

Path integral approach to the heat kernel

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Abstract¹. The heat kernel finds many applications in physics and mathematics. As a few examples one could mention the calculation of effective actions in quantum field theories, the study of anomalies, and the proofs of various index theorems. Here we describe a method of computing the heat kernel using path integrals. The advantage of using such an approach is that it is quite intuitive from the physical point of view and quite effective from the calculational point of view. Path integrals allow to visualize quantum processes by viewing them as arising from the contributions of all possible paths satisfying some boundary conditions, and allow a straightforward perturbative calculation in terms of Feynman diagrams. As an application, we discuss the perturbative calculation of the effective action of particles of spin 0, 1/2, 1, and of massless and massive antisymmetric tensor fields coupled to gravity.

1. INTRODUCTION

The Schrödinger equation is the fundamental equation describing quantum physics. Its solutions can be obtained by computing path integrals, introduced for this purpose by Feynman who arrived at his proposal by elaborating on some previous ideas due to Dirac. The analytical continuation in the time variable $t \rightarrow -i\beta$ (the so-called Wick rotation) relates the Schrödinger equation

$$(1) \quad i \frac{\partial}{\partial t} \psi = \hat{H} \psi$$

to the heat equation

$$(2) \quad -\frac{\partial}{\partial \beta} \psi = \hat{H} \psi$$

where \hat{H} is a second order differential operator. Similarly the quantum mechanical path integral can be Wick rotated to an euclidean path integral, which generates the solution of the heat equation. Such euclidean path integrals had been previously considered by Wiener in the discussion of the brownian motion. In this contribution we first review the use of such path integrals to obtain a representation of the fundamental solution to the heat equation, the so-called heat kernel, and use them to compute the asymptotic expansion for small euclidean time. Then we discuss how to apply them for calculating perturbatively the effective action of massless and massive particles of spin 0, 1/2, 1, and particles associated to antisymmetric tensor fields coupled to gravity.

2. A SIMPLE EXAMPLE

Let us first consider the simple case of a differential operator given by

$$(3) \quad \hat{H} = -\frac{1}{2} \nabla^2 + V(x)$$

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where ∇^2 is the laplacian in the cartesian coordinates of R^D , which we denote by x^μ , and $V(x)$ is an arbitrary smooth potential. Using the “bra” and “ket” language of Dirac, the heat kernel can be represented by

$$(4) \quad \psi(x, y; \beta) = \langle y | e^{-\beta \hat{H}} | x \rangle$$

which satisfies eq. (2) together with the boundary condition

$$(5) \quad \psi(x, y; 0) = \delta^D(x - y) .$$

It is well-known that the solution in the free case (i.e. for $V = 0$) is given by

$$(6) \quad \psi(x, y; \beta) = \frac{1}{(2\pi\beta)^{D/2}} e^{-(x-y)^2/2\beta} .$$

The path integral which computes (4) can formally be written as

$$(7) \quad \psi(x, y; \beta) = \int_{q(0)=x}^{q(\beta)=y} Dq e^{-S[q]}$$

where the symbol $\int_{q(0)=x}^{q(\beta)=y} Dq$ indicates the formal sum over all functions $q^\mu(t)$ which satisfy the boundary conditions $q^\mu(0) = x^\mu$ and $q^\mu(\beta) = y^\mu$, whereas the euclidean action $S[q]$ is given by

$$(8) \quad S[q] = \int_0^\beta dt \left(\frac{1}{2} \delta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + V(q) \right)$$

where a dot (as in \dot{q}^μ) represents a derivative with respect to t . The measure Dq can formally be written as

$$(9) \quad Dq = \prod_{0 < t < \beta} d^D q(t)$$

to indicate the integration over the values of $q^\mu(t)$ for each t .

The precise definition of path integrals, as usually given by physicists, requires the introduction of a regularization scheme, which means a procedure that makes sense of the integration over paths (and that usually implies a discretization). The regularization scheme must also contain the specification of certain renormalization conditions, which make sure that different regularization schemes produce the same final answer once the continuum limit is taken (i.e. once the regulating parameter is removed). We are not going to describe all the details (or the historical development) of such a procedure, which was obtained out of the study of renormalization in quantum field theory. Instead we will describe a specific regularization scheme which gives us directly the power of performing explicit calculations. One of the simplest methods is based on the Fourier expansion of all possible paths, and it is called “mode regularization”. We are going to describe this regularization next.

To start with, let us first rescale for commodity the euclidean time variable $t = \beta\tau$, so that the new time variable $\tau \in [0, 1]$. Then the action (8) can be written as

$$(10) \quad S[q] = \frac{1}{\beta} \int_0^1 d\tau \left(\frac{1}{2} \delta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + \beta^2 V(q) \right)$$

where dots (as in \dot{q}^μ) represent now derivatives with respect to τ . This rescaling is useful especially since we are going to compute the path integral in a perturbative expansion valid for small β .

One can decompose all paths to be summed over as follows

$$(11) \quad q^\mu(\tau) = q_{bg}^\mu(\tau) + \phi^\mu(\tau)$$

where $q_{bg}^\mu(\tau)$ is a fixed path (sometimes called the background path, or classical path) which is taken to satisfy the boundary conditions and, for simplicity, the classical equation of motion for $V = 0$. Thus

$$(12) \quad q_{bg}^\mu(\tau) = x^\mu + (y^\mu - x^\mu)\tau .$$

This is the classical solution for $V = 0$, as it satisfies the boundary conditions $q_{bg}^\mu(0) = x^\mu$ and $q_{bg}^\mu(1) = y^\mu$, as well as the equations of motion $\ddot{q}_{bg}^\mu(\tau) = 0$. The remaining arbitrary “quantum fluctuations” $\phi^\mu(\tau)$ must then have vanishing boundary conditions, $\phi^\mu(0) = \phi^\mu(1) = 0$. They can be expanded in a Fourier sum

$$(13) \quad \phi^\mu(\tau) = \sum_{m=1}^{\infty} \phi_m^\mu \sin(\pi m \tau) .$$

The path integral in (7) can now be concretely defined as the limit $M \rightarrow \infty$ of a big number M of Lebesgue integrations over the Fourier coefficients ϕ_m^μ suitably normalized, namely with a measure given by

$$(14) \quad \mathcal{D}q = \lim_{M \rightarrow \infty} A \prod_{m=1}^M \prod_{\mu=1}^D \sqrt{\frac{\pi m^2}{4\beta}} d\phi_m^\mu$$

where A is fixed by the consistency condition that the path integral should reproduce the correct solution to the heat equation. This requirement gives $A = (2\pi\beta)^{-D/2}$ (fixing this value in such a fashion may be considered as the renormalization condition mentioned above, and it is the only renormalization condition needed in the present case!).

Let us now make explicit the path integral using the mode expansion of the paths just described. For $V = 0$ the action (which we denote by $S_{(2)}[q]$ to differentiate it from the full action with $V \neq 0$) can be written in terms of the paths parametrized as in (11) and (12), and reads

$$(15) \quad S_{(2)}[q] = \frac{(x-y)^2}{2\beta} + \sum_{m=1}^{\infty} \frac{\pi^2 m^2}{4\beta} \phi_m^\mu \phi_m^\nu \delta_{\mu\nu}$$

where the first term is due to the classical path $q_{bg}^\mu(\tau)$, and the rest from the quantum fluctuations $\phi^\mu(\tau)$. Of course, this action can also be presented in a discretized form by including in the sum only the first M modes. The free path integral is then defined by

$$(16) \quad \begin{aligned} & \int_{q(0)=x}^{q(\beta)=y} \mathcal{D}q e^{-S_{(2)}[q]} \equiv \\ & \equiv \lim_{M \rightarrow \infty} \int A \prod_{m=1}^M \prod_{\mu=1}^D \sqrt{\frac{\pi m^2}{4\beta}} d\phi_m^\mu e^{-(x-y)^2/2\beta} e^{-\sum_{m=1}^M (\pi^2 m^2/4\beta) \phi_m^\mu \phi_m^\nu \delta_{\mu\nu}} = \\ & = A e^{(x-y)^2/2\beta} \lim_{M \rightarrow \infty} \prod_{m=1}^M \prod_{\mu=1}^D \int_{-\infty}^{\infty} d\phi_m^\mu \sqrt{\frac{\pi m^2}{4\beta}} e^{-(\pi^2 m^2/4\beta) \phi_m^\mu \phi_m^\mu} = \\ & = \frac{1}{(2\pi\beta)^{D/2}} e^{-(x-y)^2/2\beta} . \end{aligned}$$

We see that the free solution in eq. (6) is reproduced, and the term in the exponent is the action evaluated on the classical trajectory. Note that we have used repeatedly the standard gaussian integral

$$(17) \quad \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} .$$

This calculation was simple because for the free case the semiclassical approximation is exact: the exact solution is given by the exponential of the action evaluated on the classical trajectory, modified only by the overall normalization which can be considered as a one-loop correction.

The same path integral formula can be extended to the more general actions of the type given in (10). The inclusion of an arbitrary potential V makes the problem quite difficult

to solve in general. However it can be treated for a large class of potentials in perturbation theory, and the emerging solution takes the form

$$(18) \quad \psi(x, y; \beta) = \frac{1}{(2\pi\beta)^{D/2}} e^{-(x-y)^2/2\beta} \left(a_0(x, y) + a_1(x, y)\beta + a_2(x, y)\beta^2 + \dots \right)$$

where the so-called Seeley-DeWitt coefficients a_n depend on the points x^μ and y^μ and on the potential V . For $n > 1$ these coefficients contain higher loop corrections to the semiclassical approximation. It is useful to introduce as a variable the difference

$$(19) \quad \xi^\mu = (y^\mu - x^\mu)$$

whose length may be considered of order $\sqrt{\beta}$ for the brownian motion, then β controls the perturbative expansion. Of course $a_0(x, y) = 1$.

2.1. Perturbative expansion. The perturbative expansion is based on the gaussian averages of the path integral with the free quadratic actions $S_{(2)}$ (i.e. the one with $V = 0$), namely

$$(20) \quad \begin{aligned} A &\equiv \int \mathcal{D}\phi e^{-S_{(2)}[\phi]} = \frac{1}{(2\pi\beta)^{D/2}} \\ \langle \phi^\mu(\tau) \rangle &\equiv \frac{1}{A} \int \mathcal{D}\phi \phi(\tau) e^{-S_{(2)}[\phi]} = 0 \\ \langle \phi^\mu(\tau) \phi^\nu(\sigma) \rangle &\equiv \frac{1}{A} \int \mathcal{D}\phi \phi(\tau) \phi(\sigma) e^{-S_{(2)}[\phi]} = -\beta \delta^{\mu\nu} \Delta(\tau, \sigma) \\ &\dots \end{aligned}$$

where $\Delta(\tau, \sigma)$ is the Green function of the operator $\partial^2/\partial\tau^2$ on the space of functions $f(\tau)$ with vanishing boundary conditions at $\tau = 0$ and $\tau = 1$,

$$(21) \quad \frac{\partial^2}{\partial\tau^2} \Delta(\tau, \sigma) = \delta(\tau - \sigma).$$

It reads (for τ and σ in $[0, 1]$)

$$(22) \quad \begin{aligned} \Delta(\tau, \sigma) &= \sum_{m=1}^{\infty} \left[-\frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) \right] = \\ &= (\tau - 1)\sigma \theta(\tau - \sigma) + (\sigma - 1)\tau \theta(\sigma - \tau) \\ \delta(\tau - \sigma) &= \sum_{m=1}^{\infty} 2 \sin(\pi m \tau) \sin(\pi m \sigma) \end{aligned}$$

where $\theta(x)$ is the usual step function, $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1/2$ for $x = 0$. In general one may define the average of an arbitrary functional $F[\phi]$ as

$$(23) \quad \langle F[\phi] \rangle \equiv \frac{1}{A} \int \mathcal{D}\phi F[\phi] e^{-S_{(2)}[\phi]}.$$

Now the perturbative expansion is computed as follows. The action is first split into a free part which is exactly solvable plus an interaction part which is treated as a perturbation, namely

$$(24) \quad S[q] = S_{(2)}[q] + S_{int}[q]$$

where

$$(25) \quad S_{(2)}[q] = \frac{1}{\beta} \int_0^1 d\tau \frac{1}{2} \delta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$$

is the free action, and

$$(26) \quad S_{int}[q] = \beta \int_0^1 d\tau V(q)$$

is the perturbation. The path integral can now be manipulated as follows

$$(27) \quad \begin{aligned} & \int_{q(0)=x}^{q(\beta)=y} \mathcal{D}q e^{-S[q]} = \\ & = \int_{q(0)=x}^{q(\beta)=y} \mathcal{D}q e^{-(S_{(2)}[q] + S_{int}[q])} = \\ & = e^{-S_{(2)}[q_{bg}]} \int_{\phi(0)=0}^{\phi(\beta)=0} \mathcal{D}\phi e^{-S_{int}[q_{bg} + \phi]} e^{-S_{(2)}[\phi]} = \\ & = A e^{-S_{(2)}[q_{bg}]} \langle e^{-S_{int}[q_{bg} + \phi]} \rangle = \\ & = \frac{1}{(2\pi\beta)^{D/2}} e^{-(x-y)^2/2\beta} \left\langle \left(1 - S_{int}[q_{bg} + \phi] + \frac{1}{2} S_{int}^2[q_{bg} + \phi] + \dots \right) \right\rangle. \end{aligned}$$

The transition from the first to the second line is obtained by using the translation invariance of the path integral measure. Actually, we have just used the definition of the measure in (14), which could have been named by $\mathcal{D}\phi$ as well. Then we have used the notation introduced in (23) to indicate averages with the free path integral. Finally, the expansion of the exponential of the interaction part written in the last line generates the perturbative expansion.

Let us compute systematically the various terms appearing in the last line of eq. (27). The first one is trivial

$$(28) \quad \langle 1 \rangle = 1.$$

Next we have to consider $\langle S_{int}[q_{bg} + \phi] \rangle$. We can Taylor expand the potential around the initial point x^μ

$$(29) \quad \begin{aligned} S_{int}[q_{bg} + \phi] &= \beta \int_0^1 d\tau V(q_{bg} + \phi) = \\ &= \beta \int_0^1 d\tau \left(V(x) + [(y^\mu - x^\mu)\tau + \phi^\mu(\tau)] \partial_\mu V(x) + \right. \\ & \quad \left. + \frac{1}{2} [(y^\mu - x^\mu)\tau + \phi^\mu(\tau)] [(y^\nu - x^\nu)\tau + \phi^\nu(\tau)] \partial_\mu \partial_\nu V(x) + \dots \right) \end{aligned}$$

from which one obtains

$$(30) \quad \begin{aligned} \langle (-S_{int}[q_{bg} + \phi]) \rangle &= -\beta V(x) - \frac{\beta}{2} \xi^\mu \partial_\mu V(x) - \frac{\beta}{6} \xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) - \\ & \quad - \frac{\beta}{2} \partial_\mu \partial_\nu V(x) \int_0^1 d\tau \langle \phi^\mu(\tau) \phi^\nu(\tau) \rangle + \dots \end{aligned}$$

The last term is easily computed using (20) and (22)

$$(31) \quad \begin{aligned} \int_0^1 d\tau \langle \phi^\mu(\tau) \phi^\nu(\tau) \rangle &= \int_0^1 d\tau (-\beta \delta^{\mu\nu} \Delta(\tau, \tau)) = \\ &= -\beta \delta^{\mu\nu} \int_0^1 d\tau \tau(\tau - 1) = \frac{\beta}{6} \delta^{\mu\nu} \end{aligned}$$

so that

$$(32) \quad \begin{aligned} \langle (-S_{int}[q_{bg} + \phi]) \rangle &= -\beta V(x) - \frac{\beta}{2} \xi^\mu \partial_\mu V(x) - \\ & - \frac{\beta}{6} \xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) - \frac{\beta}{12} \nabla^2 V(x) + \dots \end{aligned}$$

Similarly, at lowest order one finds for the next term in (27)

$$(33) \quad \left\langle \frac{1}{2} S_{int}^2[q_{bg} + \phi] \right\rangle = \frac{\beta^2}{2} V^2(x) + \dots$$

Collecting all the terms, we find that at this order the heat kernel computed from the path integral is given by

$$(34) \quad \begin{aligned} \psi(x, y; \beta) &= \frac{1}{(2\pi\beta)^{D/2}} e^{-(x-y)^2/2\beta} \left[1 - \beta V(x) - \frac{\beta}{2} \xi^\mu \partial_\mu V(x) - \right. \\ & \left. - \frac{\beta}{6} \xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) - \frac{\beta^2}{12} \nabla^2 V(x) + \frac{\beta^2}{2} V^2(x) + \dots \right] \end{aligned}$$

from which one reads off the Seeley-DeWitt coefficients a_0, a_1 and a_2

$$(35) \quad \begin{aligned} a_0(x, y) &= 1 \\ a_1(x, y) &= -V(x) - \frac{1}{2} \xi^\mu \partial_\mu V(x) - \frac{1}{6} \xi^\mu \xi^\nu \partial_\mu \partial_\nu V(x) + \dots \\ a_2(x, y) &= \frac{1}{2} V^2(x) - \frac{1}{12} \nabla^2 V(x) + \dots \end{aligned}$$

In particular, their values at coinciding points $y^\mu = x^\mu$ (i.e. for $\xi^\mu = 0$) are given by

$$(36) \quad \begin{aligned} a_0(x, x) &= 1 \\ a_1(x, x) &= -V(x) \\ a_2(x, x) &= \frac{1}{2} V^2(x) - \frac{1}{12} \nabla^2 V(x) \end{aligned}$$

This calculation exemplifies the use of the path integral to compute the heat kernel.

The terms with quantum averages obtained in this perturbative calculation can be visualized as a set of Feynman diagrams: the propagator is the free two-point function which is drawn as a line joining the worldline points τ and σ

$$(37) \quad \langle \phi^\mu(\tau) \phi^\nu(\sigma) \rangle = -\beta \delta^{\mu\nu} \Delta(\tau, \sigma) = \text{---}$$

while the ‘‘vertices’’ are the factors obtained by expanding the interaction part of the action and indicate how the propagators are joined together. For example, the last term in eq. (30) can be graphically depicted as follow

$$(38) \quad -\frac{\beta}{2} \partial_\mu \partial_\nu V(x) \int_0^1 d\tau \langle \phi^\mu(\tau) \phi^\nu(\tau) \rangle = \text{---} \bullet \bigcirc$$

where the dot at the vertex denotes the vertex factor $-(\beta/2)\partial_\mu\partial_\nu V(x)$ together with the integration over the time location of the vertex.

To be self contained, we should perhaps address a bit more extensively how the averages in (20) (also named ‘‘correlation functions’’) are computed. Let us use a hypercondensed notation $\phi(\tau) \rightarrow \phi^i$, where the dependence on τ is indicated by the index i , and generalize the

Einstein summation convention to imply an integration for repeated indices, as for example in $\phi^i \chi_i = \int_0^1 d\tau \phi(\tau) \chi(\tau)$. Then the required formulas arise from the following gaussian integrals

$$(39) \quad \begin{aligned} Z &\equiv \int \frac{d^n \phi}{(2\pi)^{n/2}} e^{-\phi^i K_{ij} \phi^j / 2} = (\det K_{ij})^{-1/2} \\ Z[J] &\equiv \int \frac{d^n \phi}{(2\pi)^{n/2}} e^{-(\phi^i K_{ij} \phi^j / 2) + J_i \phi^i} = (\det K_{ij})^{-1/2} e^{J_i G^{ij} J_j / 2} \end{aligned}$$

where G^{ij} is the inverse of K_{ij} (i.e. $K_{ij} G^{jk} = \delta_i^k$). It corresponds to a Green function when K_{ij} is a differential operator (e.g. $K_{ij} \sim \partial_\tau \delta(\tau - \sigma)$ with the index i denoting the variable τ , and j the variable σ). The second integral in (39) is obtained by ‘‘completing the square’’ and shifting integration variables. The required averages follows from differentiating $Z[J]$

$$(40) \quad \begin{aligned} \langle \phi^{i_1} \phi^{i_2} \dots \phi^{i_n} \rangle &= \frac{1}{Z} \int D\phi \phi^{i_1} \phi^{i_2} \dots \phi^{i_n} e^{-\phi^i K_{ij} \phi^j / 2} = \\ &= \frac{1}{Z} \left. \frac{\delta^n Z[J]}{\delta J_{i_1} \delta J_{i_2} \dots \delta J_{i_n}} \right|_{J=0} = \\ &= \left. \frac{\delta^n e^{J_i G^{ij} J_j / 2}}{\delta J_{i_1} \delta J_{i_2} \dots \delta J_{i_n}} \right|_{J=0} \end{aligned}$$

obtaining in particular

$$(41) \quad \begin{aligned} \langle 1 \rangle &= 1 \\ \langle \phi^i \rangle &= 0 \\ \langle \phi^i \phi^j \rangle &= G^{ij} \\ \langle \phi^i \phi^j \phi^k \rangle &= 0 \\ \langle \phi^i \phi^j \phi^k \phi^l \rangle &= G^{ij} G^{kl} + G^{ik} G^{jl} + G^{il} G^{jk} \end{aligned}$$

and so on. In particular correlation functions of an odd number of fields vanish, while those with an even number of fields are given by sums of products of two point functions (a fact known as the ‘‘Wick theorem’’). The two-point function is also known as the ‘‘Feynman propagator’’.

3. CURVED SPACE

We now wish to discuss how to construct and compute path integrals for a nonrelativistic particle moving in a curved space. The heat equation is again the same as in eq. (2) with hamiltonian

$$(42) \quad \hat{H} = -\frac{1}{2} \nabla^2 + V(x)$$

but now we take ∇^2 as the scalar laplacian on an arbitrary curved manifold with metric $g_{\mu\nu}$.

In the operatorial approach, the quantum hamiltonian \hat{H} is obtained by quantizing the classical hamiltonian

$$(43) \quad H_{cl} = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu + V(x)$$

where p_μ are momenta conjugate to the coordinates x^μ . Reinterpreting the phase space coordinates (x^μ, p_μ) as operators $(\hat{x}^\mu, \hat{p}_\mu)$ with commutation rules

$$(44) \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu$$

produces the well-known ordering ambiguities in the identification of \hat{H} from H_{cl} . In general this has to be expected since the classical limit is irreversible and one may lose information in the passage from the quantum to the classical world (for example, all \hbar dependent terms in the potential vanish in the classical limit). Thus the reconstruction of a quantum theory from its classical limit may very well be ambiguous. The ordering ambiguities signal that several quantum theories can be constructed from a classical one. One usually requires invariance under the diffeomorphisms of the manifold, and this requirement fixes the quantum hamiltonian up to a coupling to the scalar curvature R with an unfixed parameter α

$$(45) \quad \hat{H} = -\frac{1}{2} \nabla^2 + V(x) + \alpha R(x) .$$

The fixing of a “renormalization condition” in this context essentially means fixing which value of α one chooses for the quantum theory. In the absence of other requirements, one may fix $\alpha = 0$ as “renormalization conditions” (if needed, one may always introduce an additional coupling to R by redefining the potential V to contain it).

In the path integral approach similar ambiguities appear. They take the form of ambiguous Feynman diagrams. As we have seen, Feynman diagrams correspond to integrations of various propagators joined at vertices with some factors and/or derivatives. The propagators correspond in general to distributions which cannot always be multiplied legally together: they produce ambiguities precisely when the multiplication of distributions is not well-defined. These ambiguities must then be defined by a regularization procedure with an associated renormalization chosen to satisfy the renormalization conditions. The latter specify which quantum theory one is constructing from the classical one. One can take the same renormalization conditions just discussed: one requires that the path integral produces a solution to the heat equation with a quantum hamiltonian without any coupling to the scalar curvature, i.e. with $\alpha = 0$. For the one dimensional nonlinear sigma model considered here, the “counterterms” needed to satisfy the renormalization conditions are finite. The counterterms associated with a regularization scheme consist of local terms that are added to the action (while nonlocal terms are not admissible), and perturbatively make sure that the vertices derived from them, summed with the regulated (and previously ambiguous) Feynman diagrams they correspond to, give rise to a final result that is independent of the regularization chosen. The counterterms are completely saturated at two loops, in fact our nonlinear sigma model is “super-renormalizable”. This quick description of regularization schemes will be readily exemplified with mode regularization in the next section.

The curved space generalization of the previous set up now contains the action

$$(46) \quad S[q] = \frac{1}{\beta} \int_0^1 d\tau \left(\frac{1}{2} g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu + \beta^2 V(q) \right)$$

which generalizes eq. (10) to a curved manifold. The heat kernel is then generated by the following path integral

$$(47) \quad \psi(x, y; \beta) = \int_{q(0)=x}^{q(\beta)=y} \mathcal{D}q e^{-S[q]}$$

where the formally covariant measure is given by

$$(48) \quad \begin{aligned} \mathcal{D}q &= \prod_{0 < \tau < 1} \sqrt{\det g_{\mu\nu}(q(\tau))} Dq = \\ &= \prod_{0 < \tau < 1} \sqrt{\det g_{\mu\nu}(q(\tau))} d^D q(\tau) . \end{aligned}$$

This covariant measure is not translationally invariant, i.e. not invariant under the shift $q^\mu(\tau) \rightarrow q^\mu(\tau) + e^\mu(\tau)$. This fact makes it difficult to generate the perturbative expansion: one cannot complete squares and shift integration variables to derive the propagators as

usual. A standard trick to obtain a translationally invariant measure is to introduce ghost fields and exponentiate the nontrivial factor appearing in (48)

$$(49) \quad \prod_{0 < \tau < 1} \sqrt{\det g_{\mu\nu}(q(\tau))} = \int DaDbDc e^{-S_{gh}[q,a,b,c]}$$

where

$$(50) \quad S_{gh}[q, a, b, c] = \frac{1}{\beta} \int_0^1 d\tau \frac{1}{2} g_{\mu\nu}(q)(a^\mu a^\nu + b^\mu c^\nu)$$

and with the translationally invariant measures for the ghosts variables given by

$$(51) \quad Da = \prod_{0 < \tau < 1} d^D a(\tau) , \quad Db = \prod_{0 < \tau < 1} d^D b(\tau) , \quad Dc = \prod_{0 < \tau < 1} d^D c(\tau) .$$

The ghosts a^μ are commuting variables while the ghosts b^μ and c^μ are anticommuting variables (i.e. Grassmann variables), so they reproduce the correct measure factor. Finally, to satisfy the renormalization conditions (i.e. $\alpha = 0$ in the quantum hamiltonian related to the heat kernel) one must add a counterterm V_{CT} to the action by shifting $V \rightarrow V + V_{CT}$. Once a regularization has been chosen, one can fix V_{CT} by a two-loop computation on the worldline. Higher loops will not modify the counterterm as the one-dimensional nonlinear sigma model is super-renormalizable.

Thus we have reached the following final expression for the path integral representation of the heat kernel in curved space

$$(52) \quad \psi(x, y; \beta) = \int_{q(0)=x}^{q(1)=y} DqDaDbDc e^{-S[q,a,b,c]}$$

$$S[q, a, b, c] = \frac{1}{\beta} \int_0^1 d\tau \left(\frac{1}{2} g_{\mu\nu}(q)(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu) + \beta^2 V(q) + \beta^2 V_{CT}(q) \right)$$

where the ghosts have vanishing boundary conditions ($a^\mu(0) = a^\mu(1) = 0$ etc.) since the factor $\sqrt{\det g_{\mu\nu}(q(\tau))}$ is not required for $\tau = 0, 1$ in (49). The full action has now been redefined to contain the ghosts and the counterterm $V_{CT}(q)$.

3.1. Perturbative expansion in mode regularization. A simple set up for computing the perturbative expansion makes use of mode regularization. This regularization has been already introduced in section 2 and is immediately extended to curved space. The coordinates q^μ are expanded as in section 2, splitting them into a classical part (which we take as a linear function in τ , just as in section 2: this is not a solution of the classical equations, nevertheless it takes into account the correct boundary conditions) and a quantum part. The latter is expanded into a sine series as in eq. (13). The ghost fields have vanishing boundary conditions and are expanded directly into sine series

$$(53) \quad \begin{aligned} a^\mu(\tau) &= \sum_{m=1}^{\infty} a_m^\mu \sin(\pi m \tau) \\ b^\mu(\tau) &= \sum_{m=1}^{\infty} b_m^\mu \sin(\pi m \tau) \\ c^\mu(\tau) &= \sum_{m=1}^{\infty} c_m^\mu \sin(\pi m \tau) . \end{aligned}$$

The path integral measure is defined concretely as the integration over all Fourier coefficients

$$(54) \quad DqDaDbDc = \lim_{M \rightarrow \infty} A \prod_{m=1}^M \prod_{\mu=1}^D m d\phi_m^\mu da_m^\mu db_m^\mu dc_m^\mu$$

and it is quite simple to see that this measure reproduces the free solution of section 2 after setting $g_{\mu\nu}(q) = \delta_{\mu\nu}$ and $V(q) = V_{CT}(q) = 0$.

For general metrics $g_{\mu\nu}$ and potentials V the way to implement mode regularization is simple: one restricts the integration on the Fourier coefficients for each field $(\phi^\mu, a^\mu, b^\mu, c^\mu)$ up to a finite mode number M , and computes all quantities of interest at finite M . This necessarily gives a finite and unambiguous result. Then one sends $M \rightarrow \infty$ to reach the continuum limit. This regularization is enough to resolve all ambiguities in the product of distributions that emerge in the continuum limit. One can show that no divergence arise in this limit thanks to the extra vertices contained in the path integral measure (this is most easily seen perturbatively by combining graphs, some with ghosts, that separately would be diverging). The correct counterterm that is required by this regularization is denoted by V_{MR} and is given by

$$(55) \quad V_{MR} = -\frac{1}{8} R - \frac{1}{24} g^{\mu\nu} g^{\alpha\beta} g_{\gamma\delta} \Gamma_{\mu\alpha}^\gamma \Gamma_{\nu\beta}^\delta .$$

This counterterm is necessary to satisfy the renormalization condition that we have chosen, namely the requirement that the path integral generates a solution with $\alpha = 0$ in the quantum hamiltonian. In particular, the noncovariant piece in the counterterm is necessary to restore covariance (which is broken at the regulated level), so that the complete final result is guaranteed to be covariant. This regularization is analogous to the standard momentum cut-off used in quantum field theories. We are not going to prove these statements here, but will verify that the first few terms in the perturbative expansion generate the correct solution.

For perturbative calculation one may Taylor expand the metric around a fixed point, which we choose to be the initial point x^μ (but any other choice is as good), so that the leading term in the Taylor expansion of the metric inserted in the action produces the various propagators

$$(56) \quad \begin{aligned} \langle \phi^\mu(\tau) \phi^\nu(\sigma) \rangle &= -\beta g^{\mu\nu}(x) \Delta(\tau, \sigma) \\ \langle a^\mu(\tau) a^\nu(\sigma) \rangle &= \beta g^{\mu\nu}(x) \Delta_{gh}(\tau, \sigma) \\ \langle b^\mu(\tau) c^\nu(\sigma) \rangle &= -2\beta g^{\mu\nu}(x) \Delta_{gh}(\tau, \sigma) \end{aligned}$$

where Δ and Δ_{gh} are distributions regulated by mode cut-off

$$(57) \quad \Delta(\tau, \sigma) = \sum_{m=1}^M \left[-\frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) \right]$$

$$(58) \quad \Delta_{gh}(\tau, \sigma) = \sum_{m=1}^M 2 \sin(\pi m \tau) \sin(\pi m \sigma) .$$

Note that at the regulated level (M big but finite) one has the relation $\Delta_{gh}(\tau, \sigma) = \bullet\bullet\Delta(\tau, \sigma) = \Delta\bullet\bullet(\tau, \sigma)$, where left and right dots indicate derivatives with respect to left and right variables. These functions have the following limiting value for $M \rightarrow \infty$

$$(59) \quad \Delta(\tau, \sigma) \rightarrow (\tau - 1)\sigma\theta(\tau - \sigma) + (\sigma - 1)\tau\theta(\tau - \sigma)$$

$$(60) \quad \Delta_{gh}(\tau, \sigma) \rightarrow \delta(\tau - \sigma)$$

where $\theta(x)$ is the usual step function, $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1/2$ for $x = 0$, and $\delta(x)$ is the well-known distribution called ‘‘Dirac’s delta function’’.

Using Riemann normal coordinates z^μ centered at the initial point x^μ (which then has normal coordinates $z^\mu = 0$) allows one to obtain the following Taylor expansion of the metric around the origin

$$(61) \quad g_{\mu\nu}(z) = g_{\mu\nu}(0) + \frac{1}{3} R_{\mu\alpha\beta\nu}(0) z^\alpha z^\beta + \dots .$$

Then a perturbative calculation at two loops in β produces the following result for the heat kernel

$$(62) \quad \psi(x, y; \beta) = \frac{1}{(2\pi\beta)^{D/2}} e^{-g_{\alpha\beta}\xi^\alpha\xi^\beta/2\beta} \left[1 - \frac{1}{12} \xi^\mu\xi^\nu R_{\mu\nu} - \beta \left(\frac{1}{12} R + V \right) - \frac{1}{24} \xi^\mu\xi^\nu\xi^\lambda \nabla_\lambda R_{\mu\nu} - \frac{1}{2} \beta \xi^\mu \nabla_\mu \left(\frac{1}{12} R + V \right) + O(\beta^2) \right]$$

where $\xi^\mu = y^\mu - x^\mu$, and all tensors like $g_{\mu\nu}$ and $R_{\mu\nu}$ are evaluated at the initial point x^μ . In this result we have considered ξ^μ of order $\sqrt{\beta}$, as suggested by the gaussian multiplying the terms in the square brackets, and thus we have consistently kept only terms of order less than β^2 . The Feynman diagrams that have to be computed to obtain this result are quite simple, they are only slightly more complicated than those described in section 2.1, and are left as an exercise for the reader. For the ghost diagrams one has to remember to take into account the sign arising from their Grassmann character, and then one may check that all potential divergences cancel by combining certain diagrams with corresponding ghost diagrams.

Mode regularization for path integrals in curved space was constructed and used in [1, 2, 3, 4] and described more extensively in the book [5]. In that book one may also find an extensive description of other regularization schemes, namely time slicing [6] (see also [7, 8]) and dimensional regularization [9] (see also [10, 11, 12]). The latter, though only of perturbative nature, carries a covariant counterterm $V_{DR} = -R/8$.

4. RELATIVISTIC THEORIES AND N -EXTENDED SPINNING PARTICLES

The path integrals discussed in the previous sections can be used to study relativistic particles associated with fluctuations of quantum fields. The simplest example is given by the scalar particle associated with a real Klein-Gordon field ϕ . It is convenient to discuss this specific example in some detail to get a flavor of the general situation that arises for particles with higher spin.

Therefore let us consider a real scalar field ϕ coupled to the metric $g_{\mu\nu}$. The euclidean quantum field theoretical action reads

$$(63) \quad S_{QFT}[\phi; g] = \int d^D x \sqrt{g} \frac{1}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \xi R \phi^2)$$

where m is the mass of the scalar particle, $g_{\mu\nu}$ is the background metric, and ξ is a possible nonminimal coupling to the scalar curvature R . The euclidean one-loop effective action $\Gamma[g]$ describes all possible one-loop graphs with the scalar field in the loop and any number of gravitons on the external legs. It can be depicted by the collection of Feynman diagrams, as the one drawn in Fig. 1, where external lines represent fluctuations of the metric which is not quantized.

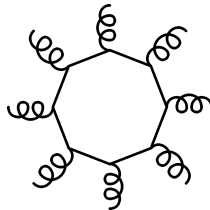


FIGURE 1. One-loop effective action induced by a scalar field.

It can be obtained by path integrating the QFT action $S_{QFT}[\phi; g]$ over ϕ , and is formally given by

$$(64) \quad e^{-\Gamma[g]} \equiv \int \mathcal{D}\phi e^{-S_{QFT}[\phi; g]} = \text{Det}^{-1/2}(-\nabla^2 + m^2 + \xi R)$$

so that

$$(65) \quad \begin{aligned} \Gamma[g] &= -\log \text{Det}^{-1/2}(-\nabla^2 + m^2 + \xi R) = \\ &= \frac{1}{2} \text{Tr} \log(-\nabla^2 + m^2 + \xi R) = \\ &= -\frac{1}{2} \int_0^\infty \frac{dT}{T} \text{Tr} e^{-T(-\nabla^2 + m^2 + \xi R)} = \\ &= -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int_{T^1} \mathcal{D}x e^{-S[x; g]} \end{aligned}$$

where in the last path integral the action that has to be used is given by²

$$(66) \quad S[x; g] = \int_0^T d\tau \left(\frac{1}{4} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + m^2 + \xi R(x) \right)$$

In the above equalities $\nabla^2 = g^{\mu\nu} \nabla_\mu \partial_\nu$ is the covariant laplacian acting on scalars. In the third line of (65) we have used the proper time representation of the logarithm ($\log(a/b) = -\int_0^\infty (dT/T)(e^{-aT} - e^{-bT})$) and dropped an additive (infinite!) constant. This provides the starting point of the heat kernel method, originally due to Schwinger, where the operator $\hat{H} = -\nabla^2 + m^2 + \xi R$ is reinterpreted as the quantum hamiltonian of a “fictitious” quantum mechanical model: that of a nonrelativistic particle in curved space with a specific coupling to the scalar curvature. This approach to the heat kernel in curved space is described extensively in [13, 14]. However, it is quite useful to reformulate this quantum mechanics using a path integral approach: this is indicated by the last equality in (65). The exponent of the path integral contains the classical action of the mechanical model whose quantization is expected to produce the quantum hamiltonian \hat{H} . The operatorial trace is obtained by using periodic boundary conditions on the worldline time $\tau \in [0, T]$, which therefore becomes a one-dimensional torus, or circle, T^1 . Redefining the proper time by $T = \beta/2$, and rescaling the quantum mechanical time as discussed in the text before eq. (10), makes contact with the normalizations used in the previous section.

It is now clear that to use this final quantum mechanical path integral formulation one has to be able to define and compute path integrals for particles moving in curved spaces. This is what we have been reviewing in section 3, and one can immediately use that setup for the present purposes.

For studying other QFT models it is useful to note that the effective action $\Gamma[g]$ induced by the scalar field ϕ can also be obtained by first-quantizing a scalar point particle with coordinates x^μ and auxiliary einbein e , described by the (euclidean) action

$$(67) \quad S[x, e; g] = \int_0^1 d\tau \frac{1}{2} [e^{-1} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e(m^2 + \xi R(x))].$$

This worldline action is invariant under reparametrizations of the worldline, with infinitesimal transformation rules given by

$$(68) \quad \begin{aligned} \delta x^\mu &= \xi \dot{x}^\mu \\ \delta e &= \xi \dot{e} + \dot{\xi} e. \end{aligned}$$

²Ghosts and counterterms described in the previous section are now conventionally hidden in the symbol $\mathcal{D}x$ which indicates the covariant measure.

In the quantization one has to take care of the volume of the gauge group by a suitable gauge fixing procedure

$$(69) \quad \Gamma[g] = \int_{T^1} \frac{\mathcal{D}e\mathcal{D}x}{\text{Vol}(\text{Gauge})} e^{-S[x,e;g]}$$

formally denoted here by the division with the (infinite) volume of the gauge group $\text{Vol}(\text{Gauge})$. One can eliminate almost completely the einbein by the gauge condition $e(\tau) = \beta$, and integrate over the remaining modular parameter β after taking into account the correct measure using the Faddeev-Popov gauge fixing procedure. The volume of the gauge group cancels out, and this reproduces the previous answer in (65) (with the rescaling $T = \beta/2$ mentioned above). Thus the ‘‘fictitious’’ quantum mechanics mentioned above is not at all fictitious: it corresponds to the first quantization of the relativistic scalar particle which makes the loop in the Feynman graph of Fig. 1.

This picture can be enlarged to include particles with spin. Extending the model with reparametrization invariance on the worldline to a model with $N = 1$ supergravity on the worldline gives a description of a spin 1/2 particle, while a model with $N = 2$ supergravity describes particles associated to vector and antisymmetric tensor fields [15, 16, 17, 18, 19]. Gauge fixed versions of these particle models were in fact used to compute gravitational and chiral anomalies in one of the most beautiful applications of the worldline approach [20, 21, 22]. Other applications of this worldline approach with external gravity include the computation of trace anomalies [1, 2, 12, 23], as well as the calculation of some amplitudes, like the one-loop correction to the graviton propagator due to loops of spin 0, 1/2, 1 and antisymmetric tensor fields [24, 25, 26, 27].

The case of spin 1/2 particles requires, on top of the bosonic path integral described above, the use of Majorana fermions on the worldline and their corresponding path integration. These worldline fermions create the quantum degrees of freedom needed to describe the spin. In section 5 we will describe directly the case of spin 1 particles, and more generally of antisymmetric tensor fields, which requires instead the use of Dirac fermions on the worldline. However, before doing that, we review the $O(N)$ extended spinning particles in flat target space, characterized by an $O(N)$ extended local supersymmetry (i.e. supergravity) on the worldline.

4.1. Spinning particles. The $O(N)$ spinning particle action is characterized by an $O(N)$ extended supergravity on the worldline. The gauge fields of the $O(N)$ supergravity contain: the einbein e which gauges worldline translations, N gravitinos χ_i with $i = 1, \dots, N$ which gauge N worldline supersymmetries, and a gauge field a_{ij} for gauging the $O(N)$ symmetry which rotates the worldline fermions and gravitinos. The einbein and the gravitinos correspond to constraints that eliminate negative norm states and make the particle model consistent with unitarity. The constraints arising from the gauge field a_{ij} makes the model irreducible, eliminating some further degrees of freedom [17, 18, 19].

The action in flat target space is most easily deduced by starting with a model with $O(N)$ extended rigid supersymmetry on the worldline, and then gauging these symmetries. The rigid model is given in phase space by the action³

$$(70) \quad S = \int dt \left[p_\mu \dot{x}^\mu + \frac{i}{2} \eta_{\mu\nu} \psi_i^\nu \dot{\psi}_i^\mu - \frac{1}{2} \eta_{\mu\nu} p^\mu p^\nu \right]$$

which implies that the graded Poisson brackets in phase space are given by $\{x^\mu, p_\nu\}_{PB} = \delta_\nu^\mu$ and $\{\psi_i^\mu, \psi_j^\nu\}_{PB} = -i\eta^{\mu\nu} \delta_{ij}$. The indices $\mu, \nu = 0, 1, \dots, D-1$ are flat spacetime indices and $\eta_{\mu\nu} \sim (-, +, +, \dots, +)$ is the Lorentz metric which is used to raise and lower spacetime indices. The indices $i, j = 1, \dots, N$ are internal indices labeling the various worldline fermion species. The action is Poincaré invariant in target space and thus describes a relativistic

³In this section we use a minkowskian time t , and only later we will perform a Wick rotation.

model. On the worldline it has rigid $O(N)$ supersymmetry generated by the charges

$$(71) \quad \begin{aligned} H &= \frac{1}{2} p_\mu p^\mu \\ Q_i &= p_\mu \psi_i^\mu \\ J_{ij} &= i\psi_i^\mu \psi_{j\mu} . \end{aligned}$$

This whole symmetry algebra can be gauged since the charges H, Q_i, J_{ij} close under Poisson brackets and they identify a set of first class constraints that can be imposed on the model

$$(72) \quad \begin{aligned} \{Q_i, Q_j\}_{PB} &= -2i\delta_{ij}H \\ \{J_{ij}, Q_k\}_{PB} &= \delta_{jk}Q_i - \delta_{ik}Q_j \\ \{J_{ij}, J_{kl}\}_{PB} &= \delta_{jk}J_{il} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk} . \end{aligned}$$

Introducing the corresponding gauge fields e, χ_i, a_{ij} produces the action for the $O(N)$ spinning particle

$$(73) \quad \begin{aligned} S &= \int dt \left[p_\mu \dot{x}^\mu + \frac{i}{2} \psi_{i\mu} \dot{\psi}_i^\mu - eH - i\chi_i Q_i - \frac{1}{2} a_{ij} J_{ij} \right] = \\ &= \int dt \left[p_\mu \dot{x}^\mu + \frac{i}{2} \psi_{i\mu} \dot{\psi}_i^\mu - \frac{1}{2} e p_\mu p^\mu - i\chi_i p_\mu \psi_i^\mu - \frac{i}{2} a_{ij} \psi_i^\mu \psi_{j\mu} \right] . \end{aligned}$$

The corresponding gauge transformations are those of the $O(N)$ extended supergravity on the worldline and are easily identified. They are given by

$$(74) \quad \begin{aligned} \delta x^\mu &= \{x^\mu, \xi H + i\epsilon_i Q_i + \frac{1}{2} \alpha_{ij} J_{ij}\}_{PB} = \xi p^\mu + i\epsilon_i \psi_i^\mu \\ \delta p_\mu &= \{p_\mu, \xi H + i\epsilon_i Q_i + \frac{1}{2} \alpha_{ij} J_{ij}\}_{PB} = 0 \\ \delta \psi_i^\mu &= \{\psi_i^\mu, \xi H + i\epsilon_j Q_j + \frac{1}{2} \alpha_{jk} J_{jk}\}_{PB} = -\epsilon_i p^\mu + \alpha_{ij} \psi_j^\mu \end{aligned}$$

and

$$(75) \quad \begin{aligned} \delta e &= \dot{\xi} + 2i\chi_i \epsilon_i \\ \delta \chi_i &= \dot{\epsilon}_i - a_{ij} \epsilon_j + \alpha_{ij} \chi_j \\ \delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im} a_{mj} + \alpha_{jm} a_{im} . \end{aligned}$$

Eliminating algebraically the momenta p_μ by using their equation of motion

$$(76) \quad p^\mu = \frac{1}{e} (\dot{x}^\mu - i\chi_i \psi_i^\mu)$$

produces the action in configuration space

$$(77) \quad S = \int dt \left[\frac{1}{2} e^{-1} (\dot{x}^\mu - i\chi_i \psi_i^\mu)^2 + \frac{i}{2} \psi_{i\mu} (\delta_{ij} \partial_t - a_{ij}) \psi_j^\mu \right] .$$

The corresponding gauge invariances can be deduced from the phase space one by using (76).

In the specific case of $N = 2$ one can set $a_{ij} \equiv a\epsilon_{ij}$, where ϵ_{ij} is the two-dimensional antisymmetric tensor ($\epsilon_{12} = -\epsilon_{21} = 1$), and add a Chern-Simon (CS) term for the gauge field of the form

$$(78) \quad S_{CS} = q \int dt a$$

which is obviously invariant under gauge transformations ($\delta a = \dot{\alpha}$ which follows from (75) after setting $\alpha_{ij} \equiv \alpha \epsilon_{ij}$). The $N = 2$ spinning particle with quantized CS coupling $q \equiv (D/2) - p - 1$ and p an integer describes an antisymmetric gauge field of rank p (and corresponding field strength of rank $p + 1$). For $N = 2$ it may be more convenient to use a $U(1)$ basis rather the $O(2)$ one. Defining

$$(79) \quad \begin{aligned} \psi^\mu &= \frac{1}{\sqrt{2}} (\psi_1^\mu + i\psi_2^\mu) & \chi &= \frac{1}{\sqrt{2}} (\chi_1 + i\chi_2) \\ \bar{\psi}^\mu &= \frac{1}{\sqrt{2}} (\psi_1^\mu - i\psi_2^\mu) & \bar{\chi} &= \frac{1}{\sqrt{2}} (\chi_1 - i\chi_2) \end{aligned}$$

the complete action for the $N = 2$ spinning particle reads

$$(80) \quad S = \int dt \left[\frac{1}{2} e^{-1} (\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu)^2 + i\bar{\psi}_\mu (\partial_t + ia)\psi^\mu + qa \right].$$

We have now described the case of massless spinning particles. A mass term can also be introduced. It is obtained starting from the massless case in one dimension higher, and then eliminating one dimension (for example x^5) by setting $p_5 = m$. The coordinate x^5 can be dropped from the action (it appears only as a total derivative), while the corresponding fermionic partners (which are present for $N > 0$) are retained. The emerging model describes a massive spinning particle.

4.2. Canonical quantization. A brief look at canonical quantization helps to identify which is the quantum field theoretical model that the $O(N)$ extended spinning particle is associated to.

The case of the $N = 0$ particle is the simplest. The phase space action for a massive $N = 0$ particle with flat target space is given by

$$(81) \quad S = \int dt \left[p_\mu \dot{x}^\mu - \frac{1}{2} e (p_\mu p^\mu + m^2) \right].$$

The einbein e imposes the constraint $H = 1/2(p_\mu p^\mu + m^2) = 0$ on the phase space with coordinates (x^μ, p_μ) . Upon canonical quantization the phase space variables are turned into operators satisfying the commutation relations (we use $\hbar = 1$)

$$(82) \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu.$$

States $|\phi\rangle$ of the full Hilbert space can be described by functions of the coordinates x^μ (which are the possible eigenvalues of the operator \hat{x}^μ)

$$(83) \quad \phi(x) \equiv \langle x | \phi \rangle.$$

The classical constraint H now become an operator \hat{H} which is used to select the physical states $|\phi_{phys}\rangle$ of the full linear space of states through the requirement $\hat{H}|\phi_{phys}\rangle = 0$. In the above representation it takes the form of a differential operator

$$(84) \quad \hat{H} = \frac{1}{2} (-\partial_\mu \partial^\mu + m^2)$$

which then reproduces the Klein-Gordon equation

$$(85) \quad \hat{H}|\phi_{phys}\rangle = 0 \Rightarrow (-\partial_\mu \partial^\mu + m^2)\phi_{phys}(x) = 0.$$

Thus the $N = 0$ spinning particle describes the propagation of a relativistic scalar particle associated to the Klein-Gordon field.

Similarly, one can treat the $N = 1$ spinning particle. The action for the massless case and with flat target space is given by

$$(86) \quad S = \int dt \left[p_\mu \dot{x}^\mu + \frac{i}{2} \psi_\mu \dot{\psi}^\mu - \frac{1}{2} e p_\mu p^\mu - i\chi p_\mu \psi^\mu \right].$$

The phase space with coordinates (x^μ, p_μ) and ψ^μ is graded, and the corresponding Poisson brackets are given by $\{x^\mu, p_\nu\}_{PB} = \delta_\nu^\mu$ and $\{\psi^\mu, \psi^\nu\}_{PB} = -i\eta^{\mu\nu}$. The gauge fields (e, χ) implement now the constraints

$$(87) \quad H = \frac{1}{2} p_\mu p^\mu = 0 \quad , \quad Q = p_\mu \psi^\mu = 0 .$$

Upon canonical quantization the phase space variables are turned into operators satisfying the (anti)-commutation relations

$$(88) \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu \quad , \quad \{\hat{\psi}^\mu, \hat{\psi}^\nu\} = \eta^{\mu\nu} .$$

As the anticommutation relations for the operator $\hat{\psi}^\mu$ produce a Clifford algebra, the $\hat{\psi}^\mu$ can be represented by the Dirac matrices γ^μ

$$(89) \quad \hat{\psi}^\mu = \frac{1}{\sqrt{2}} (\gamma^\mu)_\alpha^\beta$$

where we have explicitly shown the spinorial indices α and β on the Dirac matrices (more precisely, one identifies the matrix elements of $\hat{\psi}^\mu$ with the entries of the gamma matrices $\langle \alpha | \hat{\psi}^\mu | \beta \rangle = (1/\sqrt{2})(\gamma^\mu)_\alpha^\beta$, where the Hilbert space spanned by the states $|\alpha\rangle$ with $\alpha = 1, \dots, 2^{\lfloor D/2 \rfloor}$ is of course finite dimensional). Thus the wave function on which the operators $(\hat{x}^\mu, \hat{p}_\nu, \hat{\psi}^\mu)$ act is a spinorial wave function $\Psi_\alpha(x) = (\langle x | \otimes \langle \alpha |) | \Psi \rangle$. The classical constraints (H, Q) become differential operators

$$(90) \quad \hat{H} = -\frac{1}{2} \partial_\mu \partial^\mu \quad , \quad \hat{Q} = -\frac{i}{\sqrt{2}} (\gamma^\mu)_\alpha^\beta \partial_\mu .$$

which select the physical states by

$$(91) \quad \begin{aligned} \hat{H}|\Psi\rangle = 0 & \quad \Rightarrow \quad \partial_\mu \partial^\mu \Psi_\alpha(x) = 0 \\ \hat{Q}|\Psi\rangle = 0 & \quad \Rightarrow \quad (\gamma^\mu)_\alpha^\beta \partial_\mu \Psi_\beta(x) = 0 \end{aligned}$$

and we recognize the massless Dirac equation. The conclusion is that the $N = 1$ spinning particle describes the propagation of a particle associated to the Dirac field.

Finally, we discuss the $N = 2$ case. The action with the CS coupling can be written in the $U(1)$ basis as

$$(92) \quad S = \int dt \left[p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu - eH - i\bar{\chi}Q - i\chi\bar{Q} - a(J - q) \right]$$

where the charges $C = (H, Q, \bar{Q}, J - q)$ are given by

$$(93) \quad H = \frac{1}{2} p_\mu p^\mu \quad , \quad Q = p_\mu \psi^\mu \quad , \quad \bar{Q} = p_\mu \bar{\psi}^\mu \quad , \quad J = \bar{\psi}^\mu \psi_\mu .$$

The gauge fields $G = (e, \bar{\chi}, \chi, a)$ set these charges to zero. Canonical quantization is constructed turning the phase space variables into operators satisfying the following (anti) commutation relations

$$(94) \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu \quad , \quad \{\hat{\psi}^\mu, \hat{\psi}_\nu^\dagger\} = \delta_\nu^\mu .$$

States of the full Hilbert space can be described by functions of the ‘‘coordinates’’ x^μ and ψ^μ . By x^μ we denote the eigenvalues of the operator \hat{x}^μ , while for the fermionic variables we use bra coherent states defined by

$$(95) \quad \langle \psi | \hat{\psi}^\mu = \langle \psi | \psi^\mu = \psi^\mu \langle \psi | .$$

Any state $|\phi\rangle$ can then be described by the wave function

$$(96) \quad \phi(x, \psi) \equiv (\langle x | \otimes \langle \psi |) | \phi \rangle$$

and since ψ^μ are Grassmann variables the wave function has the following general expansion

$$(97) \quad \begin{aligned} \phi(x, \psi) &= F(x) + F_\mu(x)\psi^\mu + \frac{1}{2} F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} + \dots \\ &\dots + \frac{1}{D!} F_{\mu_1\dots\mu_D}(x)\psi^{\mu_1}\dots\psi^{\mu_D} . \end{aligned}$$

The classical constraints C now become operators \hat{C} which are used to select the physical states through the requirement $\hat{C}|\phi_{phys}\rangle = 0$. In the above representation they take the form of differential operators

$$(98) \quad \hat{H} = -\frac{1}{2} \partial_\mu \partial^\mu , \quad \hat{Q} = -i\psi^\mu \partial_\mu , \quad \hat{Q}^\dagger = -i\partial_\mu \frac{\partial}{\partial \psi_\mu} , \quad \hat{J} = -\frac{1}{2} \left[\psi^\mu , \frac{\partial}{\partial \psi^\mu} \right]$$

where we have antisymmetrized $\hat{\psi}^\mu$ and $\hat{\psi}_\mu^\dagger$ to resolve an ordering ambiguity. The constraint $(\hat{J} - q)|\phi_{phys}\rangle = 0$ selects states with only $p+1$ ψ 's (recall that $q \equiv (D/2) - p - 1$), namely

$$(99) \quad \phi_{phys}(x, \psi) = \frac{1}{(p+1)!} F_{\mu_1\dots\mu_{p+1}}(x)\psi^{\mu_1}\dots\psi^{\mu_{p+1}} .$$

The constraint $\hat{Q}|\phi_{phys}\rangle = 0$ gives the ‘‘Bianchi identities’’

$$(100) \quad \partial_{[\mu} F_{\mu_1\dots\mu_{p+1}]}(x) = 0$$

where the square brackets around indices denote complete antisymmetrization, and the constraint $\hat{Q}^\dagger|\phi_{phys}\rangle = 0$ produces the Maxwell equations

$$(101) \quad \partial^{\mu_1} F_{\mu_1\dots\mu_{p+1}}(x) = 0 .$$

The constraint $\hat{H}|\phi_{phys}\rangle = 0$ is automatically satisfied as a consequence of the algebra $\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H}$. Thus we see that the $N = 2$ spinning particle describes the propagation of the particle associated with a standard p -form gauge potential $A_{\mu_1\dots\mu_p}$ which satisfies the Maxwell equations: $F_{p+1} = dA_p$ and $d^\dagger F_{p+1} = 0$. The quantization of the CS coupling is easily understood in this canonical framework: other values admit no solution to the $\hat{J} - q$ constraint and the model becomes empty. The corresponding field theory action which gives rise to these equations is the generalization of the standard Maxwell action

$$(102) \quad S_p^{QFT}[A_p] = \int d^D x \left[-\frac{1}{2(p+1)!} F_{\mu_1\dots\mu_{p+1}}^2 \right]$$

where $F_{p+1} = dA_p$ is the field strength.

For the $N = 1$ and $N = 2$ models we have discussed the massless case. The massive case, which can be obtained by starting from the massless case in one dimension higher, proceeds along similar lines. In particular, in the $N = 2$ case one obtains the particle associated with the antisymmetric tensor field with a Proca-like action

$$(103) \quad S_p^{QFT}[A_p] = \int d^D x \left[-\frac{1}{2(p+1)!} F_{\mu_1\dots\mu_{p+1}}^2 - \frac{m^2}{2p!} A_{\mu_1\dots\mu_p}^2 \right] .$$

One can now couple these particle models to a curved target space, and use them to compute the one-loop effective action for the corresponding quantum field theories. In the next section we are going to describe the case of a differential p -form A_p with action (102) using a first quantized approach based on the $N = 2$ spinning particle action.

5. ANTISYMMETRIC TENSORS IN THE WORLDLINE APPROACH

To obtain a representation of the one-loop effective action induced by a p -form A_p coupled to gravity, we couple the $N = 2$ spinning particle action (92) to a target space metric $g_{\mu\nu}(x)$, eliminate the momenta p_μ to get the configuration space action, Wick rotate to euclidean

time ($t \rightarrow -i\tau$, and also $a \rightarrow ia$ to keep the gauge group $U(1)$ compact) and obtain the euclidean action that we want to quantize on a one dimensional torus

$$(104) \quad S = \int_0^1 d\tau \left[\frac{1}{2} e^{-1} g_{\mu\nu} (\dot{x}^\mu - \bar{\chi} \psi^\mu - \chi \bar{\psi}^\mu) (\dot{x}^\nu - \bar{\chi} \psi^\nu - \chi \bar{\psi}^\nu) + \right. \\ \left. + \bar{\psi}_a (\dot{\psi}^a + \dot{x}^\mu \omega_\mu{}^a{}_b \psi^b + ia \psi^a) - \frac{e}{2} R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d - iqa \right]$$

where $\tau \in [0, 1]$ parametrizes the torus. For simplicity we have introduced the vielbein $e^a{}_\mu$ and the corresponding spin connection $\omega_\mu{}^{ab}$ to be able to use flat indices on the worldline fermions fermion $\psi^a \equiv e^a{}_\mu \psi^\mu$.

For the purpose of gauge fixing it is useful to record the gauge transformations of the supergravity multiplet

$$(105) \quad \begin{aligned} \delta e &= \dot{\xi} + 2\bar{\chi}\epsilon + 2\chi\bar{\epsilon} \\ \delta \chi &= \dot{\epsilon} + ia\epsilon - i\alpha\chi \\ \delta \bar{\chi} &= \dot{\bar{\epsilon}} - ia\bar{\epsilon} + i\alpha\bar{\chi} \\ \delta a &= \dot{\alpha} . \end{aligned}$$

Just like the $N = 0$ model described in eq. (69), the quantization on a torus is expected to produce the one-loop effective action $\Gamma_p^{QFT}[g_{\mu\nu}]$ due to the virtual propagation of a p -form gauge field in a gravitational background

$$(106) \quad \Gamma_p^{QFT}[g_{\mu\nu}] \sim Z[g_{\mu\nu}] = \int_{T^1} \frac{\mathcal{D}G\mathcal{D}X}{\text{Vol}(\text{Gauge})} e^{-S[X,G;g_{\mu\nu}]}$$

where $G = (e, \chi, \bar{\chi}, a)$ and $X = (x^\mu, \psi^\mu, \bar{\psi}^\mu)$ indicate the dynamical fields that must be integrated over, and $S[X, G; g_{\mu\nu}]$ denotes the action in (104). Division by the volume of the gauge group reminds of the necessity of fixing the gauge symmetries.

The torus is described by taking the parameter $\tau \in [0, 1]$ and imposing periodic boundary conditions on the bosonic fields x^μ and e . The gauge field a is instead treated as a connection. As for the fermions we take antiperiodic boundary conditions. The gauge symmetries can be used to fix the supergravity multiplet to $\hat{G} = (\beta, 0, 0, \phi)$, where β and ϕ are the leftover bosonic moduli that must be integrated over. The parameter β is the usual proper time, while the parameter $\phi \in [0, 2\pi]$ is a phase that corresponds to the only modular parameter that the gauge field a can have on the torus. Note that the gravitinos χ and $\bar{\chi}$ are antiperiodic and can be completely gauged away using (105), leaving no moduli. Inserting the necessary Faddeev-Popov determinants to eliminate the volume of the gauge group gives rise to the following expression for the effective action

$$(107) \quad \Gamma_p^{QFT}[g_{\mu\nu}] = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left(2 \cos \frac{\phi}{2} \right)^{-2} \int_{T^1} \mathcal{D}X e^{-S[X, \hat{G}; g_{\mu\nu}]} .$$

This is the main formula we were looking for. We have been quite brief in deriving this final result, but to help the reader we may note that: *i*) The measure over the proper time β takes into account the effect of the symmetry generated by the Killing vector on the torus (namely the constant vector). *ii*) The Faddeev-Popov determinants due to the commuting susy ghosts are obtained from the differential operator appearing in the susy transformation laws (105) and are computed to give $\det^{-1}(\partial_\tau + i\phi) \det^{-1}(\partial_\tau - i\phi) = (2 \cos(\phi/2))^{-2}$. These are the inverse of a similar fermionic determinant which is easily obtained: antiperiodic boundary conditions produce a trace over the corresponding two-dimensional Hilbert space and thus $\det(\partial_\tau + i\phi) = e^{-i\phi/2} + e^{i\phi/2} = 2 \cos(\phi/2)$. *iii*) The other Faddeev-Popov determinants do not give rise to any moduli dependent term. *iv*) The overall normalization $-1/2$ has been inserted to match QFT results. Up to the overall sign, one could argue that this factor is due to considering a real field rather than a complex one.

Thus, up to the final integration over the moduli (β, ϕ) , one is left with a standard path integral for a nonlinear $N = 2$ supersymmetric sigma model, eq. (107). We have so reached an explicit and very useful representation of the one loop-effective action for a p -form. It contains a standard path integral in curved space of the type discussed in the previous sections. Of course, it cannot be evaluated exactly for arbitrary background metrics $g_{\mu\nu}$, but it is the starting point of various approximations schemes. In particular, one may consider an expansion in terms of the proper time β which leads to the local heat-kernel expansion of the effective action [13, 14]. It is a derivative expansion depending on the so-called Seeley-DeWitt coefficients, and reads

$$(108) \quad \Gamma_p^{QFT}[g_{\mu\nu}] = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} Z_p(\beta)$$

$$Z_p(\beta) = \int \frac{d^D x \sqrt{g(x)}}{(2\pi\beta)^{D/2}} \left(a_0(x) + a_1(x)\beta + a_2(x)\beta^2 + \dots \right)$$

where the coefficients $a_i(x)$ are the Seeley-DeWitt coefficients (in the coincidence limit). Even if convergence of the proper time integral in the upper limit is not guaranteed for the massless case, one can still compute these coefficients. They characterize the theory. For example they identify the counterterms needed to renormalize the full effective action.

To compute the effective action in the form of eq. (108) one must integrate over the modular parameter ϕ . However, at $\phi = \pi$ there is a potential divergence: in fact one can show that the integration over this modular parameter effectively interpolates between all possible fermion boundary conditions, and $\phi = \pi$ corresponds to periodic boundary conditions for which the perturbative fermion propagator develops a zero mode. To take into account this singular point we are going to use an analytic regularization in moduli space. First we change coordinates and use the Wilson loop variable $w = e^{i\phi}$ instead of ϕ . The integration region for this new variable is the unit circle γ on the complex w -plane

$$(109) \quad \int_0^{2\pi} \frac{d\phi}{2\pi} = \oint_\gamma \frac{dw}{2\pi i w}.$$

The singular point $\phi = \pi$ is now mapped to $w = -1$. The correct prescription is to use complex contour integration and deform the contour to exclude the point $w = -1$ (say by moving it outside to $w = -1 - \epsilon$ with $\epsilon > 0$, and then letting $\epsilon \rightarrow 0^+$), see figure 2.

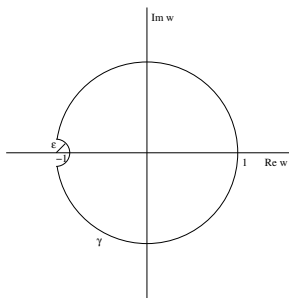


FIGURE 2. Regulated contour for the $U(1)$ modular parameter.

This prescription allows us to recover all known results for $p = 0, 1, 2$. We may note that this prescription can be interpreted as giving the worldline fermions a small mass ϵ to lift the zero modes appearing at the point $\phi = \pi$ of moduli space, i.e. replacing $\phi = \pi$ by $\phi = \pi - i\epsilon$.

This way one can compute the Seeley-DeWitt coefficients a_0, a_1, a_2 for all differential forms which satisfy Maxwell equations in all dimensions. The results and many additional details can be found in [27].

5.1. **Duality.** We have used the $N = 2$ spinning particle to compute one-loop effects due to the propagation of differential forms coupled to gravity, including as a particular case a spin 1 field (the photon). One of the most interesting points of this construction is the appearance of the $U(1)$ modulus ϕ , a parameter which does not arise in the standard derivation of Feynman rules. This point deserves further discussions. For this purpose it is convenient to switch to an operatorial picture and cast the effective action (107) in the form

$$(110) \quad \Gamma_p^{QFT}[g_{\mu\nu}] = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left(2 \cos \frac{\phi}{2}\right)^{-2} \text{Tr} [e^{i\phi(\hat{N}-(D/2)+q)} e^{-\beta\hat{H}}]$$

where the path integral on the torus has been represented by the trace in the matter sector of the Hilbert space of the spinning particle (i.e. excluding the ghost sector, which in fact we did not need to introduce). Here $\hat{N} = \hat{\psi}^\mu \hat{\psi}_\mu^\dagger$ is the (anti) fermion number operator (it counts the degree of the field strength form, and up to the ordering coincides with the current $-\hat{J}$) and \hat{H} is the quantum hamiltonian without the coupling to the gauge field, which has been explicitly factorized. Computing the trace and using the Wilson loop variable $w = e^{i\phi}$ gives an answer of the form

$$(111) \quad \Gamma_p^{QFT}[g_{\mu\nu}] = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \oint_\gamma \frac{dw}{2\pi i w} \frac{w}{(1+w)^2} \sum_{n=0}^D w^{n-(D/2)+q} t_n(\beta)$$

where the coefficients $t_n(\beta)$ arise from the trace restricted to the sector of the Hilbert space with occupation number n . From the answer written in this form one can make various comments.

If susy is not gauged then the ghost term $w/(1+w)^2$ is absent, and one obtains a simpler model. There is no pole at $w = -1$, at least for finite D , and the w integral projects onto the sector of the Hilbert space with occupation number $n - (D/2) + q = 0$, i.e. $n = p + 1$. This describes a $(p + 1)$ -form with the Beltrami-Laplace operator $dd^\dagger + d^\dagger d$ as kinetic operator. The absence of the pole at $w = -1$ means that excluding or including this point in the regularized contour γ must produce the same answer. This is related to Poincaré duality, which tells that the result for a $(p + 1)$ -form must be equivalent to that of a $(D - p - 1)$ -form. The change from $(p + 1)$ to $(D - p - 1)$ in (111) is described by $q \rightarrow -q$, which can be undone by a change of integration variable $w \rightarrow w' = 1/w$ (or $\phi \rightarrow -\phi$). This proves Poincaré equivalence, i.e. $t_n(\beta) = t_{D-n}(\beta)$.

Next consider the case of gauged susy. Now one must include the contribution of the ghost determinants $w/(1+w)^2$ and face the appearance of a possible pole at $w = -1$. The duality between a gauge p -form and a gauge $(D - p - 2)$ -form is again described by $q \rightarrow -q$ and compensated by the change of integration variable $w \rightarrow w' = 1/w$. However, the original contour which was regulated by excluding the pole at $w = -1$ gets mapped onto a contour which now includes that pole, see figure 3.

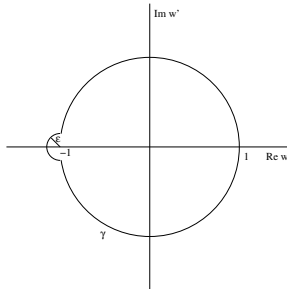


FIGURE 3. Regulated contour in the variable w' .

Therefore strict duality is not guaranteed. The mismatch corresponds to the residue at the pole $w = -1$, and can be computed as follows

$$(112) \quad \begin{aligned} & \text{Res} \left(\frac{1}{(1+w)^2} \text{Tr} [w^{\hat{N}-(D/2)+q} e^{-\beta \hat{H}}], w = -1 \right) = \\ & = (-1)^p \text{Tr} [\hat{N} (-1)^{\hat{N}} e^{-\beta \hat{H}}] + (-1)^{p+1} (p+1) \text{Tr} [(-1)^{\hat{N}} e^{-\beta \hat{H}}] \end{aligned}$$

The second term in the last line is proportional to the Witten index of the gauge fixed nonlinear sigma model [28], which is β independent and computes the topological Euler number χ of target space [20]. The first term is instead similar to an index introduced in [29] for two dimensional field theories, and which in our case can be shown to compute the partition function $Z_{D-1}(\beta)$ of a $(D-1)$ -form A_{D-1} (with the top form F_D as field strength)

$$(113) \quad \text{Tr} [\hat{N} (-1)^{\hat{N}} e^{-\beta \hat{H}}] = -Z_{D-1}(\beta).$$

The notation for the partition function $Z_{D-1}(\beta)$ is as in (108). Thus we arrive at the following equivalence between propagating p -forms and $(D-p-2)$ -forms

$$(114) \quad Z_p(\beta) = Z_{D-p-2}(\beta) + (-1)^p Z_{D-1}(\beta) + (-1)^p (p+1) \chi.$$

As the gauge field A_{D-1} does not have propagating degrees of freedom, the first Seeley-DeWitt coefficient a_0 is not spoiled by this duality.

For even D the partition function $Z_{D-1}(\beta)$ is proportional to the Euler number, namely $Z_{D-1}(\beta) = -(D/2)\chi$, as can be checked by using Poincaré duality, so that equation (114) simplifies to

$$(115) \quad Z_p(\beta) = Z_{D-p-2}(\beta) + (-1)^p \left(p+1 - \frac{D}{2} \right) \chi$$

or, in a heuristic notation,

$$(116) \quad F_{p+1} \sim F_{D-(p+1)} + (-1)^p \left(1 - \frac{2(p+1)}{D} \right) F_D.$$

The mismatch is purely topological. It was already noticed in [30] for the duality between a scalar and 2-form gauge field in $D = 4$. The β independence of χ shows, for example, that a mismatch between a scalar and a 4-form in $D = 6$ will be visible in the coefficient a_3 , and so on.

For odd D the Euler number vanishes, and one has (at the level of field strengths) a duality of the form

$$(117) \quad F_{p+1} \sim F_{D-(p+1)} + (-1)^p F_D.$$

In ref. [31] it was noted that after integrating over β , this mismatch can be related to the Ray-Singer torsion, which is also known to be a topological invariant.

These inequivalences are present at the level of unregulated effective actions. They are given by local terms that can be subtracted in the renormalization process, and thus, according to [32], they do not spoil duality.

We have seen in this section that a worldline perspective on particles of spin 1 and on antisymmetric tensor gauge fields coupled to gravity has produced a quite interesting and useful representation of the one-loop effective action. Surely, one may find some other application of this worldline approach.

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