Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 7(2008), pp. 237–247.

Some recent results and open questions in two-phase free boundary problems

Sandro Salsa

To Ermanno Lanconelli for his 65th birthday

Abstract[1]. We give a brief survey of recent results obtained in joint papers with Fausto Ferrari, concerning the regularity of the free boundary in general free boundary problems for elliptic operators. Moreover we discuss some of the relevant questions that remains open. In particular, very little is known about evolution problems of Stefan type.

1. Introduction

In this survey we present some recent results concerning free boundary problems, for both elliptic and parabolic operators. These kind of problems arise in several context and range from constraint energy minimization to phase transitions, from Finance to flow in porous media. We will be mainly concerned with problems in which the state variable can assume two phases and our emphasis will be on the optimal regularity of viscosity solutions and their free boundaries.

At the same time, we shall single out some questions that are still open, especially regarding evolution problems, and that in our opinion are quite important, also in view to the application to non linear operators.

2. Elliptic free boundary problems

Starting from the elliptic case, we are interested in the following free boundary problem (*f.b.p.* in the sequel) and in its weak solutions in the sense of viscosity.

In the unit ball $B_1 \subset \mathbb{R}^n$ we are given a continuous function u satisfying

i)

$$\mathcal{L}^1 u = 0 \quad \text{in } \Omega^+(u) = \{x \in B_1 \ : \ u(x) > 0\} \tag{1}$$

$$\mathcal{L}^2 u = 0 \quad \text{in } \Omega^-(u) = \{x \in B_1 \ : \ u(x) \leq 0\}^\circ \ . \tag{2}$$

---

[1] Author's address: S. Salsa, Politecnico di Milano, Dipartimento di Matematica, Via Bonardi 7, 20133 Milano, Italy; e-mail: `sandro.salsa@polimi.it` .

Keywords. Free boundaries, viscosity solutions, uniformly elliptic operators, continuous deformations.

AMS Subject Classification. 35R35.

Here, $\mathcal{L}^1$ and $\mathcal{L}^2$ are uniformly elliptic operators with ellipticity constant Λ, of one of the following type $(j = 1, 2)$:

$$\mathcal{L}^j u = \mathrm{Tr}\left(A^j(x)D^2u\right) + b^j(x)\cdot\nabla u$$

or

$$\mathcal{L}^j u = \Phi^j(x, \nabla u, D^2u) \tag{3}$$

or

$$\mathcal{L}^j u = \mathrm{div}\,(A^j(x)\nabla u) + b^j(x)\cdot\nabla u\ ,$$

where

$$A^j(x) = (a^j_{ik}(x)), b^j(x) = (b^j_1(x), \cdots, b^j_n(x))$$

and D^2u is the Hessian matrix of u.

ii) Along

$$F(u) \equiv \partial\Omega^+(u) \cap \mathcal{C}_1$$

(the *free boundary*), the following condition holds:
a) if at $x_0 \in F(u)$ there is a ball B such that

$$B \subset \Omega^+(u)\ , \quad \overline{B} \cap \Omega^+(u) = \{x_0\}$$

then, near x_0,

$$u^+(x) \geq \alpha\,\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B\ , \quad (\alpha > 0)\ , \tag{4}$$

$$u^-(x) \geq \beta\,\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{in } \mathcal{C}B\ , \quad (\beta \geq 0)\ , \tag{5}$$

with equality along every nontangential domain in both cases, and

$$\alpha \leq G(\beta)\ ; \tag{6}$$

b) if at $x_0 \in F(u)$ there is a ball B such that

$$B \subset \Omega^-(u)\ , \quad \overline{B} \cap \Omega^-(u) = \{x_0\}$$

then, near x_0

$$u^-(x) \geq \beta\,\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{in } B\ , \quad (\beta > 0) \tag{7}$$

$$u^+(x) \leq \alpha\,\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } \mathcal{C}B\ , \quad (\alpha \geq 0) \tag{8}$$

with equality along every nontangential domain in both cases and

$$\alpha \geq G(\beta)\ . \tag{9}$$

The conditions (4)-(9) where ν is the interior unit normal to ∂B at x_0, express the free boundary relation $u^+_\nu = G(u^-_\nu)$ in a viscosity sense; in case a) (resp.b)) we say that $x_0 \in F(u)$ is a *right (resp. left) regular point.* In particular, conditions (6) and (9) correspond to a supersolution and subsolution condition, respectively. Accordingly, we call u a viscosity solution of our *f.b.p.* (see [13], Ch. 4).

The main hypotheses on G are:

i) The function

$$z \longmapsto G = G(z, x, \nu)$$

is continuous, strictly increasing in z

ii) for some $N > 0$, the function

$$z \longmapsto z^{-N}G(z, x, \nu)$$

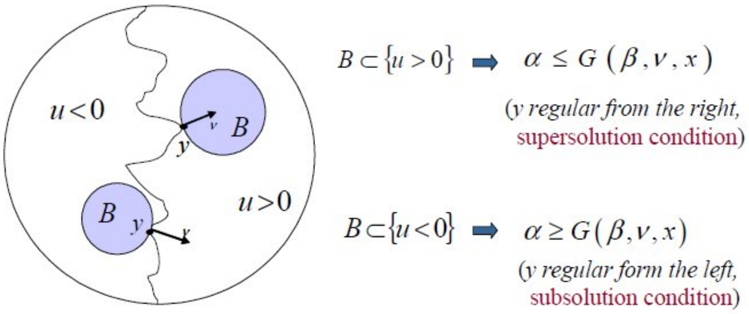

FIGURE 1.

is decreasing in $(0, +\infty)$.

Typical examples comes from constraint minimization problems as

$$\min \int_{B_1} a_{ij}(x) u_{x_i} u_{x_j} + q(x) \chi_{\{u>0\}}$$

over $u \in g + H_0^1(B_1)$, with $q \geq c > 0$ a.e. Here the free boundary condition takes the form

$$\left(u_{\nu^*}^+\right)^2 - \left(u_{\nu^*}^+\right)^2 = q(x)$$

where ν^* denotes the conormal derivative in the direction of the positive phase.

Other examples arise from singular perturbation theory, e.g. as a limiting problem when $\varepsilon \to 0$ for

$$\Delta u_\varepsilon + b(x) \cdot \nabla u_\varepsilon = \beta_\varepsilon(u)$$

where

$$\operatorname{supp}(\beta) \subset [0,1] , \quad \int_{\mathbb{R}} \beta = 1 \quad \beta_\varepsilon(s) = \varepsilon^{-1} \beta\left(\frac{s}{\varepsilon}\right) .$$

Here the free boundary condition of the limiting problem takes the form

$$\left(u_\nu^+\right)^2 = 2 .$$

3. EXISTENCE

The main issues arising in a f.b.p. are existence and optimal regularity. Due to the highly non linear nature of the problems, hardly uniqueness has to be expected (see [1]). Clearly, given the jump of the gradients across the free boundary, the optimal regularity of the solution is expected to be Lipschitz continuity.

Quite important is the analysis of the free boundary, also in order to establish how classical are viscosity solution. By a classical solution u we mean a C^1-function up to the free boundary $F(u)$ from both sides, satisfying the free boundary condition in a classical pointwise sense. We may split the analysis of $F(u)$ according the following diagram., where by *weak* results we mean the basic geometric measure properties, such as $F(u)$ to be a set of finite perimeter.

By *weak* results we mean the basic geometric measure properties, such as $F(u)$ to be a set of finite perimeter. A flatness condition may be given in several ways.

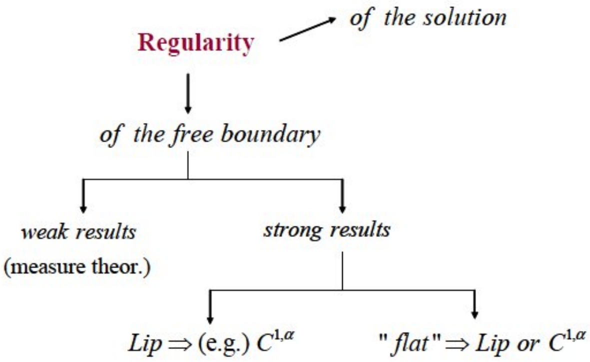

FIGURE 2.

Typically, a surface is $\varepsilon-$flat in a neighborhood of one of its point if after a blow-up can pe trapped within two hyperplane at distance ε. For instance, this happens near a differentiability point. Another way to state a flatness condition appears in Theorem 3 below.

Let us consider now the existence theory and weak regularity. In [11] Caffarelli consider the case

$$\mathcal{L}^1 = \mathcal{L}^2 = \text{div}\,(A(x)\nabla)$$

with A Hölder continuous and the non-degeneracy condition $G(0,\nu,x) \geq c > 0$. His main result is the following

Theorem 1. *Given a Lipschitz domain Ω and $g \in C(\partial\Omega)$, there exists a viscosity solution $u \in C(\overline{\Omega})$ with $u = g$ on $\partial\Omega$. Moreover, u is Lipschitz in Ω, $\Omega^+(u)$ is a set of finite perimeter and*

$$0 < \alpha_1 \leq \frac{u^+(x)}{\text{dist}\,(x, F(u))} \leq \alpha_2\ . \tag{10}$$

Notice the linear growth of the positive part of th solution expressed by the left inequality in (10). Caffarelli uses a Perron method. Indeed a solution is given by

$$u(x) = \inf_{v\in S} v(x)$$

where S is a class of continuous supersolutions v in $\overline{\Omega}$ such that $v \geq g$ on $\partial\Omega$ and that $F(v)$ is regular from the left.

This result has been extended by P.Y. Wang in [23] to the case $\mathcal{L}^1 = \mathcal{L}^2 = F\left(D^2u\right)$ with F concave.

Open problem 1. *Existence for $\mathcal{L}^1 = \mathcal{L}^2 = F\left(D^2u, Du\right)$ with F non concave in the Hessian matrix and even for the linear case $\mathcal{L}^1 = \mathcal{L}^2 = \text{Tr}\left(A(x)D^2u\right) + b(x)\cdot\nabla u$.*

Let us see what constitutes the main obstruction in extending Caffarelli's method. The key tool is the following monotonicity formula of Alt, Caffarelli and Friedman (see [2]).

Theorem 2. *Let* $u = u^+ - u^-$ *be such that* $\operatorname{div}\,(A(x)\nabla u^\pm) \geq 0$ *in* B_1*, with* $u^+(0) = u^-(0) = 0$*. Assume that* A *is Hölder continuous with exponent* α*. Then for, say,* $0 < r \leq 1/2$*, and some constant* $c > 0$*, the function*

$$\Phi(r) = r^{-4} e^{-cr^\alpha} \int_{B_r} \frac{|\nabla u^+|}{|x|^{n-2}}\, dx \int_{B_r} \frac{|\nabla u^-|}{|x|^{n-2}}\, dx \leq c(n) \|u\|^4_{L^\infty(B_1)}$$

is increasing and

$$\Phi(r) \leq c(n) \|u\|^4_{L^\infty(B_1)}\ .$$

Observe that if the supports of u^+ and u^- where separated by a smooth surface with normal ν at $x = 0$ then, by taking the limit as $r \to 0$, we could deduce that

$$\left(u^+_\nu(0)\right)^2 \left(u^-_\nu(0)\right)^2 \leq \Phi\left(\frac{1}{2}\right)$$

so that, "morally" $\Phi(r)$ gives a control in average of the product of the normal derivatives of u at the origin.

It seems to be hard to extend the above theorem to non-divergence form operators. It can be proved in some spacial cases under additional hypotheses; for instance if the two supports are separated by a Lipschitz surface then Theorem 1 holds. Thus:

Open problem 2. *To prove a monotonicity formula for nondivergence form operators with Hölder coefficients.*

4. Regularity: strong results

The regularity theory for the Laplace operator has been developed by L. Caffarelli in the well known and by now classical papers [9], [10].

In particular the *Lipschitz implies* $C^{1,\alpha}$ part is contained in [9] while the *flat implies Lipschitz* part is shown in [10]. In these papers Caffarelli sets up a general strategy to attak the regularity of the free boundary.

Briefly, for instance, the strategy of the proof in [9] consists of the following main steps. Starting from a Lipschitz graph, one shows that in a neighborhood of $F(u)$ the level sets of u are still Lipschitz graph, locally in the same direction. Then one improves the Lipschitz constant of the level sets of u away from the free boundary. Here Harnack inequality applied to directional derivatives of u plays a major role. Then the task is to carry this interior gain up to the free boundary. Here Caffarelli introduces a powerful method of continuity based on the construction of a family of continuous subsolutions, on which we will came back later on. Finally, by rescaling and iterating the last two steps, one obtains a geometric decay of the Lipschitz constant, which amounts to the $C^{1,\alpha}$ regularity of $F(u)$.

After 10 years M. Feldman (see [15]) considers anisotropic operators with constant coefficients and extends to this case the results in [9].

P. Y. Wang manages to extend the results both in [9] and [10] to a class of concave fully non linear operators of the type $F(D^2u)$ (see [24]). One year later Feldman (see [17]) considers a class of non concave fully non linear operators of the type $F(D^2u, Du)$. He shows that Lipschitz free boundaries are $C^{1,\alpha}$ thus extending the results in [9].

The first papers dealing with variable coefficient operators are by Cerutti, Ferrari, Salsa (see [12]) and by Ferrari ([15]). They consider respectively, linear elliptic operators in non-divergence form and a rather general class of fully nonlinear operators $F(D^2u, Du, x)$, including Bellman operator for instance. Basic tool is the potential theory developed in [14]. One of the main difficulty in extending the theory to variable coefficients operator is the fact that no directional derivative satisfy a nice elliptic equation. A fact on which all the above papers heavily rely.

A refinement of the techniques in [12] leads to the following results (see [18]), where the drift coefficient is merely bounded measurable.

Theorem 3. *Let u be a weak solution of our free boundary problem in B_1, where $\mathcal{L}^j u = \operatorname{Tr}\left(A^j(x)D^2u\right) + b^j(x)\cdot\nabla u$, $j = 1, 2$. Suppose $0 \in F(u)$ and that*

i) $A^j \in C^{0,a}(C_1)$, $0 < a \leq 1$, $b^j \in L^\infty(B_1)$.

ii) $0 < \alpha_1 \leq \dfrac{u^+(x)}{\operatorname{dist}(x, F(u))} \leq \alpha_2$.

iii) $G(0, \nu, x) \geq c > 0$.

There exist $0 < \overline{\theta} < \pi/2$ and $\overline{\varepsilon} > 0$ such that, if for $0 < \varepsilon < \overline{\varepsilon}$, $F(u)$ is contained in an ε-neighborhood of a graph of a Lipschitz function $x_n = g(x')$ with

$$Lip(g) \leq \tan\left(\frac{\pi}{2} - \overline{\theta}\right)$$

then $F(u)$ is a $C^{1,\alpha}$-graph in $B_{1/2}$.

Condition ii) expresses a nondegeneracy of u^+ at the free boundary while being trapped in a neighborhood of two Lipschitz graph with samall Lipschitz constant is another way to express a flatness condition. Thus flatness and nondegeneracy imply smoothness.

Under the same hypotheses on the operators, we can also prove that Lipschitz free boundary are smooth. Namely:

Theorem 4. *Let u be a weak solution of our free boundary problem in B_1, where $\mathcal{L}^j u = \operatorname{Tr}\left(A^j(x)D^2u\right) + b^j(x)\cdot\nabla u$, $j = 1, 2$. Suppose $0 \in F(u)$ and that*

i) $A^j \in C^{0,a}(C_1)$, $0 < a \leq 1$, $b^j \in L^\infty(B_1)$.

ii) $\Omega^+(u) = \{(x', x_n) : x_n > f(x')\}$ *where f is a Lipschitz continuous function with $Lip(f) \leq L$.*

iii) $G = G(z)$ *is continuous, strictly increasing and for some $N > 0$, $z^{-N}G(z)$ is decreasing in $(0, +\infty)$.*

Then, on $B'_{1/2} \subset \mathbb{R}^{n-1}$, f is a $C^{1,\gamma}$ function with $\gamma = \gamma(n, N, L, \Lambda, a)$.

Theorem has been recently extended to the same class of fully nonlinear operators considered in [15] by Argiolas and Ferrari (see [3]).

We draw two consequences from the above theorems.

Corollary 5. *The conclusion of Theorem 4 holds if $L^1 = L^2 = \operatorname{div}(A(x, u)\nabla u)$ with A Lipschitz with respect to all its arguments.*

The other application is to the minimal solution constructed in [11]:

Theorem 6. *Let u be the minimal viscosity solution constructed in Theorem 1. Assume $A = A(x)$ is Lipschitz. Then, if $x_0 \in \partial_{\text{red}}\Omega^+(u) \cap B_{1/2}$, $F(u)$ is a $C^{1,\alpha}$-graph in a neighborhood of x_0.*

Thus the theory of viscosity solution of general free boundary problems for divergence form operators can be considered quite satisfactory, at least in the case of Lipschitz coefficients.

Open problem 3. *Regularity of the free boundary in the case of divergence form operators with Hölder coefficients.*

Let us see what is the main difficulty in dealing with divergence form operators with Hölder coefficients.

Let us go back to the continuation method used by Caffarelli in [9] to carry up to the free boundary the interior decay on the Lipschitz constant. The key point is the construction of a family of subharmonic functions, constructed as the supremum of an harmonic function over balls of variable radius. Here is the main question: given a non negative function u, harmonic on its support. Let g be a smooth function, $1 \leq g \leq 2$, and define

$$v_g(x) = \sup_{B_{g(x)}(x)} u \ :$$

Under which condition on g is v_g subharmonic on its support?

The answer is the following differential inequality:

$$g\Delta g \geq C(n)|\nabla g|^2 \ . \tag{11}$$

The situation in the variable coefficient case is much more involved. For instance, if we have a non negative function u such that $\mathcal{L}u = \mathrm{Tr}\left(A(x)D^2u\right) + b(x)\cdot\nabla u = 0$ on its support, the condition that g has to satisfy in order to make v_g an $\mathcal{L}-$subsolution on its support takes the following form:

$$\mathcal{L}g \geq C(n,\Lambda)\left\{\frac{|\nabla g|^2+\omega^2}{g} + \|b\|_{L^\infty}\right\} \tag{12}$$

where ω is the modulus of continuity of A computed at $\max g/\Lambda$.

There are two main draw-backs in condition (12) with respect to its constant coefficient counterpart.

The first one is fact that it is not homogeneous with respsct to g; this causes the need of a delicate balance between rescaling and decay of the Lipschitz constant of the level sets of u in the iteration procedure to carry the interior gain to the free boundary.

The second one is that the proof is intrinsecally non-divergence and so far any attempt to find a proof for divergence form operator with Hölder continuous coefficients has failed (see [19] for a more divergence form oriented proof).

5. Evolution free boundary problems

We consider now evolution free boundary problems. Formally one seeks for a function in a space-time cylinder $C_R = B_R \times (-R^2, R^2)$ such that

$$L^1u = 0 \quad \text{in } \Omega^+(u) \quad \text{and} \quad L^2u = 0 \quad \text{in } \Omega^-(u)$$

and

$$V_\nu = -\frac{u_t^+}{\left|u_\nu^+\right|} = -G\left(u_\nu^+, u_\nu^-, \nu, x, t\right)$$

on $F(u) = \partial\Omega^+(u) \cap C_R$, the free boundary.

Here

$$L^j = \mathcal{L}^j - D_t$$

where

$$\mathcal{L}^j u = \operatorname{Tr}\left(A^j(x,t)D^2 u\right) + b^j(x,t) \cdot \nabla u$$

or

$$\mathcal{L}^j u = \Phi^j(x,t,\nabla u, D^2 u)$$

or

$$\mathcal{L}^j u = \operatorname{div}\left(A^j(x,t)\nabla u\right) + b^j(x,t) \cdot \nabla u$$

are uniformly elliptic operators. In the free boundary condition $\nu = \nabla u^+/|\nabla u^+|$, so that V_ν represents the speed of the free bondary in the positive phase direction. Typical example come from the classical two-phase Stefan problem where $G\left(u^+_\nu, u^-_\nu, \nu, x, t\right) = u^+_\nu - u^-_\nu$.

We require that G is Lipschitz continuous with respect to all its arguments, strictly increasing with respect to u^+_ν and strictly decreasing with respect to u^-_ν.

By classical supersolution resp. (subsolution) of the above problem we mean a smooth function v in both $\overline{\Omega}^+(u)$ and $\overline{\Omega}^-(u)$, $\mathcal{L}^j$-supercaloric (resp. subcaloric) whose free boundary is a smooth surface, satisfying the free boundary condition $V_\nu \geq -G\left(u^+_\nu, u^-_\nu, \nu, x, t\right)$ (resp. $\geq$) in a pointwise sense. The inequality in the free boundary condition reflect the fact that for a supersolution (subsolution) the speed V_ν has to be smaller (greater) than the one of a solution sharing the same data on the parabolic boundary $\partial_p C_R$ of C_R.

We shall deal with viscosity solutions that we introduce below.

Definition. A continuous function u in C_R is a viscosity *subsolution* (respectively *supersolution*) if for every subcylinder $Q \subset C_R$ and every classical supersolution (respectively subsolution) in Q, $u \leq v$ on $\partial_p C_R$ (respectively $u \geq v$) implies $u \leq v$ in Q (respectively $u \geq v$). The function u is a viscosity solution if it is both a viscosity sub and a super solution.

For an evolution problem we can pose the same questions on existence and regularity issues, both for the solution and the free boundary. Here other important questions are related to the asymptotic behavior for $t \to +\infty$, for instance.

The theory so far can be considered rather satisfactory only in the case of the Heat equation. In particular, existence (and also uniqueness) is known only for the classical Stefan problem. So:

Open problem 4. *To prove the existence of a viscosity solution with given data on $\partial_p C_R$.*

Now we come to the regularity questions. Also here the understanding of the problem is well developed for the heat operator. In a series of papers Athanasopoulos, Caffarelli and Salsa obtain the following results.

Optimal regularity of the solution (see [6]). If $F(u)$ is locally a Lipschitz graph both in space and time then u is Lipschitz across $F(u)$. Note that there are counterexamples showing that in general the solution in the two phase Stefan problem is not Lipschitz (see [21]). We point out that, although the heat equation scales parabolically, the free boundary condition is invariant under Hyperbolic rescaling, so that Lipschitz continuity in space and time (rather than Lipschitz in space, 1/2 Hölder in time) appears as an appropriate hypothesis for $F(u)$. We will come back on this question.

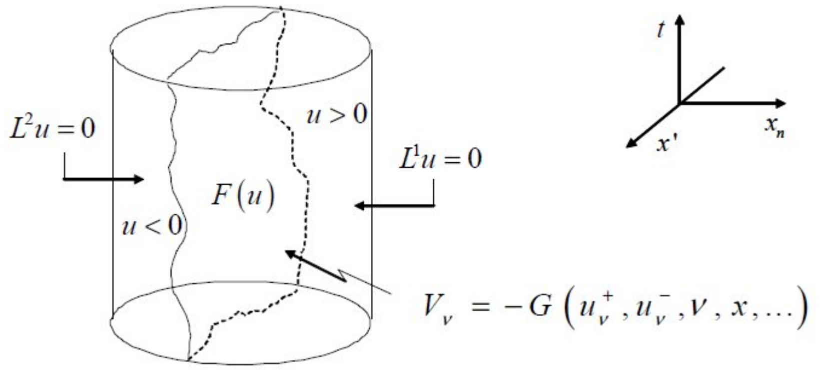

Figure 3.

Lipschitz free boundaries are smooth? This is not true in general as two counterexamples show: one is a one-phase case in dimension $n = 2$, in which $u^- \equiv 0$ (see [22]), the other one is a true two-phase Stefan problem in dimension $n = 3$ (see [7]). Thus the situation is quite different with respect to the stationary case.

Let us describe the one-phase counterexample. Consider the function

$$w(\rho, \theta, t) = \rho^{g(t)}\{\cos[g(t)\theta]\}^+$$

where ρ, θ are polar cohordinates in the plane and g is a decreasing function greater than 2.

If R is chosen sufficiently small, depending on g, then w is a supersolution of the one-phase Stefan problem in C_R. At the origin, $F(w)$ shows a persistent corner with an angle less than $\pi/2$, since $g > 2$, and the heat flux there is zero (from both sides of $F(w)$).

Let now u be the solution of the one-phase Stefan problem in C_R with $u = w$ on $\partial_p C_R$. Then $u \leq w$ in C_R forcing $F(u)$ to have a persistent corner at the origin as long as the angle stays less than $\pi/2$.

The counterexample points out the main source of difficulties in studying this kind of problems: the double homogeneity due to the parabolicity of the heat diffusion and the hyperbolicity of the free boundary condition. Indeed, the relation

$$V_\nu = -G\left(u_\nu^+, u_\nu^-, \nu, x, t\right)$$

is invariant under hyperbolic dilations of the graph of u, that is under the transformation $u \to u(\lambda x, \lambda t)/\lambda$, while a parabolic equation is invariant under the dilation $u \to u(\lambda x, \lambda^2 t)/\lambda$.

Also, a closer look to the conterexamples reveals that the obstruction to instantaneous regularization comes from two facts. The first is the simultaneous vanishing of the two fluxes from both sides of the free boundary and the second is the largeness of the Lipschitz constant in space of the free boundary.

Thus positive results can be given along two directions. First (see [7]), if $F(u)$ is locally Lipschitz and a non-degeneracy condition of the form $u_\nu^+ + u_\nu^- \geq m > 0$, in a suitable weak sense, then $F(u)$ is a C^1-surface, the time sections $F_\tau(u) = F(u) \cap (t = \tau)$ are Liapunov-Dini domains and the solution is classical. The main strategy follows the lines of the elliptic case: improvement of the Lipschitz constant of the level sets of u away from $F(u)$, propagation of this interior gain to $F(u)$, rescaling and iteration.

Flat free boundaries are smooth. The same result can be achieved if we ask that the Lipschitz constant in space is small. As a consequence it is possible to prove an instantaneous regularization from flat initial free boundary and a waiting time regularization phenomenon when the solution evolves in time towards a steady state non degenerate solution.

The extension of the above results to variable coefficients runs into complications, much more serious with respect the elliptic counterpart. Other than the underlying double homogeneity of the problem, the absence of any kind of invariance with respect to the time variables constitutes a major source of difficulties.

In a paper still in preparation, Ferrari and Salsa prove the optimal regularity of the solution under non-degeneracy and flatness conditions. The key point in a control of u_t by the spatial gradient, which is already a quite delicate estimate in the case of the heat equation. On the other hand, in the case of time independent operator this estimates carries out as in [6] and no flatness nor any non-degeneracy condition is needed.

The analysis of the free boundary requires a potential analysis apparatus, developed in two very fine papers by A. Grimaldi and R.Argiolas (see [4] and [5]). Among other results, they prove that the $L-$caloric measure in a Lipschitz (non cylindrical) domain Ω in $\mathbb{R}^{n+1}$ is an A_∞ weigth with respect to surface measure on $\partial\Omega$, a crucial fact assuring a common interior gain in the Lipschitz constant of the level set of the solution from both side. In any case, the case of time dependent coefficients remains open.

Summarizing:

Open problem 5. *Analysis of the free boundary, for linear operators with Hölder continuous time-dependent coefficients.* To prove that under the stated nondegeneracy conditions, the free boundary enjoys instantaneous regularization.

References

[1] H.W. Alt & L. Caffarelli, *Existence and regularity for a minimum problem with free bounday*, J. Reine und Angew. Math., 331(1)(1982), 105–144.

[2] H. W. Alt, L. Caffarelli & A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc., 282(2)(1984)), 431–461.

[3] R. Argiolas & F. Ferrari, to appear on Interphase and Free Boundaries.

[4] R. Argiolas & A. Grimaldi, *Green's function, caloric measure and Fatou theorems for non-divergence parabolic equations in non-cylindrical domains*, Forum Math., 20(2008), 213–237.

[5] R. Argiolas & A. Grimaldi, *The Dirichlet problem for second order parabolic operators in non-cylindrical domains*, to appear in Math. Nachrichten.

[6] Athanasopoulos, L. Caffarelli & S. Salsa, *Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems*, Ann. Math., 143(1996), 413–434.

[7] Athanasopoulos, L. Caffarelli & S. Salsa, *Regularity of the free boundary in parabolic phase-transition problems*, Acta Math., 176(2)(1996), , 245–282.

[8] Athanasopoulos, L. Caffarelli & S. Salsa, *Phase-transition problems of parabolic type: flat free boundaries are smooth*, Comm. Pure Appl. Math., 51(1998), 77–112.

[9] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries, Part* 1*: Lipschitz free boundaries are* C^1_α, Rev. Mat. Iberoamericana, 3(2)(1987), 139–162.

[10] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz*, Comm. Pure Appl. Math., 42(1)(1989), 55–78.

[11] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries, Part III: existence theory, compactness and dependence on X*, Ann. Scuola Norm. Sup. Pisa Cl. Sc. (4), 15(4)(1988), 583–602.

[12] M.C. Cerutti, F. Ferrari & S. Salsa, *Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$*, Arch. Rational Mech. Anal., 171(2004), 329–348

[13] L. Caffarelli & S. Salsa, *A geometric approach to free boundary problems* Graduate Studies in Mathematics, 68, American Mathematical Society, Providence, RI, 2005. x+270 pp..

[14] E. Fabes, N. Garofalo, S. Marin-Malave & S. Salsa, *Fatou theorems for some nonlinear elliptic equations*, Rev. Mat. Iberoamericana, 4(2)(1988), 227–251.

[15] F. Ferrari, *Two-phase problems for a class of fully nonlinear elliptic operators. Lipschitz free boundaries are $C^{1,\gamma}$*, Amer. J. Math., 128(3)(2006), 541–571.

[16] M. Feldman, *Regularity for nonisotropic two-phase problems with Lipschitz free boundaries*, Differential Integral Equations, 10(6)(1997), 1171–1179.

[17] M. Feldman, *Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations*, Indiana Univ. Math. J., 50(3)(2001), 1171–1200.

[18] F. Ferrari & S. Salsa, *Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$*, Arch. Rat. Mech. Anal., 171(2004), 329–348.

[19] F. Ferrari & S. Salsa, *Regularity of the free boundary in two-phase problems for linear elliptic operators*, Adv. in Math, 214(2007), 288–322.

[20] F. Ferrari & S. Salsa, *Subsolutions of elliptic operators in divergence form and application to two-phase free boundary problem*, Bound. Value Probl., 2007, ID 21425.

[21] A. Friedman, *Variational principles and free boundary problems*, Wiley, New York, 1970.

[22] S. Salsa, *Regularity in free boundary problems*, Conferenze del Seminario di Matematica, Università di Bari, 270, 1997.

[23] P.Y. Wang, *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. I. Lipschitz free boundaries are $C^{1,\alpha}$*, Comm. Pure Appl. Math., 53(7)(2000), 799–810.

[24] P.Y. Wang, *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. II. Flat free boundaries are Lipschitz*, Comm. Partial Differential Equations, 27(7&8)(2002), 1497–1514.

[25] P.Y. Wang, *Existence of solutions of two-phase free boundary for fully non linear equations of second order*, J. Geom. Anal., 7(2000), 1497–1514.