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Parton distributions, logarithmic expansions and kinetic evolution

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Abstract[1]. Aspects of the QCD parton densities are briefly reviewed, drawing some parallels to the density matrix formulation of quantum mechanics, exemplified by Wigner functions. We elaborate on the solution of their evolution equations using logarithmic expansions and overview their kinetic interpretation. We illustrate how a Fokker-Planck equation can be derived using the master formulation of the same equations and its construction in the case of the transverse spin distributions. A simple connection of the leading order DGLAP equation to fractional diffusion using fractional calculus is also briefly outlined.

1. Introduction

In Quantum Chromodynamics, or QCD, the accepted theory of the strong interactions, the description of high energy collisions is formulated with the help of some important theorems, commonly refered to as *factorization theorems* [1]. According to these theorems it is possible to describe certain hadronic processes in three steps, the first involving the initial state, followed by a second intermediate state, commonly known as "the hard scattering", and the third which involves the final state. Both the initial and the final state are described with the help of some non-local operators supported on the light cone, whose matrix elements, termed "parton distributions" and "fragmentation functions" respectively, can't be computed from first principle, at least at this time, in a second quantized theory. It is possible, however, to parametrize their functional form by combining a vast amount of experimental data with numerical fits.

The separation between the initial and the intermediate state is perfomed with the introduction of an intermediate scale, the factorization scale, denoted by Q, where Q is the amount of energy/virtuality involved in the actual separation.

Parton distributions depend both on the choice of Q and on a second variable, x (Bjorken x) which is supported on the interval $(0, 1)$. This variable can be interpreted as the fraction of the momentum of the particles in the initial state that flows into the intermediate state. Parton distributions, at least some of them, have a simple interpretation in terms of densities of quarks and gluons (the partons) in a nucleon. They are probability distributions that change (evolve) with the scale Q and their evolution is described by a set of equations known as DGLAP equations [2], which are a special type of renormalization group equation (RGE). They are, by now, textbook material in high energy physics. Mathematicians may find a thorough discussion of their origin in [1] and in other books in Quantum Field Theory and particle phenomenology.

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On the contrary of other RGE's, such as those describing the evolution of other local operators in the Standard Model which are ordinary differential equations, the RGE's of the parton densities are integro-differential equations. They are defined in terms of some kernels ($P(x)$) which are not regular functions but distributions, finite after integration over x.

The objective of this work is to present a short review of previous studies by us on this subject. Here we will discuss first a direct way to solve these equations and then move to describe their kinetic interpretation. We will show how one can construct a hierarchy of equations starting from the original integro-differential equations and illustrate the procedure at the lowest order and identify a Fokker-Planck approximation to the same equations. A simple connection between the leading order equations and the formalism of fractional calculus is also briefly pointed out.

2. Parton distributions as Wigner functions

Evolution equations describing the high energy behaviour of scattering amplitudes carry significant information on the factorization/renormalization scale dependence of such amplitudes, and allow to link the behaviour of processes at a given energy scale to collisions taking place at another (usually much higher) scale.

In QCD confinement forbids the detection of the fundamental states of the theory, such as quarks and gluons. However, asymptotic freedom, the smallness of the coupling constant as the energy raises (or approaching smaller distances), allows to separate the perturbative dynamics at short distances from the non-perturbative one, due to confinement, through factorization theorems and the introduction of the parton distributions. We will very briefly introduce them below, and we will exploit the density matrix formulation of quantum mechanics as an analogy to illustrate the topic.

The mathematical construct which is the closest to a parton distribution function (p.d.f.) $q(x, Q)$ is a *Wigner function*, as first noticed in [2]. The analogy is, of course, limited.

We recall that Wigner's description of quantum mechanics via quasi-probabilities of phase space $(\mathbf{x}, \mathbf{p})$

$$f(\mathbf{x}, \mathbf{p}) = \frac{1}{2\pi} \int \mathbf{dy} \psi^*(\mathbf{x} - \frac{\hbar}{2}\mathbf{y}) e^{i\mathbf{p}\cdot\mathbf{y}} \psi(\mathbf{x} + \frac{\hbar}{2}\mathbf{y}) \tag{1}$$

is fully equivalent to Schrodinger's formulation [3]. Its dynamics is specified by Moyal's equation, that extends Liouville's formulation of classical mechanics for a classical hamiltonian $H(\mathbf{x}, \mathbf{p})$, $\partial_t f + \{f, H\} = 0$ to quantum mechanics

$$\frac{\partial f}{\partial t} = \frac{H * f - f * H}{i\hbar} \tag{2}$$

where the star-product (*) is

$$* \equiv e^{i\hbar/2 \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x \right)} . \tag{3}$$

The nonlocality of the construct appears immediately in its dynamics, with derivatives in its evolution equation (2) that extend to all orders. In practice one can use translation of its arguments

$$f(x,p) * g(x,p) = f\left(x + i\frac{\hbar}{2} \overrightarrow{\partial}_p, p - i\frac{\hbar}{2} \overrightarrow{\partial}_x \right) g(x,p) \tag{4}$$

for the explicit evaluation of this product.

Differently from Wigner functions, in a p.d.f. the variable x takes now the role of the momentum "p". Also, parton distributions are correlation functions of a special type, being defined just on the light-cone. In this sense they are not generic nonlocal correlators. This limitation, due to the special nature of high energy collisions in asymptotically free theories, sets the boundary of validity of the parton model approach to QCD.

Another observation that may help in the distinction between a p.d.f. and a Wigner function is the different nature of the respective equations. For the p.d.f. the nonlocality

in the evolution is all expressed in a renormalization group equation rather than in the Hamiltonian formalism of (2). However, also in this case some interesting analogies remain. Once the evolution equation of a p.d.f. is written in the form of a master equation, a hierarchy of equations can be identified quite straighforwardly using the Kramers-Moyal expansion of the transition probabilities. This point will be discussed next. From this analysis, that we believe has limited phenomenological applications, emerge however some amusing features which may be generic for other types of nonlocal operators in field theory.

2.1. **The "Wiggies" of QCD.** In QCD one starts by introducing, via arguments based on unitarity, the hadronic tensor, which is the key construct describing the collision

$$W^{\mu\nu} = \int \frac{d^4x}{(2\pi)^4}\, e^{iq\cdot x} \langle P_A S_A; P_B S_B | \left[J^\mu(0), J^\nu(x) \right] | P_A S_A; P_B S_B \rangle \ , \tag{5}$$

with P_A and P_B being the momenta of the colliding hadrons and S_A and S_B their covariant spins. The J's are electromagnetic currents.

The distribution functions that emerge -at leading order- from this factorized picture are correlation functions of non-local operators in configuration space. They are the quark-quark and the quark-antiquark correlators.

Their expression simplifies in the axial gauge, in which the gauge link is removed by the gauge condition. For instance, the quark-quark correlator takes the form

$$\left(\Phi_{a/A}\right)_{\alpha\beta}(P,S,k) = \int \frac{d^4z}{(2\pi)^4}\, e^{ik\cdot z} \langle P,S|\overline{\psi}^{(a)}_\beta(0)\psi^{(a)}_\alpha(z)|PS\rangle \ . \tag{6}$$

In (6) we have included the quark flavour index a and an index A for the hadron, as usual. Fields are not time ordered since they can be described by the good light cone components of ψ and by A_T, a transverse component of the gauge field A_μ, as discussed in [4]. The non-perturbative information in a collision is carried by matrix elements of this type.

Further considerations allow to show that the leading contributions to (6) come from the light-cone region. The leading expansion of the quark-quark correlator then is of the form

$$\int \frac{d\lambda}{2\pi}\, e^{i\lambda x} \langle PS|\overline{\psi}(0)\psi(\lambda n)|PS\rangle = \frac{1}{2}\left(\not{p} f_1(x) + \lambda\gamma_5 \not{p} g_1(x) + \gamma_5 \not{S}_T \not{p} h_1(x) \right) \ , \tag{7}$$

where we have used all the four vectors at our disposal (spin S, momentum P of the hadron) and introduced invariant amplitudes (parton distributions) f_1, g_1, h_1, now expressed in terms of a scaling variable x (Bjorken x). n^μ is a light-cone four-vector ($n^2 = 0$), approximately perpendicular to the hadron momentum.

The definition of p.d.f.'s in (7) involves also an underlying physical scale Q ($Q >> \Lambda_{QCD}$, with Λ_{QCD} being the scale of confinement), not apparent from that equation and characterizing the energy scale at which these matrix elements, summarized by (7), are defined. Truly: $f_1 = f_1(x, Q^2)$, $h_1 = h_1(x, Q^2)$ and so on.

The role of the various renormalization group equations is to describe the perturbative change in these functions as the scale Q is raised (lowered). Each equation involves kernels ($P(x)$) of various types, of well known form, and asymptotic expansions of the solutions exist (see for instance [5] and the implementation given in [6]). The structure to all orders of the solution has been discussed by us recently using alternative methods [7].

2.2. **Factorization and evolution.** Factorization theorems play a crucial role in the application of perturbation theory to hadronic reactions. The proof of these theorems and the actual implementation of their implications has spanned a long time and has allowed to put the parton model under a stringent experimental test. Prior to embark on the discussion of our contributions to the study of evolution algorithms in Bjorken x-space, we provide here a brief background on the topic in order to make our treatment self-contained. We refer to [1] for a thorough overview of the topic.

In sufficiently inclusive cross section, leading power factorization theorems allow to write down a hadronic cross section in terms of parton distributions and of some hard scatterings,

the latter being calculable at a given order in perturbation theory using the fundamental QCD lagrangean. Specifically, for a hadronic cross section, for instance a proton proton cross section σ_{pp}, the result of the calculation can be summarized by the formula

$$\sigma_{pp} = \sum_f \int_0^1 dx_1 \int_0^1 dx_2 f_{h_1\to f}(x_1,Q^2) f_{h_2\to f}(x_2,Q^2)\hat{\sigma}(x_1,x_2,\hat{s},\hat{t},Q^2)\,, \tag{8}$$

where the integral is a function of some variables x_1 and x_2 which describe the QCD dynamics at parton level in the Deep Inelastic Scattering (DIS) limit or, equivalently, at large energy and momentum transfers. These variables are termed "Bjorken variables" and are scale invariant. This formula is a statement about the computability of a hadronic collision in terms of some "building blocks" of easier definition.

The variable Q^2, in the equation above, can be identified with the factorization scale of the process. $\hat{\sigma}$ can be computed at a given order in perturbation theory in an expansion in α_s, the QCD coupling, while the $f_{h_i\to f}(x,Q^2)$ are the parton distributions. These describe the probability for a hadron h to prepare for the scattering a parton f, which undergoes the collision. An equivalent interpretation of the functions $f_{h_i\to f}(x,Q^2)$ is to characterize the density of partons of type f into a hadron of type h. A familiar notation, which simplifies the previous notations shown above, is to denote by $q_i(x,Q^2)$ the density of quarks in a hadron (a proton in this case) of flavour i and by $g(x,Q^2)$ the corresponding density of gluons. For instance, the annihilation channel of a quark with an antiquark in a generic process is accounted for by the contribution

$$\int_0^1 dx_1 \int_0^2 dx_2 q(x_1,Q^2)\overline{q}(x_2,Q^2)\hat{\sigma}_{q\overline{q}}(x_1,x_2,Q^2)\,, \tag{9}$$

and so on for the other contributions, such as the quark-gluon sector (qg) or the gluon-gluon sector (gg) each of them characterized by a specific hard scattering cross section $\hat{\sigma}_{qg}$, or $\hat{\sigma}_{qg}$. In this separation of the cross section into contributions of hard scatterings $\hat{\sigma}$ and parton distributions $f(x,Q^2)$ the scale at which the separation occurs is artificial, in the sense that the hadronic cross section σ should not depend on Q^2 or

$$\frac{d\sigma}{dQ^2} = 0\,. \tag{10}$$

However, a perturbative computation performed by using the factorization formula, however, shows that this is not the case, since the perturbative expansion of $\hat{\sigma}$

$$\hat{\sigma} = \hat{\sigma}^{(0)} + \alpha_s(Q^2)^2\hat{\sigma}^{(1)} + \alpha_s(Q^2)\hat{\sigma}^{(2)} \tag{11}$$

naturally brings in the dependence on the factorization scale Q. This dependence is weaker if we are able to push our computation of the hard scattering to a sufficiently high order in α_s. The order at which the perturbative expansion stops is also classified as a "leading order" (LO), "next-to-leading order" (NLO) or, even better, a "next-to-next-to-leading" (NNLO) contribution if more and more terms in the expansion are included.

At the same time, as we have already mentioned, the parton distributions $f(x,Q^2)$ also carry a similar dependence on the scale Q, which is summarized by some RGE's. The equations resum the logarithmic violations to the lowest order scale invariance, induced by the perturbative expansion. Also in this case we need to quantify this effect and reduce its impact on the prediction of the cross section. Solving the RGE's for the p.d.f.'s to higher order and, at the same time, computing the hard scatterings to higher orders reduces the spurious dependence on the factorization scale Q and improves the theoretical prediction of the real physical phenomenon.

2.3. **Parton dynamics at NLO.** As we have mentioned, within the framework of the parton model, where the "partons" are the quarks and gluons inside a hadron, evolution equations of DGLAP-type - and the corresponding initial conditions on the parton distributions - are among the most important blocks which characterize the description of the

quark-gluon interaction. Other parts of this description require the computation of the hard scatterings (what we have called "the second stage" of the collision) with high accuracy. Here we illustrate an implementation to NLO of a method based on an ansatz which allows to rewrite the evolution equations as a set of recursion relations for some scale invariant functions, $A_n(x)$ and $B_n(x)$, which appear in the expansion. The advantage, compared to others, of using these recursion relations is that just few iterates of these are necessary in order to obtain a stable solution. One of the advantages of the method is its analytical base, since the recursion relations can be solved explicitely in terms of the initial conditions. We have recently shown conclusively that these methods are equivalent to other methods that also give the exact solution using Mellin transforms. Our approach follows closely the Mellin method in the demonstration of the correctness of the recursion relations, but then is implemented directly in x-space.

3. Running coupling

In QCD the coupling constant changes with the change of the energy scale and there is a RGE associated to it as well, computed within a certain level of accuracy, also classified as LO, NLO, NNLO etc. The solution of the RGE for the running coupling constant in the NLO approximation can be expressed in terms of two coefficients β_0 and β_1, describing the beta function of the theory and a renormalization group invariant scale Λ_{QCD} that sets the scale for confinement. The NNLO extension of the equation involves a third coefficient, β_2. The RGE of the coupling is

$$\beta(\alpha_s) = \frac{d\alpha_s(Q^2)}{d\log Q^2} \ , \tag{12}$$

and its expansion, for instance to NNLO, is given by [10] [11]

$$\beta(\alpha_s) = -\frac{\beta_0}{4\pi}\,\alpha_s^2 - \frac{\beta_1}{16\pi^2}\,\alpha_s^3 - \frac{\beta_2}{64\pi^3}\,\alpha_s^4 + O(\alpha_s^5) \ . \tag{13}$$

For the moment we work in the NLO approximation (only β_0 and β_1 are included), and in this case an explicit solution of this equation is given by

$$\begin{aligned} \alpha_s(Q^2) = {} & \frac{4\pi}{\beta_0}\,\frac{1}{\log(Q^2/\Lambda^2_{\overline{MS}})}\,\times \\ & \times\left[1 - \frac{\beta_1}{\beta_0^2}\,\frac{\log\log(Q^2/\Lambda^2_{\overline{MS}})}{\log(Q^2/\Lambda^2_{\overline{MS}})} + O\left(\frac{1}{\log^2(Q^2/\Lambda^2_{\overline{MS}})}\right)\right] \ , \end{aligned} \tag{14}$$

where

$$\beta_0 = \frac{11}{3}\,N_C - \frac{4}{3}\,T_f \ \ , \ \ \beta_1 = \frac{34}{3}\,N_C^2 - \frac{10}{3}\,N_C n_f - 2C_F n_f \ , \tag{15}$$

and

$$N_C = 3 \ \ , \ \ C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3} \ \ , \ \ T_f = T_R n_f = \frac{1}{2}\,n_f \ , \tag{16}$$

where N_C is the number of colors, n_f is the number of active flavors, which is fixed by the number of quarks with $m_q \leq Q$. Λ_{QCD} is a scale that depends on the number of active quarks included in the evolution (number of flavours). We have taken for the quark masses (charm, bottom and top respectively) $m_c = 1.5\,\mathrm{GeV}, m_b = 4.5\,\mathrm{GeV}$ and $m_t = 175\,\mathrm{GeV}$, these are necessary in order to identify the thresholds at which the number of flavours n_f is raised as we increase the final evolution scale.

Λ_{QCD} should be more correctly denoted as $\Lambda^{(n_f)}_{\overline{MS}}$, given its flavour dependence, and is given by

$$\Lambda^{(3,4,5,6)}_{\overline{MS}} = 0.248,\, 0.200,\, 0.131,\, 0.050\,\mathrm{GeV} \ . \tag{17}$$

The label "$\overline{MS}$" denotes conventionally the modified minimal subtraction scheme and denotes a specific regularization scheme of the perturbative expansion.

We also define the distribution of a given helicity ($\pm$), $f^{\pm}(x,Q^2)$, which is the probability of finding a parton of type f at a scale Q, where $f = q_i, \overline{q_i}, g$, in a longitudinally polarized proton with the spin aligned (+) or anti-aligned (-) respect to the proton spin and carrying a fraction x of the proton's momentum.

We introduce the longitudinally polarized parton distribution of the proton

$$\Delta f(x,Q^2) \equiv f^{+}(x,Q^2) - f^{-}(x,Q^2)\ . \tag{18}$$

We also introduce another type of parton density, termed *transverse spin distribution*, which is defined as the probability of finding a parton of type f in a transversely polarized proton with its spin parallel ($\uparrow$) minus the probability of finding it antiparallel ($\downarrow$) to the proton spin

$$\Delta_T f(x,Q^2) \equiv f^{\uparrow}(x,Q^2) - f^{\downarrow}(x,Q^2)\ . \tag{19}$$

Similarly, the unpolarized (spin averaged) parton distribution of the proton is given by

$$f(x,Q^2) \equiv f^{+}(x,Q^2) + f^{-}(x,Q^2) = f^{\uparrow}(x,Q^2) + f^{\downarrow}(x,Q^2)\ . \tag{20}$$

We also recall, if not obvious, that taking linear combinations of Equations (20) and (18), one recovers the parton distributions of a given helicity

$$f^{\pm}(x,Q^2) = \frac{f(x,Q^2) \pm \Delta f(x,Q^2)}{2}\ . \tag{21}$$

In regard to the kernels, the notations P, ΔP, $\Delta_T P$, $P^{+\pm}$, will be used to denote the kernels in the unpolarized, longitudinally polarized, transversely polarized, and the positive (negative) helicity cases respectively.

The DGLAP equation is an integro-differential equation whose general mathematical structure is

$$\frac{d}{d\log Q^2}\ f(x,Q^2) = P(x,\alpha_s(Q^2)) \otimes f(x,Q^2)\ , \tag{22}$$

where the convolution product is defined by

$$[a \otimes b]\,(x) = \int_x^1 \frac{dy}{y}\ a\left(\frac{x}{y}\right) b(y) = \int_x^1 \frac{dy}{y}\ a(y) b\left(\frac{x}{y}\right)\ . \tag{23}$$

Let us now turn to the evolution equations, starting from the unpolarized case. Defining

$$q_i^{(\pm)} = q_i \pm \overline{q_i}\ \ ,\ \ q^{(+)} = \sum_{i=1}^{n_f} q_i^{(+)}\ \ ,\ \ \chi_i = q_i^{(+)} - \frac{1}{n_f}\ q^{(+)}\ , \tag{24}$$

the evolution equations are

$$\frac{d}{d\log Q^2}\ q_i^{(-)}(x,Q^2) = P_{NS^-}(x,\alpha_s(Q^2)) \otimes q_i^{(-)}(x,Q^2)\ , \tag{25}$$

$$\frac{d}{d\log Q^2}\ \chi_i(x,Q^2) = P_{NS^+}(x,\alpha_s(Q^2)) \otimes \chi_i(x,Q^2)\ , \tag{26}$$

for the non-singlet sector and

$$\begin{aligned} &\frac{d}{d\log Q^2} \begin{pmatrix} q^{(+)}(x,Q^2) \\ \\ g(x,Q^2) \end{pmatrix} = \\ &= \begin{pmatrix} P_{qq}(x,\alpha_s(Q^2)) & P_{qg}(x,\alpha_s(Q^2)) \\ \\ P_{gq}(x,\alpha_s(Q^2)) & P_{gg}(x,\alpha_s(Q^2)) \end{pmatrix} \otimes \begin{pmatrix} q^{(+)}(x,Q^2) \\ \\ g(x,Q^2) \end{pmatrix} \end{aligned} \tag{27}$$

for the singlet sector. Equations analogous to (24)-(27), with just a change of notation, are valid in the longitudinally polarized case and, due to the linearity of the evolution equations, also for the distributions in the helicity basis. In the transverse case instead, there is no coupling between gluons and quarks, so the singlet sector (27) is missing. In this case we will solve just the nonsinglet equations

$$\frac{d}{d\log Q^2}\,\Delta_T q_i^{(-)}(x,Q^2) = \Delta_T P_{NS^-}(x,\alpha_s(Q^2)) \otimes \Delta_T q_i^{(-)}(x,Q^2)\,, \tag{28}$$

$$\frac{d}{d\log Q^2}\,\Delta_T q_i^{(+)}(x,Q^2) = \Delta_T P_{NS^+}(x,\alpha_s(Q^2)) \otimes \Delta_T q_i^{(+)}(x,Q^2)\,. \tag{29}$$

We also recall that the perturbative expansion, up to next-to-leading order, of the kernels is

$$P(x,\alpha_s) = \left(\frac{\alpha_s}{2\pi}\right) P^{(0)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P^{(1)}(x) + \dots\,. \tag{30}$$

Kernels of fixed helicity can also be introduced with $P_{++}(z) = (P(z) + \Delta P(z))/2$ and $P_{+-}(z) = (P(z) - \Delta P(z))/2$ denoting splitting functions of fixed helicity, which will be used below.

The equations, in the helicity basis, are

$$\begin{aligned}
\frac{dq_+(x)}{dt} &= \frac{\alpha_s}{2\pi}(P^{qq}_{++}(\frac{x}{y}) \otimes q_+(y) + P^{qq}_{+-}(\frac{x}{y}) \otimes q_-(y) + \\
&\quad + P^{qg}_{++}(\frac{x}{y}) \otimes g_+(y) + P^{qg}_{+-}(\frac{x}{y}) \otimes g_-(y))\,, \\
\frac{dq_-(x)}{dt} &= \frac{\alpha_s}{2\pi}(P_{+-}(\frac{x}{y}) \otimes q_+(y) + P_{++}(\frac{x}{y}) \otimes q_-(y) + \\
&\quad + P^{qg}_{+-}(\frac{x}{y}) \otimes g_+(y) + P^{qg}_{++}(\frac{x}{y}) \otimes g_-(y))\,, \\
\frac{dg_+(x)}{dt} &= \frac{\alpha_s}{2\pi}(P^{gq}_{++}(\frac{x}{y}) \otimes q_+(y) + P^{gq}_{+-}(\frac{x}{y}) \otimes q_-(y) + \\
&\quad + P^{gg}_{++}(\frac{x}{y}) \otimes g_+(y) + P^{gg}_{+-}(\frac{x}{y}) \otimes g_-(y))\,, \\
\frac{dg_-(x)}{dt} &= \frac{\alpha_s}{2\pi}(P^{gq}_{+-}(\frac{x}{y}) \otimes q_+(y) + P^{gq}_{++}(\frac{x}{y}) \otimes q_-(y) + \\
&\quad + P^{g}_{+-}(\frac{x}{y}) \otimes g_+(y) + P^{gg}_{++}(\frac{x}{y}) \otimes g_-(y))\,.
\end{aligned} \tag{31}$$

The non-singlet (valence) analogue of this equation is also easy to write down

$$\begin{aligned}
\frac{dq_{+,V}(x)}{dt} &= \frac{\alpha_s}{2\pi}(P_{++}(\frac{x}{y}) \otimes q_{+,V}(y) + P_{+-}(\frac{x}{y}) \otimes q_{-,V}(y))\,, \\
\frac{dq_{-,V}(x)}{dt} &= \frac{\alpha_s}{2\pi}(P_{+-}(\frac{x}{y}) \otimes q_{+,V}(y) + P_{++}(\frac{x}{y}) \otimes q_{-,V}(y))\,.
\end{aligned} \tag{32}$$

where the $q_{\pm,V} = q_\pm - \bar{q}_\pm$ are the valence components of fixed helicity. The kernels in this basis are given by

$$
\begin{aligned}
P^{(0)}_{NS\pm,++} &= P^{(0)}_{qq,++} = P^{(0)}_{qq} \\
P^{(0)}_{qq,+-} &= P^{(0)}_{qq,-+} = 0 \\
P^{(0)}_{qg,++} &= n_f x^2 \\
P_{qg,+-} &= P_{qg,-+} = n_f (x-1)^2 \\
P_{gq,++} &= P_{gq,--} = C_F \, \frac{1}{x} \\
P^{(0)}_{gg,++} &= P^{(0)}_{gg,++} = N_c \left(\frac{2}{(1-x)_+} + \frac{1}{x} - 1 - x - x^2 \right) + \beta_0 \, \delta(1-x) \\
P^{(0)}_{gg,+-} &= N_c \left(3x + \frac{1}{x} - 3 - x^2 \right) \, .
\end{aligned}
\tag{33}
$$

Taking linear combinations of these equations (adding and subtracting), one recovers the usual evolutions for unpolarized $q(x)$ and longitudinally polarized $\Delta q(x)$ distributions.

4. The mathematical structure of the kernel

Here we try to illustrate some simple manipulations on the kernels which are very useful in order to simplify the equations. Their typical form is the following

$$
P(x) = P_1(x) + \frac{P_2(x)}{(1-x)_+} + P_3 \delta(1-x) \, , \tag{34}
$$

with a regular part $P_1(x)$, a "plus distribution" part $P_2(x)/(1-x)_+$ and a part P_3 which multiplies a Dirac delta function. For a generic function $\alpha(x)$ defined in the $[0,1)$ interval and singular in $x=1$, the plus distribution $[\alpha(x)]_+$ is defined by

$$
\int_0^1 f(x) [\alpha(x)]_+ \, dx = \int_0^1 \left(f(x) - f(1) \right) \alpha(x) \, dx \, , \tag{35}
$$

where $f(x)$ is a regular test function. Alternatively, an operative definition (in the sense of distributions) is the following

$$
[\alpha(x)]_+ = \alpha(x) - \delta(1-x) \int_0^1 \alpha(y) \, dy \, . \tag{36}
$$

From (36) it follows immediately that each plus distribution integrate to zero in the $[0,1]$ interval

$$
\int_0^1 [\alpha(x)]_+ \, dx = 0 \, . \tag{37}
$$

We want to make the convolution of the generic kernel (34) with a function $f(x)$. We introduce the notation $\bar{f}(x) = x f(x)$, the factor x being introduced to stabilize the evolution for small x values. The treatment of the regular and the delta-function parts is trivial

$$
P_1(x) \otimes \bar{f}(x) = x P_1(x) \otimes f(x) = x \int_x^1 \frac{dy}{y} \, P_1(y) f\left(\frac{x}{y} \right) = \int_x^1 dy P_1(y) \bar{f}\left(\frac{x}{y} \right) \tag{38}
$$

$$
P_3 \delta(1-x) \otimes \bar{f}(x) = \int_x^1 \frac{dy}{y} \, P_3 \delta(1-y) \bar{f}\left(\frac{x}{y} \right) \, . \tag{39}
$$

Let us now treat the more involved case of the plus distribution part

$$\frac{P_2(x)}{(1-x)_+} \otimes f(x) =$$

$$= \frac{P_2(x)}{1-x} \otimes f(x) - \left(\int_0^1 \frac{dy}{1-y} \right) P_2(x)\delta(1-x) \otimes f(x) =$$

$$= \int_x^1 \frac{dy}{y} \frac{P_2(y)}{1-y} f\left(\frac{x}{y}\right) - \int_0^1 \frac{dy}{1-y} \int_x^1 \frac{dy}{y} P_2(y)\delta(1-y) f\left(\frac{x}{y}\right) =$$

$$= \int_x^1 \frac{dy}{y} \frac{P_2(y)}{1-y} f\left(\frac{x}{y}\right) - P_2(1)f(x) \int_0^1 \frac{dy}{1-y} = \tag{40}$$

$$= \int_x^1 \frac{dy}{y} \frac{P_2(y)}{1-y} f\left(\frac{x}{y}\right) - P_2(1)f(x) \int_x^1 \frac{dy}{1-y} -$$

$$-P_2(1)f(x) \int_0^x \frac{dy}{1-y} =$$

$$= \int_x^1 \frac{dy}{y} \frac{P_2(y)f(x/y) - yP_2(1)f(x)}{1-y} + f(x)\log(1-x) \, ,$$

which yields

$$\frac{P_2(x)}{(1-x)_+} \otimes \bar{f}(x) = \int_x^1 dy \frac{P_2(y)\bar{f}(x/y) - P_2(1)\bar{f}(x)}{1-y} + \bar{f}(x)\log(1-x) \, . \tag{41}$$

From Feynman diagrams calculations one can get just the regular part $P_1(x)$ of each kernel. The remaining distributional parts (plus distribution and delta distribution) emerge from a procedure of regularization, that introduce the plus distribution part to regularize the eventual singularity in $x = 1$ and the delta distribution to fulfil some physical constraints, the *sum rules*.

The first one is the *baryon number sum rule* (BNSR), asserting that the baryon number (number of quarks less number of antiquarks) of the hadron must remain equal to its initial value (3 in the case of the proton) throughout the evolution, i.e. for each value of Q^2

$$q_1^{(-)}(Q^2) = \int_0^1 q^{(-)}(x, Q^2)\, dx = 3 \, . \tag{42}$$

Deriving (42) with respect to $\log(Q^2)$ and having in mind that $q^{(-)}$ evolves with P_{NS}^V, we get

$$\int_0^1 dx \left[P_{NS}^V(Q^2) \otimes q^{(-)}(Q^2) \right](x) = 0 \, . \tag{43}$$

Making use of the property of the Mellin moment of a convolution (75) this implies

$$\left(\int_0^1 P_{NS}^V(x, Q^2)\, dx \right) \left(\int_0^1 q^{(-)}(x, Q^2)\, dx \right) = 0 \, , \tag{44}$$

from which, using (42), we find the BNSR condition on the kernel

$$\int_0^1 P_{NS}^V(x)\, dx = 0 \, . \tag{45}$$

The other constraint is the *momentum sum rule* (MSR), asserting that the total momentum of the hadron is constant throughout the evolution. Having in mind that x is the

fraction of momentum carried out by each parton, this concept is translated into the relation

$$\int_0^1 \left(x q^{(+)}(x,Q^2) + x g(x,Q^2) \right) \, dx = 1 \tag{46}$$

that must hold for each value of Q^2. Deriving with respect to $\log(Q^2)$ and using the singlet DGLAP equation one obtains

$$\begin{aligned} \int_0^1 dx\, x \Big\{ \Big[P_{qq}(Q^2) \otimes q^{(+)}(Q^2) \Big] (x) + \big[P_{qg}(Q^2) \otimes g(Q^2) \big] (x) + \\ + \Big[P_{gq}(Q^2) \otimes q^{(+)}(Q^2) \Big] (x) + \big[P_{gg}(Q^2) \otimes g(Q^2) \big] (x) \Big\} = 0 \, . \end{aligned} \tag{47}$$

In a similar fashion, using (75) we also obtain

$$\begin{aligned} \left[\int_0^1 x \left(P_{qq}(x,Q^2) + P_{gq}(x,Q^2) \right) \, dx \right] \left[\int_0^1 x q^{(+)}(x,Q^2) \, dx \right] + \\ + \left[\int_0^1 x \left(P_{qg}(x,Q^2) + P_{gg}(x,Q^2) \right) \, dx \right] \left[\int_0^1 x g(x,Q^2) \, dx \right] = 0 \, , \end{aligned} \tag{48}$$

from which we find the MSR conditions on the singlet kernels

$$\int_0^1 x \left(P_{qq}(x,Q^2) + P_{gq}(x,Q^2) \right) \, dx = 0 \, , \tag{49}$$

$$\int_0^1 x \left(P_{qg}(x,Q^2) + P_{gg}(x,Q^2) \right) \, dx = 0 \, . \tag{50}$$

4.1. **The kernels and their regularization.** We illustrate now an example of the regularization procedure of the DGLAP kernels through the sum rules. The LO kernels computed by diagrammatic techniques for $x < 1$ are

$$P_{qq}^{(0)}(x) = P_{NS}^{(0)}(x) = C_F \left[\frac{1+x^2}{1-x} \right] = C_F \left[\frac{2}{1-x} - 1 - x \right] \tag{51}$$

$$P_{qg}^{(0)}(x) = 2 T_f \left[x^2 + (1-x)^2 \right] \tag{52}$$

$$P_{gq}^{(0)}(x) = C_F \left[\frac{1+(1-x)^2}{x} \right] \tag{53}$$

$$P_{gg}^{(0)}(x) = 2 N_C \left[\frac{1}{1-x} + \frac{1}{x} - 2 + x(1-x) \right] \, . \tag{54}$$

We want to analitically continue these kernels to $x = 1$ curing the ultraviolet singularities in $P_{qq}^{(0)}(x)$ and $P_{gg}^{(0)}(x)$. We start introducing the plus distribution prescription in $P_{qq}^{(0)}(x)$. We make the replacement

$$\frac{1}{1-x} \longrightarrow \frac{1}{(1-x)_+} \tag{55}$$

to avoid the singularity and we add a term $k\delta(1-x)$ (where k has to be determined) to fulfill the BNSR (45). So we have

$$P_{qq}^{(0)}(x) \longrightarrow C_F \left[\frac{2}{(1-x)_+} - 1 - x + k\delta(1-x) \right] \, . \tag{56}$$

Imposing by the BNSR that $P_{qq}^{(0)}(x)$ integrates to zero in $[0,1]$ and remembering that the plus distribution integrates to zero we get

$$\int_0^1 P_{qq}^{(0)}(x) \, dx = C_F \left[-1 - \frac{1}{2} + k \right] = 0 \, , \tag{57}$$

hence $k = 3/2$, and the regularized form of the kernel is

$$P_{qq}^{(0)}(x) = C_F \left[\frac{2}{(1-x)_+} - 1 - x + \frac{3}{2}\delta(1-x) \right] . \tag{58}$$

Noticing that

$$\int_0^1 \frac{x}{(1-x)_+}\, dx = \int_0^1 \frac{x-1+1}{(1-x)_+}\, dx = \int_0^1 \left(-1 + \frac{1}{(1-x)_+} \right) dx = -1 \tag{59}$$

it can be easily proved that the MSR (49) is satisfied. Let us now regularize $P_{gg}(x)$. We make the replacement

$$P_{gg}^{(0)}(x) \longrightarrow 2N_C \left[\frac{1}{(1-x)_+} + \frac{1}{x} - 2 + x(1-x) \right] + k\delta(1-x) . \tag{60}$$

Imposing the other MSR (50) we get

$$\int_0^1 \left\{ 2N_C \left[\frac{x}{(1-x)_+} + 1 - 2x + x^2(1-x) \right] + kx\delta(1-x) + \right. \\ \left. + 2T_f \left[x^3 + x(1-x)^2 \right] \right\} dx = 0 , \tag{61}$$

from which we find

$$k = \frac{11}{6}\, N_C - \frac{2}{3}\, T_f = \frac{\beta_0}{2} , \tag{62}$$

so the regularized form of the kernel is

$$P_{gg}^{(0)}(x) = 2N_C \left[\frac{1}{(1-x)_+} + \frac{1}{x} - 2 + x(1-x) \right] + \frac{\beta_0}{2}\, \delta(1-x) . \tag{63}$$

5. First view: an ansatz from x-space and some examples

In order to solve the evolution equations directly in x-space, we assume solutions of the form

$$f(x,Q^2) = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!}\, \log^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} + \alpha_s(Q^2) \sum_{n=0}^{\infty} \frac{B_n(x)}{n!}\, \log^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} , \tag{64}$$

for each parton distribution f, where Q_0 defines the initial evolution scale. The justification of this ansatz can be found, at least in the case of the photon structure function, in the original work of Rossi [12], and its connection to the ordinary solutions of the DGLAP equations is most easily worked out by taking moments of the scale invariant coefficient functions A_n and B_n and comparing them to the corresponding moments of the parton distributions, as we are going to illustrate in section 5. The link between Rossi's expansion and the solution of the evolution equations (which are ordinary differential equations) in the space of the moments up to NLO will be discussed in that section, from which it will be clear that Rossi's ansatz involves a resummation of the ordinary Mellin moments of the parton distributions.

Setting $Q = Q_0$ in (64) we get

$$f(x,Q_0^2) = A_0(x) + \alpha_s(Q_0^2) B_0(x) . \tag{65}$$

Inserting (64) in the evolution equations, we obtain the following recursion relations for the coefficients A_n and B_n

$$A_{n+1}(x) = -\frac{2}{\beta_0}\, P^{(0)}(x) \otimes A_n(x) , \tag{66}$$

$$
\begin{aligned}
B_{n+1}(x) &= -B_n(x) - \frac{\beta_1}{4\pi\beta_0}\, A_{n+1}(x) - \\
&- \frac{2}{\beta_0}\, P^{(0)}(x) \otimes B_n(x) - \frac{1}{\pi\beta_0}\, P^{(1)}(x) \otimes A_n(x)
\end{aligned}
\tag{67}
$$

obtained by equating left-hand sides and right-hand-side of the equation of the same logarithmic power in $\log^n \alpha_s(Q^2)$ and $\alpha_s \log^n \alpha_s(Q^2)$. Any boundary condition satisfying (65) can be chosen at the lowest scale Q_0 and in our case we choose

$$
B_0(x) = 0 \ , \ \ f(x, Q_0^2) = A_0(x) \ . \tag{68}
$$

In the numerical implementations of this procedure one has to be careful in the manipulations of the numerical integrals, being all the kernels defined as distributions. Since the distributions are integrated, there are various ways to render the integrals finite, as discussed in the previous literature on the method [13] in the case of the photon structure function. In these previous studies the edge-point contributions - i.e. the terms which multiply $\delta(1-x)$ in the kernels - are approximated using a sequence of functions converging to the δ function in a distributional sense.

This technique is not very efficient. We think that the best way to proceed is to actually perform the integrals explicitly in the recursion relations and let the subtracting terms appear under the same integral together with the bulk contributions ($x < 1$). This procedure is best exemplified by the integral relation

$$
\int_x^1 \frac{dy}{y(1-y)_+}\, f(x/y) = \int_x^1 \frac{dy}{y}\, \frac{y f(y) - x f(x)}{y - x} - \log(1-x) f(x) \tag{69}
$$

in which, on the right hand side, regularity of both the first and the second term is explicit. For instance, the evolution equations become (prior to separation between singlet and non-singlet sectors) in the unpolarized case

$$
\begin{aligned}
\frac{dq_i(x)}{d\log(Q^2)} &= 2C_F \int \frac{dy}{y}\, \frac{y q_i(y) - x q_i(x)}{y - x} + \\
&+ 2C_F \log(1-x) q_i(x) - \int_x^1 \frac{dy}{y}\, (1+z)\, q_i(y) + \frac{3}{2}\, C_F q(x) + \\
&+ n_f \int_x^1 \frac{dy}{y} \left(z^2 + (1-z)^2\right) g(y) \\
\frac{dg(x)}{d\log(Q^2)} &= C_F \int_x^1 \frac{dy}{y}\, \frac{1 + (1-z)^2}{z} q_i(y) + \\
&+ 2N_c \int_x^1 \frac{dy}{y}\, \frac{y f(y) - x f(x)}{y - x}\, g(y) + \\
&+ 2N_c \log(1-x) g(x) + 2N_c \int_x^1 \frac{dy}{y} \left(\frac{1}{z} - 2 + z(1-z)\right) g(y) + \frac{\beta_0}{2}\, g(x)
\end{aligned}
\tag{70}
$$

with $z \equiv x/y$. Here q are fixed flavour distributions.

6. The evolution of the transverse spin distributions

We show here an example, worked out in some detail, that illustrates the kind of manipulations that are involved in the computation of the recursion relations.

In fact, leading order (LO) and NLO recursion relations for the coefficients of the expansion can be worked out quite easily. We illustrate here in detail the implementation of a

non-singlet evolutions, such as those involving transverse spin distributions. For the first recursion relation (66) in this case we have

$$A^{\pm}_{n+1}(x) = -\frac{2}{\beta_0}\,\Delta_T P^{(0)}_{qq}(x)\otimes A^{\pm}_n(x) =$$

$$= C_F\left(-\frac{4}{\beta_0}\right)\left[\int_x^1 \frac{dy}{y}\,\frac{yA^{\pm}_n(y)-xA^{\pm}_n(x)}{y-x} + A^{\pm}_n(x)\log(1-x)\right] + \tag{71}$$

$$+C_F\left(\frac{4}{\beta_0}\right)\left(\int_x^1 \frac{dy}{y}\,A^{\pm}_n(y)\right) + C_F\left(-\frac{2}{\beta_0}\right)\frac{3}{2}\,A^{\pm}_n(x)\,.$$

As we move to NLO, it is convenient to summarize the structure of the transverse kernel $\Delta_T P^{\pm,(1)}_{qq}(x)$ [14] as

$$\Delta_T P^{\pm,(1)}_{qq}(x) = K^{\pm}_1(x)\delta(1-x) + K^{\pm}_2(x)S_2(x) + K^{\pm}_3(x)\log(x) + \tag{72}$$

$$+K^{\pm}_4(x)\log^2(x) + K^{\pm}_5(x)\log(x)\log(1-x) + K^{\pm}_6(x)\frac{1}{(1-x)_+} + K^{\pm}_7(x)\,.$$

Hence, for the (+) case we have

$$\Delta_T P^{+,(1)}_{qq}(x)\otimes A^{+}_n(x) = K^{+}_1 A^{+}_n(x) +$$

$$+\int_x^1 \frac{dy}{y}\left[K^{+}_2(z)S_2(z) + K^{+}_3(z)\log(z) + \log^2(z)K^{+}_4(z) + \right.$$

$$\left. +\log(z)\log(1-z)K^{+}_5(z)\right]A^{+}_n(y) + \tag{73}$$

$$+K^{+}_6\left\{\int_x^1 \frac{dy}{y}\,\frac{yA^{+}_n(y)-xA^{+}_n(x)}{y-x} + A^{+}_n(x)\log(1-x)\right\} +$$

$$+K^{+}_7\int_x^1 \frac{dy}{y}\,A^{+}_n(y)\,,$$

where $z = x/y$. For the $(-)$ case we get a similar expression.
For the $B^{\pm}_{n+1}(x)$ we get (for the (+) case)

$$B^{+}_{n+1}(x) = -B^{+}_n(x) + \frac{\beta_1}{2\beta_0^2}\left\{2C_F\left[\int_x^1 \frac{dy}{y}\,\frac{yA^{+}_n(y)-xA^{+}_n(x)}{y-x} + A^{+}_n(x)\log(1-x)\right] - \right.$$

$$\left. -2C_F\left(\int_x^1 \frac{dy}{y}\,A^{+}_n(y)\right) + C_F\frac{3}{2}\,A^{+}_n(x)\right\} - \frac{1}{4\pi\beta_0}\,K^{+}_1 A^{+}_n(x) + \int_x^1 \frac{dy}{y}\left[K^{+}_2(z)S_2(z) + \right.$$

$$\left. +\,K^{+}_3(z)\log(z) + \log^2(z)K^{+}_4(z) + \log(z)\log(1-z)K^{+}_5(z)\right]\left(-\frac{1}{4\pi\beta_0}\right)A^{+}_n(y) +$$

$$+K^{+}_6\left(-\frac{1}{4\pi\beta_0}\right)\left\{\left[\int_x^1 \frac{dy}{y}\,\frac{yA^{+}_n(y)-xA^{+}_n(x)}{y-x} + A^{+}_n(x)\log(1-x)\right] + \right.$$

$$\left. +K^{+}_7\int_x^1 \frac{dy}{y}\,A^{+}_n(y)\right\} - C_F\left(-\frac{4}{\beta_0}\right)\left[\int_x^1 \frac{dy}{y}\,\frac{yB^{\pm}_n(y)-xB^{\pm}_n(x)}{y-x} + \right.$$

$$\left. +B^{\pm}_n(x)\log(1-x)\right] + C_F\left(\frac{4}{\beta_0}\right)\left(\int_x^1 \frac{dy}{y}\,B^{\pm}_n(y)\right) + C_F\left(-\frac{2}{\beta_0}\right)\frac{3}{2}\,B^{\pm}_n(x).$$

The explicit expressions of the K_i can be found in [15].

6.1. **Solutions in moments space and comparisons.** It is interesting to compare the recursive solution with the solution in moment space, that is easy to derive. In moment space the equations become algebraic and can be readily solved. For this we need some definitions.

The n-th Mellin moment of a function of the Bjorken variable $f(x)$ is defined by

$$f_n = \int_0^1 x^{n-1} f(x)\, dx \,. \tag{74}$$

An important property of Mellin moments is that the Mellin moment of the convolution of two functions is equal to the product of the individual Mellin moments

$$[f \otimes g]_n = f_n g_n \,. \tag{75}$$

Let us prove it.

$$[f \otimes g]_n = \int_0^1 dx\, x^{n-1} \left[f \otimes g\right](x) = \int_0^1 dx\, x^{n-1} \int_x^1 \frac{dy}{y}\, f(y) g\left(\frac{x}{y}\right) \,. \tag{76}$$

Exchanging the x and y integrations

$$[f \otimes g]_n = \int_0^1 dy\, f(y) \int_0^y \frac{dx}{y}\, x^{n-1} g\left(\frac{x}{y}\right) \,, \tag{77}$$

and introducing the new variable $z = x/y$

$$[f \otimes g]_n = \int_0^1 dy\, f(y) y^{n-1} \int_0^1 dz\, z^{n-1} g(z) = f_n g_n \,. \tag{78}$$

This leads to an alternative formulation of DGLAP equation, that is also the most widely used to solve numerically the evolution equations. By taking the first Mellin moment of both sides of the integro-differential equation (22) we are left with the differential equation

$$\frac{d}{d \log Q^2}\, f_1(Q^2) = P_1(Q^2) f_1(Q^2) \tag{79}$$

that can be easily solved to give

$$f_1(Q^2) = \int_0^1 f(x, Q^2)\, dx \,. \tag{80}$$

To get the desired solution $f(x, Q^2)$ there is a last step, the inverse Mellin transform of the first moment of the parton distributions, involving a numerical integration on the complex plane. This is the most difficult (and time-consuming) task that the algorithms of solution of DGLAP equation based on Mellin transformation – by far the most widely used – must accomplish. It is particularly instructing to illustrate here briefly the relation between the Mellin moments of the parton distributions, which evolve with standard ordinary differential equations, and those of the arbitrary coefficient $A_n(x)$ and $B_n(x)$ which characterize Rossi's expansion up to next-to leading order (NLO). This relation, as we are going to show, involves a resummation of the ordinary moments of the parton distributions.

Specifically, here we will be dealing with the relation between the Mellin moments of the coefficients appearing in the expansion

$$\begin{aligned} A(N) = &\quad \textstyle\int_0^1 dx\, x^{N-1} A(x) \\ B(N) = &\quad \textstyle\int_0^1 dx\, x^{N-1} B(x) \end{aligned} \tag{81}$$

and those of the distributions

$$\Delta_T q^{(\pm)}(N, Q^2) = \int_0^1 dx\, x^{N-1} \Delta_T q^{(\pm)}(x, Q^2)) \,. \tag{82}$$

For this purpose we recall that the general (non-singlet) solution to NLO for the latter moments is given by

$$\Delta_T q_\pm(N,Q^2) = K(Q_0^2,Q^2,N)\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)^{-2\Delta_T P_{qq}^{(0)}(N)/\beta_0} \Delta_T q_\pm(N,Q_0^2)$$

with the input distributions $\Delta_T q_\pm^n(Q_0^2)$ at the input scale Q_0. We also have set

$$K(Q_0^2,Q^2,N) = 1 + \frac{\alpha_s(Q_0^2)-\alpha_s(Q^2)}{\pi\beta_0}\left[\Delta_T P_{qq,\pm}^{(1)}(N) - \frac{\beta_1}{2\beta_0}\,\Delta_T P_{qq\pm}^{(0)}(N)\right] . \tag{83}$$

In the expressions above we have introduced the corresponding moments for the LO and NLO kernels ($\Delta_T P_{qq}^{(0),N}$, $\Delta_T P_{qq,\pm}^{(1),N}$).

The relation between the moments of the coefficients of the non-singlet $x-$space expansion and those of the parton distributions at any Q, as expressed by equation (83) can be easily written down

$$A_n(N) + \alpha_s B_n(N) = \Delta_T q_\pm(N,Q_0^2) K(Q_0,Q,N)\left(\frac{-2\Delta_T P_{qq}(N)}{\beta_0}\right)^n . \tag{84}$$

As a check of this expression, notice that the initial condition is easily obtained from (84) setting $Q \to Q_0, n \to 0$, thereby obtaining

$$A_0^{NS}(N) + \alpha_s B_0^{NS}(N) = \Delta_T q_\pm(N,Q_0^2) , \tag{85}$$

which can be solved with $A_0^{NS}(N) = \Delta_T q_\pm(N,Q_0^2)$ and $B_0^{NS}(N) = 0$.

It is then evident that the expansion (64) involves a resummation of the logarithmic contributions, as shown in equation (84).

In the singlet sector we can work out a similar relation both to LO

$$A_n(N) = e_1\left(\frac{-2\lambda_1}{\beta_0}\right)^n + e_2\left(\frac{-2\lambda_2}{\beta_0}\right)^n \tag{86}$$

with

$$\begin{aligned} e_1 &= \frac{1}{\lambda_1-\lambda_2}\left(P^{(0)}(N) - \lambda_2\mathbf{1}\right) \\ e_2 &= \frac{1}{\lambda_2-\lambda_1}\left(-P^{(0)}(N) + \lambda_1\mathbf{1}\right) \\ \lambda_{1,2} &= \frac{1}{2}\Bigg(P_{qq}^{(0)}(N) + P_{gg}^{(0)}(N) \pm \\ &\pm\sqrt{\left(P_{qq}^{(0)}(N) - P_{gg}^{(0)}(N)\right)^2 + 4P_{qg}^{(0)}(N)P_{gq}^{(0)}(N)}\Bigg) , \end{aligned} \tag{87}$$

and to NLO

$$A_n(N) + \alpha_s B_n(N) = \chi_1\left(\frac{-2\lambda_1}{\beta_0}\right)^n + \chi_2\left(\frac{-2\lambda_2}{\beta_0}\right)^n , \tag{88}$$

where

$$\begin{aligned} \chi_1 &= e_1 + \frac{\alpha}{2\pi}\left(\frac{-2}{\beta_0}\, e_1\mathbf{R}e_1 + \frac{e_2\mathbf{R}e_1}{\lambda_1-\lambda_2-\beta_0/2}\right) \\ \chi_2 &= e_2 + \frac{\alpha}{2\pi}\left(\frac{-2}{\beta_0}\, e_2\mathbf{R}e_2 + \frac{e_1\mathbf{R}e_2}{\lambda_2-\lambda_1-\beta_0/2}\right) \end{aligned} \tag{89}$$

with

$$\mathbf{R} = P^{(1)}(N) - \frac{\beta_1}{2\beta_0}\, P^{(0)}(N) . \tag{90}$$

We remark, if not obvious, that $A_n(N)$ and $B_n(N)$, $P^{(0)}(N)$, $P^{(1)}(N)$, in this case, are all 2-by-2 singlet matrices.

7. Moving to higher orders

The structure of the solution to higher orders can be worked out in generality. More details can be found in [7], here we just outline the procedure.

The perturbative expansion of the kernels now includes the NNLO contributions and is given by

$$P(x,\alpha_s) = \left(\frac{\alpha_s}{2\pi}\right) P^{(0)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P^{(1)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^3 P^{(2)}(x) + \dots \ . \tag{91}$$

whose specific form can be found in the original literature [16, 17].

We solve Eq. (22) directly in x-space, assuming a solution of the form

$$\begin{aligned} f(x,Q^2) = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!} \ \log^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} + \alpha_s(Q^2) \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} \ \log^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} + \\ + \big(\alpha_s(Q^2)\big)^2 \sum_{n=0}^{\infty} \frac{C_n(x)}{n!} \ \log^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \end{aligned} \tag{92}$$

for each parton distribution f, where Q_0 defines the initial evolution scale.

As in the previous examples, also in this case we derive the following recursion relations for the coefficients A_n, B_n and C_n

$$A_{n+1}(x) = -\frac{2}{\beta_0} \ P^{(0)}(x) \otimes A_n(x) \ , \tag{93}$$

$$\begin{aligned} B_{n+1}(x) = -B_n(x) - \frac{\beta_1}{4\pi\beta_0} \ A_{n+1}(x) - \\ -\frac{2}{\beta_0} \ P^{(0)}(x) \otimes B_n(x) - \frac{1}{\pi\beta_0} \ P^{(1)}(x) \otimes A_n(x) \ , \end{aligned} \tag{94}$$

$$\begin{aligned} C_{n+1}(x) = -2C_n(x) - \frac{\beta_1}{4\pi\beta_0} \ B_n(x) - \frac{\beta_1}{4\pi\beta_0} \ B_{n+1}(x) - \\ -\frac{\beta_2}{16\pi^2\beta_0} \ A_{n+1}(x) - \frac{2}{\beta_0} \ P^{(0)}(x) \otimes C_n(x) - \\ -\frac{1}{\pi\beta_0} \ P^{(1)}(x) \otimes B_n(x) - \frac{1}{2\pi^2\beta_0} \ P^{(2)}(x) \otimes A_n(x) \ . \end{aligned} \tag{95}$$

It is an easy exercise to derive the recursion relations for the coefficients of the expansion. We illustrate the derivation for the interested reader.

We introduce the notation

$$L(Q^2) = \log \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \tag{96}$$

and, making use of the beta function definition (12), we compute its derivative

$$\frac{dL(Q^2)}{d\log Q^2} = \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \ \frac{d}{d\log Q^2} \ \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} = \frac{1}{\alpha_s(Q^2)} \ \frac{d\alpha_s(Q^2)}{d\log Q^2} = \frac{\beta(\alpha_s)}{\alpha_s(Q^2)} \tag{97}$$

Inserting our ansatz (92) for the solution into the DGLAP equation (22) we get for the LHS

$$\sum_{n=1}^{\infty}\left\{\frac{A_n(x)}{n!}\,n\,L^{n-1}\frac{\beta(\alpha_s)}{\alpha_s}+\alpha_s\frac{B_n(x)}{n!}\,n\,L^{n-1}\frac{\beta(\alpha_s)}{\alpha_s}+\right.$$

$$\left.+\alpha_s^2\frac{C_n(x)}{n!}\,n\,L^{n-1}\frac{\beta(\alpha_s)}{\alpha_s}\right\}+ \tag{98}$$

$$+\sum_{n=0}^{\infty}\left\{\beta(\alpha_s)\frac{B_n(x)}{n!}\,L^n+2\alpha_s\beta(\alpha_s)\frac{C_n(x)}{n!}\,L^n\right\}\,.$$

Sending $n\to n-1$ in the first sum, using the three-loop expansion of the beta function (13) and neglecting all the terms of order α_s^4 or more, the previous formula becomes

$$\sum_{n=0}^{\infty}\left\{\frac{A_{n+1}(x)}{n!}\,L^n\left(-\frac{\beta_0}{4\pi}\,\alpha_s-\frac{\beta_1}{16\pi^2}\,\alpha_s^2-\frac{\beta_2}{64\pi^3}\,\alpha_s^3\right)+\right.$$

$$+\frac{B_{n+1}(x)}{n!}\,L^n\left(-\frac{\beta_0}{4\pi}\,\alpha_s^2-\frac{\beta_1}{16\pi^2}\,\alpha_s^3\right)+\frac{C_{n+1}(x)}{n!}\,L^n\left(-\frac{\beta_0}{4\pi}\,\alpha_s^3\right)+ \tag{99}$$

$$\left.+\frac{B_n(x)}{n!}\,L^n\left(-\frac{\beta_0}{4\pi}\,\alpha_s^2-\frac{\beta_1}{16\pi^2}\,\alpha_s^3\right)+2\frac{C_n(x)}{n!}\,L^n\left(-\frac{\beta_0}{4\pi}\,\alpha_s^3\right)\right\}\,.$$

Using the expansion of the kernels (91), we get for the RHS

$$\sum_{n=0}^{\infty}\frac{L^n}{n!}\left\{\frac{\alpha_s}{2\pi}\left[P^{(0)}\otimes A_n\right](x)+\frac{\alpha_s^2}{4\pi^2}\left[P^{(1)}\otimes A_n\right](x)+\right.$$

$$+\frac{\alpha_s^3}{8\pi^3}\left[P^{(2)}\otimes A_n\right](x)+\frac{\alpha_s^2}{2\pi}\left[P^{(0)}\otimes B_n\right](x)+ \tag{100}$$

$$\left.+\frac{\alpha_s^3}{4\pi^2}\left[P^{(1)}\otimes B_n\right](x)+\frac{\alpha_s^3}{2\pi}\left[P^{(0)}\otimes C_n\right](x)\right\}\,.$$

Equating (99) and (100) term by term and grouping the terms proportional respectively to α_s, α_s^2 and α_s^3 we get the three desired recursion relations (93), (94) and (95).

Setting $Q=Q_0$ in (92) we get

$$f(x,Q_0^2)=A_0(x)+\alpha_s(Q_0^2)B_0(x)+\left(\alpha_s(Q^2)\right)^2C_0(x)\,. \tag{101}$$

We solve these relations using a boundary condition satisfying (101). In our case we choose

$$B_0(x)=C_0(x)=0\ \ ,\ \ f(x,Q_0^2)=A_0(x)\,. \tag{102}$$

The procedure studied in the previous section can be generalized and applied to obtain solutions that retain higher order logarithmic contributions in the NLO/NNLO singlet cases. The same procedure is also the one that has been implemented in all the existing codes for the singlet: one has to truncate the equation and then try to reach the exact solution by a sufficiently high number of iterates. These arguments can be extended to even higher orders. We have recently shown [7] that the exact solution to all orders of the RGE's of the p.d.f.'s can be written down in close form by the introduction of a generalized logarithmic expansion that includes contributions of all powers of the strong coupling constant α_s and are summarized by the expression

$$f(x,Q^2)=\sum_{n=0}^{\infty}\frac{1}{n!}\,C_n(x,\alpha_s)\log^n\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right) \tag{103}$$

where the function

$$C_n(x,\alpha_s,\alpha_0) = \sum_{k=0}^{\infty} \alpha_s^k A_n^{(k)}(x) \tag{104}$$

is determined perturbatively by the recursion relations extracted from the RGE. The analysis is quite involved and can be found in the original literature.

8. Other solutions in x-space: the Laguerre expansion

Another possible way to solve the evolution equations is to use an operatorial formalism and observe that the distributions can be expanded in a suitable orthonormal basis (Laguerre polynomials). This method has had limited applications due to the difficulty to handle numerically the small-x behavior of the algorithm that constructs the solution. The implementation of the method is discussed in [6]. One starts by writing the expression of the solution in operatorial form in terms of two singlet evolution operators $E_\pm(t,x)$ and initial conditions $\tilde{q}_\pm(x,t=0) \equiv \tilde{q}_\pm(x)$ as

$$\frac{d}{dt}E_\pm = P_\pm \otimes E_\pm \ , \tag{105}$$

whose solutions are given by

$$\begin{aligned} q_i^{(-)}(t,x) &= E_{(-)} \otimes \tilde{q}_i^{(-)} \\ \chi_i(t,x) &= E_{(+)} \otimes \tilde{\chi}_i(x) \ . \end{aligned} \tag{106}$$

The singlet evolution for the matrix operator $E(t,x)$

$$\begin{pmatrix} E_{FF} & E_{FG} \\ E_{GF} & E_{GG} \end{pmatrix} \tag{107}$$

$$\frac{dE}{dt} = P \otimes E \tag{108}$$

is solved similarly as

$$\begin{pmatrix} q^{(+)}(t,x) \\ G(t,x) \end{pmatrix} = E(t,x) \otimes \begin{pmatrix} \tilde{q}^{(+)}(x) \\ \tilde{G}(x) \end{pmatrix} \ . \tag{109}$$

This method, proposed in [8], requires an expansion of the splitting functions and of the parton distributions in the basis of the Laguerre polynomials

$$L_n(y) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k \frac{y^k}{k!} \tag{110}$$

which satisfies the property of closure under a convolution

$$L_n(y) \otimes L_m(y) = L_{n+m}(y) - L_{n+m+1}(y) \ . \tag{111}$$

In order to improve the small-x behaviour of the algorithm, from now on, the evolution is applied to the modified kernel $xP(x)$, which, for simplicity, is still denoted as in all the equations above, i.e. by $P(x)$. At a second step, the $0 < x < 1$ interval is mapped into an infinite interval $0 < y < \infty$ by a change of variable $x = e^{-y}$ and all the integrations are performed in this last interval. We start from the non-singlet case by defining the Laguerre

expansion of the kernels and the corresponding (Laguerre) moments to lowest order

$$P_V^{(0)}(y) = \sum_{n=0}^{\infty} P_n^{(0)} L_n(y) \ ,$$
$$P_n^{(0)} = \int_0^{\infty} dy\, e^{-y} L_n(y)\, P^{(0)}(y) \tag{112}$$

and to NLO

$$R(y) = \sum_{n=0}^{\infty} R_n L_n(y) \ . \tag{113}$$

One defines also the difference of moments

$$\begin{aligned} p_i^{(0)} &= P_i^{(0)} - P_{i-1}^{(0)} \qquad (P_{-1}^{(0)} = 0) \\ r_i &= R_i - R_{i-1} \qquad R_{-1} = 0 \ . \end{aligned} \tag{114}$$

A similar expansion is set up for the evolution operators $E(t, y)$

$$\begin{aligned} E^{(0)}(t,y) &= \sum_{n=0}^{\infty} E_n^{(0)}(t) L_n(y) \\ E(t,y) &= \sum_{n=0}^{\infty} E_n(t) L_n(y) \ , \end{aligned} \tag{115}$$

where all the information on the t evolution is contained in the moments $E_n(t)$. The solution to NLO is expressed as

$$E_n(t) = E_n^{(0)}(t) - \frac{2}{\beta_0} \frac{\alpha(t) - \alpha(0)}{2\pi} E_n^{(1)}(t) \ , \tag{116}$$

where

$$E_n^{(0)}(t) = e^{P_0^{(0} t} \sum_{k=0}^{n} \frac{A_n^{(k)} t^k}{k!} \tag{117}$$

$$E_n^{(1)}(t) = \sum_{i}^{n} r_{n-i} E_i^{(0)}(t) \ , \tag{118}$$

and the coefficients $A_n^{(k)}$ are determined recursively from the moments of the lowest order kernel $P^{(0)}$

$$\begin{aligned} A_n^{(0)} &= 1 \\ A_n^{(k+1)} &= \sum_{i=k}^{n-1} p_{n-i}^{(0)} A_i^{(k)} \quad (k = 0, 1, 2, \cdots, n-1) \ . \end{aligned} \tag{119}$$

In the singlet case one proceeds in a similar way. The solution is expressed in terms of a 2-by-2 matrix operator

$$E^{(0)}(t,y) = \sum_{n=0}^{\infty} E_n^{(0)}(t) L_n(y) \ . \tag{120}$$

The solution (at leading order) is written down in terms of 2 projection matrices and one eigenvalue (λ) of the $P^{(0)}$ (matrix) kernel

$$e_1 = \frac{1}{\lambda} P^{(0)} \ , \ \ e_2 = \frac{1}{\lambda} \left(-P^{(0)} + \lambda \mathbf{1} \right) \ , \tag{121}$$

where

$$\lambda = -(\frac{4}{3}\, C_F + \frac{2}{3}\, n_f\, T_R)\ , \tag{122}$$

in the form

$$E_n^{(0)}(t) = \sum_{k=0}^{n} \frac{t^k}{k!} \left(A_n^{(k)} + B_n^{(k)} e^{\lambda t} \right)\ . \tag{123}$$

The recursion relations which allow to build $A_n^{(k)}$ and $B_n^{(k)}$ are solved in two steps as follows. One solves first for two sets of matrices $a_n^{(k)}$ and $b_n^{(k)}$ by the relations

$$\begin{aligned} a_n^{(0)} &= \quad 0 \\ a_n^{(k+1)} &= \quad \lambda e_1 a_n^{(k)} + \sum_{i=k}^{n-1} p_{n-i}^{(0)} a_i^{(k)} \\ b_n^{(0)} &= \quad 0 \\ b_n^{(k+1)} &= \quad -\lambda e_2 b_n^{(k)} + \sum_{i=1}^{n-1} p_{n-i}^{(0)} b_i^{(k)}\ , \end{aligned} \tag{124}$$

which are used to construct the matrices $A_n^{(0)}$ and $B_n^{(0)}$

$$\begin{aligned} A_n^{(0)} &= \quad e_2 - \frac{1}{\lambda^n} \left(e_1 a_n^{(n)} - (-1)^n e_2 b_n^{(n)} \right) \\ B_n^{(0)} &= \quad e_1 + \frac{1}{\lambda^n} \left(e_1 a_n^{(n)} - (-1)^n e_2 b_n^{(n)} \right)\ . \end{aligned} \tag{125}$$

These matrices are then input in the recursion relations

$$\begin{aligned} A_0^{(0)} &= \quad e_2 \qquad\qquad B_0^{(0)} = e_1 \\ A_n^{(k+1)} &= \quad \lambda e_1 A_n^{(k)} + \sum_{i=k}^{n-1} p_{n-i}^{(0)} A_i^k \\ B_n^{k+1} &= \quad -\lambda e_2 B_n^{(k)} + \sum_{i=k}^{n-1} p_{n-i}^{(0)} B_i^{(k)} \end{aligned} \tag{126}$$

with $n > 0$ and $k = 0, 1, \cdots, n-1$, which generates the coefficients of the matrix-valued operator $E^{(0)}$ (i.e. the leading order solution). The NLO part of the evolution is obtained from

$$E^{(1)}(t, y) = \sum_{n=0}^{\infty} E_n^{(1)}(t) L_n(y)\ , \tag{127}$$

with

$$E_n^{(1)}(t) = \tilde{E}_n^{(1)}(t) - 2\tilde{E}_{n-1}^{(1)}(t) + \tilde{E}_{n-2}^{(1)}(t) \tag{128}$$

where

$$\tilde{E}_n^{(1)}(t) = \int_0^t d\tau e^{-\beta_0 \tau/2} \sum_{ijk} E^{(0)}(t-\tau) R_j E_k^{(0)}(\tau) \delta(n-i-j-k)\ . \tag{129}$$

The expressions of $E^{(0)}$ and $E^{(1)}$ are inserted into eq. (116) thereby providing a complete NLO solution of the singlet sector. The implementation of this method to NLO is quite involved and provides an accuracy which is comparable to other methods of solutions.

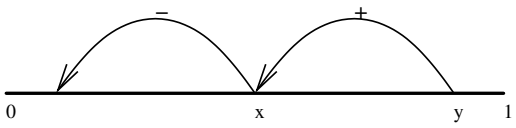

FIGURE 1. The constrained random walk of the parton densities.

9. Second view: the master form of the evolution equations and a hierarchy

A second part of our brief review is dedicated to the discussion of the kinetic structure that underlines an evolution equation of DGLAP type. We recall that master equations are introduced in statistical mechanics in order to describe the evolution in time of some diffusive processes. One can take as an example an ensemble of random walkers moving on a given lattice. The density of walkers present in a certain point on the lattice can be described by a master equation and expressed in terms of transition probabilities. The master form of the DGLAP equation can be derived quite simply by performing some manipulations on the kernels, having separated the bulk contributions ($x < 1$) from the edge points ($x = 1$). Below we show how this is obtained and how one can contruct, by a formal Kramers-Moyal expansion of the transition probabilities, a hierarchy of PDE's of arbitrarily high orders.

Let's start considering a generic 1-D master equation for transition probabilities $w(x|x')$ which we interpret as the probability of making a transition to a point x given a starting point x' for a given physical system. The picture we have in mind is that of a gas of particles making collisions in 1-D and entering the interval $(x, x + dx)$ with a probability $w(x|x')$ per single transition, or leaving it with a transition probability $w(x'|x)$. In general one writes down a master equation

$$\frac{\partial}{\partial\tau} f(x,\tau) = \int dx' \left(w(x|x') f(x',\tau) - w(x'|x) f(x,\tau) \right) dx' \,. \tag{130}$$

describing the time τ evolution of the density of the gas undergoing collisions or the motion of a many replicas of walkers of density $f(x, \tau)$ jumping with a pre-assigned probability, according to taste.

The result of Collins a Qiu, who were after a derivation of the DGLAP equation that could include automatically also the "edge point" contributions (or x=1 terms of the DGLAP kernels) is in pointing out the existence of a probabilistic picture of the DGLAP dynamics. These edge point terms had been always introduced in the past only by hand and serve to enforce the baryon number sum rule and the momentum sum rule as Q, the momentum scale, varies.

The kinetic interpretation was used in [18] to provide an alternative proof of Soffer's inequality. We recall that this inequality

$$|h_1(x)| < q^+(x) \tag{131}$$

famous by now, sets a bound on the transverse spin distribution $h_1(x)$ in terms of the components of the positive helicity component of the quarks, for a given flavour. The inequality has to be respected by the evolution. We recall that h_1, also denoted by the symbol

$$\Delta_T q(x, Q^2) \equiv q^\uparrow(x, Q^2) - q^\downarrow(x, Q^2) \,, \tag{132}$$

has the property of being purely non-singlet and of appearing at leading twist. It is identifiable in transversely polarized hadron-hadron collisions and not in Deep Inelastic Scattering (DIS), where can appear only through an insertion of the electron mass in the unitarity graph of DIS.

The connection between the Collins-Qiu form of the DGLAP equation and the master equation is established as follows. The DGLAP equation, in its original formulation is generically written as

$$\frac{dq(x,Q^2)}{d\log(Q^2)} = \int_x^1 \frac{dy}{y} P(x/y) q(y,Q^2) \ , \tag{133}$$

where we are assuming a scalar form of the equation, such as in the non-singlet sector. The generalization to the singlet sector of the arguments given below is, of course, quite straightforward. To arrive at a probabilistic picture of the equation we start reinterpreting $\tau = \log(Q^2)$ as a time variable, while the parton density $q(x,\tau)$ lives in a one dimensional (Bjorken) x space.

We recall that the kernels are defined as "plus" distributions. Conservation of baryon number, for instance, is enforced by the addition of edge-point contributions proportional to $\delta(1-x)$.

We start with the following form of the kernel

$$P(z) = \hat{P}(z) - \delta(1-z) \int_0^1 \hat{P}(z)\, dz \ , \tag{134}$$

where we have separated the edge point contributions from the rest of the kernel, here called $\hat{P}(z)$. This manipulation is understood in all the equations that follow. The equation is rewritten in the following form

$$\frac{d}{d\tau} q(x,\tau) = \int_x^1 dy \hat{P}\left(\frac{x}{y}\right) \frac{q(y,\tau)}{y} - \int_0^x \frac{dy}{y}\ \hat{P}\left(\frac{y}{x}\right) \frac{q(x,\tau)}{x} \tag{135}$$

Now, if we define

$$w(x|y) = \frac{\alpha_s}{2\pi}\ \hat{P}(x/y) \frac{\theta(y>x)}{y} \tag{136}$$

(135) becomes a master equation for the probability function $q(x,\tau)$

$$\frac{\partial}{\partial \tau}\ q(x,\tau) = \int dx' \left(w(x|x') q(x',\tau) - w(x'|x) q(x,\tau)\right)\ dx' \ . \tag{137}$$

There are some interesting features of this special master equation. Differently from other master equations, where transitions are allowed from a given x both toward $y > x$ and $y < x$, in this case, transitions toward x take place only from values $y > x$ and leave the momentum cell $(x, x+dx)$ only toward smaller y values (see Figure (1)).

Clearly, this sets a direction of the kinetic evolution of the densities from large x values toward smaller-x values as τ, the fictitious "time" variable, increases.

Probably this is the simplest illustration of the fact that parton densities, at large final evolution scales, are dominated by their small-x behaviour. As the "randomly moving partons" reach the $x \approx 0$ region of momentum space, they find no space where to go, while other partons tend to pile up toward the same region from above. This is the picture of a random walk biased to move downward (toward the small-x region) and is illustrated in Figure (1).

10. Probabilistic kernels

We briefly discuss some salient features of the structure of the kernels in this approach and comment on the type of regularization involved in order to define them appropriately.

We recall that unpolarized and polarized kernels, in leading order, are given by

$$
\begin{aligned}
P_{NS}^{(0)} = \quad & P_{qq}^{(0)} = C_F \left(\frac{2}{(1-x)_+} - 1 - x + \frac{3}{2}\,\delta(1-x) \right) \\
P_{qg}^{(0)} = \quad & 2T_f \left(x^2 + (1-x)^2 \right) \big) \\
P_{gq}^{(0)} = \quad & C_F \frac{1+(1-x)^2}{x} \\
P_{gg}^{(0)} = \quad & 2N_c \left(\frac{1}{(1-x)_+} + \frac{1}{x} - 2 + x(1-x) \right) + \frac{\beta_0}{2}\,\delta(1-x)
\end{aligned}
\tag{138}
$$

where

$$
C_F = \frac{N_C^2 - 1}{2N_C} \quad , \quad T_f = T_R n_f = \frac{1}{2}\,n_f \quad , \quad \beta_0 = \frac{11}{3}\,N_C - \frac{4}{3}\,T_f
$$

and

$$
\begin{aligned}
\Delta P_{NS}^{(0)} &= \Delta P_{qq}^{(0)} \\
\Delta P_{qq}^{(0)} &= C_F \left(\frac{2}{(1-x)_+} - 1 - x + \frac{3}{2}\,\delta(1-x) \right) \\
\Delta P_{qg}^{(0)} &= 2T_f(2x-1) \\
\Delta P_{gq}^{(0)} &= C_F(2-x) \\
\Delta P_{gg}^{(0)} &= 2N_c \left(\frac{1}{(1-x)_+} - 2x + 1 \right) + \delta(1-x)\frac{\beta_0}{2}\,,
\end{aligned}
\tag{139}
$$

while the LO transverse kernels are given by

$$
\Delta_T P_{qq}^{(0)} = C_F \left(\frac{2}{(1-x)_+} - 2 + \frac{3}{2}\,\delta(1-x) \right)\,. \tag{140}
$$

The unpolarized kernels should be compared with the Collins-Qiu form

$$
\begin{aligned}
P_{qq} &= \gamma_{qq} - \delta(1-x) \int_0^1 dz \gamma_{qq} \\
P_{gg} &= \gamma_{gg} - \left(n_f \int_0^1 dz\,\gamma_{qg} + \frac{1}{2} \int_0^1 dz\,\gamma_{gg} \right) \delta(1-x) \\
P_{qg} &= \gamma_{qg} \\
P_{gq} &= \gamma_{gq}
\end{aligned}
\tag{141}
$$

where

$$
\begin{aligned}
\gamma_{qq} &= C_F \left(\frac{2}{1-x} - 1 - x \right) \\
\gamma_{qg} &= (2x-1) \\
\gamma_{gq} &= C_F(2-x) \\
\gamma_{gg} &= 2N_c \left(\frac{1}{1-x} + \frac{1}{x} - 2 + x(1-x) \right)\,.
\end{aligned}
\tag{142}
$$

These kernels need a suitable regularization to be well defined. Below we will analyze the implicit regularization underlying eq. (141). One observation is however almost immediate: the component P_{gg} is not of the form given by eq. (134). In general, therefore, in the singlet case, the generalization of eq. (134) is given by

$$P(x) = \hat{P}_1(x) - \delta(1-x)\int_0^1 \hat{P}_2(z)\,dz \tag{143}$$

and a probabilistic interpretation is more complex compared to the non-singlet case and has been discussed in the original literature [9].

11. Convolutions and master form of the singlet

Distributions are folded with the kernels and the result rearranged in order to simplify the structure of the equations. Since in the previous literature this is done in a rather involuted way [19] we provide here a simplificaton, from which the equivalence of the various forms of the kernel, in the various regularizations adopted, will be apparent. All we need is to apply (69) and the evolution equations become

$$\frac{dq}{d\log(Q^2)} = 2C_F\int\frac{dy}{y}\,\frac{yq(y)-xq(x)}{y-x} + 2C_F\log(1-x)\,q(x) -$$

$$-\int_x^1\frac{dy}{y}\,(1+z)\,q(y) + \frac{3}{2}\,C_F q(x) + n_f\int_x^1\frac{dy}{y}\left(z^2+(1-z)^2\right)g(y)$$

$$\frac{dg}{d\log(Q^2)} = C_F\int_x^1\frac{dy}{y}\,\frac{1+(1-z)^2}{z}\,q(y) + \tag{144}$$

$$+2N_c\int_x^1\frac{dy}{y}\,\frac{yf(y)-xf(x)}{y-x}\,g(y) + 2N_c\log(1-x)g(x) +$$

$$+2N_c\int_x^1\frac{dy}{y}\left(\frac{1}{z}-2+z(1-z)\right)g(y) + \frac{\beta_0}{2}\,g(x)$$

with $z \equiv x/y$. The same simplified form is obtained from the probabilistic version, having defined a suitable regularization of the edge point singularities in the integrals over the components $\gamma_{ff'}$ in eq. (142). The canonical expressions of the kernels (139), expressed in terms of "+" distributions, can also be rearranged to look like their equivalent probabilistic form by isolating the edge-point contributions hidden in their "+" distributions. We get the expressions

$$P_{qq}^{(0)}{}_{NS} = P_{qq}^{(0)} = C_F\left(\frac{2}{(1-x)} - 1 - x\right) -$$

$$-\left(C_F\int_0^1\frac{dz}{1-z} - \frac{3}{2}\right)\delta(1-x)$$

$$P_{gg}^{(0)} = 2N_c\left(\frac{1}{(1-x)} + \frac{1}{x} - 2 + x(1-x)\right) - \tag{145}$$

$$-\left(2N_c\int_0^1\frac{dz}{1-z} - \frac{\beta_0}{2}\right)\delta(1-x)$$

and

$$\begin{aligned}\Delta P_{qq}^{(0)} &= C_F\left(\frac{2}{(1-x)}-1-x\right)-C_F\left(\int_0^1\frac{dz}{1-z}-\frac{3}{2}\right)\delta(1-x)\\ \Delta P_{gg}^{(0)} &= 2N_c\left(\frac{1}{1-x}-2x+1\right)-\left(2N_c\int_0^1\frac{dz}{1-z}-\frac{\beta_0}{2}\right)\delta(1-x)\ ,\end{aligned}\tag{146}$$

the other expressions remaining invariant.

A master form of the singlet (unpolarized) equation is obtained by a straightforward change of variable in the decreasing terms. We obtain

$$\begin{aligned}\frac{dq}{d\tau} &= \int_x^{1-\Lambda}\frac{dy}{y}\,\gamma_{qq}(x/y)q(y)-\int_0^{x-\Lambda}\frac{dy}{y}\,\gamma_{qq}(y/x)q(x)\\ \frac{dg}{d\tau} &= \int_x^{1-\Lambda}\frac{dy}{y}\,\gamma_{gg}(x/y)-n_f\int_0^x\gamma_{qg}(y/x)g(x)-\\ &\quad-\frac{1}{2}\int_\Lambda^{x-\Lambda}\gamma_{gg}(y/x)g(x)+\int_x^1\frac{dy}{y}\,\gamma_{gq}(x/y)q(y)\end{aligned}\tag{147}$$

with a suitable (unique) cutoff Λ needed to cast the equation in the form (144). The (regulated) transition probabilities are then given by

$$\begin{aligned}w_{qq}(x|y) &= \gamma_{qq}(x/y)\,\frac{\theta(y>x)\theta(y<1-\Lambda)}{y}\\ w_{qq}(y|x) &= \gamma_{qq}(y/x)\,\frac{\theta(y<x-\Lambda)\theta(y>0)}{x}\\ w_{gg}(x|y) &= \gamma_{gg}(x/y)\,\frac{\theta(y>x)\theta(y<1-\Lambda)}{y}\\ w_{qq}(y|x) &= \left(n_f\gamma_{qg}(y/x)-\frac{1}{2}\,\gamma_{gg}(y/x)\right)\frac{\theta(y<x-\Lambda)\theta(y>0)}{x}\\ w_{gq}(y|x) &= \gamma_{gq}(x/y)\,\frac{\theta(y>x)\theta(y<1-\Lambda)}{y}\\ w_{gq}(x|y) &= 0\ ,\end{aligned}\tag{148}$$

as one can easily deduct from the form of eq. (137).

12. A Kramers-Moyal expansion for the DGLAP equation

A way to get rid of the integrals that appear in the master equation is the Kramers-Moyal (KM) expansion. These expansions (backward or forward) are sometimes useful in order to gain insight into the master equation itself, since they may provide a complementary view of the underlying dynamics.

The expansion introduces differential operator of arbitrary order, due to the nonlocal structure of the equation. For the approximation to be useful, one has to stop the expansion after the first few orders and in many cases this turns out to be possibile. Examples of processes of this type are special Langevin processes and processes described by a Fokker-Planck operator. In these cases the probabilistic interpretation allows us to write down a fictituous lagrangean, a corresponding path integral and solve for the propagators using the Feynman-Kac fomula. For definitess we take the integral to cover all the real axis in the

variable x'

$$\frac{\partial}{\partial\tau}\, q(x,\tau) = \int_{-\infty}^{\infty} dx'\, (w(x|x')q(x',\tau) - w(x'|x)q(x,\tau))\, dx' \,. \tag{149}$$

As we will see below, in the DGLAP case some modifications to the usual form of the KM expansion will appear. At this point we perform a KM expansion of the equation in the usual way. We make the substitutions in the master equation $y \to x - y$ in the first term and $y \to x + y$ in the second term

$$\frac{\partial}{\partial\tau}\, q(x,\tau) = \int_{-\infty}^{\infty} dy\, (w(x|x-y)q(x-y,\tau) - w(x+y|x)q(x,\tau)) \,, \tag{150}$$

identically equal to

$$\begin{aligned} &\frac{\partial}{\partial\tau}\, q(x,\tau) = \\ &= \int_{-\infty}^{\infty} dy\, (w(x+y-y'|x-y')q(x-y',\tau) - w(x+y'|x)q(x,\tau)) \,, \end{aligned} \tag{151}$$

with $y = y'$. First and second term in the equation above differ by a shift (in $-y'$) and can be related using a Taylor (or KM) expansion of the first term

$$\frac{\partial}{\partial\tau}\, q(x,\tau) = \int_{-\infty}^{\infty} dy \sum_{n=1}^{\infty} \frac{(-y)^n}{n!}\, \frac{\partial^n}{\partial x^n}\, (w(x+y|x)q(x,\tau)) \tag{152}$$

where the $n = 0$ term has canceled between the first and the second contribution coming from (151). The result can be written in the form

$$\frac{\partial}{\partial\tau}\, q(x,\tau) = \sum_{n=1}^{\infty} \frac{(-y)^n}{n!}\, \frac{\partial^n}{\partial x^n}\, (a_n(x)q(x,\tau)) \tag{153}$$

where

$$a_n(x) = \int_{-\infty}^{\infty} dy (y-x)^n w(y|x) \,. \tag{154}$$

In the DGLAP case we need to amend the former derivation, due to the presence of boundaries ($0 < x < 1$) in the Bjorken variable x. For simplicity we will focus on the non-singlet case. We rewrite the master equation using the same change of variables used above

$$\begin{aligned} &\frac{\partial}{\partial\tau}\, q(x,\tau) = \int_x^1 dy\; w(x|y)q(y,\tau) - \int_0^x dy\; w(y|x)q(x,\tau) - \\ &- \int_0^{\alpha(x)} dy\; w(x+y|x) * q(x,\tau) + \int_0^{-x} dy\; w(x+y|x)q(x,\tau) \,, \end{aligned} \tag{155}$$

where the Moyal product gets simplified

$$w(x+y|x) * q(x) \equiv w(x+y|x) e^{-y\left(\overleftarrow{\partial}_x + \overrightarrow{\partial}_x\right)} q(x,\tau) \tag{156}$$

and $\alpha(x) = x - 1$. The expansion is of the form

$$\begin{aligned} &\frac{\partial}{\partial\tau}\, q(x,\tau) = \int_{\alpha(x)}^{-x} dy\; w(x+y|x)q(x,\tau) - \\ &- \sum_{n=1}^{\infty} \int_0^{\alpha(x)} dy\; \frac{(-y)^n}{n!}\; {\partial_x}^n\, (w(x+y|x)q(x,\tau)) \end{aligned} \tag{157}$$

which can be reduced to a differential equation of arbitrary order using simple manipulations. We recall that the Fokker-Planck approximation is obtained stopping the expansion at the second order

$$\frac{\partial}{\partial\tau}\, q(x,\tau) = a_0(x) - \partial_x\left(a_1(x)q(x)\right) + \frac{1}{2}\,\partial_x^2\left(a_2(x)q(x,\tau)\right) \tag{158}$$

with

$$a_n(x) = \int dy\, y^n\, w(x+y,x) \tag{159}$$

being moments of the transition probability function w. Given the boundary conditions on the Bjorken variable x, even in the Fokker-Planck approximation, the Fokker-Planck version of the DGLAP equation is slightly more involved than Eq. (158) and the coefficients $a_n(x)$ need to be redefined.

13. The Fokker-Planck approximation

Having identified a probabilistic description of the DGLAP equation in terms of a master equation, it is then natural to try to investigate the role of the Fokker-Planck (FP) approximation to it. In the context of a random walk, an all-order derivative expansion of the master equation can be arrested to the first few terms either if the conditions of Pawula's theorem are satisfied -in which case the FP approximation turns out to be exact- or if the transition probabilities show an exponential decay above a certain distance allowed to the random walk. Since the DGLAP kernels show only an algebraic decay in x, and there isn't any explicit scale in the kernel themselves, the expansion is questionable. However, from a formal viewpoint, it is still allowed. With these caveats in mind we proceed to investigate the features of this expansion.

We redefine

$$\begin{aligned}
\tilde{a}_0(x) &= \int_{\alpha(x)}^{-x} dy\, w(x+y|x)q(x,\tau) \\
a_n(x) &= \int_0^{\alpha(x)} dy\, y^n w(x+y|x)q(x,\tau) \\
\tilde{a}_n(x) &= \int_0^{\alpha(x)} dy\, y^n \partial_x{}^n\left(w(x+y|x)q(x,\tau)\right) \qquad n=1,2,\cdots
\end{aligned} \tag{160}$$

For the first two terms ($n=1,2$) one can easily work out the relations

$$\begin{aligned}
\tilde{a}_1(x) &= \partial_x a_1(x) - \alpha(x)\partial_x\alpha(x) w(x+\alpha(x)|x)q(x,\tau) \\
\tilde{a}_2(x) &= \partial_x^2 a_2(x) - 2\alpha(x)(\partial_x\alpha(x))^2 w(x+\alpha(x)|x)q(x,\tau) - \\
&\quad -\alpha(x)^2\partial_x\alpha(x)\partial_x\left(w(x+\alpha(x)|x)q(x,\tau)\right) - \\
&\quad -\alpha^2(x)\partial_x\alpha(x)\partial_x\left(w(x+y|x)q(x,\tau)\right)|_{y=\alpha(x)}
\end{aligned} \tag{161}$$

Let's see what happens when we arrest the expansion (157) to the first 3 terms. The Fokker-Planck version of the equation is obtained by including in the approximation only $\tilde{a}_n$ with $n=0,1,2$.

The Fokker-Planck limit of the (non-singlet) equation is then given by

$$\frac{\partial}{\partial\tau}\, q(x,\tau) = \tilde{a}_0(x) + \tilde{a}_1(x) - \frac{1}{2}\,\tilde{a}_2(x) \tag{162}$$

which we rewrite explicitly as

$$\frac{\partial}{\partial\tau}\,q(x,\tau) = C_F\left(\frac{85}{12}+\frac{3}{4x^4}-\frac{13}{3x^3}+\frac{10}{x^2}-\frac{12}{x}+2\log\left(\frac{1-x}{x}\right)\right)q(x)+$$

$$+C_F\left(9-\frac{1}{2x^3}+\frac{3}{x^2}-\frac{7}{x}-\frac{9}{2}\right)\partial_x q(x,\tau)+ \tag{163}$$

$$+C_F\left(\frac{9}{4}+\frac{1}{8x^2}-\frac{5}{6x}-\frac{5x}{2}+\frac{23x^2}{24}\right)\partial_x^2 q(x,\tau)\ .$$

A similar approach can be followed also for other cases, for which a probabilistic picture (a derivation of Collins-Qiu type) has not been established yet, such as for h_1. We describe briefly how to proceed in this case.

First of all, we rewrite the evolution equation for the transversity in a suitable master form. This is possible since the subtraction terms can be written as integrals of a positive function. A possibility is to choose the transition probabilities

$$w_1[x|y] = \frac{C_F}{y}\left(\frac{2}{1-x/y}-2\right)\theta(y>x)\theta(y<1)$$

$$w_2[y|x] = \frac{C_F}{x}\left(\frac{2}{1-y/x}-\frac{3}{2}\right)\theta(y>-x)\theta(y<0) \tag{164}$$

which reproduce the evolution equation for h_1 in master form

$$\frac{dh_1}{d\tau} = \int_0^1 dy\, w_1(x|y)h_1(y,\tau) - \int_0^1 dy\, w_2(y|x)h_1(x,\tau)\ . \tag{165}$$

The Kramers-Moyal expansion is derived as before, with some slight modifications. The result is obtained introducing an intermediate cutoff which is removed at the end. In this case we get

$$\frac{dh_1}{d\tau} = C_F\left(\frac{17}{3}-\frac{2}{3x^3}+\frac{3}{x^2}-\frac{6}{x}+2\log\left(\frac{1-x}{x}\right)\right)h_1(x,\tau)+$$

$$+C_F\left(6+\frac{2}{3x^2}-\frac{3}{x}-\frac{11x}{3}\right)\partial_x h_1(x,\tau)+ \tag{166}$$

$$+C_F\left(\frac{3}{2}-\frac{1}{3x}-2x+\frac{5x^2}{6}\right)\partial_x^2 h_1(x,\tau)\ .$$

Notice that compared to the standard Fokker-Planck approximation, the boundary now generates a term on the left-hand-side of the equation proportional to $q(x)$ which is absent in eq. (158). This and higher order approximations to the DGLAP equation can be studied systematically both analytically and numerically and it is possible to assess the validity of the approximation [15].

13.1. **Links to fractional diffusion.** A formal connection of the LO DGLAP equation to fractional diffusion can also be easily worked out [22] in dimensional regularization. Since the literature on fractional calculus and anomalous diffusion is quite vast [23], we need just few essential definitions to make our brief discussion self-contained. The n-th primitive of a distribution function $f(x)$ can be written as

$$J^n f(x) = \frac{1}{(n-1)!}\int_x^1 (x-y)^{n-1} f(y)\,dy \tag{167}$$

which can be easily analitically continued for any real $\alpha > 0$

$$J^\alpha f(x) = \frac{1}{\Gamma[\alpha]}\int_x^1 f(y)(x-y)^{\alpha-1}\,dy \tag{168}$$

thereby defining the *fractional* integral of order α of the function $f(x)$. We also recall that the *fractional derivative* of a function can be defined by a suitable analytic continuation

$$D^{\beta} f(x) = \frac{1}{\Gamma[n-\beta]} \frac{d^n}{dx^n} \int_x^1 \frac{f(y)}{(x-y)^{\beta-n+1}} \, dy \tag{169}$$

where $n-1 < \beta < n$. In our case the role of β is taken by the parameter ϵ of dimensional regularization. One can also formally define $D^{-\alpha} \equiv J^{\alpha}$ to denote the corresponding fractional primitive.

Now, since the kernels contain "+" distributions and Dirac delta functions, is is convenient to use the relation

$$\left(\frac{1}{1-w}\right)^{1+\epsilon} = \frac{1}{(1-w)_+} - \epsilon \left(\frac{\log(1-w)}{1-w}\right)_+ - \frac{1}{\epsilon} \, \delta(1-w) + O(\epsilon^2) \, , \tag{170}$$

valid in dimensional regularization, with $\epsilon > 0$, from which one obtains

$$\frac{1}{2}\left(\frac{1}{1-w}\right)^{1+\epsilon} + \frac{1}{2}\left(\frac{1}{1-w}\right)^{1-\epsilon} = \frac{1}{(1-w)_+} \, . \tag{171}$$

Notice that we need to keep ϵ strictly positive in order to manipulate the "+" distribution consistently in all the equations below. The equation for the transverse spin distribution, for instance, can be easily recast in the form

$$\frac{\partial h_1}{\partial \tau} = -C_F \left(D^{\epsilon} + D^{-\epsilon}\right) h_1(x) - 2 C_F D^{-1} \left(\frac{h_1(x)}{x}\right) + \frac{3}{2} \, C_F h_1(1) \tag{172}$$

with similar expressions for all the other non singlet equations. This equation is a possible starting point for the analysis of anomalous diffusion in the context of such equations. Equations of this type provide probability functions belonging to the class of stable distributions, such as those describing Levy processes and continuous time random walks.

14. Conclusions

We have reviewed previous work of us on the determination of the solutions of renormalization group equations in QCD from x-space and on the KM expansion of the master formulation of these equations in analogy to the theory of stochastic processes. Parton distributions represent the first important example of nonlocal operators in a realistic field theory that have been studied extensively for almost three decades. Although we are unable to compute from first principles these fundamental field theory structures, the study of their identification and classification at leading twist and higher has allowed to break substantial new ground in field theory and QCD phenomenology.

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