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Degenerate Sobolev spaces and regularity of Monge–Ampère and subelliptic equations

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Dedicated to Ermanno Lanconelli

Abstract[1]. We summarize some recent regularity results for degenerate Monge-Ampère equations, quasilinear equations and linear subelliptic equations. The results for linear equations form a basis for the others and generalize ones obtained jointly with E. Sawyer by extending the definition of weak solution to functions in a Hilbert space associated with a nonnegative locally integrable quadratic form.

The results I will describe are joint work with Eric Sawyer and Cristian Rios. They involve elementary facts about Sobolev spaces associated with degenerate quadratic forms, and they include regularity results for solutions of degenerate linear, quasilinear and nonlinear equations. The linear theory for subelliptic equations with rough coefficients underlies everything, and for this we were greatly inspired by work initiated by Ermanno Lanconelli and Bruno Franchi.

I will start by summarizing some new results of Monge-Ampère type, then indicate how rough degenerate quasilinear equations arise in their study, and finally turn to the background theory about degenerate Sobolev spaces and regularity for solutions of rough linear subelliptic equations.

1. Degenerate Monge-Ampére and quasilinear equations

Let Ω be a strictly convex domain in $\mathbb{R}^n$ with smooth (C^∞) boundary. Let $k \geq 0$ (k may vanish) be a smooth function of its variables $(x, r, \rho) \in \Omega \times \mathbb{R}^1 \times \mathbb{R}^n$. For a smooth function φ on $\partial\Omega$, consider the Monge–Ampère Dirichlet problem

$$\det D^2 u = k(x, u, Du) \quad \text{in } \Omega$$

$$u = \varphi \quad \text{on } \partial\Omega \,.$$

Here u is a convex function which we assume is a generalized solution in the Alexandrov sense.

We will also be interested in *local* Monge–Ampère solutions, i.e., generalized solutions of $\det D^2 u = k$ in Ω without imposing any boundary condition.

Main question: Is u smooth?

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In case k is strictly positive, then there is a unique solution $u \in C^\infty$ for the Dirichlet problem, as shown by Caffarelli-Nirenberg-Spruck [2].

Moreover, when $k > 0$, any *local* solution of $\det D^2u = k$ which belongs to $C^{1,1-(2/n)+\varepsilon}$ for some $\varepsilon > 0$ is smooth: see Urbas [18] and Caffarelli [1].

However, as shown by the example $n = 2$ and

$$u(x,y) = |(x,y)|^3 = (x^2+y^2)^{3/2} ,$$

$$u_{xx}u_{yy} - u_{xy}^2 = 18(x^2+y^2) ,$$

u may fail to be smooth even when k vanishes to second order at a single point. This function u is C^2 with second derivatives 0 at $(0,0)$, but it is not C^3 at $(0,0)$.

Some history about $k \geq 0$.
• Due to work of Guan [12] and of Guan-Trudinger-Wang [13], we know:

If $k = k(x)$ and $k(x)^{1/(n-1)} + A|x|^2$ is subharmonic for some constant A, then there exists a unique convex solution $u(x)$ in $C^{1,1}(\bar{\Omega})$ for the Dirichlet problem.

No better regularity is true in general as shown by an example due to Sibony reported in Guan [11] and [13].

• On the other hand, assuming only that $k \geq 0$, Guan [11] proved there is better regularity than $C^{1,1}$ when $n = 2$ provided more conditions on k and u are imposed. He showed:
If $n = 2$ and $k = k(x) \geq 0$ vanishes only at a single point and is of (some particular) finite type, then a convex local $C^{1,1}$ solution u is smooth if one fixed principal curvature of u is bounded away from 0 (e.g., if $u_{yy} \geq c > 0$).
An easy corollary of Guan's result is that if $n = 2$ and $k(x) \approx |x|^2$ (so k vanishes only at $x = 0$), then a convex local C^2 solution u is smooth in Ω if and only if its mean curvature $\Delta u > 0$ in Ω.

• When $n \geq 3$, Guan's result was generalized in [15] by showing that a convex $C^{2,1}$ local solution u is smooth if the $(n-1) \times (n-1)$ minor

$$\det[u_{ij}]_{i,j\geq 2} > 0 \quad \text{and}$$

$$k \text{ is of (some particular) finite type, e.g., } k(x,r,\rho) \approx x_1^{2m} + \psi(x)$$

where ψ is smooth and $\psi^{1/(2m)}$ is Lipschitz.

New results.
In [16] and [17], we relax the restriction that $u \in C^{2,1}$ but require growth conditions and/or structural conditions on k. Here are 3 examples:

(A) If $k(x,r,\rho) \approx |x|^2$, then a convex C^2 local solution u is smooth if and only if the symmetric curvature κ_{n-1} of order $n-1$ of u is strictly positive in Ω.

(B) If $k(x,r,\rho) \approx |x|^{2m}, m = 2,3,\cdots$, is a sum of squares of smooth functions,

$$k(x,r,\rho) = \sum_{j=1}^{N} P_j(x,r,\rho)^2 ,$$

and if for each r, ρ, the functions $\{P_j^{[m]}(x, r, \rho)\}_{j=1}^N$ span (linearly) the entire space of homogeneous polynomials of degree m on $\mathbb{R}^n$, then any convex C^2 solution is smooth provided $\kappa_{n-1}(0) > 0$.

Here we denote

$$f^{[m]} = \text{the homogeneous polynomial of degree } m \text{ in the Maclaurin series of } f\ .$$

For example, if $n = m = 2$ and $k(x, y) = (x^2 + y^2)^2$, then these polynomials are simply x^2, xy and y^2 and clearly span the space of homogeneous polynomials of degree 2. In fact, for any n and m, the conditions are satisfied if

$$k = |x|^{2m} g(x, u, Du) \quad \text{with } g > 0 \text{ and smooth}\ .$$

(C) If $k(x, r, \rho) \approx |x|^{2m}, m = 2, 3, \cdots$, is *not* a sum of squares, then u is smooth provided

$$u \in W^{3,n-(m/(m+1))+\varepsilon} \quad \text{for some } \varepsilon > 0\ .$$

For small ε, the space $W^{3,n-(m/(m+1))+\varepsilon}$ is not contained in C^2.

The proofs are complicated. One basic step is to use the partial Legendre transform (assuming that $\det[\partial_{ij}u]_{i,j\geq 2} \geq c > 0$)

$$\begin{aligned} s &= x_1 \\ t_\ell &= \partial_\ell u(x)\ , \quad \ell = 2, \cdots, n \end{aligned} \tag{1.1}$$

to reduce to a divergence-form quasilinear system of the type

$$\left\{\frac{\partial^2}{\partial s^2} + \operatorname{div}_t kM(\mathbf{p})\nabla_t\right\}\mathbf{p} = \mathbf{f}((s,t), \mathbf{v}, \mathbf{p}, D\mathbf{p})\ , \quad t = (t_2, \cdots, t_n)\ , \tag{1.2}$$

where

$$\mathbf{v} = (x_\ell(s,t))_{\ell=2}^n \quad \text{is the inverse of the pLt}\ ,$$

$$\mathbf{p} = \left(\frac{\partial x_\ell}{\partial t_j}\right)_{\ell\geq 2, j\geq 1}\ , \quad t_1 = s\ ,$$

and $\mathbf{f}$ is quadratic in $D\mathbf{p}$ with multiple k. Here $M(\mathbf{p})$ is the matrix of cofactors of the Jacobian of $\mathbf{v}$ with respect to t. In case $n = 2$, then $M(p) = 1$ and the system is a single quasilinear equation.

In any dimension, the coefficient k in (1.2) is not smooth since k is composed with the solution.

Main point: Regularity of solutions of rough quasilinear systems leads to regularity of Monge–Ampère solutions and is in turn related to regularity of solutions of single subelliptic equations with rough coefficients. In fact, since $\mathbf{v}$ and $\mathbf{p}$ are fixed vectors (arising from a given u), we can apply regularity results for *linear* subelliptic equations with rough coefficients as in [14].

Remark 1.1. Our arguments involve approximation of rough vector fields by first order Taylor expansions, as also done in Citti-Lanconelli-Montanari [5], [3] and [4]. However, the quadratic dependence of $\mathbf{f}$ on $D\mathbf{p}$ caused us difficulty, and we do not know if it is possible to also use their approach.

2. Linear subelliptic equations and degenerate Sobolev spaces

We now turn to regularity results for a single subelliptic equation with rough coefficients. The main reference for the results we need is [14], but the spirit is closely connected to work of Franchi-Lanconelli [8], Franchi [7], and Fabes-Kenig-Serapioni [6]. The paper [6] studies weighted degenerate elliptic equations rather than subelliptic ones, but we are indebted to it because it suggested an axiomatic approach and because it gave an example when the notion of weak gradient is not unique.

One result in [14] gives a list of axioms which guarantee when weak solutions of linear equations of the form

$$\operatorname{div} Q\nabla u + \text{lower order } = f + T'g \tag{2.3}$$

are locally Hölder continuous. Here Q is a nonnegative semidefinite symmetric matrix and T' is the adjoint of a vector field

$$T = v \cdot \nabla$$

that is subunit with respect to Q, i.e.,

$$(v(x) \cdot \xi)^2 \leq \xi' Q(x)\xi \ , \quad \xi \in \mathbb{R}^n \ .$$

The lower order terms are not important for applications to Monge–Ampère equations, but the inclusion of the nonhomogeneous term $T'g$ is crucial.

In [14], the coefficient matrix Q in (2.3) is assumed to be bounded and the definition of weak solution requires that $u \in W^{1,2}(\Omega)$, the classical Sobolev space. These hypotheses are adequate for our applications to Monge-Ampère equations, but they are unesthetic.

The main new fact is that the axiomatic regularity result from [14] still holds if Q is merely locally integrable and if the definition of weak solution is extended to a degenerate Sobolev space $W^{1,2}_Q(\Omega)$ which incorporates Q and is defined below. There is *no* hypothesis that Q be controlled by vector fields that are Lipschitz continuous or of class $W^{1,2}(\Omega)$. However, we do require the existence of an underlying quasimetric and space of homogeneous type.

Thus suppose that $Q = Q(x)$ is a nonnegative semidefinite symmetric matrix which is locally integrable in Ω:

$$\int_L \|Q(x)\| \, dx < \infty \ , \quad \text{for every compact set } L \subset \Omega \ ,$$

and define the corresponding nonnegative quadratic form

$$Q(x,\xi) = \xi \cdot Q(x)\xi \ , \quad \xi \in \mathbb{R}^n \ .$$

Definition 2.2. Let

$$\mathcal{L}^2(\Omega, Q) = \Bigg\{ \mathbf{f} = (f_1, \cdots, f_n) \text{ measurable on } \Omega \ : \\ \|\mathbf{f}\|_{\mathcal{L}^2(\Omega,Q)} = \left(\int_\Omega Q(x, \mathbf{f}(x)) \, dx \right)^{1/2} < \infty \Bigg\} \ .$$

If $\|\mathbf{f} - \mathbf{g}\|_{\mathcal{L}^2(\Omega,Q)} = 0$ for two elements $\mathbf{f}, \mathbf{g} \in \mathcal{L}^2(\Omega, Q)$, we identify $\mathbf{f} = \mathbf{g}$ and think of $\mathcal{L}^2(\Omega, Q)$ as a collection of equivalence classes. A characterization of both $\|\mathbf{f}\|_{\mathcal{L}^2(\Omega,Q)}$ and of when $\|\mathbf{f} - \mathbf{g}\|_{\mathcal{L}^2(\Omega,Q)} = 0$ is given below.

Theorem 2.3. *$\mathcal{L}^2(\Omega, Q)$ is a Hilbert space with inner product*

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{L}^2(\Omega,Q)} = \int_\Omega \mathbf{f}(x)' Q(x) \mathbf{g}(x)\, dx \,.$$

There is a simple way to realize $\mathcal{L}^2(\Omega, Q)$ in terms of weighted L^2 spaces with weights equal to the eigenvalues of Q. In fact, let

$$\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x)$$

be the (nonnegative, measurable and locally integrable) eigenvalues of $Q(x)$. One can choose corresponding measurable orthonormal eigenvectors (orthonormal with respect to the Euclidean inner product)

$$\boldsymbol{\eta}_1(x), \boldsymbol{\eta}_2(x), \cdots, \boldsymbol{\eta}_n(x) \,.$$

For any $\mathbf{f} \in \mathcal{L}^2(\Omega, Q)$, we let $f_j(x)$ be the components of $\mathbf{f}$ with respect to the basis of eigenvectors $\{\boldsymbol{\eta}_j(x)\}$, i.e.,

$$\mathbf{f}(x) = \sum_{j=1}^n f_j(x) \boldsymbol{\eta}_j(x) \,, \quad x \in \Omega \,.$$

Then

$$Q(x, \mathbf{f}(x)) = \mathbf{f}(x)' Q(x) \mathbf{f}(x) = \left(\sum_{j=1}^n f_j \boldsymbol{\eta}_j \right) \cdot \left(\sum_{j=1}^n f_j Q \boldsymbol{\eta}_j \right) = \sum_{j=1}^n |f_j(x)|^2 \lambda_j(x) \,.$$

Hence,

$$\|f\|^2_{\mathcal{L}^2(\Omega,Q)} = \sum_{j=1}^n \int_\Omega |f_j(x)|^2 \lambda_j(x)\, dx = \sum_{j=1}^n \|f_j\|^2_{L^2(\lambda_j dx)} \,. \tag{2.4}$$

Remark 2.4. It follows that two functions $\mathbf{f} = \sum_{j=1}^n f_j \boldsymbol{\eta}_j$ and $\mathbf{g} = \sum_{j=1}^n g_j \boldsymbol{\eta}_j$ are equal in $\mathcal{L}^2(\Omega, Q)$ (i.e., $\mathbf{f}$ and $\mathbf{g}$ lie in the same equivalence class) if and only if

$$f_j(x) = g_j(x) \quad \text{for } \lambda_j - \text{a.e. } x \in \Omega, 1 \le j \le n \,.$$

In particular, the values of $\mathbf{f}$ and $\mathbf{g}$ on the sets where $\lambda_j = 0$ have no effect on whether $\mathbf{f}$ and $\mathbf{g}$ are equal.

Definition 2.5. If $u \in \mathrm{Lip}(\Omega)$, let

$$\|u\|_Q = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{\mathcal{L}^2(\Omega,Q)} \right)^{1/2} \,,$$

$$\mathrm{Lip}_Q(\Omega) = \{u \in \mathrm{Lip}(\Omega) \;:\; \|u\|_Q < \infty\}$$

(note that Ω may not be bounded), and

$$W^{1,2}_Q(\Omega) = \text{ Completion of } \mathrm{Lip}_Q(\Omega) \text{ in the metric} \|\cdot\|_Q \,.$$

An element of $W^{1,2}_Q(\Omega)$ is an equivalence class of Cauchy sequences of pairs $\{(u_k, \nabla u_k)\}$ in $L^2(\Omega) \times \mathcal{L}^2(\Omega, Q)$, where $u_k \in \mathrm{Lip}(\Omega)$. Thus

$$u_k \to u \quad \text{in } L^2(\Omega) \,, \quad \text{and } \nabla u_k \to \mathbf{w} \text{ in } \mathcal{L}^2(\Omega, Q) \,.$$

The pair $(u, \mathbf{w})$ represents the equivalence class, **but** the vector $\mathbf{w}$ is not generally uniquely determined in $\mathcal{L}^2(\Omega, Q)$ by the first component u.

Example. Fabes-Kenig-Serapioni [6] give an example of a quadratic form $Q(x,\xi) = q(x)\xi^2$ with $x \in (0,1), \xi \in (-\infty,\infty)$ and $0 < q(x) \leq 1$, together with a sequence $\{u_k\}$ in $\mathrm{Lip}(0,1)$ such that

$$u_k \to 0 \text{ in } L^2(0,1) \quad (\text{even } u_k \to 0 \text{ uniformly}), \text{ but}$$

$$\frac{d}{dx}u_k \to 1 \text{ in } \mathcal{L}^2((0,1),Q) = L^2((0,1),qdx)\ .$$

Thus the pair $(0,1) \in W_Q^{1,2}(0,1)$. So does the pair $(0,0)$. In fact, one can show that the pair $(0,\mathbf{w}) \in W_Q^{1,2}(0,1)$ for *every* $\mathbf{w} \in \mathcal{L}^2(\Omega,Q)$!

However, if $\{\mathbf{v}_j(x)\}_{j=1}^m$ is a collection of $H_{\mathrm{div}}^{1,2}(\Omega)$ vector fields, i.e., if $\mathbf{v}_j \in L^2(\Omega)$ and $\operatorname{div}\mathbf{v}_j \in L^2(\Omega)$, and if $Q(x,\xi)$ is the corresponding quadratic form,

$$Q(x,\xi) = \sum_{j=1}^m (\mathbf{v}_j(x)\cdot\xi)^2\ ,$$

then u does uniquely determine $\mathbf{w}$. Hence $\sqrt{q(x)}$ does not have a derivative in $L^2((0,1))$ in the example from [6].

• We do not need to have uniqueness of gradients in order to have a regularity theory.

Remark 2.6. In the case of a collection of vector fields $\mathbf{v}_j \in H_{\mathrm{div}}^{1,2}(\Omega)$, we can define the corresponding collection $\mathcal{X} = \{X_j\}$ of differential operators

$$X_j f = \mathbf{v}_j \cdot \nabla f\ , \quad f \in \mathrm{Lip}(\Omega)\ ,$$

and consider the Hilbert space

$$H_{\mathcal{X}}^{1,2}(\Omega) = \left\{f\ :\ f \in L^2(\Omega) \text{ and } X_j f \in L^2(\Omega) \text{ in the weak sense}\right\}\ .$$

If we consider the associated quadratic form

$$Q(x,\xi) = \mathcal{X}(x,\xi) = \sum (\mathbf{v}_j(x)\cdot\xi)^2\ ,$$

then

$$W_Q^{1,2}(\Omega) = W_{\mathcal{X}}^{1,2}(\Omega) \subset H_{\mathcal{X}}^{1,2}(\Omega)\ .$$

We do not know exactly when $W_Q^{1,2}(\Omega) = H_{\mathcal{X}}^{1,2}(\Omega)$ in case $Q(x,\xi) = \mathcal{X}(x,\xi)$, but we have 3 related results:

• They are equal in case $n = 1$ for any $H_{\mathrm{div}}^{1,2}$ weight. ($H_{\mathrm{div}}^{1,2} = H^{1,2}$ when $n = 1$.)

• They are equal if $n > 1$ for any $H^{1,2\sigma'}$ vector fields, where σ is the Sobolev gain and $1/\sigma + 1/\sigma' = 1$. This generalizes independent results of Franchi-Serapioni-Serra Cassano [9] and Garofalo-Nhieu [10], where the authors assume the vector fields are Lipschitz continuous. In the classical case of the gradient in $\mathbb{R}^n$, we have $\sigma = n/(n-2)$ and consequently $2\sigma' = n$.

• They are equal for $H^{1,2}$ vector fields which are merely *comparable* locally to Lipschitz vector fields off the common 0-set Z of the vector fields, and Lipschitz *at* Z.

Definition 2.7. A weak solution of

$$\operatorname{div} Q(x)\nabla u + \text{lower order terms } = f + T'g$$

is a pair $(u, \nabla u) \in W_Q^{1,2}(\Omega)$ such that for all $w \in \mathrm{Lip}_0(\Omega)$,

$$-\int \nabla w \cdot Q\nabla u + \cdots = \int fw + \int gTw \, .$$

The leftmost integral here converges absolutely since $|\sqrt{Q}\nabla u| \in L^2(\Omega)$. Also, $\|\sqrt{Q}\,\| \in L^2_{\mathrm{loc}}(\Omega)$.

We define $\left(W_Q^{1,2}\right)_0(\Omega)$ in the same way as $W_Q^{1,2}(\Omega)$ except that the sequences $\{u_k\}$ consist of Lipschitz functions each with compact support in Ω.

Theorem 2.8 (Axiomatic regularity for rough subelliptic equations)**.** *Assume that Q is symmetric, nonnegative and in $L^1_{\mathrm{loc}}(\Omega)$ in a bounded open set $\Omega \subset \mathbb{R}^n$. Let $d(x,y)$ be a symmetric quasimetric in Ω which is Lebesgue measurable in each variable and satisfies*

$$d(x,y) \geq c|x-y| \, .$$

Suppose also that

(1) *d-balls are doubling with respect to Lebesgue measure*
(2) *the Fefferman–Phong condition $d(x,y) \leq C|x-y|^{\varepsilon}$ holds for some $\varepsilon > 0$*
(3) *the Q-version of Sobolev's inequality holds with gain σ, i.e., there exists $\sigma > 1$ such that*

$$\left(\frac{1}{|B|}\int_B |w|^{2\sigma}\right)^{1/2\sigma} \leq cr(B)\left(\frac{1}{|B|}\int_B Q(x,\nabla w)\right)^{1/2} + c\left(\frac{1}{|B|}\int_B w^2\right)^{1/2}$$

for all $(w,\nabla w) \in \left(W_Q^{1,2}\right)_0(\Omega)$ and all d-balls $B \subset \Omega$

(4) *the Q-version of a "straight-across" (no gain) Poincaré inequality holds, i.e.,*

$$\frac{1}{|B|}\int_B |w-w_B|^2 \leq cr(B)^2 \frac{1}{|B^*|}\int_{B^*} Q(x,\nabla w) \, , \quad B^* = CB \, ,$$

for all $(w,\nabla w) \in W_Q^{1,2}(\Omega)$ and all d-balls B with $B^* \subset \Omega$

(5) *there exist Q-versions of accumulating sequences of Lipschitz cutoff functions.*

Conclusion: Any weak solution is locally Hölder continuous if $f \in L^{q/2}(\Omega)$ and $g \in L^q(\Omega)$ for some q with

$$q > \max\{2\sigma', D^*\} \, , \quad \textit{with}$$

$$D^* = \limsup_{r\to 0} \max_{x\in\Omega} \frac{\log|B(x,r)|}{\log r} \quad (\leq \ \textit{doubling order}) \, .$$

Remark 2.9.

(1) The Fefferman-Phong condition (2) can be deleted if Hölder continuity is measured with respect to $d(x,y)$ instead of $|x-y|$.
(2) In case Q is *continuous* and $d(x,y)$ is the control metric relative to Q-subunit curves $\gamma(t)$ ($(\gamma'(t)\cdot\xi)^2 \leq Q(\gamma(t),\xi)$), then assumption (5) about cutoff functions is automatically true.

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