to a new class of second order fully nonlinear equations which are not elliptic at any point. However, they have the following redeeming feature: the missing ellipticity direction can be recovered by suitable commutation relations. We shall use this property to study the Dirichlet problem for graphs with prescribed Levi curvature.

The paper is organized as follows. In Section 2 we introduce notations and we precisely define the Levi form and the Levi curvatures for a real hypersurface in \mathbb{C}^{n+1} . Then, we show that the Levi curvatures contain information about the measure of a bounded domain by showing the isoperimetric estimates (2). In Section 3 we use our isoperimetric estimate to prove a characterization result for spheres of Alexandrov type. We then analyze why the moving plane technique in the complex setting seems to fail, and we recall some recent symmetry results due to different authors for closed hypersurfaces with some constant Levi curvatures. In Section 4 we are concerned with the Dirichlet problem for the prescribed total Levi curvature equation and we show existence and uniqueness of viscosity solutions of the problem under sharp conditions on the boundary data. In Section 5 we are concerned with

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the problem of local regularity of viscosity solutions of the prescribed Levi curvature equations. We shall show that different hypotheses on the zeros of the prescribed curvatures will lead to different regularity properties for the viscosity solutions of the prescribed Levi curvature equation.

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2. NOTATIONS, DEFINITIONS AND ISOPERIMETRIC INEQUALITIES

Let Σ be an oriented hypersurface of \mathbb{C}^{n+1} and p be a point in Σ . We shall denote by $T_p\Sigma$ the real tangent space of Σ at p and by J be the standard complex structure map for $T_p\mathbb{C}^{n+1}$ corresponding to the product by i , i.e. for all $j = 1, \dots, n + 1$

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Let us denote by $H_p\Sigma = T_p\Sigma \cap JT_p\Sigma$ the complex tangent space. Precisely a vector field $X \in H_p\Sigma$ iff $X \in T_p\Sigma$ and $JX \in T_p\Sigma$. It is a standard fact that the complexified holomorphic tangent space $\mathbb{C}H_p\Sigma = \mathbb{C} \otimes H_p\Sigma$ admits a decomposition as a direct sum of holomorphic and anti-holomorphic tangent vectors $\mathbb{C}H_p\Sigma = H_p^{1,0}\Sigma \oplus H_p^{0,1}\Sigma$.

Definition 1 (Levi Form). If Σ is represented in a neighborhood of p by $f = 0$ with $(\nabla f)(p) \neq 0$ and the orientation of Σ is so that $\nabla f/|\nabla f|$ is the outward pointing normal, then the Levi form at p is the Hermitian form on $H_p^{1,0}\Sigma \times H_p^{1,0}\Sigma$

$$L_p(Z, W) = \bar{\partial}\partial f(p)(Z, \bar{W}), \quad Z, W \in H_p^{1,0}\Sigma.$$

where $\bar{\partial}\partial f$ is the transpose of the complex Hessian matrix of f .

It is quite easy to recognize that L_p contains less information on the shape of Σ than the second fundamental form II_p . Indeed, by [2] for every unitary vector fields $Z = X - iY \in H_p^{1,0}\Sigma$

$$L_p(Z, Z) = II_p(X, X) + II_p(Y, Y),$$

Definition 2 (Pseudoconvexity). A domain D is *pseudoconvex* if the Levi form of its boundary is positive semidefinite at every point of ∂D .

Every convex set is pseudoconvex, while the converse does not hold. Moreover, it is easy to see that pseudoconvexity is invariant under a biholomorphic change of variables. Indeed, the Levi form is the biholomorphic invariant part of the real Hessian matrix of f .

By analogy with the Gauss curvature, which is defined in terms of the second fundamental form, we give the following definition

Definition 3 (Total Levi curvature). We define the *total Levi Curvature* of ∂D at $p \in \partial D$ as the real number

$$k_{\partial D}(p) = \prod_{j=1}^n \lambda_j,$$

with λ_j the eigenvalues of the normalized Levi form

$$\frac{L_p}{|\partial f|} = \frac{\bar{\partial}\partial f(p)}{|\partial f|}.$$

In [22] we explicitly wrote it in terms second order derivatives of a defining function f of ∂D as

$$k_{\partial D}(p) = -\frac{1}{|\partial_p f|^{n+2}} \det \begin{pmatrix} 0 & f_{\bar{z}} \\ f_z & f_{z,\bar{z}} \end{pmatrix}$$

and we showed that the right hand side is independent of the defining function f .

Analogously, we can define q -pseudoconvexity as follows.

Definition 4 (q -pseudoconvexity and q -Levi curvature). A domain D of \mathbb{C}^{n+1} is q -pseudoconvex if, for every $j \in \{1, \dots, q\}$, the j -th elementary symmetric function in the eigenvalues of the normalized Levi form of its boundary

$$\sigma^{(j)}(\lambda_1, \dots, \lambda_n) := \sum_{1 \leq i_1 < \dots < i_j \leq n} \lambda_{i_1} \cdots \lambda_{i_j}$$

is non negative at every point $p \in \partial D$.

In this situation, for every $q \in \{1, \dots, n\}$ we define q -Levi curvature of ∂D as

$$k_{\partial D}^{(q)}(p) = \frac{\sigma^{(q)}(\lambda)}{\binom{n}{q}}.$$

Precisely, $k^{(n)}$ is the total Levi curvature and $k^{(1)}$ is the mean Levi curvature

$$k_{\partial D}^{(1)} = \frac{\lambda_1 + \dots + \lambda_n}{n}.$$

Example 1 (q -Levi curvature of a sphere). Let $f(z) = |z|^2 - r^2$ be a defining function for the ball B_r of radius r . Then the Levi form of its boundary is the identity matrix and the normalized Levi form is

$$\frac{1}{r} I_n, \quad \forall p \in \partial B_r.$$

Hence, the eigenvalues of the normalized Levi form are all equal to $1/r$, and

$$k_{\partial B_r}^{(q)}(p) = \frac{1}{r^q}, \quad \forall p \in \partial B_r.$$

In a joint work with Vittorio Martino [19] we proved that the Levi curvatures, which have been first introduced by Slodkowski and Tomassini by analogy with Euclidean curvatures, contain information about the shape of a closed hypersurface. To show this, let us first recall the following link between elementary symmetric functions in the eigenvalues of the complex Hessian $\partial\bar{\partial}f$ of the defining function f of a bounded domain and the Levi curvatures of its boundary.

Theorem 1 (Integral formulas). *Let Ω be a bounded domain of \mathbb{C}^{n+1} with boundary a real hypersurface of class C^2 . For every defining function f of class C^2 of $\Omega = \{f(z) < 0\}$ and for every $j = 1, \dots, n$ we have*

$$(1) \quad \int_{\Omega} \sigma_{j+1}(\partial\bar{\partial}f) dx = \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} k_{\partial\Omega}^{(j)}(z) |\partial f|^{j+1} d\sigma(x),$$

where $k_{\partial\Omega}^{(j)}$ is the j -th Levi curvature of $\partial\Omega$.

Starting from equality (1) in [19] we proved the following *Isoperimetric inequality* for a bounded domain $\Omega \subset \mathbb{C}^{n+1}$ with C^2 boundary.

Theorem 2 (Isoperimetric inequality). *If $k_{\partial\Omega}^{(j)}$ is non negative at every point of $\partial\Omega$ then*

$$(2) \quad \int_{\partial\Omega} \left(\frac{1}{k_{\partial\Omega}^{(j)}(x)} \right)^{1/j} d\sigma(x) \geq 2(n+1)|\Omega|$$

where $|\Omega|$ is the Lebesgue measure of Ω .

Moreover, if $k_{\partial\Omega}^{(j)}$ is constant, then the equality in (2) holds iff Ω is a ball of radius $(1/(k_{\partial\Omega}^{(j)}))^{1/j}$.

Sketch of the proof. If

$$\int_{\partial\Omega} \left(\frac{1}{k_{\partial\Omega}^{(j)}(x)} \right)^{1/j} d\sigma(x) = \infty$$

the inequality is trivial, so in the sequel we shall assume that

$$\int_{\partial\Omega} \left(\frac{1}{k_{\partial\Omega}^{(j)}(x)} \right)^{1/j} d\sigma(x) < \infty.$$

The main ingredients of the proof are the following.

1. By Theorem 1, for every defining function $f \in C^2(\bar{\Omega})$

$$\int_{\Omega} \sigma_{j+1}(\partial\bar{\partial}f) dx = \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} k_{\partial\Omega}^{(j)}(z) |\partial f|^{j+1} d\sigma(x),$$

2. By the Cauchy-Schwarz inequality, for every defining function $f \in C^2(\bar{\Omega})$ we have

$$\begin{aligned} & \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} k^{(j)} |\partial f|^{j+1} d\sigma(x) \geq \\ & \geq \frac{\left(\binom{n+1}{j+1} \left(\int_{\partial\Omega} |\partial f| d\sigma(x) \right)^{j+1} \right)}{2(n+1) \left(\int_{\partial\Omega} (1/k^{(j)})^{1/j} d\sigma(x) \right)^j} = \\ & = \frac{\left(\binom{n+1}{j+1} \right) (2|\Omega|)^{j+1}}{2(n+1) \left(\int_{\partial\Omega} (1/k^{(j)})^{1/j} d\sigma(x) \right)^j}. \end{aligned}$$

3. By the Newton inequality

$$\sigma_j(\partial\bar{\partial}f) \leq \binom{n+1}{j} \left(\frac{\text{trace } \partial\bar{\partial}f}{n+1} \right)^j$$

for all $j = 2, \dots, n+1$, with equality holding iff $(\partial\bar{\partial}f)$ is proportional to the identity matrix.

4. By choosing the defining function f as follows. Let $f : \bar{\Omega} \rightarrow \mathbb{R}$ be the solution of the Dirichlet problem

$$(3) \quad \begin{cases} \text{trace } \partial\bar{\partial}f = 1, & \text{in } \Omega; \\ f = 0, & \text{on } \partial\Omega. \end{cases}$$

where $\text{trace } \partial\bar{\partial} = (1/4)\Delta$, with Δ the Laplacian operator in $\mathbb{R}^{2(n+1)}$.

We now use the Reilly approach and the previous ingredients to get

$$\begin{aligned} \int_{\Omega} \sigma_{j+1}(\partial\bar{\partial}f) \, dx &\leq \binom{n+1}{j+1} \frac{1}{(n+1)^{j+1}} \int_{\Omega} (\text{trace}(\partial\bar{\partial}f))^{j+1} \, dx = \\ &= \binom{n+1}{j+1} \frac{|\Omega|}{(n+1)^{j+1}} \end{aligned}$$

and

$$\frac{(2|\Omega|)^j}{\left(\int_{\partial\Omega} (1/k^{(j)})^{1/j} \, d\sigma(x)\right)^j} \leq \frac{1}{(n+1)^j} .$$

Hence, we get an upper bound for the Lebesgue measure of Ω in terms of the boundary integral of $(1/k^{(j)})^{1/j}$. Precisely,

$$\int_{\partial\Omega} \left(\frac{1}{k^{(j)}}\right)^{1/j} \, d\sigma(x) \geq 2(n+1)|\Omega| .$$

In particular, if equality holds in the isoperimetric estimate and $k^{(j)}$ is a positive constant then $\partial\bar{\partial}f = (1/(n+1)) Id$ in $\bar{\Omega}$ and $|\partial f| = 1/(n+1) (k^{(j)})^{1/j}$ is constant on $\partial\Omega$. Then the Dirichlet problem in (3) is over-determinate and by Serrin's Theorem [24] Ω is a ball and $\partial\Omega$ is a sphere. □

Let us remark that there are non spherical sets which satisfy the equality in (2).

Example 2. In \mathbb{C}^2 let us take the function

$$(4) \quad f(z) = -R^2 + |z|^2 + \frac{\text{Re}(z_1^2 + z_2^2)}{2}$$

as a defining function for Ω . Then the set of zeros of the function f in (4) is not a sphere, it is an ellipsoid. However, equality holds in our isoperimetric inequality because the function $\text{Re}(z_1^2 + z_2^2)$ is a pluriharmonic function and therefore $\partial\bar{\partial}f = Id$.

Thus, the class of sets which satisfy the equality in (2) is larger than the class of sets which satisfy the equality in the classical isoperimetric inequality (see [12], [14]).

3. SYMMETRY RESULTS

As an application of our isoperimetric inequality we get an Alexandrov type result. If Ω is a starlike domain then by the Minkowski formula

$$\begin{aligned} \text{meas}(\partial\Omega) = \int_{\partial\Omega} d\sigma(x) &= \int_{\partial\Omega} H(x)\langle x, \nu(x)\rangle \, d\sigma(x) \leq \\ &\leq 2(n+1)|\Omega| \max_{\partial\Omega} H , \end{aligned}$$

where H is the Euclidean mean curvature of $\partial\Omega$ and ν is the outward unit normal. Then

$$\frac{\text{meas}(\partial\Omega)}{2(n+1)|\Omega|} \leq \max_{\partial\Omega} H .$$

If $k_{\partial\Omega}^{(j)}$ is constant, then

$$(5) \quad \left(k_{\partial\Omega}^{(j)}\right)^{1/j} \leq \frac{\text{meas}(\partial\Omega)}{2(n+1)|\Omega|}$$

and Aleksandrov Theorem holds for starlike domains whose mean curvature is bounded from above by the j -th Levi curvature to $1/j$.

Let us recall that the Alexandrov Soap Bubble Theorem received in the past a great amount of attention. Liebmann in 1899 in [18] proved that the spheres are the only compact surfaces in \mathbb{R}^3 with constant Gauss curvature. In [27] Süss in 1952 showed that a compact convex hypersurface in the Euclidean space must be a sphere, provided that for some r the r -th elementary symmetric polynomial in the principal curvatures is constant. Hsiung in 1954 in [17] proved that the “convexity” assumption can be relaxed to the “star-shapedness”. A breakdown paper in this context was made by Alexandrov, who in 1956 proved that the sphere is the only compact hypersurface embedded into the Euclidean space with constant mean curvature. Alexandrov in [1] proved that every direction is a direction of symmetry by introducing the celebrated *moving plane technique*, which is based on the following ingredients:

- (i) The prescribed mean curvature equation is invariant with respect to hyperplane reflections.
- (ii) The prescribed mean curvature equation is uniformly elliptic and therefore the strong comparison principle holds.
- (iii) Since the equation is uniformly elliptic then the boundary comparison principle holds.

In a joint paper with Ermanno Lanconelli [22] we proved that, even if the prescribed Levi curvature equations are not elliptic, the following strong comparison principle holds.

Theorem 3 (Strong comparison principle). *Let D and D' be strictly q -pseudoconvex domains of \mathbb{C}^{n+1} with connected boundaries. If*

- (1) $D' \subseteq D$ and $\partial D \cap \partial D' \neq \emptyset$
- (2) $k_{\partial D'}^{(q)}(p') \leq k_{\partial D}^{(q)}(p)$, $\forall p \in \partial D$ and $p' \in \partial D'$

then $D' = D$.

As an application of our strong comparison principle we get the following identification result.

Corollary 1 (Identification result). *Let $D \subseteq \mathbb{C}^{n+1}$ be a strictly q -pseudoconvex domain with connected boundary, $1 \leq q \leq n$. Let $B_r(z_0) \subseteq D$ be a tangent sphere to ∂D at some point of ∂D . If*

$$k_{\partial D}^{(q)}(p) \geq \frac{1}{r^q}, \quad \forall p \in \partial D,$$

then $D = B_r(z_0)$.

However, it does not seem possible to apply the Alexandrov moving plane method to the prescribed Levi curvature equations because they are not invariant with respect to hyperplane reflections and there is an obstacle to the boundary comparison principle. Indeed, Hounie and Lanconelli in [15] proved that in general sums of squares of Hörmander vector fields do not satisfy Hopf Lemma. In particular, in [15] they exhibit the following counterexample.

Example 3 (Hounie, Lanconelli). In \mathbb{R}^2 consider the second order partial differential operator $L = X^2 + Y^2$, with

$$X = y \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},$$

then

$$[X, Y] = -\frac{\partial}{\partial x}.$$

The function $u(x, y) = y^4 - 6(x + y^2/2)^2$ satisfies

- (1) $Lu = 0$
- (2) $u < 0$ in the half plane $x \geq 0$ except at the origin where $u(0, 0) = 0$. Then, the restriction of u to the half plane $x \geq 0$ has a strict maximum at the origin
- (3) $\nabla u(0, 0) = 0$.

Hence, the Hopf boundary Lemma does not hold.

However, in [15] Hounie and Lanconelli proved Aleksandrov for Reinhardt domains of \mathbb{C}^2 , i.e. for domain D such that

$$(z_1, z_2) \in D \Rightarrow (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in D \quad \forall \theta_1, \theta_2.$$

In this case in a neighborhood of a point there is a defining function only depending on radius $r_1 = |z_1|, r_2 = |z_2|, F(r_1, r_2) = f(r_2^2) - r_1^2$ with f satisfying the degenerate ODE

$$s f f'' = s f'^2 - k(f + s f'^2)^{3/2} - f f'$$

and they proved that Alexandrov follows from uniqueness of the solution of the ODE. Later on in [16] they extended this result to a particular class of Reinhardt domain in \mathbb{C}^n . Moreover, Monti and Morbidelli in [21] proved a Darboux type theorem in \mathbb{C}^n for $n > 2$ for Levi umbilical hypersurfaces, i.e. $\lambda_1 = \dots = \lambda_{n-1}$. Precisely, they proved that if all Levi curvatures are constant then the hypersurface is a sphere or a cylinder. In a recent paper with Vittorio Martino we also prove the following property of a non Levi flat real hypersurface in \mathbb{C}^{n+1} : if the unit characteristic direction T is a geodesic, then it is an eigenvector of the second fundamental form and the relative eigenvalue is constant. As an application in [20] we get a symmetry result, of Alexandrov type, for compact hypersurfaces in \mathbb{C}^{n+1} with positive constant Levi mean curvature.

4. THE DIRICHLET PROBLEM FOR THE LMA

In this section we are concerned with the problem of finding a hypersurface with prescribed Levi curvature and passing through a prescribed contour. We can localize the problem as follows.

Definition 5 (Levi convexity and the LMA operator). If M is locally the graph of a function $u : \Omega \rightarrow \mathbb{R}, \Omega \subset \mathbb{C}^n \times \mathbb{R}$, we say that u is Levi convex in Ω if its epigraph

$$\text{Epi}(u) = \{(z, w) \in \mathbb{C}^{n+1} : \text{Im } w > u(z, \text{Re } w)\}$$

is pseudoconvex at every point of M and in this situation the total Levi curvature of M writes as

$$(6) \quad k_M = -\frac{2^{n+2}}{(1 + |Du|^2)^{(n+2)/2}} \det \begin{pmatrix} 0 & u_{\bar{z}} & u_{\bar{w}} - i/2 \\ u_z & u_{z\bar{z}} & u_{z\bar{w}} \\ u_w + \frac{i}{2} & u_{w\bar{z}} & u_{w\bar{w}} \end{pmatrix}.$$

We shall say that minus the determinant on the right hand side of (6) is the Levi Monge-Ampère operator and in the sequel we shall denote it by $LMA(u)$.

Given $\varphi \in C(\partial\Omega)$ and $k \geq 0$, we want to find a Levi convex solution u of the Dirichlet Problem

$$(DP) \quad \begin{cases} LMA(u) = k(\cdot, u)(1 + |Du|^2)^{(n+2)/2} & \text{in } \Omega, \\ u(\xi) = \varphi(\xi) & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet Problem (DP) has been first considered by Slodkowski and Tomassini in [25], [26]. Recently, in a joint work with Francesca Da Lio [11] we proved existence and uniqueness of a viscosity solution of (DP) under sharp conditions on the boundary data and on the prescribed function k . Precisely, in [11] we assume that the prescribed curvature function $k : \bar{\Omega} \times \mathbb{R} \rightarrow [0, +\infty)$ is bounded and continuous and it satisfies the following hypotheses

- (H1) Strictly monotonicity
for every $R > 0$, there is $\ell_R > 0$, such that, for all $x \in \bar{\Omega}$, and $-R \leq v \leq u \leq R$ $\ell_R(u - v) \leq k(\cdot, u) - k(\cdot, v)$
- (H2) Modulus of continuity
for every $R > 0$, there is ω_R such that $\omega_R(s) \rightarrow 0$ for $s \rightarrow 0^+$ and $|k(x, u) - k(y, u)| \leq \omega_R(|x - y|)$ for every $(x, y) \in \bar{\Omega}$ and $|u| \leq R$.
- (H3) Boundary condition
 $\Omega \times i\mathbb{R}$ is strictly pseudoconvex and $\sup_{\bar{\Omega} \times \mathbb{R}} k < k_{\partial\Omega \times i\mathbb{R}}(x_0)$ for every $x_0 \in \partial\Omega$.

Under these conditions in [11] we proved the following comparison principle for viscosity solutions of (DP).

Theorem 4 (Comparison principle). *Assume (H1)–(H3), u sub- and v supersolution of (DP) then*

$$u \leq v \quad \text{in } \bar{\Omega}.$$

By using the Perron method we then get existence and uniqueness of a viscosity solution on (DP). Precisely

Theorem 5 (Uniqueness and existence theorem). *Assume (H1)–(H3). Then, for every $\varphi \in C(\partial\Omega)$ there exists a unique u viscosity solution of (DP). Moreover, if $\varphi \in \text{Lip}(\partial\Omega)$ and $k \in \text{Lip}(\Omega \times \mathbb{R})$ then $u \in \text{Lip}(\Omega)$.*

5. REGULARITY

In this section we are concerned with the problem of local regularity of solutions of the Levi Monge-Ampère equation in (DP). Even if the Levi Monge-Ampère operator LMA is not elliptic, in a joint paper with Francesca Lascialfari [23] we proved that it has the following hypoellipticity property

Theorem 6 (Hypoellipticity). *If $u \in C^{2,\alpha}(\Omega)$ is a Levi convex solution of*

$$LMA(u) = q(\cdot, u, Du)$$

where $\Omega \subset \mathbb{R}^{2n+1}$ and $q \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{2n+1})$, $q > 0$, then $u \in C^\infty(\Omega)$.

Sketch of the proof. Since the Levi Monge-Ampère equation presents formal similarities with the real and complex Monge-Ampère equations, which are elliptic PDE's if evaluated on strictly convex and plurisubharmonic functions respectively (see [13], [3]), we would like to briefly recall how the smoothness follows from the classical Schauder theory for the real Monge-Ampère equation. The real Monge-Ampère equation in a domain $\Omega \subset \mathbb{R}^n$ is of the form $\det(D^2u) = \psi(x, u, Du)$. If $u \in C^{2,\alpha}(\Omega)$ is a strictly convex solution to this equation, then the linearized operator L (at u) is elliptic with C^α coefficients, and Du satisfies a linear uniformly elliptic equation of the form $L(Du) = f \in C^\alpha(\Omega)$. By the Schauder theory, $Du \in C^{2,\alpha}(\Omega)$. Repeating this procedure one proves $u \in C^\infty(\Omega)$. In our case it is not possible to argue in the same way, because the Levi Monge-Ampère operator $LMA(u)$ is not elliptic at any point, also when restricted to the class of strictly Levi convex functions. Indeed, as we prove in [23], if we call D^2u the Euclidean Hessian matrix of u , then there exists a smooth function $F = F(p, r)$, with $p = Du$ and $r = D^2u$, such that

$$LMA(u) = F(Du, D^2u)$$

and

$$\frac{\partial F}{\partial r_{ij}}(Du, D^2u) \geq 0 .$$

Moreover, in [23] we proved that the minimum eigenvalue of the real matrix $(\partial F / \partial r_{ij})_{i,j=1}^{2n+1}$ is identically zero. Hence in [23] we were forced to develop a new technique, which takes into account the CR structure of the hypersurface M .

In order to present our technique we introduce here some notations.

If $u \in C^{2,\alpha}(\Omega)$ is a strictly Levi convex solution, then in [23] we proved that we can write the Levi Monge-Ampère operator on u as

$$LMA(u) = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u ,$$

where $X_j = \partial_{x_j} + b_j(x) \partial_{x_{2n+1}}$ are first order partial differential operators whose coefficients are smooth functions of the gradient of u and therefore $b_j(x) = b_j(Du(x)) \in C^{1,\alpha}(\Omega)$ and $a_{ij}(x) = a_{ij}(X^2u) = a_{ij}(Du(x), D^2u(x)) \in C^{0,\alpha}(\Omega)$.

Since u is strictly Levi convex, in [23] we proved that for all $K \subset\subset \Omega$ there exists a positive constant $C > 0$ such that

$$\sum_{i,j=1}^{2n} a_{ij}(x) \xi_i \xi_j \geq C \sum_{j=1}^{2n} \xi_j^2 , \quad \forall \xi \in \mathbb{R}^{2n} , \quad \forall x \in K .$$

Hence $LMA(u)$ is elliptic in $2n$ linearly independent directions and there is a lack of ellipticity in one direction. However, in [23] we proved that we can recover the lack of ellipticity by considering the brackets

$$[X_i, X_j] = h_{ij}(x) \partial_{x_{2n+1}} ,$$

and we proved that if u is a strictly Levi convex solution then at every point $x \in \Omega$ then there is a couple (i, j) such that $h_{ij}(x) \neq 0$. Therefore

$$\dim(\text{Span}\{X_i, [X_i, X_j], i, j = 1, \dots, 2n\}) = 2n + 1$$

at every point of Ω . This condition, which is formally analogous to Hörmander condition of hypoellipticity for sum of squares of smooth vector fields, guarantees the following connectivity property: for every $x, y \in \Omega$ there is an absolutely continuous curve γ which connects x, y and which is piecewise integral curve of the vector fields $X_j, j = 1, \dots, 2n$. This allows us to prove a bootstrap argument and smooth regularity by starting from a fixed strictly Levi convex solution of class $C^{2,\alpha}$. □

The $C^{2,\alpha}$ regularity of Lipschitz continuous viscosity solutions of the prescribed total Levi curvature equation is still an open problem in the fully nonlinear case $n \geq 2$. In the quasilinear case $n = 1$, the Levi operator is

$$LMA(u) = \text{trace}(A(Du)D^2u)$$

where

$$(7) \quad A(p) = \begin{pmatrix} 1 + p_3^2 & 0 & p_2 - p_1p_3 \\ 0 & 1 + p_3^2 & -p_1 - p_2p_3 \\ p_2 - p_1p_3 & -p_1 - p_2p_3 & p_1^2 + p_2^2 \end{pmatrix}$$

for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ and a simple calculation shows that we can rewrite it as

$$LMA(u) = a(Du)(X_1^2u + X_2^2u)$$

where $a(p) = (1 + p_3^2)^2$, and

$$\begin{aligned} X_1 &= \partial_{x_1} + b_1(p)\partial_{x_3}, & b_1(p) &= \frac{p_2 - p_1p_3}{1 + p_3^2}, \\ X_2 &= \partial_{x_2} + b_2(p)\partial_{x_3}, & b_2(p) &= \frac{-p_1 - p_2p_3}{1 + p_3^2} \end{aligned}$$

for all $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. Moreover, $X_ju = (\partial_{x_j} + b_j(Du)\partial_{x_3})u$ and $X_j^2u = X_j(X_ju) = (\partial_{x_j} + b_j(Du)\partial_{x_3})(X_ju)$ for $j = 1, 2$. In a joint work with Giovanna Citti [7], by differentiation of the equation and integration by parts, we obtained the following first regularity achievement

Theorem 7 (Strong solutions). *If $k \in \text{Lip}(\Omega \times \mathbb{R})$, every viscosity solution $u \in \text{Lip}(\Omega)$ of the prescribed Levi curvature equation*

$$(8) \quad X_1^2u + X_2^2u = k(x, u) \frac{(1 + |Du|^2)^{3/2}}{(1 + (\partial_{x_3}u)^2)^2}$$

is a strong solution, i.e. $X_1u, X_2u \in H_{\text{loc}}^1(\Omega)$ and u punctually solves the equation a.e.

Successively, in [8] we proved the following Frobenius type result for strong solutions of the Levi flat equation.

Theorem 8. *Every strong solution u of the Levi equation (8) with $k \equiv 0$, is foliated by analytic curves and u is analytic on every leaf.*

On the other side, if $k \neq 0$ at every point, in a joint work with Citti and Lanconelli [6] we obtained the following regularity result.

Theorem 9. *Every strong solution u of the Levi equation (8), with $k \neq 0$, $k \in C^\infty(\Omega)$, is smooth.*

Sketch of the proof. In an open set Ω we fix a solution u of the regularized equation

$$(9) \quad \mathcal{L}_\varepsilon u = k(x, u) \frac{(1 + |Du|^2)^{3/2}}{(1 + (\partial_{x_3} u)^2)^2}$$

where \mathcal{L}_ε denotes the elliptic operator

$$(10) \quad \mathcal{L}_\varepsilon u := \mathcal{L}u + \varepsilon^2 \frac{u_{x_3 x_3}}{(1 + u_{x_3}^2)^2}$$

and

$$(11) \quad \mathcal{L}u := X_1^2 u + X_2^2 u .$$

The uniform limit as ε goes to 0 of a sequence of solutions of equation (9) is a viscosity solution of the equation (8) in the weak sense of Crandall-Ishii-Lions (see [10]). Since the minimum eigenvalue of $A(p)$ in (7) is equal to zero for every $p \in \mathbb{R}^3$, the operator \mathcal{L} is not elliptic at any point and the regularity results for viscosity solutions of non-linear elliptic [4] and parabolic equations [28] [29] cannot be applied to our case. We have to introduce a completely different procedure, based on the particular structure of the Levi equation. Let us remark that in this situation the Lie-bracket of the first order differential operators X_1 and X_2 is

$$(12) \quad [X_1, X_2] = -\frac{\mathcal{L}u}{1 + u_{x_3}^2} \partial_{x_3} .$$

In the sequel we shall denote by L_ε a linear operator formally defined as \mathcal{L}_ε :

$$L_\varepsilon = X_1^2 + X_2^2 + T_\varepsilon^2 ,$$

where $T_\varepsilon = \varepsilon(1 + u_{x_3}^2)^{-1} \partial_{x_3}$, and the coefficients of the vector fields X_1 and X_2 depend on the fixed solution u of (9). Then we prove that the coefficients b_1 and b_2 of the vector fields and the two functions

$$\omega = \partial_{x_3} u , \quad \text{and} \quad v = \arctan u_{x_3}$$

are solutions of

$$(13) \quad L_\varepsilon z = f ,$$

with different functions f .

The proof of Theorem 9 is based on the regularity of the solutions of this linear equation in some Sobolev Spaces $W_\varepsilon^{m,p}$ naturally defined in terms of the vector fields X_1 , X_2 and T_ε . The classical elliptic regularization procedure is based on Sobolev inequalities and on a priori estimates of Caccioppoli type. In the present situation neither the Caccioppoli inequality holds, since the vector fields are not selfadjoint, nor the Sobolev inequality, since the coefficients of the vector fields are only bounded.

To overcome these difficulties we first prove an interpolation inequality, which will play a role similar to the Sobolev one in the classical setting.

Let C be a positive constant such that

$$\|b_1\|_\infty + \|b_2\|_\infty + \|v\|_\infty \leq C ,$$

then for every function $z \in C^\infty$, $\varphi \in C_0^\infty$, we have

$$(14) \quad \int |X_1 z|^3 \varphi^6 \leq c \int |\nabla_\varepsilon(X_1 z)|^2 \varphi^6 + c \int (|\nabla_\varepsilon \varphi|^6 + \varphi^6) (1 + z^6)$$

where ∇_ε is the vector $(X_1, X_2, T_\varepsilon)$ and the constant $c > 0$ only depends on C and k . An analogous inequality is also satisfied if we replace $X_1 z$ with $X_2 z$ or $T_\varepsilon z$.

Only if the coefficients are much more regular we can establish a Sobolev type inequality with optimal exponent. Thus, we establish some a priori estimates in the intrinsic directions X_1 and X_2 , weaker than the classical Caccioppoli one. Using these inequalities together with the interpolation ones, we prove a priori estimates in $W_\varepsilon^{m,p}$, for solutions z of (13) which holds under very general assumptions on the commutators of the vector fields, but requires some strong a priori estimates on the derivative $\partial_{x_3} z$. We then conclude the proof of Theorem 9, starting with the estimates of the derivative ∂_{x_3} , which, by equality (12), can be expressed in term of the commutator of the vector fields. We also use in an essential way the nonlinearity of the equation: the interpolation and Caccioppoli inequalities for the derivative ∂_{x_3} provide a gain of regularity only if applied to the function $\partial_{x_3} v$. In this way we obtain a L^2 estimate for $X_1 v_{x_3}$ and $X_2 v_{x_3}$. Since $v_{x_3} = u_{x_3 x_3} / (1 + u_{x_3}^2)$, then v_{x_3} has to be considered a derivative of weight 4 of u , while $X_1 v_{x_3}$ and $X_2 v_{x_3}$ are derivatives of weight 5 of the same function. Once proved the summability of these derivatives with respect to x_3 , we obtain analogous estimates for any derivation of weight 5 and 4. In particular the coefficients b_1 and b_2 of the vector fields are now regular, and we can apply a Sobolev type inequality. It then follows that the derivatives of weight 4 belong to L^4 , of weight 3 to L^p for every p , the derivatives of weight 2 belong to suitable classes C^α for every $\alpha \in]0, 1[$. Now, using the results in [5], we deduce that $u \in C^{2,\alpha}$. □

The smoothness of classical solutions of the prescribed Levi curvature equation with smooth prescribed curvature k with some first order zeros has been studied in [9].

Recently, in a joint paper with Vittorio Martino, we also proved some a priori interior gradient bound for solutions of (8) and the local Lipschitz continuity of continuous viscosity solutions in the strictly monotone case $\partial k / \partial u > 0$. Precisely, in [21] we obtained the following interior regularity result

Theorem 10 (Strictly monotone case). *If $k \in C^1(\Omega \times \mathbb{R})$ and $\partial k / \partial u > 0$ then every continuous viscosity solution of (8) is locally Lipschitz continuous.*

In the Levi flat case ($k \equiv 0$) we have a counterexample. Indeed, the function $u(x) = \varphi(x_3)$, for every continuous function φ , is a viscosity solution of (8) with $k \equiv 0$.

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