

Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 7(2008), pp. 249–267.

Rectifiable sets in Carnot groups

Raul SERAPIONI

A Ermanno Lanconelli, amico e maestro

Abstract¹. A general definition of intrinsic C^1 submanifolds and of Lipschitz submanifolds in Carnot groups is proposed. Intrinsic rectifiable sets, of general dimension and codimension, are consequently defined.

1. INTRODUCTION

The notion of rectifiable set is a key one in calculus of variations and in geometric measure theory. To develop a satisfactory theory of rectifiable sets inside Carnot groups has been the object of much research in the last ten years (see e.g. [1], [2], [6], [7], [10], [11], [12], [16], [17], [19], [22]).

In this note a general approach to rectifiable sets in homogeneous stratified groups (Carnot groups) is suggested. This approach generalizes the one introduced by Franchi, Serra Cassano and the author in [12].

Rectifiable sets, in Euclidean spaces, are natural generalizations of C^1 submanifolds; in one of the possible equivalent definitions (see e.g. [8, 18]), k -dimensional rectifiable sets in \mathbb{R}^n , $1 \leq k \leq n$, are sets contained, but for a negligible subset, in the countable union of k -dimensional C^1 submanifolds of \mathbb{R}^n . In such a definition, key roles are played both by the notion of C^1 submanifold and by the choice of the measure with which the exceptional negligible set is evaluated. The measure used is the k -dimensional Hausdorff measure. This is natural because relative open and bounded subsets of C^1 submanifolds of \mathbb{R}^n have finite and positive k -dimensional Hausdorff measure.

Hence, understanding C^1 submanifolds in Carnot groups is a preliminary task in developing a satisfactory theory of rectifiable sets.

In this note (see Definition 5.1) intrinsic C^1 submanifolds of a Carnot group \mathbb{G} are defined as subsets of \mathbb{G} that are, locally about any point, *uniformly intrinsic differentiable graphs of functions acting between complementary subgroups of \mathbb{G}* . Let us then explain more precisely this idea.

¹Author's address: R. Serapioni, Università degli Studi di Trento, Dipartimento di Matematica, Via Sommarive 14, 38050 Povo (Trento), Italy; e-mail: serapion@science.unitn.it.

The author is supported by GALA project of the 6th framework programme of the E.U. and by University of Trento and MIUR, Italy.

Keywords. Carnot groups, Heisenberg groups, intrinsic rectifiable sets, intrinsic graphs, intrinsic Lipschitz functions.

AMS Subject Classification. 28A75, 28C10.

Intrinsic graphs appeared naturally in [10], while studying level sets of P-differentiable functions from \mathbb{H}^n to \mathbb{R} (see Definition 2.2). These level sets, indeed, locally are intrinsic graphs. The definition is the following one: let \mathbb{M} and \mathbb{H} be *complementary subgroups* of \mathbb{G} , that is homogeneous subgroups, such that $\mathbb{M} \cap \mathbb{H} = e$ and $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ (here \cdot indicates the group operation in \mathbb{G} and $e = (0, \dots, 0)$ is the unit element), then the intrinsic (left) graph of $f : \mathbb{M} \rightarrow \mathbb{H}$ is the set

$$\text{graph}(f) = \{m \cdot f(m) : m \in \mathbb{M}\} .$$

In this case we say that $\text{graph}(f)$ is a graph over \mathbb{M} in direction \mathbb{H} .

Intrinsic Lipschitz continuity and intrinsic differentiability are given respecting the geometric structure of the ambient space \mathbb{G} .

$f : \mathbb{M} \rightarrow \mathbb{H}$ is *intrinsic Lipschitz* if, at each $p \in \text{graph}(f)$, there is an intrinsic cone with vertex p , axis \mathbb{H} and fixed opening, intersecting $\text{graph}(f)$ only at p (see Definitions 4.1 and 4.2).

$f : \mathbb{M} \rightarrow \mathbb{H}$ is *intrinsic differentiable at* $m \in \mathbb{M}$ if there is a homogeneous subgroup \mathbb{N} of \mathbb{G} such that, at $p = m \cdot f(m) \in \text{graph}(f)$, the left coset $p \cdot \mathbb{N}$ is the limit of group dilations of $\text{graph}(f)$ centered in p , or, in other words, if $p \cdot \mathbb{N}$ is the tangent plane to $\text{graph}(f)$ in p (see Definition 4.11). A uniform version of intrinsic differentiability is introduced in Definition 4.14.

To motivate this definition of intrinsic C^1 submanifolds, recall that, in Heisenberg groups \mathbb{H}^n (see definition in Section 2), a class of surfaces that proved to be a good generalization to the group setting of euclidean C^1 submanifolds are the so called H -regular submanifolds, (see Definition 6.5 and the references [12, 1, 6, 17, 24]). H -regular submanifolds can be considered as C^1 submanifolds because (i) they have a tangent plane at each point, the tangent plane being the coset of a homogeneous subgroup of the ambient space \mathbb{H}^n , (ii) the tangent planes depend continuously on the tangency point, (iii) provided we define the Hausdorff measures \mathcal{S}_d^k using the Carnot-Carathéodory metric of \mathbb{H}^n , H -regular submanifolds have integer Hausdorff dimension and locally finite Hausdorff measure, that can also be obtained by integration with appropriate area type formulas (see [12]).

H -regular submanifolds in \mathbb{H}^n are defined in two different ways according to their topological dimension k . If $1 \leq k \leq n$, a k -dimensional or *low dimensional* H -regular submanifold is, locally, the image in \mathbb{H}^n of an open set of \mathbb{R}^k , through an injective, P-differentiable function; a k -codimensional or *low codimensional* H -regular submanifold is, locally, the non critical level set of a P-differentiable function from \mathbb{H}^n to \mathbb{R}^k .

Differently from \mathbb{R}^n , where C^1 embedded submanifolds are equivalently defined as non-critical level sets of differentiable functions or as images of injective differentiable maps (or as *graphs of C^1 functions*), in \mathbb{H}^n , low dimensional H -regular surfaces cannot be seen as non critical level sets and low codimensional ones cannot be seen as (bilipschitz) images of open sets. Nevertheless, a common characterization of H -regular surfaces exists. Both low dimensional and low codimensional H -regular surfaces are, locally, *uniformly, intrinsic differentiable graphs* of functions acting between *complementary homogeneous subgroups* of \mathbb{H}^n (see [3]). This characterization extends naturally to general Carnot groups and it is used here as a definition of C^1 submanifold.

About the dimension of intrinsic differentiable submanifolds, notice that in \mathbb{H}^n , the topological dimension of a H -regular surface determines also its metric dimension. Indeed, low dimensional ones have topological and metric dimension k ,

$1 \leq k \leq n$, while low codimensional ones have topological dimension $2n + 1 - k$ and metric dimension $2n + 2 - k$ (see [12]).

This is not anymore the case in general Carnot groups. In a given Carnot group \mathbb{G} , intrinsic differentiable graphs with equal topological dimension but different metric dimension may exist. Indeed, bounded and relatively open subsets of an intrinsic Lipschitz or uniformly intrinsic differentiable graph over a subgroup \mathbb{M} have the same topological and metric dimensions of the group \mathbb{M} and there are homogeneous subgroups with the same topological dimension but different metric dimension (see Example 3.6 and Proposition 4.8).

Hence it is natural to distinguish graphs and submanifolds according with their topological *and* metric dimensions. We consider here (h, k) -graphs or (h, k) -submanifolds, meaning intrinsic graphs or submanifolds of topological dimension h and of metric dimension k ; here h, k will be integers with $h \leq k$.

Correspondingly $E \subset \mathbb{G}$ is said to be (h, k) -rectifiable if

$$(1) \quad \mathcal{S}_d^k(E \setminus \bigcup_{i \in \mathbb{N}} S_i) = 0$$

and S_i are *uniformly intrinsic differentiable (h, k) -submanifolds* (see Definition 5.2).

Recall also that, in the definition of euclidean rectifiable sets, C^1 submanifolds can be equivalently substituted by Lipschitz graphs of the same dimension. The equivalence of the two definitions relies mainly on almost everywhere differentiability of Lipschitz functions (Rademacher’s theorem) and on Whitney’s extension theorem.

It is then natural to consider also the class of $(h, k)_L$ -rectifiable sets: $E \subset \mathbb{G}$ is $(h, k)_L$ -rectifiable if (1) holds with S_i intrinsic Lipschitz (h, k) -graphs (see Definition 5.2).

The equivalence of these two definitions of intrinsic rectifiable sets is still an open problem, even in Heisenberg groups. In fact, generalizations of Rademacher’s theorem, of Lipschitz extension theorem and of Whitney’s extension theorem, to the setting of intrinsic Lipschitz or intrinsic differentiable functions, are largely unknown. Only some partial results have been proved, for 1-codimensional intrinsic graphs in Heisenberg groups (see [13]) and, in turn, it is known that, in \mathbb{H}^n , $(2n, 2n + 1)_L$ -rectifiable sets are the same as $(2n, 2n + 1)$ -rectifiable sets.

The general definition of rectifiable sets, proposed in this paper, follows from work of the author together with Bruno Franchi, Pertti Mattila and Francesco Serra Cassano. Nevertheless, the author takes full responsibility of any inaccuracy or mistake introduced in this note.

2. NOTATIONS AND DEFINITIONS

Carnot groups. Here we only fix a few notations. For a general account, see e.g. [4, 9, 14]. A *graded group* of step k is a connected, simply connected Lie group \mathbb{G} whose Lie algebra \mathfrak{g} , of dimension n , is the direct sum of k subspaces \mathfrak{g}_i , $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

for $1 \leq i, j \leq k$ and $\mathfrak{g}_i = 0$ for $i > k$. A *Carnot group* \mathbb{G} of step k is a graded group of step k , where \mathfrak{g}_1 generates all of \mathfrak{g} . That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, k$. The exponential map is a one to one diffeomorphism from \mathfrak{g} to \mathbb{G} . Let X_1, \dots, X_n be a base for \mathfrak{g} such that X_1, \dots, X_{m_1} is a base for \mathfrak{g}_1 and, for $1 < j \leq k$,

$X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for \mathfrak{g}_j .

Any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1 X_1 + \dots + p_n X_n)$ and we will always identify p with the n -tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and \mathbb{G} with (\mathbb{R}^n, \cdot) . The explicit expression of the group operation \cdot , determined by the Campbell-Hausdorff formula, has the form

$$(2) \quad x \cdot y = x + y + \mathcal{Q}(x, y), \quad \forall x, y \in \mathbb{R}^n$$

where $\mathcal{Q}(x, y) = (\mathcal{Q}_1(x, y), \dots, \mathcal{Q}_n(x, y)) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are polynomials. Moreover, $\mathcal{Q}_i(x, y) = 0$, for $i = 1, \dots, m_1$ and

$$\mathcal{Q}_i(x, y) = \mathcal{Q}_i(x_1, \dots, x_{m_{j-1}}, y_1, \dots, y_{m_{j-1}}),$$

for $1 < j \leq k$ and $m_{j-1} + 1 \leq i \leq m_j$. It is useful to think $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \dots \oplus \mathbb{G}^k$, where $\mathbb{G}^i = \exp(\mathfrak{g}_i) = \mathbb{R}^{m_i - m_{i-1}}$ is the i^{th} layer of \mathbb{G} and to write $p \in \mathbb{G}$ as (p^1, \dots, p^k) , with $p^i \in \mathbb{G}^i$. According to this, we write also, for all $p, q \in \mathbb{G}$,

$$(3) \quad p \cdot q = (p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p, q), \dots, p^k + q^k + \mathcal{Q}^k(p, q)).$$

The inverse of $p \in \mathbb{G}$ is $p^{-1} = (-p_1, \dots, -p_n)$ and $e = (0, \dots, 0)$ is the identity of \mathbb{G} . If \mathbb{G} is a graded group, for all $\lambda > 0$, the (*non isotropic*) dilations $\delta_\lambda : \mathbb{G} \rightarrow \mathbb{G}$ are automorphisms of \mathbb{G} defined as $\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \dots, \lambda^{\alpha_n} x_n)$, where $\alpha_i = j$, if $m_{j-1} < i \leq m_j$. Given any homogeneous norm $\|\cdot\|$, the distance in \mathbb{G} is defined as

$$(4) \quad d(x, y) = d(y^{-1} \cdot x, 0) = \|y^{-1} \cdot x\|, \quad \text{for all } x, y \in \mathbb{G}.$$

For an explicit family of homogeneous norms in a Carnot group see [11, Theorem 5.1]). The distance d in (4) is comparable with the Carnot Carathéodory distance of \mathbb{G} and

$$(5) \quad d(z \cdot x, z \cdot y) = d(x, y), \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$$

for all $x, y, z \in \mathbb{G}$ and all $\lambda > 0$. For $r > 0$ and $p \in \mathbb{G}$, $B(p, r)$ is the open ball associated with d . With the distance d , (spherical) Hausdorff measures are obtained following Carathéodory's construction (see [8, Section 2.10.2.]). Precisely, if $m \geq 0$ and $\mathcal{A} \subset \mathbb{G}$, the (*intrinsic spherical*) Hausdorff measure \mathcal{S}_d^m is,

$$\mathcal{S}_d^m(\mathcal{A}) := \lim_{\delta \rightarrow 0} \mathcal{S}_{d, \delta}^m(\mathcal{A}),$$

where $\mathcal{S}_{d, \delta}^m(\mathcal{A}) = \inf \{ \sum_i r_i^m : \mathcal{A} \subset \bigcup_i B(p_i, r_i), r_i \leq \delta \}$.

Translation invariance and dilation homogeneity of Hausdorff measures follow from (5) and, for $\mathcal{A} \subseteq \mathbb{G}$, $p \in \mathbb{G}$ and $r \in [0, \infty)$,

$$\mathcal{S}_d^m(p \cdot \mathcal{A}) = \mathcal{S}_d^m(\mathcal{A}) \quad \text{and} \quad \mathcal{S}_d^m(\delta_r \mathcal{A}) = r^m \mathcal{S}_d^m(\mathcal{A}).$$

A *homogeneous subgroup* of a Carnot group \mathbb{G} (see [23, 5.2.4]) is a subgroup \mathbb{H} such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda > 0$. Homogeneous subgroups are linear subspaces of \mathbb{G} , when it is identified with \mathbb{R}^n . The *topological dimension* of a (sub)group is the dimension of its Lie algebra. The *metric dimension* of a subset is its Hausdorff dimension, with respect to the Hausdorff measures \mathcal{S}_d^k . The metric dimension of a homogeneous subgroup is an integer usually larger than its topological dimension (see [20]).

Definition 2.1. \mathbb{M} is a (h, k) -subgroup of \mathbb{G} if \mathbb{M} is a homogeneous subgroup of \mathbb{G} with linear or topological dimension h and metric dimension $k \geq h$.

The notion of *P-differentiability* for functions acting between graded groups, was introduced by Pansu in [21].

Definition 2.2. Let $\mathbb{G}_1, \mathbb{G}_2$ be graded groups, with homogeneous norms $\|\cdot\|_1, \|\cdot\|_2$ and dilations $\delta_\lambda^1, \delta_\lambda^2$.

$L : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *H-linear* if L is a *homogeneous homomorphism*, that is if L is a group homomorphism from \mathbb{G}_1 to \mathbb{G}_2 such that

$$L(\delta_\lambda^1 g) = \delta_\lambda^2 L(g), \quad \text{for all } g \in \mathbb{G}_1 \text{ and } \lambda > 0.$$

$f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *P-differentiable* in $g_0 \in \mathcal{A}$ if there is a *H-linear* function $df_{g_0} : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ such that

$$\left\| (df_{g_0}(g_0^{-1} \cdot g))^{-1} \cdot f(g_0)^{-1} \cdot f(g) \right\|_2 = o(\|g_0^{-1} \cdot g\|_1),$$

as $\|g_0^{-1} \cdot g\|_1 \rightarrow 0$, where $o(t)/t \rightarrow 0$ as $t \rightarrow 0^+$. The *H-linear* function df_{g_0} is called the *P-differential* of f in g_0 .

Heisenberg groups. Once more we only fix a few notations. For a general account, see e.g. [4, 5, 23].

The n -dimensional Heisenberg group \mathbb{H}^n is identified with \mathbb{R}^{2n+1} through exponential coordinates and $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) \in \mathbb{R}^{2n+1}$. For $p, q \in \mathbb{H}^n$, the group operation is

$$(6) \quad \begin{aligned} p \cdot q &= (p_1 + q_1, \dots, p_{2n} + q_{2n}, \\ & p_{2n+1} + q_{2n+1} + \sum_{i=1}^n (p_i q_{i+n} - p_{i+n} q_i) / 2). \end{aligned}$$

For $\lambda > 0$, the dilations $\delta_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are

$$\delta_\lambda p := (\lambda p_1, \dots, \lambda p_{2n}, \lambda^2 p_{2n+1}), \quad \text{for all } p \in \mathbb{H}^n.$$

The standard base of the Lie algebra \mathfrak{h} of \mathbb{H}^n is given by the left invariant vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$, where, for $i = 1, \dots, n$,

$$X_i(p) := \partial_i - \frac{1}{2} p_{i+n} \partial_{2n+1}, \quad Y_i(p) := \partial_{i+n} + \frac{1}{2} p_i \partial_{2n+1}, \quad T(p) := \partial_{2n+1}.$$

The *horizontal subspace* \mathfrak{h}_1 is the subspace of \mathfrak{h} spanned by $X_1, \dots, X_n, Y_1, \dots, Y_n$. Denoting by \mathfrak{h}_2 the linear span of T , then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2$.

The centre of \mathbb{H}^n is the subgroup $\mathbb{T} := \exp(\mathfrak{h}_2) = \{(0, \dots, 0, p_{2n+1})\}$.

The *horizontal bundle* $H\mathbb{H}^n$ is the sub-bundle of the tangent bundle $T\mathbb{H}^n$ whose fibers $H\mathbb{H}_p^n$ are spanned by the horizontal vectors $X_1(p), \dots, Y_n(p)$. In particular

$$H\mathbb{H}_e^n = \{(p_1, \dots, p_{2n}, 0)\}.$$

We will use the homogeneous norm

$$(7) \quad \|p\| := \max \{ \|(p_1, \dots, p_{2n})\|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2} \},$$

for $p \in \mathbb{H}^n$, while the distance d is defined as in (4).

3. COMPLEMENTARY SUBGROUPS AND GRAPHS

From now on \mathbb{G} will always be a homogeneous stratified group.

Definition 3.1. Two homogeneous subgroups \mathbb{M}, \mathbb{H} of \mathbb{G} are *complementary subgroups* in \mathbb{G} , if $\mathbb{M} \cap \mathbb{H} = \{e\}$ and if $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, that is for all $g \in \mathbb{G}$, there are $m \in \mathbb{M}, h \in \mathbb{H}$ such that $g = m \cdot h$. If \mathbb{M}, \mathbb{H} are complementary subgroups in \mathbb{G} and one of them is a normal subgroup then \mathbb{G} is said the *semi-direct product* of \mathbb{M} and \mathbb{H} .

Proposition 3.2. *If \mathbb{M}, \mathbb{H} are complementary subgroups in \mathbb{G} as in Definition 3.1, each $g \in \mathbb{G}$ has unique components $g_{\mathbb{M}} \in \mathbb{M}, g_{\mathbb{H}} \in \mathbb{H}$, such that $g = g_{\mathbb{M}} \cdot g_{\mathbb{H}}$. The projection maps*

$$(9) \quad \Pi_{\mathbb{M}} : \mathbb{G} \rightarrow \mathbb{M} \text{ and } \Pi_{\mathbb{H}} : \mathbb{G} \rightarrow \mathbb{H}$$

defined as $\Pi_{\mathbb{M}}(g) = g_{\mathbb{M}}$ and $\Pi_{\mathbb{H}}(g) = g_{\mathbb{H}}$ are continuous. Moreover, there is $c_0 = c_0(\mathbb{M}, \mathbb{H}) > 0$ such that

$$(9) \quad c_0 (\|g_{\mathbb{M}}\| + \|g_{\mathbb{H}}\|) \leq \|g\| \leq \|g_{\mathbb{M}}\| + \|g_{\mathbb{H}}\| .$$

Proof. See [3, 17]. □

Recall the following elementary fact in group theory (see e.g. [15, Lemma 2.8])

Proposition 3.3. *If $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ then it is also true that*

$$\mathbb{G} = \mathbb{H} \cdot \mathbb{M} ,$$

that is, each $g \in \mathbb{G}$ can be written as $g = h \cdot m$, with $m \in \mathbb{M}, h \in \mathbb{H}$. Once more $g = h \cdot m$ in a unique way because $\mathbb{H} \cap \mathbb{M} = e$,

The notation $\Pi_{\mathbb{M}}$ and $\Pi_{\mathbb{H}}$, for the projections defined in Lemma 3.1, is somehow non correct. Indeed each projection function $\Pi_{\mathbb{M}}$ and $\Pi_{\mathbb{H}}$ depends on both the complementary subgroups \mathbb{M} and \mathbb{H} . One should write, more correctly, something like $\Pi_{\mathbb{M}, \mathbb{H}}$ and $\Pi_{\mathbb{H}, \mathbb{M}}$.

Remark 3.4. Notice that, differently from Euclidean spaces, a map like $\Pi_{\mathbb{M}}$, in general, is not a Lipschitz map from \mathbb{G} to \mathbb{M} . Here we endow \mathbb{M} with the distance given by the restriction of the distance d of \mathbb{G} .

Example 3.5. Let \mathbb{G} be the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot)$ with the group law as in (6). For any integer $h, 1 \leq h \leq n$, let

$$\begin{aligned} \mathbb{V} &= \{(x_1, \dots, x_h, 0, \dots, 0) : x_1, \dots, x_h \in \mathbb{R}\} , \\ \mathbb{W} &= \{(0, \dots, 0, x_{h+1}, \dots, x_{2n+1}) : x_{h+1}, \dots, x_{2n+1} \in \mathbb{R}\} . \end{aligned}$$

Then \mathbb{V}, \mathbb{W} are homogeneous subgroups and are complementary subgroups in \mathbb{H}^n . \mathbb{V} is a (h, h) -subgroup while \mathbb{W} is a $(2n + 1 - h, 2n + 2 - h)$ -subgroup.

Notice that \mathbb{V} is a commutative subgroup, isomorphic with \mathbb{R}^h , while \mathbb{W} is a normal subgroup; hence \mathbb{H}^n is the semi-direct product of \mathbb{V} and \mathbb{W} . See also Proposition 6.1.

Example 3.6. Let \mathbb{G} be the Engel group \mathbb{E} . That is $\mathbb{E} = (\mathbb{R}^4, \cdot)$ with the group law defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + (x_1 y_2 + x_2 y_1)/2 \\ x_4 + y_4 + [(x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2)]/2 + \\ + (x_1 - y_1 + x_2 - y_2)(x_1 y_2 - x_2 y_1)/12 \end{pmatrix}$$

and the family of dilation

$$\delta_\lambda(p_1, p_2, p_3, p_4) = (\lambda p_1, \lambda p_2, \lambda^2 p_3, \lambda^3 p_4).$$

Then

$$\mathbb{G}_1 = \{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\}$$

$$\mathbb{G}_2 = \{(0, x_2, x_3, x_4) : x_2, x_3, x_4 \in \mathbb{R}\},$$

are homogeneous complementary subgroups in \mathbb{E} . \mathbb{G}_1 is a $(1, 1)$ -subgroup while \mathbb{G}_2 is a $(3, 6)$ -subgroup.

Another couple of complementary subgroups is given by

$$\mathbb{E}_1 = \{(x_1, -x_1, x_3, 0) : x_1, x_3 \in \mathbb{R}\}$$

$$\mathbb{E}_2 = \{(x_1, 0, 0, x_4) : x_1, x_4 \in \mathbb{R}\}.$$

\mathbb{E}_1 is a $(2, 3)$ -subgroup and \mathbb{E}_2 is a $(2, 4)$ -subgroup.

Notice that, in the first couple, \mathbb{G}_2 is a normal subgroup, hence \mathbb{E} is the semi-direct product of \mathbb{G}_1 and \mathbb{G}_2 . On the contrary, neither \mathbb{E}_1 nor \mathbb{E}_2 is a normal subgroup, as can be checked with a direct computation. In this case the product is not semi-direct.

Definition 3.7. Assume that \mathbb{H} is a homogeneous subgroup of \mathbb{G} . A set $S \subset \mathbb{G}$ is a (*left*) \mathbb{H} -graph (or a left graph in direction \mathbb{H}) if S intersects each left coset of \mathbb{H} in at most one point.

Remark 3.8. If $A \subset \mathbb{G}$ parametrizes the left cosets of \mathbb{H} , that is if A is a \mathbb{H} -graph that intersects each left coset of \mathbb{H} , and if S is another \mathbb{H} -graph, then there is a unique function $\varphi : \mathcal{E} \subset A \rightarrow \mathbb{H}$ such that S is the graph of φ , that is

$$S = \text{graph}(\varphi) := \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}.$$

Conversely, for any $\psi : \mathcal{D} \subset A \rightarrow \mathbb{H}$ the set $\text{graph}(\psi)$ is an \mathbb{H} -graph.

One has an important special case when \mathbb{H} admits a complementary subgroup \mathbb{M} . Indeed, in this case, \mathbb{M} parametrizes the left cosets of \mathbb{H} and we have that

$$S \text{ is a } \mathbb{H}\text{-graph if and only if } S = \text{graph}(\varphi)$$

for $\varphi : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$. By uniqueness of the components along \mathbb{M} and \mathbb{H} , if $S = \text{graph}(\varphi)$ then φ is uniquely determined.

From now only graphs of functions acting between complementary subgroups will be considered here. Nevertheless it is relevant to mention that well known examples of \mathbb{H} -graphs that are not graphs of functions acting between complementary subgroups are given by sets as

$$S = \{(x_1, \dots, y_n, \varphi(x_1, \dots, y_n))\} \subset \mathbb{H}^n.$$

In our notation S is a \mathbb{T} -graph. The left cosets of \mathbb{T} are parametrized over the set $A = H\mathbb{H}_e^n$. We recall that \mathbb{T} has no complementary subgroup in \mathbb{H}^n (see Proposition 6.1), hence in general there isn't a couple of complementary subgroups \mathbb{M}, \mathbb{H} of \mathbb{H}^n and a function $\psi : \mathbb{M} \rightarrow \mathbb{H}$ such that $S = \text{graph}(\psi)$.

The importance of intrinsic graphs inside groups relies on the following invariance property.

Proposition 3.9. *Assume that \mathbb{H} is a homogeneous subgroup of \mathbb{G} . If S is a \mathbb{H} -graph then, for all $\lambda > 0$ and $q \in \mathbb{G}$, $\delta_\lambda S$ and $q \cdot S$ are \mathbb{H} -graphs. In particular, if \mathbb{M}, \mathbb{H} are complementary subgroups in \mathbb{G} , if $S = \text{graph}(\varphi)$ with $\varphi : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$, then for all $\lambda > 0$,*

$$\delta_\lambda S = \text{graph}(\varphi_\lambda), \quad \text{with } \delta_\lambda \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$$

and $\varphi_\lambda = \delta_\lambda \circ \varphi \circ \delta_{1/\lambda}$; for any $q \in \mathbb{G}$,

$$q \cdot S = \text{graph}(\varphi_q), \quad \text{with } \varphi_q : \mathcal{E}_q \subset \mathbb{M} \rightarrow \mathbb{H}.$$

This last statement can be made explicit when \mathbb{G} is a semi-direct product of \mathbb{M}, \mathbb{H} . Precisely, if \mathbb{M} is normal in \mathbb{G} then $\mathcal{E}_q := q \cdot \mathcal{E} \cdot (q_{\mathbb{H}})^{-1} \subset \mathbb{M}$ and

$$(10) \quad \varphi_q(y) = q_{\mathbb{H}} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{H}}), \quad \text{for } y \in \mathcal{E}_q;$$

if \mathbb{H} is normal in \mathbb{G} then $\mathcal{E}_q := q_{\mathbb{M}} \cdot \mathcal{E} \subset \mathbb{M}$ and

$$(11) \quad \varphi_q(y) = y^{-1} \cdot q \cdot q_{\mathbb{M}}^{-1} \cdot y \cdot \varphi(q_{\mathbb{M}}^{-1} \cdot y), \quad \text{for } y \in \mathcal{E}_q.$$

Proof. If $x, x' \in S$ and $x \neq x'$ then, by definition of \mathbb{H} -graph, x, x' belong to different left cosets of \mathbb{H} . Then it is easy to see that $\delta_\lambda x, \delta_\lambda x'$ belong to different cosets of \mathbb{H} , because \mathbb{H} is a homogeneous subgroup, and that $q \cdot x, q \cdot x'$ belong to different cosets of \mathbb{H} , by elementary properties of cosets (see e.g. [15, chapter 2, section 4]). This, by definition, proves that both $\delta_\lambda S$ and $q \cdot S$ are \mathbb{H} -graphs and, consequently, that there are φ_λ and φ_q such that $\delta_\lambda S = \text{graph}(\varphi_\lambda)$ and $q \cdot S = \text{graph}(\varphi_q)$.

Then observe that, by uniqueness of the components, $\delta_\lambda(m \cdot \varphi(m)) = m' \cdot \varphi(m')$ implies that $\delta_\lambda m = m'$ and that $\varphi_\lambda = \delta_\lambda \circ \varphi \circ \delta_{1/\lambda}$.

The proof of (10) and (11) is in [3, Proposition 3.6]. □

4. INTRINSIC LIPSCHITZ AND INTRINSIC DIFFERENTIABLE GRAPHS

Intrinsic Lipschitz functions. The definition of *intrinsic Lipschitz graph* in \mathbb{G} respects strictly the geometry of the ambient group \mathbb{G} . Precisely a \mathbb{H} -graph S is said to be an intrinsic Lipschitz \mathbb{H} -graph if S intersects *intrinsic cones* with axis \mathbb{H} , fixed opening and vertex on S only in the vertex. Intrinsic Lipschitz functions appeared for the first time in [10] and were studied, more diffusely, in [12, 13].

We begin with two definitions of intrinsic (closed) cones.

Definition 4.1. Let \mathbb{H} be a homogeneous subgroup of \mathbb{G} , $q \in \mathbb{G}$, $\alpha > 0$.

The cone $X(q, \mathbb{H}, \alpha)$ with axis \mathbb{H} , vertex q , opening α is

$$X(q, \mathbb{H}, \alpha) = q \cdot X(e, \mathbb{H}, \alpha), \quad \text{where}$$

$$X(e, \mathbb{H}, \alpha) = \{p : \text{dist}(p, \mathbb{H}) \leq \alpha \|p\|\}.$$

Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} . The cone $C_{\mathbb{M},\mathbb{H}}(q, \alpha)$ with base \mathbb{M} , axis \mathbb{H} , vertex q , opening α is

$$C_{\mathbb{M},\mathbb{H}}(q, \alpha) = q \cdot C_{\mathbb{M},\mathbb{H}}(e, \alpha), \quad \text{where}$$

$$C_{\mathbb{M},\mathbb{H}}(e, \alpha) = \{p \in \mathbb{G} : \|p_{\mathbb{M}}\| \leq \alpha \|p_{\mathbb{H}}\|\} .$$

It is not difficult to prove that, given $\alpha > 0$ there are α_1, α_2 , depending on \mathbb{M} and \mathbb{H} , such that $0 < \alpha_1 < \alpha < \alpha_2$ and

$$(12) \quad C_{\mathbb{M},\mathbb{H}}(q, \alpha_1) \subset X(q, \mathbb{H}, \alpha) \subset C_{\mathbb{M},\mathbb{H}}(q, \alpha_2) .$$

Definition 4.2. Let \mathbb{H} be a homogeneous subgroup of \mathbb{G} .

A \mathbb{H} -graph S is an *intrinsic Lipschitz \mathbb{H} -graph* if there is $L > 0$ such that, for all $p \in S$,

$$S \cap X(p, \mathbb{H}, L) = \{p\} .$$

Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} and $f : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$. Then f is *intrinsic Lipschitz* in \mathcal{E} , if there is $M > 0$ such that, for all $q \in \text{graph}(f)$,

$$(13) \quad C_{\mathbb{M},\mathbb{H}}(q, 1/M) \cap \text{graph}(f) = \{q\} .$$

The Lipschitz constant of f in \mathcal{E} is the infimum of the numbers M such that (13) holds. An intrinsic Lipschitz function, with Lipschitz constant not exceeding $L > 0$, is an *intrinsic L -Lipschitz function*.

If \mathbb{M}, \mathbb{H} are complementary subgroups in \mathbb{G} then S is a Lipschitz \mathbb{H} -graph if and only if there is an intrinsic Lipschitz function $f : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$ such that

$$S = \text{graph}(f) .$$

Being an intrinsic Lipschitz function is conserved after left translations of the graph. This is stated in the following Proposition whose proof is immediate.

Proposition 4.3. *Let \mathbb{H} be a homogeneous subgroup of \mathbb{G} . If $S \subset \mathbb{G}$ is an intrinsic Lipschitz \mathbb{H} -graph then, for all $q \in \mathbb{G}$, $q \cdot S$ is an intrinsic Lipschitz \mathbb{H} -graph. If \mathbb{M}, \mathbb{H} are complementary subgroups, then, for all $q \in \mathbb{G}$, $f : \mathbb{M} \rightarrow \mathbb{H}$ is intrinsic L -Lipschitz, if and only if $f_q : \mathbb{M} \rightarrow \mathbb{H}$ is intrinsic L -Lipschitz.*

There are also algebraic characterizations of intrinsic Lipschitz functions. One of them is given in the following Proposition whose proof can be found in [3].

Proposition 4.4. *Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} . Then $f : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$ is intrinsic Lipschitz in \mathcal{E} , if and only if there is $L > 0$ such that, for all $q \in \text{graph}(f)$ and for all $m \in \mathcal{E}_{q^{-1}}$,*

$$(14) \quad \|f_{q^{-1}}(m)\| \leq L \|m\| .$$

Proposition 4.5. *If $f : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$ is intrinsic Lipschitz in \mathcal{E} , then it is continuous in \mathcal{E} .*

Proof. By Definition 4.2 and (12), if f is intrinsic Lipschitz in $m \in \mathcal{E} \subset \mathbb{M}$, then there is $\alpha > 0$ such that

$$X(m \cdot f(m), \mathbb{H}, \alpha) \cap \text{graph}(f) = \{m \cdot f(m)\} .$$

Hence, for all $m' \in \mathcal{E}$,

$$(15) \quad d(m \cdot f(m), m' \cdot f(m')) \leq \frac{1}{\alpha} \text{dist}(m' \cdot f(m'), m \cdot f(m) \cdot \mathbb{H}) .$$

Observe that, because $f(m), f(m') \in \mathbb{H}$,

$$\begin{aligned} \text{dist}(m' \cdot f(m'), m \cdot f(m) \cdot \mathbb{H}) &= \text{dist}(m' \cdot f(m'), m \cdot \mathbb{H}) \leq \\ &\leq d(m' \cdot f(m'), m \cdot f(m')) \rightarrow 0 \end{aligned}$$

if $d(m, m') \rightarrow 0$. Hence, by (15), if $d(m, m') \rightarrow 0$ then also $d(m \cdot f(m), m' \cdot f(m')) \rightarrow 0$ and consequently $d(f(m), f(m')) \rightarrow 0$ and the continuity is proved. □

Remark 4.6. If \mathbb{M}, \mathbb{H} are homogeneous subgroups they are metric spaces as subsets of \mathbb{G} . Hence it makes sense to consider also *Lipschitz functions* from \mathbb{M} to \mathbb{H} . As usual, $f : \mathbb{M} \rightarrow \mathbb{H}$ is Lipschitz if there is $L > 0$ such that, for all $m, \bar{m} \in \mathbb{M}$,

$$(16) \quad \|f(m)^{-1} \cdot f(\bar{m})\| \leq L \|m^{-1} \cdot \bar{m}\| .$$

The classes of intrinsic Lipschitz functions and of Lipschitz functions are different ones. The following examples show that neither *intrinsic Lipschitz* implies *Lipschitz* nor the opposite. In particular notice that condition (16) is not conserved after left translations of the graph of f . It is precisely the invariance under graph translations (Proposition 4.3) that makes intrinsic Lipschitz functions useful to define geometric objects in \mathbb{G} .

Example. In $\mathbb{G} = \mathbb{H}^1 = \mathbb{R}^3$, consider the complementary subgroups \mathbb{V} and \mathbb{W} defined as

$$\mathbb{V} = \{v = (v_1, 0, 0)\} \text{ and } \mathbb{W} = \{w = (0, w_2, w_3)\} .$$

Because \mathbb{W} is a normal subgroup in \mathbb{H}^1 , the algebraic form of the translated functions are as in Proposition 3.9. The norm in (7) becomes $\|w\| = \max\{|w_2|, |w_3|^{1/2}\}$, for $w \in \mathbb{W}$ and $\|v\| = |v_1|$, for $v \in \mathbb{V}$.

$\varphi : \mathbb{W} \rightarrow \mathbb{V}$, defined as $\varphi(w) := (1 + |w_3|^{1/2}, 0, 0)$, satisfies (16) with $L = 1$, hence φ is Lipschitz. On the contrary φ is not intrinsic Lipschitz. Indeed, let $p := (1, 0, 0) \in \text{graph}(\varphi)$, then $\varphi_{p^{-1}}(w) = (|w_2 + w_3|^{1/2}, 0, 0)$. For $\varphi_{p^{-1}}$, (14) does not hold. This example shows also that condition (16)) is not invariant under graph translations.

$\psi : \mathbb{W} \rightarrow \mathbb{V}$, defined as $\psi(w) := (1 + |w_3 - w_2|^{1/2}, 0, 0)$, is intrinsic Lipschitz; indeed, with $p = (1, 0, 0)$ and $\varphi(w) := (|w_3|^{1/2}, 0, 0)$ we have $\psi(w) = \varphi_p(w)$, so that ψ is intrinsic Lipschitz because φ is intrinsic Lipschitz. On the contrary ψ is not Lipschitz, in the sense of (16), as can be easily observed.

Proposition 4.8 shows that the metric dimension of the graph of an intrinsic Lipschitz function is bounded above by the metric dimension of the domain (see also [13, Theorem 4.12]). We begin with a Lemma whose proof is not as trivial as in Euclidean spaces because projections are not Lipschitz maps (see Remark 3.4).

Lemma 4.7. *Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} . Assume that \mathbb{M} is a (h, k) -group, that is \mathbb{M} has linear dimension h and metric dimension $k \geq h$. Then there is $c = c(\mathbb{M}, \mathbb{H}) > 0$ such that*

$$(17) \quad \mathcal{L}^h(\Pi_{\mathbb{M}}B(p, r)) = cr^k ,$$

for all $p \in \mathbb{G}$ and $r > 0$.

Proof. Define

$$c(\mathbb{M}, \mathbb{H}) := \mathcal{L}^h(B(e, 1)) ,$$

and observe that $c(\mathbb{M}, \mathbb{H}) > 0$, because $\Pi_{\mathbb{M}}$ is an open mapping. By group dilations $\Pi_{\mathbb{M}}(B(e, r)) = \Pi_{\mathbb{M}}(\delta_r B(e, 1)) = \delta_r \Pi_{\mathbb{M}}(B(e, r))$ hence

$$\mathcal{L}^h(\Pi_{\mathbb{M}}(B(e, r))) = c(\mathbb{M}, \mathbb{H})r^k .$$

Now to prove (17) it is enough to observe that, for any measurable $\mathcal{E} \subset \mathbb{G}$ and for any $p \in \mathbb{G}$,

$$(18) \quad \mathcal{L}^h(\Pi_{\mathbb{M}}(p \cdot \mathcal{E})) = \mathcal{L}^h(\Pi_{\mathbb{M}}\mathcal{E}) .$$

In turn, to prove (18) it is enough to show that, for any $p \in \mathbb{G}$

$$(19) \quad \begin{aligned} \tau_p : \mathbb{M} &\rightarrow \mathbb{M} \text{ defined as } \tau_p(m) := \Pi_{\mathbb{M}}(p \cdot m) , \\ &\text{has Jacobian identically equal to } 1 . \end{aligned}$$

Indeed this fact yields that, for $q \in \mathcal{E}$, the map

$$(20) \quad \begin{aligned} \Pi_{\mathbb{M}}(\mathcal{E}) &\rightarrow \Pi_{\mathbb{M}}(p \cdot \mathcal{E}) \text{ defined as } m = \Pi_{\mathbb{M}}(q) \mapsto \Pi_{\mathbb{M}}(p \cdot q) , \\ &\text{has Jacobian identically equal to } 1 . \end{aligned}$$

To see that (19) implies (20) just observe that, if $m = \Pi_{\mathbb{M}}(q)$,

$$\begin{aligned} p \cdot q &= \Pi_{\mathbb{M}}(p \cdot q) \cdot \Pi_{\mathbb{H}}(p \cdot q) \\ p \cdot q &= p \cdot m \cdot \Pi_{\mathbb{H}}(q) = \Pi_{\mathbb{M}}(p \cdot m) \cdot \Pi_{\mathbb{H}}(p \cdot m) \cdot \Pi_{\mathbb{H}}(q) \end{aligned}$$

hence, by uniqueness of the components, $\Pi_{\mathbb{M}}(p \cdot m) = \Pi_{\mathbb{M}}(p \cdot q)$ if $m = \Pi_{\mathbb{M}}(q)$.

Let us then prove (19).

Using the notations in (3) if $p = (p^1, \dots, p^k)$, $m = (m^1, \dots, m^k)$, then

$$p \cdot m = (p^1 + m^1, p^2 + m^2 + \mathcal{Q}^2(p^1, m^1), \dots, p^k + m^k + \mathcal{Q}^k(\dots)) ,$$

where, for $2 \leq i \leq k$, \mathcal{Q}^i depends only on p^1, \dots, p^{i-1} and m^1, \dots, m^{i-1} . Hence the k components of $\Pi_{\mathbb{M}}(p \cdot m)$ are

$$\begin{aligned} (\Pi_{\mathbb{M}}(p \cdot m))^1 &= (\Pi_{\mathbb{M}}(p^1))^1 + m^1 \\ (\Pi_{\mathbb{M}}(p \cdot m))^2 &= (\Pi_{\mathbb{M}}(p^2 + \mathcal{Q}^2(p^1, m^1)))^2 + m^2 \\ &\vdots \\ (\Pi_{\mathbb{M}}(p \cdot m))^k &= (\Pi_{\mathbb{M}}(p^k + \mathcal{Q}^k(\dots)))^k + m^k . \end{aligned}$$

Hence

$$J\tau_p = \begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 \\ * & I_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ * & \cdots & * & I_{k-1} & 0 \\ * & \cdots & * & * & I_k \end{pmatrix}$$

where I_1, \dots, I_k are the identity matrices, respectively, in \mathbb{R}^{m_1} and in $\mathbb{R}^{m_i - m_{i-1}}$, for $1 < i \leq k$. Hence $J\tau_p$ has determinant equal to 1.

□

Proposition 4.8. *Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} . Assume that \mathbb{M} is a (h, k) -group, that is \mathbb{M} has linear dimension h and metric dimension $k \geq h$. Let \mathcal{A} be a relatively open subset of \mathbb{M} and $f : \mathcal{A} \subset \mathbb{M} \rightarrow \mathbb{H}$ be an intrinsic L -Lipschitz function. Then there is $c = c(\mathbb{M}, \mathbb{H}, L) > 0$ such that*

$$(21) \quad \mathcal{L}^h(\mathcal{A}) \leq \mathcal{S}_d^k(\text{graph}(f)) \leq c \mathcal{L}^h(\mathcal{A}) .$$

Proof. The lower bound for $\mathcal{S}_d^k(\text{graph}(f))$ is a consequence of Lemma 4.7. Indeed, assume $\mathcal{S}_d^k(\text{graph}(f)) < \infty$. Fix $\varepsilon > 0$, $r = r(\varepsilon) > 0$ and a covering of $\text{graph}(f)$ with closed balls $B_i = B(p_i, r_i)$ such that $r_i \leq r$ and

$$\sum_i r_i^k \leq \mathcal{S}_d^k(\text{graph}(f)) + \varepsilon .$$

Then, by (17),

$$\mathcal{L}^h(\mathcal{A}) \leq \sum_i \mathcal{L}^h(\Pi_{\mathbb{M}}(B_i)) = c \sum_i r_i^k \leq c \mathcal{S}_d^k(\text{graph}(f)) + \varepsilon .$$

To get the upper bound (21), it is enough to prove that there is $c = c(\mathbb{M}, \mathbb{H}, L, R) > 0$ such that, for any $p \in \text{graph}(f)$, $R > 0$ and $\varepsilon > 0$, it is possible to cover $\text{graph}(f) \cap B(p, R)$ with less than $N := c(1/\varepsilon)^k$ metric balls of radius less than ε . Without loss of generality, we assume that $p = e$. Then we fix ε , $0 < \varepsilon < 1$. Using a Vitali covering argument we choose a covering of $\text{graph}(f) \cap B(e, R)$ with metric balls $B(p_i, 5\varepsilon)$, $p_i \in \text{graph}(f)$, such that the concentric smaller balls $B_i := B(p_i, \varepsilon)$ are pairwise disjoint. We estimate the number N of balls B_i in this Vitali covering. Observe that $B_i \subset B(e, R + 1)$ for $i = 1, \dots, N$. Hence

$$(22) \quad E_i := \Pi_{\mathbb{M}}(B_i) \subset \Pi_{\mathbb{M}}(B(e, R + 1)) , \text{ for } i = 1, \dots, N$$

and, because the sets E_i are pairwise disjoint, to estimate N it is enough to evaluate from below $\mathcal{L}^h(E_i)$.

First observe that if $f(e) = e$ and f is intrinsic L Lipschitz then

$$(23) \quad \Pi_{\mathbb{M}}\left(B\left(e, \frac{c_0 r}{1 + L}\right)\right) \subset \Pi_{\mathbb{M}}(B(e, r) \cap \text{graph}(f)) ,$$

where c_0 is the constant in (9). Indeed, let $m \in \Pi_{\mathbb{M}}(B(e, c_0 r/(1 + L)))$, then there is $h \in \mathbb{H}$ such that $\|m \cdot h\| < c_0 r/(1 + L)$. Hence

$$\|m\| \leq \|m\| + \|h\| \leq \frac{1}{c_0} \|m \cdot h\| < \frac{r}{1 + L} ;$$

it follows

$$\|m \cdot f(m)\| \leq \|m\| + \|f(m)\| \leq (1 + L) \|m\| < r ,$$

that is $m \in \Pi_{\mathbb{M}}(B(e, r) \cap \text{graph}(f))$. The proof of (23) is completed.

Now observe that for all $A, B \subset \mathbb{G}$ and for all $p \in \mathbb{G}$

$$(24) \quad \Pi_{\mathbb{M}}(A) \subset \Pi_{\mathbb{M}}(B) \implies \Pi_{\mathbb{M}}(p \cdot A) \subset \Pi_{\mathbb{M}}(p \cdot B) .$$

Indeed, observe that $m \in \Pi_{\mathbb{M}}(p \cdot A) \iff m \cdot \mathbb{H} \cap p \cdot A \neq \emptyset \iff (p^{-1} \cdot m) \cdot \mathbb{H} \cap A \neq \emptyset$. Hence

$$\begin{aligned} m \in \Pi_{\mathbb{M}}(p \cdot A) &\iff \Pi_{\mathbb{M}}(p^{-1} \cdot m) \in \Pi_{\mathbb{M}}(A) \implies \\ &\implies \Pi_{\mathbb{M}}(p^{-1} \cdot m) \in \Pi_{\mathbb{M}}(B) \iff m \in \Pi_{\mathbb{M}}(p \cdot B) . \end{aligned}$$

Now, from (23), (24) it follows that for all $p \in \text{graph}(f)$,

$$(25) \quad \Pi_{\mathbb{M}} \left(B(p, \frac{c_0 r}{1+L}) \right) \subset \Pi_{\mathbb{M}}(B(p, r) \cap \text{graph}(f)) .$$

Indeed, $B(p, r) \cap \text{graph}(f) = p \cdot (B(e, r) \cap \text{graph}(f_{p^{-1}}))$ while $f_{p^{-1}}(e) = e$ and $f_{p^{-1}}$ is intrinsic L Lipschitz.

From (25) and Lemma 4.7 we have

$$\mathcal{L}^h(E_i) \geq \mathcal{L}^h \left(\Pi_{\mathbb{M}} \left(B(p, \frac{c_0 r}{1+L}) \right) \right) = cr^k ,$$

where $c = c(\mathbb{M}, \mathbb{H}, L) > 0$ and from (22), we finally have

$$N \leq c \left(\frac{1}{\varepsilon} \right)^k .$$

where c depends on $\mathbb{M}, \mathbb{H}, L, R$.

□

Intrinsic differentiable functions. A function $f : \mathbb{M} \rightarrow \mathbb{H}$, acting between complementary subgroups of \mathbb{G} , is *intrinsic differentiable* in a point $m \in \mathbb{M}$ if the graph of f has a tangent homogeneous subgroup in $m \cdot f(m) \in \text{graph}(f)$. Naturally, such a notion of intrinsic differentiability is equivalent with the existence of an approximating linear function. Intrinsic linear functions, acting between complementary subgroups, are functions whose graphs are homogeneous subgroups.

Definition 4.9. Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} . Then $L : \mathbb{M} \rightarrow \mathbb{H}$ is an *intrinsic linear function* if L is defined on all of \mathbb{M} and if $\text{graph}(L) = \{g \cdot L(g) : g \in \mathbb{M}\}$ is an homogeneous subgroup of \mathbb{G} .

A complete characterization of intrinsic linear functions seems to be known in \mathbb{H}^n only (see Proposition 6.3).

Remark 4.10. Intrinsic linear functions are not necessarily group homomorphism between their domains and codomains, as the following example shows.

Example. Let \mathbb{V}, \mathbb{W} be the complementary subgroups of \mathbb{H}^1 defined as $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$.

For any fixed $a \in \mathbb{R}$, the function $L : \mathbb{V} \rightarrow \mathbb{W}$ defined as

$$L(v) = (0, av_1, -av_1^2/2)$$

is intrinsic linear because $\text{graph}(L) = \{(t, at, 0) : t \in \mathbb{R}\}$ is a 1-dimensional subgroup of \mathbb{H}^1 . This L is not a group homomorphism from \mathbb{V} to \mathbb{W} .

Definition 4.11. Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} and $f : \mathcal{A} \subset \mathbb{M} \rightarrow \mathbb{H}$ with \mathcal{A} relatively open in \mathbb{M} . For $\bar{m} \in \mathcal{A}$ and $\bar{p} := \bar{m} \cdot f(\bar{m}) \in \text{graph}(f)$ let $f_{\bar{p}^{-1}} : \mathcal{A}_{\bar{p}^{-1}} \subset \mathbb{M} \rightarrow \mathbb{H}$ be the translated function, (see Proposition 3.9 and remember that $e \in \mathcal{A}_{\bar{p}^{-1}}$ and $f_{\bar{p}^{-1}}(e) = e$). We say that f is *intrinsic differentiable in $\bar{m} \in \mathcal{A}$* if there is an intrinsic linear map $df_{\bar{m}} : \mathbb{M} \rightarrow \mathbb{H}$ such that, for all $m \in \mathcal{A}_{\bar{p}^{-1}}$,

$$\|df_{\bar{m}}(m)^{-1} \cdot f_{\bar{p}^{-1}}(m)\| = o(\|m\|) , \quad \text{as } \|m\| \rightarrow 0 ,$$

where $o(t)/t \rightarrow 0$ as $t \rightarrow 0^+$. The intrinsic linear map $df_{\bar{m}}$ is called the *intrinsic differential* of f .

Remark 4.12. P-differentiability and intrinsic differentiability are *different*. Indeed, assume $\mathbb{G} = \mathbb{H}^n$, $\mathbb{M} = \mathbb{W} = \{w = (0, p_2, \dots, p_{2n+1})\}$ and $\mathbb{H} = \mathbb{V} = \{v = (p_1, 0, \dots, 0)\}$. Then, by (ii) of Proposition 6.4, we know that $f : \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic differentiable in $w \in \mathbb{W}$ if, for all $w' \in \mathbb{W}$,

$$\|df_w(w^{-1} \cdot w')^{-1} \cdot f(w)^{-1} \cdot f(w')\| = o(\|f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w)\|) ;$$

while f is P-differentiable in $w \in \mathbb{W}$ if, for all $w' \in \mathbb{W}$,

$$\|df_w(w^{-1} \cdot w')^{-1} \cdot f(w)^{-1} \cdot f(w')\| = o(\|w^{-1} \cdot w'\|) .$$

On the contrary, if $\mathbb{G} := \mathbb{M} \times \mathbb{H}$, it is easy to convince oneself that

$$f : \mathbb{M} \rightarrow \mathbb{H} \text{ is } P\text{-differentiable}$$

if and only if

$$f : \mathbb{M} \rightarrow \mathbb{H} \text{ is intrinsic differentiable .}$$

Hence, intrinsic differentiability is a generalization of P-differentiability.

Remark 4.13. Intrinsic differentiability is invariant by left translations of the graph. Indeed, let $q_1 = m_1 \cdot f(m_1)$ and $q_2 = m_2 \cdot f(m_2) \in \text{graph}(f)$; then f is intrinsic differentiable in $m_1 \in \mathbb{M}$ if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e . Consequently, f is intrinsic differentiable in m_1 if and only if $f_{q_2 \cdot q_1^{-1}} \equiv (f_{q_1^{-1}})_{q_2}$ is intrinsic differentiable in m_2 .

Definition 4.14. Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} . We say that $f : \mathcal{A} \subset \mathbb{M} \rightarrow \mathbb{H}$ is *uniformly intrinsic differentiable* in \mathcal{A} if

- (i) f is intrinsic differentiable at each $\bar{g} \in \mathcal{A}$;
- (ii) $df_{\bar{g}} : \mathbb{M} \rightarrow \mathbb{H}$ depends continuously on \bar{g} , that is, for each compact $\mathcal{K} \subset \mathcal{A}$, there is $\eta = \eta_{\mathcal{K}} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, with $\eta(t) \rightarrow 0$ as $t \downarrow 0$ such that

$$\sup_{g \in \mathcal{K}} \|df_{\bar{g}_1}(g)^{-1} \cdot df_{\bar{g}_2}(g)\| \leq \eta(\|g_1^{-1} \cdot g_2\|) ;$$

- (iii) for each compact $\mathcal{K} \subset \mathcal{A}$, there is $\varepsilon = \varepsilon_{\mathcal{A}, \mathcal{K}} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, with $\varepsilon(t) \rightarrow 0$ as $t \downarrow 0$, and such that, for all $g \in \mathcal{K}_{\bar{p}^{-1}}$ and for all $\bar{g} \in \mathcal{K}$,

$$\|df_{\bar{g}}(g)^{-1} \cdot f_{\bar{p}^{-1}}(g)\| \leq \varepsilon(\|g\|) \|g\| ,$$

$$\text{where } \bar{p} = \bar{g} \cdot f(\bar{g}).$$

Uniform intrinsic differentiability implies local intrinsic Lipschitz continuity.

Proposition 4.15. *Let \mathbb{M}, \mathbb{H} be complementary subgroups in \mathbb{G} and $f : \mathcal{A} \subset \mathbb{M} \rightarrow \mathbb{H}$ be uniformly intrinsic differentiable in \mathcal{A} . Then, for all $p \in \mathcal{A}$ there is $r > 0$ such that f is intrinsic Lipschitz in $\mathcal{A} \cap B(p, r)$.*

The proof is elementary.

Remark 4.16. The graph of a uniform intrinsic differentiable function, defined on a relatively open subset of a (h, k) -subgroup \mathbb{M} , has topological dimension h and metric dimension k .

5. INTRINSIC SUBMANIFOLDS AND INTRINSIC RECTIFIABLE SETS

Intrinsic submanifolds. In Euclidean spaces h -dimensional C^1 submanifolds are locally graphs of C^1 functions defined on open sets of h -dimensional linear subspaces. It is very natural, using uniform intrinsic differentiable functions, to follow the same approach to define intrinsic submanifolds in Carnot groups.

There is one difference worth to be noticed. A single number is not enough to specify the dimension of the graph or of the intrinsic submanifold. Not only because the topological dimension of a graph can be different from its metric dimension, but also because it is possible to have intrinsic differentiable graphs with the same topological dimension but with different metric dimension.

Definition 5.1. Let \mathbb{G} be a Carnot group.

- (i) $S \subset \mathbb{G}$ is an *intrinsic Lipschitz (h, k) -submanifold* if for each $p \in S$ there are $r > 0$, complementary subgroups \mathbb{M}, \mathbb{H} of \mathbb{G} , where \mathbb{M} is a (h, k) -subgroup, and an intrinsic Lipschitz function $f : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$ such that

$$S \cap B(p, r) = \text{graph}(f) .$$

- (ii) $S \subset \mathbb{G}$ is a *regular (h, k) -submanifold* or an *intrinsic (h, k) -submanifold* if for each $p \in S$ there are $r > 0$, complementary subgroups \mathbb{M}, \mathbb{H} of \mathbb{G} , where \mathbb{M} is a (h, k) -subgroup, and a uniform intrinsic differentiable $f : \mathcal{E} \subset \mathbb{M} \rightarrow \mathbb{H}$ such that

$$S \cap B(p, r) = \text{graph}(f) .$$

By definition, (h, k) -submanifolds exist in \mathbb{G} if and only if in \mathbb{G} there is at least a couple of complementary subgroups \mathbb{M}, \mathbb{H} and one of the two is a (h, k) -subgroup. As an example we mention that in Heisenberg groups \mathbb{H}^n there are only (k, k) -submanifolds, with $1 \leq k \leq n$ and $(k, k + 1)$ -submanifolds, with $n + 1 \leq k \leq 2n$ (see Proposition 6.1).

Intrinsic rectifiable sets. In Euclidean spaces, k -dimensional rectifiable sets are generalizations of k -dimensional Lipschitz or of C^1 submanifolds. Following the original definition of Federer and Fleming, they are sets contained, but for a negligible subset, in the countable union of k -dimensional Lipschitz or of C^1 submanifolds and the measure involved is the k -dimensional Hausdorff measure. Once more it is possible to follow closely this approach using (h, k) -submanifolds and the naturally associated measure \mathcal{S}_d^k , as suggested by Proposition 4.8.

Definition 5.2. Let \mathbb{G} be a Carnot group. Then

- (i) $E \subset \mathbb{G}$ is *$(h, k)_L$ -rectifiable* or *intrinsic Lipschitz rectifiable* if there are intrinsic Lipschitz (h, k) -submanifolds S_i such that

$$\mathcal{S}_d^k(E \setminus \bigcup_{i \in \mathbb{N}} S_i) = 0 ;$$

- (ii) $E \subset \mathbb{G}$ is *(h, k) -rectifiable* or *intrinsic rectifiable* if there are intrinsic (h, k) -submanifolds S_i such that

$$\mathcal{S}_d^k(E \setminus \bigcup_{i \in \mathbb{N}} S_i) = 0 .$$

We recall that if \mathbb{G} is an Euclidean spaces these two definitions are equivalent. We do not know, in general, the relations among these different definitions, but for the obvious fact that if E is (h, k) -rectifiable then it is $(h, k)_L$ -rectifiable.

6. HEISENBERG GROUPS

In Heisenberg groups \mathbb{H}^n , all the notions, described in the preceding sections for general Carnot groups, can be made more explicit. One of the key point is the structure of complementary subgroups of \mathbb{H}^n (see [3, 17]).

Proposition 6.1. *All homogeneous subgroups of \mathbb{H}^n are either horizontal, that is contained in the horizontal fiber $H\mathbb{H}_e^n$, or vertical, that is containing the subgroup \mathbb{T} . Horizontal subgroups are (k, k) -subgroups, for $1 \leq k \leq n$, and they are algebraically isomorphic and isometric to \mathbb{R}^k . Vertical subgroups are $(k, k + 1)$ -subgroups, with $1 \leq k \leq 2n + 1$, and they are normal subgroups. If \mathbb{V}, \mathbb{W} are complementary subgroups of \mathbb{H}^n then, for $1 \leq k \leq n$,*

$$\begin{aligned} \mathbb{V} & \text{ is a horizontal } (k, k)\text{-subgroup,} \\ \mathbb{W} & \text{ is a vertical } (2n + 1 - k, 2n + 2 - k)\text{-subgroup.} \end{aligned}$$

In particular notice that, for $1 \leq k \leq n$, any horizontal (k, k) -subgroup has a complementary vertical subgroup and that any vertical $(2n + 1 - k, 2n + 2 - k)$ -subgroup has a complementary horizontal subgroup (see [12]). On the contrary, vertical $(k, k + 1)$ -subgroups, of topological dimension $k \leq n$, do not have complementary subgroups.

In Heisenberg groups intrinsic Lipschitz functions, intrinsic linear functions and intrinsic differentiability can be characterized as follows (see [3]).

Proposition 6.2. *If \mathbb{W}, \mathbb{V} are complementary subgroups in \mathbb{H}^n as in Proposition 6.1, then $f : \mathcal{A} \subset \mathbb{V} \rightarrow \mathbb{W}$ is intrinsic Lipschitz, if and only if $\Phi_f : \mathcal{A} \rightarrow \mathbb{H}^n$, defined as $\Phi_f(v) := v \cdot f(v)$, is Lipschitz in \mathcal{A} , that is if and only if there is $\alpha > 0$ such that, for all $v, \bar{v} \in \mathcal{A}$,*

$$\|\Phi_f(\bar{v})^{-1} \cdot \Phi_f(v)\| \leq \alpha \|\bar{v}^{-1} \cdot v\| ,$$

$f : \mathcal{A} \subset \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic Lipschitz, if and only if there is $\alpha > 0$ such that, for all $w, w' \in \mathcal{A}$,

$$\|f(w)^{-1} \cdot f(w')\| \leq \alpha \|f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w)\| .$$

Proposition 6.3. *If \mathbb{W}, \mathbb{V} are complementary subgroups in \mathbb{H}^n as in Proposition 6.1, then $L : \mathbb{V} \rightarrow \mathbb{W}$ is intrinsic linear if and only if $\Phi_L : \mathbb{V} \rightarrow \mathbb{H}^n$, defined as $\Phi_L(v) := v \cdot L(v)$, is H -linear; $L : \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic linear if and only if L is H -linear.*

Proposition 6.4. *If \mathbb{W}, \mathbb{V} are complementary subgroups in \mathbb{H}^n as in Proposition 6.1, then $f : \mathcal{A} \subset \mathbb{V} \rightarrow \mathbb{W}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if $\Phi_f : \mathcal{A} \rightarrow \mathbb{H}^n$, $\Phi_f(g) := g \cdot f(g)$, is P -differentiable in \bar{g} and*

$$(d\Phi_f)_{\bar{g}}(g) = g \cdot df_{\bar{g}}(g) , \quad \text{for all } g \in \mathbb{V} .$$

$f : \mathcal{A} \subset \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{W} \rightarrow \mathbb{V}$, such that

$$\|df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g)\| = o(\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\|)$$

for $g \in \mathcal{A}$ and $\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\| \rightarrow 0$.

H -regular surfaces, of dimension k or of codimension k , is a class of submanifolds of \mathbb{H}^n playing the role of C^1 submanifolds of Euclidean spaces. They were introduced in [10], more explicitly in [12], and are now the object of much research (see [24] and the bibliography therein). We recall here the definition.

Definition 6.5. Let k be an integer, $1 \leq k \leq n$. $S \subset \mathbb{H}^n$ is a k -dimensional H -regular submanifold if for each $p \in S$ there are an open $\mathcal{U} \subset \mathbb{H}^n$ with $p \in \mathcal{U}$, an open $\mathcal{A} \subset \mathbb{R}^k$ and an injective, continuously P-differentiable $f : \mathcal{A} \rightarrow \mathcal{U}$, with injective P-differential, such that

$$S \cap \mathcal{U} = f(\mathcal{A}) .$$

$S \subset \mathbb{H}^n$ is a k -codimensional H -regular submanifold if for each $p \in S$ there are an open $\mathcal{U} \subset \mathbb{H}^n$, with $p \in \mathcal{U}$, and $f : \mathcal{U} \rightarrow \mathbb{R}^k$, f continuously P-differentiable with surjective P-differential, such that

$$S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\} .$$

H -regular submanifolds coincide with intrinsic submanifolds of \mathbb{H}^n , as defined in Definition 5.1. Indeed the following characterization holds (see [3, Theorem 4.2] and also [12]).

Theorem 6.6. *The following statements are equivalent*

- (i) $S \subset \mathbb{H}^n$ is a H -regular submanifold.
- (ii) S is an intrinsic submanifold.

Precisely for $1 \leq k \leq n$, S is a k -dimensional H -regular surface if and only if S is a (k, k) -submanifold; for $n + 1 \leq k \leq 2n$, S is a $(2n + 1 - k)$ -codimensional H -regular surface if and only if S is a $(k, k + 1)$ -submanifold.

According to Proposition 6.1 and Definition 5.2, in a Heisenberg group \mathbb{H}^n we have only $(h, h)_L$ -rectifiable and (h, h) -rectifiable sets, for $1 \leq h \leq n$ and $(h, h + 1)_L$ -rectifiable and $(h, h + 1)$ -rectifiable sets, for $n + 1 \leq h \leq 2n$.

As mentioned before, the equivalence of the definitions of intrinsic Lipschitz rectifiable sets and of intrinsic rectifiable sets is still an open problem. It is only known, in \mathbb{H}^n and in the end point case, that $(2n, 2n + 1)_L$ -rectifiable is the same as $(2n, 2n + 1)$ -rectifiable (see [13]).

Finally we conclude with a recent characterization of intrinsic rectifiable sets of \mathbb{H}^n in terms of almost every where existence of an approximate tangent group, see the definition below, or of almost every where existence of weak tangent measures. These characterizations, being quite similar to the classical ones in Euclidean spaces (see e.g. [8, 18]), support the above definition of intrinsic rectifiable sets.

The *approximate tangent group* of $E \subset \mathbb{H}^n$ at p was introduced in [19, Definition 3.8], following ideas in [18]. We reproduce here the same definition with a little change of notation.

Definition 6.7. Let k be an integer with $1 \leq k \leq 2n + 1$ and let $E \subset \mathbb{H}^n$ be \mathcal{S}_d^k -measurable. We say that the homogeneous (h, k) -subgroup \mathbb{T}_p of \mathbb{H}^n is a (h, k) -approximate tangent group to E at p if the upper density $\Theta^{*k}(E, p)$ is positive, i.e.

$$\Theta^{*k}(E, p) := \limsup_{r \rightarrow 0} \frac{\mathcal{S}_d^k(E \cap B(p, r))}{r^k} > 0$$

and if, for all $0 < s < 1$,

$$\lim_{r \rightarrow 0} \frac{\mathcal{S}_d^k(E \cap B(p, r) \setminus X(p, \mathbb{T}_p, s))}{r^k} = 0 .$$

If there is only one (h, k) -approximate tangent group \mathbb{T}_p to E at p , we write $\mathbb{T}_p = \text{apTan}_{\mathbb{H}}^{(h, k)}(E, p)$.

Notice that, not only in Euclidean spaces but also in \mathbb{H}^n , approximate tangent groups are unique almost everywhere (see [19, Proposition 3.11]).

If μ is an outer measure in \mathbb{H}^n , the image $f_{\#}\mu$ under a map $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is the measure on \mathbb{H}^n defined by

$$f_{\#}\mu(\mathcal{A}) = \mu(f^{-1}(\mathcal{A})) , \quad \text{for all } \mathcal{A} \subset \mathbb{H}^n .$$

For $p \in \mathbb{H}^n$ and $r > 0$, $T_{p,r} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is

$$T_{p,r}(q) := \delta_{1/r}(p \cdot q) \quad \text{for all } q \in \mathbb{H}^n .$$

The following theorems are proved in [19]. The first one characterizes low dimensional $(h, h)_L$ -rectifiable sets, $1 \leq h \leq n$, the second one low codimensional $(h, h + 1)$ -rectifiable sets, $n + 1 \leq h \leq 2n$.

Theorem 6.8. *Let $1 \leq h \leq n$ be an integer and $E \subset \mathbb{H}^n$ be a Borel set such that $\mathcal{S}_d^h \llcorner E$ is locally finite. Then are equivalent*

- (i) E is $(h, h)_L$ -rectifiable.
- (ii) For \mathcal{S}_d^h -a.e. $p \in E$ there is a (h, h) -subgroup \mathbb{T}_p of \mathbb{H}^n such that

$$\frac{1}{r^h} T_{p,r\#}(\mathcal{S}_d^h \llcorner E) \rightharpoonup \mathcal{S}_d^h \llcorner \mathbb{T}_p , \quad \text{as } r \rightarrow 0 ,$$

weakly in the sense of measures.

- (iii) For \mathcal{S}_d^h -a.e. $p \in E$ there is a (h, h) -subgroup \mathbb{T}_p of \mathbb{H}^n such that

$$\mathbb{T}_p = \text{apTan}_{\mathbb{H}}^{(h,h)}(E, p) .$$

Theorem 6.9. *Let $n + 1 \leq h \leq 2n$ be an integer and $E \subset \mathbb{H}^n$ be a Borel set such that $\mathcal{S}_d^{h+1} \llcorner E$ is locally finite. Then are equivalent*

- (i) E is $(h, h + 1)$ -rectifiable.
- (ii) For \mathcal{S}_d^{h+1} -a.e. $p \in E$ there is a $(h, h + 1)$ -subgroup \mathbb{T}_p of \mathbb{H}^n with

$$\frac{1}{r^{d+1}} T_{p,r\#}(\mathcal{S}_d^{h+1} \llcorner E) \rightharpoonup \mathcal{S}_d^{h+1} \llcorner \mathbb{T}_p , \quad \text{as } r \rightarrow 0 ,$$

weakly in the sense of measures.

- (iii) For \mathcal{S}_d^{h+1} -a.e. $p \in E$

$$\Theta_*^{h+1}(E, p) := \liminf_{r \rightarrow 0} \frac{\mathcal{S}_d^{h+1}(E \cap B(p, r))}{r^{h+1}} > 0$$

and there is a $(h, h + 1)$ -subgroup \mathbb{T}_p of \mathbb{H}^n such that

$$\mathbb{T}_p = \text{apTan}_{\mathbb{H}}^{(h,h+1)}(E, p) .$$

REFERENCES

- [1] L. Ambrosio & B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann., 318(2000), 527–555.
- [2] L. Ambrosio & B. Kirchheim, *Currents in metric spaces*, Acta Math., 185(2000), 1–80.
- [3] G. Arena & R. Serapioni, *H-regular submanifolds in Heisenberg groups are intrinsic graphs*, to appear on Calculus of Variation and Partial Differential Equations, (2008).
- [4] A. Bonfiglioli, E. Lanconelli & F. Uguzzoni, *Stratified Lie Groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, New York, 2007.
- [5] L. Capogna, D. Danielli, S.D. Pauls & J.T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Birkhäuser Verlag, Basel, 2007.
- [6] G. Citti & M. Manfredini, *Implicit function theorem in Carnot-Carathéodory spaces*, Commun. Contemp. Math., 8(5)(2006), 657–680.

- [7] G. Citti & M. Manfredini, *Blow-up in non homogeneous Lie groups and rectifiability*, Houston J. Math., 31(2)(2005), 333–353.
- [8] H. Federer, *Geometric measure theory*, Springer, 1969.
- [9] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., 13(1975), 161–207.
- [10] B. Franchi, R. Serapioni & F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann., 321(2001), 479–531.
- [11] B. Franchi, R. Serapioni & F. Serra Cassano, *On the structure of finite perimeter sets in step 2 Carnot groups*, J. Geom. Anal., 13(3)(2003), 421–466.
- [12] B. Franchi, R. Serapioni & F. Serra Cassano, *Regular submanifolds, graphs and area formula in Heisenberg Groups*, Adv. in Math., 211(1)(2007), 152–203.
- [13] B. Franchi, R. Serapioni & F. Serra Cassano, *Rademacher theorem for intrinsic Lipschitz continuous functions*, Preprint 2008.
- [14] M. Gromov, *Metric structures for Riemannian and non Riemannian spaces*, Progress in Mathematics, 152, Birkhauser Verlag, Boston, 1999.
- [15] I.N. Herstein, *Topics in algebra*, Blaisdell Publishing Company, London, 1964.
- [16] V. Magnani, *Elements of geometric measure theory on sub-Riemannian groups*, Tesi di Perfezionamento, Scuola Normale Superiore, Pisa, 2003.
- [17] V. Magnani, *Towards differential calculus in stratified groups* Preprint, 2007.
- [18] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press, 1995.
- [19] P. Mattila, R. Serapioni & F. Serra Cassano, *Characterizations of intrinsic rectifiability in Heisenberg groups*, Preprint, 2008.
- [20] J. Mitchell, *On Carnot-Carathéodory metrics*, J. Differential Geom., 21(1985), 35–45.
- [21] P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. Math., 129(1989), 1–60.
- [22] S.D. Pauls, *A notion of rectifiability modelled on Carnot groups*, Indiana Univ. Math. J., 53(1)(2004), 49–81.
- [23] E.M. Stein, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton, 1993.
- [24] D. Vittone, *Submanifolds in Carnot groups*, Edizioni della Normale, Scuola Normale Superiore, Pisa, 2008.