Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 7(2008), pp. 269–281.

A class of nonvariational Hörmander operators

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Dedicated to Ermanno Lanconelli, on the occasion of his 65th birthday

Abstract[1]. We are concerned with linear second order partial differential operators of the type $H = \partial_t - L = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x)X_iX_j - \sum_{k=1}^{q} a_k(t,x)X_k - a_0(t,x)$, where $X_1, X_2, \cdots, X_q$ is a system of real Hörmander's vector fields in some bounded domain $\Omega \subseteq \mathbb{R}^n$, $A = \{a_{ij}(t,x)\}_{i,j=1}^{q}$ is a real symmetric uniformly positive definite matrix. The coefficients a_{ij}, a_k, a_0 are Hölder continuous with respect to the parabolic CC-metric. We are interested in Harnack inequalities and in the constuction of a fundamental solution for H, satisfying natural properties and sharp Gaussian bounds w.r.t. the CC-metric.

1. INTRODUCTION

In this note we report some results obtained in collaboration with M. Bramanti, L. Brandolini and E. Lanconelli [10]. Let us consider the heat-type operator in $\mathbb{R}^{n+1}$

$$H = \partial_t - L = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x)X_iX_j - \sum_{k=1}^{q} a_k(t,x)X_k - a_0(t,x) \tag{1.1}$$

structured on a system of real smooth vector fields $X_1, X_2, \cdots, X_q$ defined in some bounded domain $\Omega \subseteq \mathbb{R}^n$ and satisfying Hörmander's condition in Ω: rank $\mathrm{Lie}\{X_i, i = 1, 2, \cdots, q\} = n$ at any point of Ω. Here $A = \{a_{ij}(t,x)\}_{i,j=1}^{q}$ is a real symmetric uniformly positive definite matrix satisfying, for some positive constant λ,

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(t,x)\xi_i\xi_j \le \lambda|\xi|^2 ,$$

for every $\xi \in \mathbb{R}^q$, $x \in \Omega$, $t \in (T_1, T_2)$, for some $T_1 < T_2$. If $d(x,y)$ denotes the Carnot-Carathéodory metric generated in Ω by the X_i's and $d_P((t,x),(s,y)) = \sqrt{d(x,y)^2 + |t-s|}$ is its "parabolic" counterpart in $\mathbb{R}\times\Omega$, we assume that a_{ij}, a_k, a_0 are Hölder continuos on $\mathcal{C} = (T_1,T_2) \times \Omega$ with respect to the distance d_P.

We prove the existence and basic properties of a fundamental solution h for the operator H, including a representation formula for solutions to the Cauchy problem,

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Keywords. Hörmander's vector fields, heat kernels, Gaussian bounds, Harnack inequalities.
AMS Subject Classification. 35H20, 35A08, 35K65, 35H10, 35A17.

a "reproduction property" for h, and regularity results: namely, we show that h is locally Hölder continuous, far off the pole, together with its derivatives $X_j h, X_i X_j h$, $\partial_t h$. The operator H is defined only on the cylinder $\mathcal{C}$ but it is convenient to work with an operator defined on the whole space. For this reason we extend the operator H to the whole $\mathbb{R}^{n+1}$, in such a way that, outside a compact set in the space variables, it coincides with the classical heat operator, and henceforth we study the fundamental solution for this extended operator.

Strictly related to the proof of the existence of h, and of independent interest, are sharp Gaussian bounds for h

$$\frac{\exp(-cd(x,y)^2/(t-s))}{c|B(x,\sqrt{t-s})|} \leq h(t,x;s,y) \leq c\,\frac{\exp(-d(x,y)^2/c(t-s))}{|B(x,\sqrt{t-s})|} \tag{1.2}$$

and for its derivatives

$$|X_i h(t,x;s,y)| \leq \frac{c}{\sqrt{t-s}}\,\frac{\exp(-d(x,y)^2/c(t-s))}{|B(x,\sqrt{t-s})|}\,, \tag{1.3}$$

$$|X_i X_j h(t,x;s,y)| + |\partial_t h(t,x;s,y)| \leq \frac{c}{t-s}\,\frac{\exp(-d(x,y)^2/c(t-s))}{|B(x,\sqrt{t-s})|}\,.$$

Here $x,y \in \mathbb{R}^n$, $0 < t-s < T$ and $|B(x,r)|$ denotes Lebesgue measure of the d-ball $B(x,r)$. The constant c in these estimates depends on the coefficients a_{ij}, a_k, a_0 only through their Hölder moduli of continuity and the ellipticity constant λ.

A precise list of the results we prove about h is contained in Theorem 3.1 below. A remarkable consequence of these bounds is a scaling invariant Harnack inequality for H, and for its stationary counterpart L in (1.1). Precise results are stated in Theorems 4.1 and 4.2. As we mentioned before, all the results we have described so far are proved for an operator defined on the whole space, which extends H, initially defined only locally. We can also show how to come back to the original operator, deducing local results from the above global theorems.

Our operators arise in many theoretical and applied settings sharing a sub-Riemannian underlying geometry: e.g. in diffusion theory, mathematical models for finance and for human vision, control theory, geometric theory of several complex variables. In particular the *nonvariational* operators in (1.1) appear as linearizations of the following fully nonlinear second order PDE, known as the Levi curvature equation:

$$\sum_{i,j=1}^{2n} a_{i,j}(Du, D^2u) X_i X_j u = K(x,u,Du) \quad \text{in } \mathbb{R}^{2n+1}\,. \tag{1.4}$$

(In (1.4), the vector fields X_j also depend on Du). This (non-elliptic) equation is the complex-analogue of the classical Monge-Ampère one, as its solutions are functions u whose graphs are CR manifolds in $\mathbb{C}^{n+1}$ with prescribed Levi curvature K (see e.g. [27] and references therein). Existence of classical solutions to (1.4) is still a widely open problem. This paper is part of a project aimed to provide the linear framework for the Levi-Monge-Ampère equation (1.4). Several results concerning the divergence form counterpart of (1.1) are present in the literature. On the contrary (at least to the authors knowledge) very few papers are devoted to the non-divergence form operators (1.1).

Gaussian estimates for the fundamental solution of second order partial differential operators of parabolic type, have a long history, starting with Aronson's work

[1]. The relevance of two-sided Gaussian estimates to get scaling invariant Harnack inequalities for positive solutions was firstly pointed out by Nash in the Appendix of his celebrated paper [28]. However, a complete implementation of the method outlined by Nash was given much later by Fabes and Stroock in [15], also inspired by some ideas of Krylov and Safonov (see [19], [20], [31]). Since then, the full strength of Gaussian estimates has been enlightened by several authors, showing their deep relationship not only with the scaling invariant Harnack inequality, but also with the ultracontractivity property of heat diffusion semigroups, with inequalities of Nash, Sobolev or Poincaré type, and with the doubling property of the measure of "intrinsic" balls. We directly refer to the recent monograph by Saloff-Coste [32] for a beautiful exposition of this circle of ideas, and for an exhaustive list of references on these subjects. Here we explicitly recall just the results in literature strictly close to the core of our work.

For heat operators of the kind $H = \partial_t - \sum_{i=1}^q X_i^2$ with X_i left invariant homogeneous vector fields on a Carnot group in $\mathbb{R}^n$, Gaussian bounds have been proved by Varopoulos ([35], [36], see also [37]). Gaussian estimates and a scaling invariant Harnack inequality for the operator

$$H = \partial_t - \sum_{i,j=1}^{q} X_i(a_{ij}X_j)$$

have been proved by Saloff-Coste and Stroock in [33], where $\{a_{ij}\}$ is a uniformly positive matrix with measurable entries, and the vector fields X_i are left invariant with respect to a connected unimodular Lie group with polynomial growth. In absence of a group structure, Gaussian bounds for operators $H = \partial_t - \sum_{i=1}^q X_i^2$ have been proved, on a compact manifold and for finite time, by Jerison-Sanchez-Calle [18], with an analytic approach (see also the previous partial result in [34]), and, on the whole $\mathbb{R}^{n+1}$, by Kusuoka-Stroock, [21], [22], using the Malliavin stochastic calculus.

Unlike the study of "sum of squares" Hörmander's operators, the investigation of non-divergence operators of Hörmander type has a relatively recent history. Stationary operators of kind

$$L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j \tag{1.5}$$

with $X_1, \cdots, X_q$ system of Hörmander's vector fields have been studied by Xu [38], Bramanti, Brandolini [7], [8], Capogna, Han [11]. A first attempt to study Cordes and/or Alexandrov-Bakelman-Pucci estimates for operators (1.5) with measurable coefficients a_{ij} and particular classes of vector fields X_i are contained in [12], [13], [14]. Evolution operators of kind (1.1) have been considered by Bonfiglioli, Lanconelli, Uguzzoni [2], [3], [5], Bramanti, Brandolini [9]. In these papers, the matrix $\{a_{ij}\}$ is assumed symmetric and uniformly elliptic, and the entries a_{ij} typically belong to some function space defined in terms of the vector fields X_i and the metric they induce. In particular, these operators do not have smooth coefficients, so they are no longer hypoelliptic. Therefore the mere existence of a fundamental solution is troublesome. For the operators (1.1) (without lower order terms) with X_i left invariant homogeneous Hörmander's vector fields on a stratified Lie group, it has been proved by Bonfiglioli, Lanconelli, Uguzzoni in [2], [3], [4] that the fundamental solution h exists and satisfies Gaussian bounds. As a consequence of these estimates, in [5] it is proved a scaling invariant Harnack inequality for the operator H. A particular class of related operators, namely ultraparabolic operators of Kolmogorov-Fokker-Planck type, has been studied by Pascucci and Polidoro in relation both with Harnack inequality and Gaussian bounds for the fundamental solution (see [29]).

Previous results about Harnack inequality for general Hörmander's operators date back to Bony's seminal paper [6], where a first qualitative version of this result is proved. A first scaling invariant Harnack inequality for heat-type Hörmander's operators was proved later by Kusuoka-Stroock [21].

Our study proceeds in three steps. In the first step we consider operators of kind (1.1) with *constant* coefficients a_{ij}, and no lower order terms. For these operators, existence and basic properties of the fundamental solution h_A are guaranteed by known results. Here the point is to prove sharp Gaussian bounds on h_A, which have to be *uniform* in the ellipticity class of the matrix $A = \{a_{ij}\}$.

In the second step we study operators with variable Hölder continuous coefficients a_{ij}, a_k, a_0, and we establish existence and Gaussian bounds for the fundamental solution applying the uniform estimates of the frozen operators proved in the first step. This is accomplished by a suitable adaptation to our subelliptic context of the classical Levi's parametrix method.

Finally, in the third step, the proof of a Harnack inequality for H can follow the lines drawn in [5], and inspired by Fabes-Stroock's paper [15].

Acknowledgment. I would like to take this opportunity in the occasion of 65th birthday of Ermanno Lanconelli, to thank him for his constant advice and encouragement. I am deeply grateful to him for his teachings an his help.

2. CONSTANT COEFFICIENTS OPERATORS

Let us consider the heat-type operator in $\mathbb{R}^{n+1}$

$$H_A = \partial_t - L_A = \partial_t - \sum_{i,j=1}^{q} a_{ij} X_i X_j \tag{2.1}$$

where $X_1, X_2, \cdots, X_q$ is a system of real smooth vector fields which are defined in some bounded domain $\Omega \subseteq \mathbb{R}^n$ and satisfy Hörmander's condition of some step s in Ω. Explicitly, this means that $X_i = \sum_{k=1}^{n} b_{ik}(x)\partial_{x_k}$ with $b_{ik} \in C^\infty(\Omega)$, and the vector space spanned at every point of Ω by the fields X_i and their commutators up to some step s, is the whole $\mathbb{R}^n$. We assume that $A = \{a_{ij}\}_{i,j=1}^{q}$ is a real symmetric positive definite matrix with constant entries, and $\lambda > 0$ a constant such that:

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}\xi_i\xi_j \le \lambda|\xi|^2$$

for every $\xi \in \mathbb{R}^q$. When this condition is fulfilled, we will say briefly that $A \in \mathcal{E}_\lambda$. As we have already noted, for these operators (or, more precisely, for a suitable extension of these operators to the whole $\mathbb{R}^{n+1}$), existence and basic properties of the fundamental solution h_A are guaranteed by known results. Our first goal is to prove the following uniform Gaussian bounds on h_A.

Theorem 2.1. *For any $T > 0$ there exists $c > 0$ such that, for any $t \in (0, T)$, $x, y \in \mathbb{R}^n$ the following bounds hold:*

1. Upper and lower bounds on h_A*:*

$$\frac{1}{c|B(x,\sqrt{t})|}\, e^{-cd(x,y)^2/t} \le h_A(t,x,y) \le \frac{c}{|B(x,\sqrt{t})|}\, e^{-d(x,y)^2/ct} \tag{2.2}$$

2. Upper bounds on the derivatives of h_A *of arbitrary order:*

$$|X_x^I X_y^J \partial_t^i h_A(t,x,y)| \le \frac{c}{t^{i+(|I|+|J|)/2}\,|B(x,\sqrt{t})|}\, e^{-d(x,y)^2/ct} \tag{2.3}$$

3. Estimate on the difference of the fundamental solutions of two operators (*and their derivatives*)*:*

$$\begin{aligned} &|X_x^I X_y^J \partial_t^i h_A(t,x,y) - X_x^I X_y^J \partial_t^i h_B(t,x,y)| \le \\ &\le \frac{c\|A-B\|}{t^{i+(|I|+|J|)/2}\,|B(x,\sqrt{t})|}\, e^{-d(x,y)^2/ct} \end{aligned} \tag{2.4}$$

(*here* I, J *are arbitrary multiindices,* $A, B \in \mathcal{E}_\lambda$)*. The constants depend on the matrix* A *only through the number* λ*; in* (2.3), (2.4)*, the constant also depends on the multiindices. The same estimates hold for* $h_A(t,y,x)$*.*

To prove Theorem 2.1, the techniques used in [2] for homogeneous left invariant vector fields are not suitable. Instead, we follow an approach that just uses the results in [2] and which is basically inspired to the work of Jerison and Sanchez-Calle [18], integrated with several other devices to overcome the new difficulties. The main of them are the following: first, we have to take into account the dependence on the matrix A, getting estimates depending on A only through the number λ; second, our estimates have to be global in space, while in [18] the Authors work on a compact manifold; third, our estimates on the difference of the fundamental solutions of two operators have no analogue in [18].

Our strategy is as follows. We first show how to extend to the whole space the vector fields X_i, so that the distance induced in $\mathbb{R}^n$ by these extended vector fields could enjoy suitable global properties, which will be used throughout. In particular, the Lebesgue measure will satisfy globally the doubling condition w.r.t. metric balls. Consequently, we will extend to the whole space $\mathbb{R}^{n+1}$ the operator H_A, in order to assure the existence of a global fundamental solution h_A, satisfying natural properties. Then we start to prove the uniform Gaussian bounds (2.2), (2.3), (2.4). The hardest step is the proof of the upper bound in (2.2). The strategy is the following. First, one proves the upper bound for $t \in (0,1)$ and $\varepsilon < d(x,y) < R$. In this range, the bound is equivalent to $h_A(t,x,y) \le ce^{-1/ct}$ and is proved by means of estimates of Gevray type. This means that the exponential decay of h_A for vanishing t is deduced by a control on the supremum of the time derivative of any order of a solution to $H_A u = 0$. This technique makes the constant c depend on

$$\sup_{y\in\mathbb{R}^n} \int_0^T d\tau \int_{\varepsilon<d(x,y)<R} h_A(\tau,x,y)\,dx\,.$$

So the next problem is to prove a uniform upper bound on this quantity (i.e., depending on A only through λ). This is accomplished exploiting suitable estimates on fractional and singular integrals on spaces of homogeneous type, and uniform subelliptic estimates. Next, one has to prove the upper bound in (2.2) for $t \in (0,1)$ and $d(x,y) < \varepsilon$. This is performed applying Rothschild-Stein's technique of "lifting and approximation". This allows, by a rather involved procedure, to deduce the desired uniform bound from the analogous result proved, in the context of homogeneous groups, by Bonfiglioli-Lanconelli-Uguzzoni [2], and therefore completes the proof of the upper bound in (2.2) for $t \in (0,1)$ and $d(x,y) < R$. To prove the same upper bound for any $x, y \in \mathbb{R}^n$ and $t \in (0,T)$, we use a comparison argument, exploiting the *ad hoc* extension of the operator H_A performed before.

We then prove the lower bound in (2.2), exploiting again the Rothschild-Stein's lifting technique. Again, uniformity of the lower bound relies on the analogous uniform lower bound which holds in the case of homogeneous groups.

The Gaussian bound (2.3) on the derivatives of h_A is deduced by the upper bound on h_A, applying a powerful result proved by Fefferman and Sanchez-Calle [16], which assures the existence of a local change of coordinates which is a good substitute of dilations (which in our context do not generally exist).

Finally, we prove our estimate (2.4). The basic estimate, on the difference of two fundamental solutions $h_A - h_B$, relies on a suitable use of basic properties of the fundamental solution and on the uniform bound (2.3) on the derivatives of h_A. The estimate on the difference of derivatives of two fundamental solution is then derived by the basic estimate, using again the Fefferman and Sanchez-Calle result.

3. HÖLDER COEFFICIENTS OPERATORS

In this section, we deal with variable-coefficient complete operators, of the kind:

$$H = \partial_t - \sum_{i,j=1}^{m} a_{i,j}(t,x) X_i X_j - \sum_{k=1}^{m} a_k(t,x) X_k - a_0(t,x)\ . \tag{3.1}$$

To exploit the results in the previous section, we will make the same assumptions on the vector fields and the structure of the matrix of the coefficients in the principal part. Moreover, the coefficients a_{ij}, a_k, a_0 will be assumed globally defined and Hölder continuous with respect to the parabolic CC-distance d_P; the matrix $\{a_{ij}\}_{i,j=1}^m$ will be assumed symmetric and uniformly positive definite. Under these assumptions (which we will state more precisely in a while) we will prove the existence of a (global) fundamental solution for H, satisfying natural basic properties and sharp Gaussian bounds. A precise list of our results is contained in Theorem 3.1 here below. Before stating it, we need to introduce some precise definitions, notation and assumptions.

We will assume that $X = (X_1, X_2, \cdots, X_m)$ $(m = n + q)$ is a fixed system of Hörmander's vector fields defined in the whole $\mathbb{R}^n$, and such that

$$X = (0, 0, \cdots, 0, \partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}) \quad \text{in } \mathbb{R}^n \setminus \Omega_0 \tag{3.2}$$

where Ω_0 is a fixed bounded domain.

The intrinsic-derivative along the vector field X_j of a function $v(x)$ at a point $x_0 \in \mathbb{R}^n$, is defined to be $X_j v(x_0) = (d/d\sigma)|_{\sigma=0}\, v(\gamma(\sigma))$ (if such derivative exists), where γ is the solution to $\dot\gamma(\sigma) = X_j(\gamma(\sigma))$, $\gamma(0) = x_0$. We can now introduce some function spaces which will be useful in the following. Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. We denote by $\mathfrak{C}^2(U)$ the class of functions $u(t,x)$ defined on U which are continuous in U w.r.t. the pair (t,x) and such that $u(t,\cdot)$ has continuous intrinsic-derivatives up to second order along the vector fields $X_1, \cdots, X_m$ (w.r.t. x, for every fixed t) and $u(\cdot, x)$ has continuous derivative (w.r.t. t, for every fixed x), in their respective domains of definition.

We will denote by d the Carnot-Carathéodory distance induced by the system $\{X_i\}_{i=1}^m$ in the whole $\mathbb{R}^n$ and by $B(x,r)$ the balls in the metric d. Moreover d_P will be the corresponding "parabolic-CC-distance". We can introduce "parabolic CC-Hölder spaces" $C^{k,\alpha}(U)$ related to d_P, by setting

$$|u|_{C^\alpha(U)} = \sup \frac{|u(t,x)-u(s,y)|}{d_P((t,x),(s,y))^\alpha}\ ,$$

$\|u\|_{C^\alpha(U)} = |u|_{C^\alpha(U)} + \|u\|_{L^\infty(U)}$ and $\|u\|_{C^{k,\alpha}(U)} = \sum_{|I|+2h\le k} \left\|\partial_t^h X^I u\right\|_{C^\alpha(U)}$, where, for any multiindex $I=(i_1,i_2,\cdots,i_s)$, with $1\le i_j\le q$, we say that $|I|=s$ and $X^I u = X_{i_1}X_{i_2}\cdots X_{i_s}u$. We explicitly remark that in the definition of $C^{k,\alpha}$ we are assuming that the derivatives of u involved exist as intrinsic derivatives.

Throughout Sections 3 and 4 we assume that the matrix $\{a_{ij}\}_{i,j=1}^m$ has the following structure

$$A = \{a_{ij}\}_{i,j=1}^m = \begin{bmatrix} \{a_{ij}\}_{i,j=1}^q & 0 \\ 0 & I_n \end{bmatrix} \tag{3.3}$$

where the functions $a_{ij}=a_{ji}$, a_k, a_0 are defined on $\mathbb{R}^{n+1}$ and satisfy, for some $\alpha\in(0,1]$ and for some positive constants λ, K,

$$\begin{aligned} \lambda^{-1}|w|^2 \ &\le \sum_{i,j=1}^q a_{ij}(t,x)w_iw_j \le \lambda|w|^2 \quad \forall w\in\mathbb{R}^q,\ (t,x)\in\mathbb{R}^{n+1} \\ &\|a_{ij}\|_{C^\alpha(\mathbb{R}^{n+1})} + \|a_k\|_{C^\alpha(\mathbb{R}^{n+1})} + \|a_0\|_{C^\alpha(\mathbb{R}^{n+1})} \le K\ . \end{aligned} \tag{3.4}$$

We shall denote by $\mathbf{c}$ any positive constant only depending on $X_1,\cdots,X_m$ and the parameters λ, K, α appearing in (3.4). Moreover, we will write $\mathbf{c}(f_1,\cdots,f_p)$ if $\mathbf{c}$ also depends on $f_1,\cdots,f_p$. The points of $\mathbb{R}^{n+1}=\mathbb{R}\times\mathbb{R}^n$ will be denoted by $z=(t,x)$; $\zeta=(\tau,\xi)$; $\eta=(s,y)$. For the sake of brevity, we shall use the notation

$$\mathbf{E}(x,\xi,t) = |B(x,\sqrt{t})|^{-1}\exp(-d(x,\xi)^2/t)\ ,\quad x,\xi\in\mathbb{R}^n\ ,\quad t>0\ .$$

We can now state the main result of this section.

Theorem 3.1. *Let H be as in* (3.1). *Under the above assumptions, there exists a function $h:\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}$ such that:*

- *i) h is continuous away from the diagonal of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$;*
- *ii) $h(z,\zeta)$ is nonnegative, and vanishes for $t\le\tau$;*
- *iii) for every fixed $\zeta\in\mathbb{R}^{n+1}$, we have $h(\cdot\,;\zeta)\in C^{2,\alpha}_{loc}(\mathbb{R}^{n+1}\setminus\{\zeta\})$ and $H(h(\cdot\,;\zeta)) = 0$ in $\mathbb{R}^{n+1}\setminus\{\zeta\}$;*
- *iv) the following estimates hold for every $T>0$, $z=(t,x)$, $\zeta=(\tau,\xi)\in\mathbb{R}^{n+1}$, $0<t-\tau\le T$:*

$$\begin{aligned} \mathbf{c}(T)^{-1}\mathbf{E}(x,\xi,\mathbf{c}^{-1}(t-\tau)) \ &\le h(z;\zeta) \le \mathbf{c}(T)\,\mathbf{E}(x,\xi,\mathbf{c}(t-\tau))\ , \\ |X_j(h(\cdot\,;\zeta))(z)| \ &\le \mathbf{c}(T)\,(t-\tau)^{-1/2}\,\mathbf{E}(x,\xi,\mathbf{c}(t-\tau))\ ; \\ |X_iX_j(h(\cdot\,;\zeta))(z)| + |\partial_t(h(\cdot\,;\zeta))(z)| \ &\le \mathbf{c}(T)\,(t-\tau)^{-1}\,\mathbf{E}(x,\xi,\mathbf{c}(t-\tau))\ ; \end{aligned}$$

- *v) for any $f\in C^\alpha(\mathbb{R}^{n+1})$, $g\in C(\mathbb{R}^n)$, both satisfying suitable growth condition at infinity, $T\in\mathbb{R}$, the function*

$$u(t,x) = \int_{\mathbb{R}^n} h(t,x;T,\xi)\,g(\xi)\,d\xi + \int_{[T,t]\times\mathbb{R}^n} h(t,x;\tau,\xi)\,f(\tau,\xi)d\tau\,d\xi$$

is a $C^{2,\alpha}_{loc}$ *solution to the following Cauchy problem*

$$\begin{cases} Hu = f & in\ (T,\infty)\times\mathbb{R}^n\ , \\ u(T,\cdot\,) = g & in\ \mathbb{R}^n \end{cases}$$

vi) the following reproduction formula holds

$$h(t,x;\tau,\xi) = \int_{\mathbb{R}^n} h(t,x;s,y)\,h(s,y;\tau,\xi)\,dy\ ,$$

for $t > s > \tau$ *and* $x,\ \xi\in\mathbb{R}^n$.

Our assumptions (3.2), (3.3), as well as the fact that both the vectors fields and the coefficients are defined on the whole space, are just made to have a convenient setting to prove the existence of a global fundamental solution, but are not really restrictive. Namely, assume we have an operator

$$H_{loc} = \partial_t - \sum_{i,j=1}^{q} a_{i,j}(t,x)X_iX_j - \sum_{k=1}^{q} a_k(t,x)X_k - a_0(t,x)$$

where $X_1, X_2, \cdots, X_q$ is a system of Hörmander's vector fields defined in a bounded domain Ω of $\mathbb{R}^n$, the coefficients a_{ij}, a_k, a_0 are defined and Hölder continuous in some domain $U\subset\mathbb{R}\times\Omega$, and the matrix $\{a_{ij}\}_{i,j=1}^q$ satisfies

$$\lambda^{-1}|w|^2 \le \sum_{i,j=1}^{q} a_{ij}(t,x)w_iw_j \le \lambda|w|^2\ \forall w\in\mathbb{R}^q,\ (t,x)\in U\ .$$

Then, we can define a new operator H, satisfying all the assumptions we have made above and such that, in some compact subdomain of U, H coincides with H_{loc}, and the CC-distances of the two operators are equivalent. This fact will allow to deduce local results for the operator H_{loc} from the results we have proved for our globally defined operator H.

The proofs of all the results collected in Theorem 3.1, is strictly based on the achievements of Section 2. Namely, let

$$H_{\zeta_0} \equiv \partial_t - L_{\zeta_0} \equiv \partial_t - \sum_{i,j=1}^{m} a_{i,j}(\zeta_0)X_iX_j \tag{3.5}$$

be the operator obtained from the "principal part" of H by freezing the coefficients a_{ij} (but not the vector fields X_j) at any point $\zeta_0\in\mathbb{R}^{n+1}$. By the assumptions we have made above on H, the operator H_{ζ_0} fits the assumptions of the theory developed in Section 2. Let us denote by h_{ζ_0} its fundamental solution (with the notation of Section 2, $h_{\zeta_0}(z,\zeta) = h_A(z,\zeta)$ where $A = (a_{ij}(\zeta_0))_{i,j=1}^m$).

For the reader's convenience, we now recall the properties of h_{ζ_0} that play a crucial role. The function h_{ζ_0} is smooth away from the diagonal of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and

$$H_{\zeta_0}(h_{\zeta_0}(\,\cdot\,,\zeta)) = 0 \quad \text{in } \mathbb{R}^{n+1}\setminus\{\zeta\}\ . \tag{3.6}$$

Moreover, for every $\zeta_0\in\mathbb{R}^{n+1}$ and $T>0$ we have

$$\int_{\mathbb{R}^n} h_{\zeta_0}(t,x,\tau,\xi)\,d\xi \quad = 1\ ,\ \text{if } t>\tau,\ x\in\mathbb{R}^n\ , \tag{3.7}$$

$$\int_{\mathbb{R}^n} h_{\zeta_0}(t,x,\tau,\xi)\,dx \quad \le \mathbf{c}(T),\ \text{if } 0<t-\tau<T,\ \xi\in\mathbb{R}^n\ , \tag{3.8}$$

$$h_{\zeta_0}(t,x,\tau,\xi) \quad = \int_{\mathbb{R}^n} h_{\zeta_0}(t,x,s,y) h_{\zeta_0}(s,y,\tau,\xi)\, dy \text{ if } t > s > \tau,\ x,\xi \in \mathbb{R}^n \ . \tag{3.9}$$

The following uniform Gaussian bounds hold: for every $T > 0$ and for every nonnegative integers p, q, we have

$$\mathbf{c}(T)^{-1}\mathbf{E}(x,\xi,\mathbf{c}^{-1}(t-\tau)) \le h_{\zeta_0}(t,x,\tau,\xi) \le \mathbf{c}(T)\mathbf{E}(x,\xi,\mathbf{c}(t-\tau)) \ ,$$

$$\begin{aligned} &|X_{i_1}\cdots X_{i_p}(\partial_t)^q\, h_{\zeta_0}(\,\cdot\,,\tau,\xi)(t,x)| \le \\ &\le \mathbf{c}(T,p,q)(t-\tau)^{-(p+2q)/2}\mathbf{E}(x,\xi,\mathbf{c}(t-\tau)) \ , \end{aligned} \tag{3.10}$$

$$\begin{aligned} &|X_{i_1}\cdots X_{i_p}(\partial_t)^q(h_{\zeta_0}(\cdot\,,\tau,\xi) - h_{\zeta_1}(\cdot\,,\tau,\xi))(t,x)| \le \\ &\le \mathbf{c}(T,p,q)\, d_P(\zeta_0,\zeta_1)^\alpha (t-\tau)^{-(p+2q)/2}\mathbf{E}(x,\xi,\mathbf{c}(t-\tau)) \ , \end{aligned} \tag{3.11}$$

if $0 < t-\tau < T$, $x,\xi \in \mathbb{R}^n$ and $\zeta_0 = (\tau_0,\xi_0)$, $\zeta_1 = (\tau_1,\xi_1) \in \mathbb{R}^{n+1}$.

The above results will be the key ingredient to prove the existence of a fundamental solution for the operator (3.1), using the Levi parametrix method. Other tools will be some geometric properties of d plus some other miscellaneous properties which we collect here below. Once these properties are established the Levi method can be applied in a somehow axiomatic way.

i) There exists $\mathbf{c}$ such that

$$\mathbf{E}(x,\xi,t) \le \mathbf{c}\ \beta^{Q/2}\ \mathbf{E}(x,\xi,\beta t) \tag{3.12}$$

for every $\beta \ge 1$, $x,\xi \in \mathbb{R}^n$, $t > 0$.

ii) For any $\mu \ge 0$, there exists $\mathbf{c}(\mu)$ such that

$$(d(x,\xi)^2/t)^\mu\ \mathbf{E}(x,\xi,\lambda t) \le \mathbf{c}(\mu)\ \lambda^\mu\ \mathbf{E}(x,\xi,2\lambda t) \tag{3.13}$$

for every $\lambda > 0$, $x,\xi \in \mathbb{R}^n$, $t > 0$.

iii) For any $\varepsilon > 0$ and $\mu \ge 0$, there exists $\mathbf{c}(\mu,\varepsilon)$ such that

$$t^{-\mu}\ \mathbf{E}(x,\xi,t) \le \mathbf{c}(\varepsilon,\mu) \tag{3.14}$$

for every $x,\xi \in \mathbb{R}^n$, $t > 0$ such that $d(x,\xi)^2 + t \ge \varepsilon$.

iv) There exists a positive constant $\delta = \mathbf{c}^{-1}$ such that, if $0 \le \mu \le \delta/T$, then

$$\mathbf{E}(x,\xi,t)\ \exp(\mu|\xi|^2) \le \mathbf{c}\ \mathbf{E}(x,\xi,2t)\ \exp(2\mu|x|^2) \tag{3.15}$$

for every $x,\xi \in \mathbb{R}^n$ and $0 < t \le T$.

The "Levi method" is a classical technique that allows to construct the fundamental solution of a variable coefficient differential operator, starting from the fundamental solution of the corresponding operator with constant coefficients. This method was originally developed by E. E. Levi at the beginning of 20th century to study uniformly elliptic equations of order $2n$ (see [25], [26]) and later extended to uniformly parabolic operators (see e.g. [17]).

In the context of hypoelliptic ultraparabolic operators of Kolmogorov-Fokker-Planck type, Polidoro in [30] managed to adapt this method, thanks to the knowledge of an explicit expression for the fundamental solution of the "frozen" operator, which had been constructed in [23]. For operators of type (3.1), structured on homogeneous and invariant vector fields on Carnot groups, no explicit fundamental solution is available in general. Nevertheless, Bonfiglioli, Lanconelli, Uguzzoni in [3] showed how to adapt the same method, exploiting suitable sharp uniform Gaussian bounds on the fundamental solutions of the frozen operators. Here we follow the same line, thanks to the results of Section 2.

We now give a brief outline of the scheme of Levi method. Let us consider the fundamental solution $h_{\zeta_0}(z,\zeta)$ of the "frozen" operator H_{ζ_0}; the function $z \mapsto h_\zeta(z,\zeta)$ is called *parametrix*. The idea of the Levi method is to look for a fundamental solution $h(z,\zeta)$ for H, which could be written in the form:

$$h(z,\zeta) = h_\zeta(z,\zeta) + \int_\tau^t \int_{\mathbb{R}^n} h_\eta(z,\eta)\Phi(\eta,\zeta)\, d\eta \tag{3.16}$$

for a suitable, unknown kernel $\Phi(z,\zeta)$. This seems reasonable because we expect h to be a small perturbation of the parametrix, as the integral equation (3.16) expresses. The following formal computation suggests how to guess the right form of $\Phi(z,\zeta)$. If we set

$$Z_1(z;\zeta) = -H(z \mapsto h_\zeta(z,\zeta))(z)\ , \quad z \neq \zeta \in \mathbb{R}^{n+1}$$

and apply the operator H to both sides of (3.16) for $z \neq \zeta$, we find:

$$0 = -Z_1(z;\zeta) + \Phi(z,\zeta) - \int_\tau^t \int_{\mathbb{R}^n} Z_1(z,\eta)\Phi(\eta,\zeta)\, d\eta\ .$$

This means that Φ solves the integral equation

$$Z_1(z,\zeta) = \Phi(z;\zeta) - \int_\tau^t \int_{\mathbb{R}^n} Z_1(z,\eta)\Phi(\eta,\zeta)\, d\eta$$

which, defining the integral operator T with kernel Z_1, can be rewritten as $Z_1 = (I-T)\Phi$ whence, formally,

$$\Phi = \sum_{k=0}^\infty T^k Z_1 \equiv \sum_{k=0}^\infty Z_k\ .$$

To make the above idea rigorous, one has to reverse the order of the previous steps: to start studying the properties of the function Z_1, then Z_k, then $\Phi = \sum_{k=0}^\infty Z_k$, then

$$J(z,\zeta) = \int_\tau^t \int_{\mathbb{R}^n} h_\eta(z,\eta)\Phi(\eta,\zeta)\, d\eta$$

and finally $h(z,\zeta) = h_\zeta(z,\zeta) + J(z,\zeta)$.

4. HARNACK INEQUALITY

Let us consider the evolutionary and stationary operators of the kind

$$H = \partial_t - \sum_{i,j=1}^m a_{i,j}(t,x)X_iX_j - \sum_{k=1}^m a_k(t,x)X_k \tag{4.1}$$

or

$$L = \sum_{i,j=1}^m a_{i,j}(x)X_iX_j + \sum_{k=1}^m a_k(x)X_k\ , \tag{4.2}$$

respectively. Our assumptions are the same as in Section 3, except for the vanishing of the zero-order term (a_0 in (3.1)), that here will be assumed.

Our main result is the following invariant Harnack inequality.

Theorem 4.1. *Let H be as above. Let $R_0 > 0$, $0 < h_1 < h_2 < 1$ and $\gamma \in (0,1)$. There exists a positive constant $M = \mathbf{c}(h_1, h_2, \gamma, R_0)$ such that for every $(\tau_0, \xi_0) \in \mathbb{R}^{n+1}$, $R \in (0, R_0]$ and every*

$$u \in \mathfrak{C}^2((\tau_0 - R^2, \tau_0) \times B(\xi_0, R)) \cap C([\tau_0 - R^2, \tau_0] \times \overline{B(\xi_0, R)})$$

satisfying $Hu = 0$, $u \geq 0$ in $(\tau_0 - R^2, \tau_0) \times B(\xi_0, R)$, we have:

$$\max_{[\tau_0 - h_2 R^2, \tau_0 - h_1 R^2] \times \overline{B(\xi_0, \gamma R)}} u \leq M\, u(\tau_0, \xi_0)\ .$$

The above theorem immediately implies the stationary version:

Theorem 4.2. *Let L be as above. Let $R_0 > 0$. There exists a positive constant $M = \mathbf{c}(R_0)$ such that for every $\xi_0 \in \mathbb{R}^n$, $R \in (0, R_0]$ and every $u \in \mathfrak{C}^2(B(\xi_0, 3R))$ satisfying*

$$Lu = 0\ ,\ u \geq 0\ , \quad in\ \ B(\xi_0, 3R)$$

one has

$$\max_{\overline{B(\xi_0, R)}} u \leq M \min_{\overline{B(\xi_0, R)}} u\ .$$

The strategy we follow to prove "parabolic" Harnack inequality for operators (4.1) is inspired to the paper by Fabes-Stroock [15], who, in turn, exploited the original ideas by Krylov-Safanov about parabolic operators in nondivergence form (see [19], [20], [31]). In Fabes-Stroock's paper, Harnack inequality is derived by a fairly short but clever combination of estimates based only on the Gaussian bounds (from above and below) on the Green function for a cylinder. The radius of the cylinder incorporates the essential geometrical information, giving dilation invariance to the Harnack estimate. About at the same time of Fabes-Stroock paper, the same deep ideas were applied by Kusuoka-Stroock [21] in the context of Hörmander's operators $\partial_t - \sum_{i=1}^q X_i^2$. Much more recently, this general strategy has been adapted by Bonfiglioli, Uguzzoni [5] to study nonvariational operators structured on Hörmander's vector fields in Carnot groups.

Here we will follow the same line. The striking feature of this proof is the "axiomatic" nature of its core: it depends only on the suitable Gaussian estimates for the Green function, a maximum principle for H, the fact that constants are solutions to $Hu = 0$ (absence of the zero order term), and some geometric properties of CC-distance and balls, like the doubling property for the Lebesgue measure of metric balls. Then, also in our subriemannian setting, and for operators in nondivergence form, we recover an axiomatic link between Gaussian bounds, scaling invariant Harnack inequality, and properties of the underlying metric structure, as the one stressed by Saloff-Coste and Grigor'yan for divergence form parabolic operators on Riemannian manifolds (see the book [32] and references therein).

However, in our setting, a first problem arises: for our operator H with Hölder continuous coefficients, the existence of the Green function is not yet granted. Therefore it is convenient, as a first step, to make the qualitative assumption of smoothness on the coefficients, study the Green function in this setting and derive the Harnack inequality for operators with smooth coefficients. Since the Gaussian bounds on the Green function will be derived by the analogous bounds on the fundamental solution, all the constants will depend on the coefficients a_{ij}, a_k only through their C^α-norms and ellipticity (the constants K, λ defined in (3.4)). This will allow to get, by a limiting procedure, Harnack inequality in the non-smooth case. But a second problem arises: even for the operator with smooth coefficients, which is hypoelliptic and fits the assumptions of the classical theory developed by Bony [6], the cylinder based on a metric ball could be a bad domain for the Dirichlet problem, so that the existence of the Green functions for this kind of domain is still troublesome. Nevertheless, Lanconelli and Uguzzoni have recently proved in [24] that given two metric balls $B(\xi, \delta R), B(\xi, R)$, (with $\delta \in (0,1)$), there always exists a domain $A(\xi, R)$, regular for the (stationary) Dirichlet problem, and such that $B(\xi, \delta R) \subseteq A(\xi, R) \subseteq B(\xi, R)$. The Green function for H on $\mathbb{R} \times A(\xi, R)$ must be thought as the natural substitute of the Green function for the cylinder $\mathbb{R} \times B(\xi, R)$. In the final limiting procedure, we will also use the fact that the domain $A(\xi, R)$ can be suitably chosen in order for it to be "uniformly regular" for the family of approximating operators H_ε. This is another fact proved in [24].

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