

An application of Green-function methods to gravitational radiation theory

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Abstract¹. Previous work in the literature has studied gravitational radiation in black-hole collisions at the speed of light. In particular, it had been proved that the perturbative field equations may all be reduced to equations in only two independent variables, by virtue of a conformal symmetry at each order in perturbation theory. The Green function for the perturbative field equations is here analyzed by studying the corresponding second-order hyperbolic operator with variable coefficients, instead of using the reduction method from the retarded flat-space Green function in four dimensions. After reduction to canonical form of this hyperbolic operator, the integral representation of the solution in terms of the Riemann function is obtained. The Riemann function solves a characteristic initial-value problem for which analytic formulae leading to the numerical solution are derived.

1. INTRODUCTION

In the theory of gravitational radiation, it remains true what R. Sachs stressed in his 1963 Les Houches lectures, i.e. that we understand the following three main features [1]:

(i) We can give a description of radiation at large distances from its sources in an asymptotically flat universe; this description is geometrically elegant and sufficiently detailed to analyze all conceptual experiments concerning the behaviour of test particles or test absorbers in the far field.

(ii) How the exact theory relates the far field to the near field and the sources or non-gravitational fields.

(iii) We have approximation methods that make it possible to obtain numerical results for the amount of radiation emitted in a particular situation, or for scattering cross-sections, etc.; these approximation methods do not have a profound geometrical nature, but are very important in comparing theory with the experiment, in case the latter succeeds in finding observational evidence in favour of gravitational waves.

Within this framework, it is often desirable to use Green-function methods, since the construction of suitable inverses of differential operators lies still at the very heart of many profound properties in classical and quantum field theory. For example, the theory of small disturbances in local field theory can only be built if suitable invertible operators are considered [2]. In a functional-integral formulation, these correspond to the gauge-field and ghost operators, respectively [3, 4]. Moreover, the Peierls bracket on the space of physical observables, which is a Poisson bracket preserving the invariance under the full infinite-dimensional symmetry group of the theory, is obtained from the advanced and retarded Green functions

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of the theory via the supercommutator function [2, 3, 4, 5] and leads possibly to a deeper approach to quantization. Last, but not least, a perturbation approach to classical general relativity relies heavily on a careful construction of Green functions of operators of hyperbolic [6, 7, 8, 9] and elliptic [10, 11] type. In particular, following [6, 7, 8, 9], we shall be concerned with the axisymmetric collision of two black holes travelling at the speed of light, each described in the centre-of-mass frame before the collision by an impulsive plane-fronted shock wave with energy μ . One then passes to a new frame to which a large Lorentz boost is applied. There the energy $\nu = \mu e^\alpha$ of the incoming shock 1 obeys $\nu \gg \lambda$, where $\lambda = \mu e^{-\alpha}$ is the energy of the incoming shock 2 and $e^\alpha \equiv \sqrt{(1+\beta)/(1-\beta)}$ (β being the usual relativistic parameter). In the boosted frame, to the future of the strong shock 1, the metric can be expanded in the form [7, 9]

$$(1.1) \quad g_{ab} \sim \nu^2 \left[\eta_{ab} + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\nu} \right)^i h_{ab}^{(i)} \right],$$

where η_{ab} is the standard notation for the Minkowski metric. The task of solving the Einstein field equations becomes then a problem in singular perturbation theory, having to find $h_{ab}^{(1)}, h_{ab}^{(2)}, \dots$ by solving the linearized field equations at first, second, ... order respectively in λ/ν , once that characteristic initial data are given just to the future of the strong shock 1. The perturbation series (1.1) is physically relevant because, on boosting back to the centre-of-mass frame, it is found to give an accurate description of space-time geometry where gravitational radiation propagates at small angles away from the forward symmetry axis $\hat{\theta} = 0$. The news function c_0 (see appendix), which describes gravitational radiation arriving at future null infinity in the centre-of-mass frame, is expected to have the convergent series expansion [7, 9]

$$(1.2) \quad c_0(\hat{\tau}, \hat{\theta}) = \sum_{n=0}^{\infty} a_{2n}(\hat{\tau}/\mu) (\sin \hat{\theta})^{2n},$$

with $\hat{\tau}$ a suitable retarded time coordinate, and μ the energy of each incoming black hole in the centre-of-mass frame. In [7, 9] a very useful analytic expression of $a_2(\hat{\tau}/\mu)$ was derived, exploiting the property that perturbative field equations may all be reduced to equations in only two independent variables, by virtue of a remarkable conformal symmetry at each order in perturbation theory. The Green function for perturbative field equations was then found by reduction from the retarded flat-space Green function in four dimensions.

However, a *direct* approach to the evaluation of Green functions appears both desirable and helpful in general, and it has been our aim to pursue such a line of investigation. For this purpose, following hereafter our work in [12], reduction to two dimensions with the associated hyperbolic operator is studied again in section 2. Section 3 performs reduction to canonical form with the associated Riemann function. Equations for the Goursat problem obeyed by the Riemann function are derived in section 4, while the corresponding numerical algorithm is discussed in section 5. Some background material is described in the appendix.

2. REDUCTION TO TWO DIMENSIONS AND THE ASSOCIATED OPERATOR

As is well known from the work in [7] and [9], the field equations for the first-order correction $h_{ab}^{(1)}$ in the expansion (1.1) are particular cases of the general system given by the flat-space wave equation (here $u \equiv (1/\sqrt{2})(z+t)$, $v \equiv (1/\sqrt{2})(z-t)$)

$$(2.1) \quad \square \psi = 2 \frac{\partial^2 \psi}{\partial u \partial v} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} = 0,$$

supplemented by the boundary condition

$$(2.2) \quad \psi(u=0) = e^{im\phi} \rho^{-n} f[8 \log(v\rho) - \sqrt{2}v],$$

$$(2.3) \quad f(x) = 0 \quad \forall x < 0.$$

Moreover, ψ should be of the form $e^{im\phi}\rho^{-n}\chi(q, r)$ for $u \geq 0$, where

$$(2.4) \quad q \equiv u\rho^{-2},$$

$$(2.5) \quad r \equiv 8 \log(\nu\rho) - \sqrt{2}v.$$

For the homogeneous wave equation (2.1) there is no advantage in eliminating ρ and ϕ from the differential equation. However, the higher-order metric perturbations turn out to obey inhomogeneous flat-space wave equations of the form

$$(2.6) \quad \square \psi = S,$$

where S is a source term equal to $e^{im\phi}\rho^{-(n+2)}H(q, r)$. This leads to the following equation for $\chi \equiv e^{-im\phi}\rho^n\psi$:

$$(2.7) \quad \mathcal{L}_{m,n}\chi(q, r) = H(q, r),$$

where $\mathcal{L}_{m,n}$ is an hyperbolic operator in the independent variables q and r , and takes the form [7, 9]

$$(2.8) \quad \begin{aligned} \mathcal{L}_{m,n} = & -(2\sqrt{2} + 32q) \frac{\partial^2}{\partial q \partial r} + 4q^2 \frac{\partial^2}{\partial q^2} + 64 \frac{\partial^2}{\partial r^2} + \\ & + 4(n+1)q \frac{\partial}{\partial q} - 16n \frac{\partial}{\partial r} + n^2 - m^2. \end{aligned}$$

The proof of hyperbolicity of $\mathcal{L}_{m,n}$, with the associated normal hyperbolic form, can be found in section 3 of [7], and in [9]. The advantage of studying Eq. (2.7) is twofold: to evaluate the solution at some space-time point one has simply to integrate the product of H and the Green function $G_{m,n}$ of $\mathcal{L}_{m,n}$:

$$(2.9) \quad \chi(q, r) = \int G_{m,n}(q, r; q_0, r_0) H(q_0, r_0) dq_0 dr_0,$$

and the resulting numerical calculation of the solution is now feasible [8, 9].

If one defines the variables

$$(2.10) \quad X \equiv \log(q) + \frac{r}{4}, \quad Y \equiv \log(q) - \frac{r}{4},$$

the operator $\mathcal{L}_{m,n}$ is turned into

$$(2.11) \quad T_{m,n} \equiv 16 \frac{\partial^2}{\partial Y^2} + 8n \frac{\partial}{\partial Y} + n^2 - m^2 - \frac{1}{\sqrt{2}} e^{-(X+Y)/2} \left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} \right).$$

The operator $T_{m,n}$ is the ‘sum’ of an elliptic operator in the Y variable and a two-dimensional wave operator ‘weighted’ with the exponential $e^{-(X+Y)/2}$, which is the main source of technical complications in these variables.

3. REDUCTION TO CANONICAL FORM AND THE RIEMANN FUNCTION

It is therefore more convenient, in our general analysis, to reduce first Eq. (2.7) to canonical form, and then find an integral representation of the solution. Reduction to canonical form means that new coordinates $x = x(q, r)$ and $y = y(q, r)$ are introduced such that the coefficients of $\partial^2/(\partial x^2)$ and $\partial^2/(\partial y^2)$ vanish. As is shown in [7, 9], this is achieved if

$$(3.1) \quad \frac{\partial x}{\partial r} = \frac{\partial y}{\partial r} = 1,$$

$$(3.2) \quad \frac{\partial x}{\partial q} = \frac{1 + 8q\sqrt{2} + \sqrt{1 + 16q\sqrt{2}}}{2\sqrt{2}q^2},$$

$$(3.3) \quad \frac{\partial y}{\partial q} = \frac{1 + 8q\sqrt{2} - \sqrt{1 + 16q\sqrt{2}}}{2\sqrt{2}q^2}.$$

The resulting formulae are considerably simplified if one defines

$$(3.4) \quad t \equiv \sqrt{1 + 16q\sqrt{2}} = t(x, y) .$$

The dependence of t on x and y is obtained implicitly by solving the system [7, 9]

$$(3.5) \quad x = r + \log \left(\frac{t-1}{2} \right) - \frac{8}{t-1} - 4 ,$$

$$(3.6) \quad y = r + \log \left(\frac{t+1}{2} \right) + \frac{8}{t+1} - 4 .$$

This leads to the equation

$$(3.7) \quad \log \frac{t-1}{t+1} - \frac{2t}{t^2-1} = \frac{x-y}{8} ,$$

which can be cast in the form

$$(3.8) \quad \frac{t-1}{t+1} e^{(2t)/(1-t^2)} = e^{(x-y)/8} .$$

This suggests defining

$$(3.9) \quad w \equiv \frac{t-1}{t+1} ,$$

so that one first has to solve the transcendental equation

$$(3.10) \quad w e^{(w^2-1)/2w} = e^{(x-y)/8} ,$$

to obtain $w = w(x-y)$, from which one gets

$$(3.11) \quad t = \frac{1+w}{1-w} = t(x-y) .$$

On denoting by $g(w)$ the left-hand side of Eq. (3.10), one finds that, in the plane $(w, g(w))$, the right-hand side of Eq. (3.10) is a line parallel to the w -axis, which intersects $g(w)$ at no more than one point for each value of $x-y$. For example, when $w=1$, $g(w)$ intersects the line taking the constant value 1, for which $x-y=0$. The function

$$g : w \rightarrow g(w) = w e^{(w^2-1)/2w}$$

is asymmetric and has the limiting behaviour described by

$$(3.12) \quad \lim_{w \rightarrow 0^-} g(w) = -\infty , \quad \lim_{w \rightarrow 0^+} g(w) = 0 ,$$

$$(3.13) \quad \lim_{w \rightarrow -\infty} g(w) = 0 , \quad \lim_{w \rightarrow +\infty} g(w) = \infty .$$

Thus, in the lower half-plane, g has an horizontal asymptote given by the w -axis, and a vertical asymptote given by the line $w=0$, while it has no asymptotes in the upper half-plane, since

$$\lim_{w \rightarrow \infty} \frac{g(w)}{w} = \infty$$

in addition to (3.13). The first derivative of g reads as

$$(3.14) \quad g'(w) = \frac{(w+1)^2}{2w} e^{(w^2-1)/2w} .$$

One therefore has $g'(w) > 0$ for all $w > 0$, and $g'(w) < 0$ for all $w \in (-\infty, 0) - \{-1\}$, and g is monotonically decreasing for negative w and monotonically increasing for positive w . The point $w = -1$, at which $g'(w)$ vanishes, is neither a maximum nor a minimum point, because

$$(3.15) \quad g''(w) = \left(\frac{1}{4w^3} + \frac{1}{2w} + 1 + \frac{w}{4} \right) e^{(w^2-1)/2w} ,$$

$$(3.16) \quad g'''(w) = \left(\frac{1}{8w^5} - \frac{3}{4w^4} + \frac{3}{8w^3} + \frac{3}{8w} + \frac{3}{4} + \frac{w}{8} \right) e^{(w^2-1)/2w}.$$

These formulae imply that $g''(-1) = 0$ but $g'''(-1) = -1 \neq 0$, and hence $w = -1$ yields a flex of $g(w)$ (see Fig. 1).

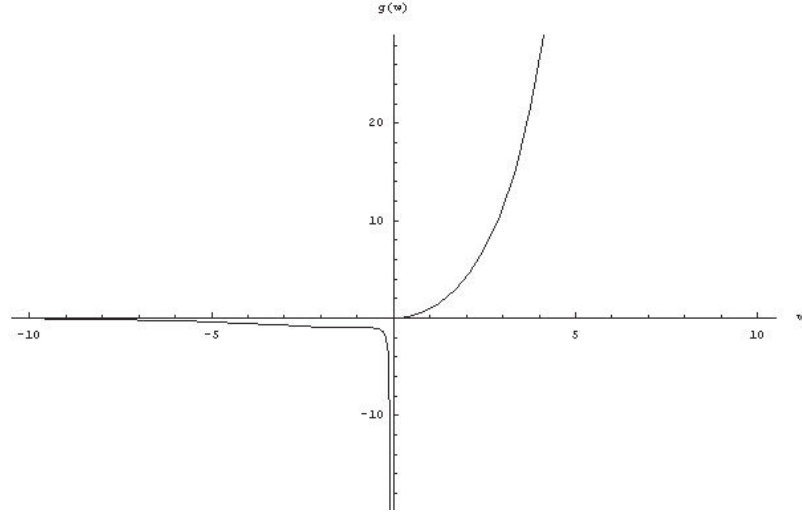


FIGURE 1. Plot of the function $g : w \rightarrow g(w) = we^{(w^2-1)/2w}$.

In the (x, y) variables, the operator $\mathcal{L}_{m,n}$ therefore reads

$$(3.17) \quad \mathcal{L}_{m,n} = f(x, y) \frac{\partial^2}{\partial x \partial y} + g(x, y) \frac{\partial}{\partial x} + h(x, y) \frac{\partial}{\partial y} + n^2 - m^2,$$

where, exploiting the formulae

$$(3.18) \quad \frac{\partial x}{\partial q} = \frac{64\sqrt{2}}{(t-1)^2},$$

$$(3.19) \quad \frac{\partial y}{\partial q} = \frac{64\sqrt{2}}{(t+1)^2},$$

one finds

$$(3.20) \quad \begin{aligned} f(x, y) &= -(2\sqrt{2} + 32q) \left(\frac{\partial x}{\partial q} + \frac{\partial y}{\partial q} \right) + 8q^2 \frac{\partial x}{\partial q} \frac{\partial y}{\partial q} + 128 = \\ &= 256 \left[1 - \frac{2t^2(t^2+1)}{(t-1)^2(t+1)^2} \right], \end{aligned}$$

$$(3.21) \quad g(x, y) = 4(n+1)q \frac{\partial x}{\partial q} - 16n = 16 \left[1 + \frac{2(n+1)}{t-1} \right],$$

$$(3.22) \quad h(x, y) = 4(n+1)q \frac{\partial y}{\partial q} - 16n = 16 \left[1 - \frac{2(n+1)}{t+1} \right].$$

The resulting canonical form of Eq. (2.7) is

$$(3.23) \quad \begin{aligned} L[\chi] &= \left(\frac{\partial^2}{\partial x \partial y} + a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y} + c(x, y) \right) \chi(x, y) = \\ &= \tilde{H}(x, y) \end{aligned}$$

where

$$(3.24) \quad a(x, y) \equiv \frac{g(x, y)}{f(x, y)} = \frac{1}{16} \frac{(1-t)(t+1)^2(2n+1+t)}{t^4 + 4t^2 - 1},$$

$$(3.25) \quad b(x, y) \equiv \frac{h(x, y)}{f(x, y)} = \frac{1}{16} \frac{(t+1)(t-1)^2(2n+1-t)}{t^4 + 4t^2 - 1},$$

$$(3.26) \quad c(x, y) \equiv \frac{n^2 - m^2}{f(x, y)} = \frac{m^2 - n^2}{256} \frac{(t-1)^2(t+1)^2}{t^4 + 4t^2 - 1},$$

$$(3.27) \quad \tilde{H}(x, y) \equiv \frac{H(x, y)}{f(x, y)} = -\frac{H(x, y)}{256} \frac{(t-1)^2(t+1)^2}{t^4 + 4t^2 - 1}.$$

Note that $a(-t) = b(t)$, $b(-t) = a(t)$, $c(-t) = c(t)$, $\tilde{H}(-t) = \tilde{H}(t)$.

For an hyperbolic equation in the form (3.23), we can use the Riemann integral representation of the solution. For this purpose, recall from [13] that, on denoting by L^\dagger the adjoint of the operator L in (3.23), which acts according to

$$(3.28) \quad L^\dagger[\chi] = \chi_{xy} - (a\chi)_x - (b\chi)_y + c\chi,$$

one has to find a ‘function’ $R(x, y; \xi, \eta)$ (actually a kernel) subject to the following conditions ((ξ, η) being the coordinates of a point P such that characteristics through it intersect a curve C at points A and B , AP being a segment with constant y , and BP being a segment with constant x , as is shown in Fig. 2):

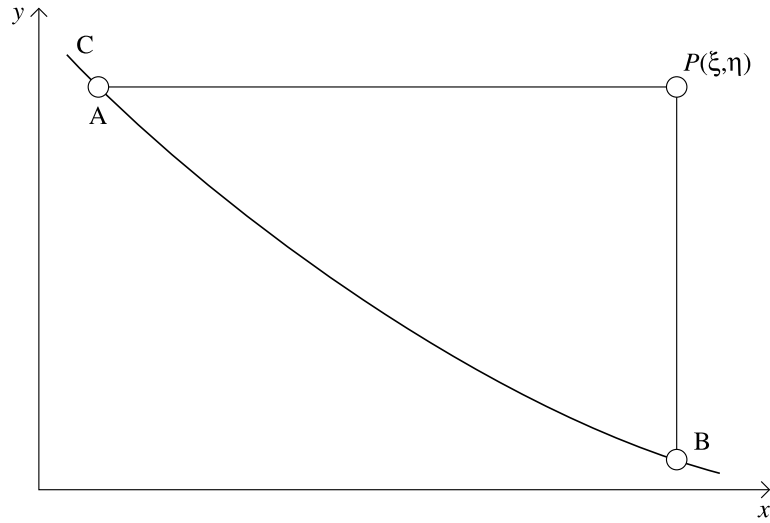


FIGURE 2. Geometry of the characteristic initial-value problem in two independent variables.

(i) As a function of x and y , R satisfies the adjoint equation

$$(3.29) \quad L_{(x,y)}^\dagger[R] = 0,$$

(ii) $R_x = bR$ on AP , i.e.

$$(3.30) \quad R_x(x, y; \xi, \eta) = b(x, \eta)R(x, y; \xi, \eta) \quad \text{on } y = \eta,$$

and $R_y = aR$ on BP , i.e.

$$(3.31) \quad R_y(x, y; \xi, \eta) = a(\xi, y)R(x, y; \xi, \eta) \quad \text{on } x = \xi,$$

(iii) R equals 1 at P , i.e.

$$(3.32) \quad R(\xi, \eta; \xi, \eta) = 1 .$$

It is then possible to express the solution of Eq. (3.23) in the form

$$(3.33) \quad \begin{aligned} \chi(P) = & \frac{1}{2}[\chi(A)R(A) + \chi(B)R(B)] + \\ & + \int_{AB} \left(\left[\frac{R}{2} \chi_x + \left(bR - \frac{1}{2} R_x \right) \chi \right] dx - \right. \\ & \left. - \left[\frac{R}{2} \chi_y + \left(aR - \frac{1}{2} R_y \right) \chi \right] dy \right) + \\ & + \iint_{\Omega} R(x, y; \xi, \eta) \tilde{H}(x, y) dx dy , \end{aligned}$$

where Ω is a domain with boundary.

Note that Eqs. (3.30) and (3.31) are ordinary differential equations for the Riemann function $R(x, y; \xi, \eta)$ along the characteristics parallel to the coordinate axes. By virtue of (3.32), their integration yields

$$(3.34) \quad R(x, \eta; \xi, \eta) = \exp \int_{\xi}^x b(\lambda, \eta) d\lambda ,$$

$$(3.35) \quad R(\xi, y; \xi, \eta) = \exp \int_{\eta}^y a(\lambda, \xi) d\lambda ,$$

which are the values of R along the characteristics through P . Equation (3.33) yields instead the solution of Eq. (3.23) for arbitrary initial values given along an arbitrary non-characteristic curve C , by means of a solution R of the adjoint equation (3.29) which depends on x, y and two parameters ξ, η . Unlike χ , the Riemann function R solves a characteristic initial-value problem.

4. GOURSAT PROBLEM FOR THE RIEMANN FUNCTION

By fully exploiting the reduction to canonical form of Eq. (2.7) we have considered novel features with respect to the analysis in [7, 9], because the Riemann formula (3.33) also contains the integral along the piece of curve C from A to B , and the term $(1/2)[\chi(A)R(A) + \chi(B)R(B)]$. This representation of the solution might be more appropriate for the numerical purposes considered in [8], but the task of finding the Riemann function R remains extremely difficult. One can however use approximate methods for solving Eq. (3.29). For this purpose, we first point out that, by virtue of Eq. (3.28), Eq. (3.29) is a canonical hyperbolic equation of the form

$$(4.1) \quad \left(\frac{\partial^2}{\partial x \partial y} + A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \right) R(x, y; \xi, \eta) = 0 ,$$

where

$$(4.2) \quad A \equiv -a ,$$

$$(4.3) \quad B \equiv -b ,$$

$$(4.4) \quad C \equiv c - a_x - b_y .$$

Thus, on defining

$$(4.5) \quad U \equiv R ,$$

$$(4.6) \quad V \equiv R_x + BR ,$$

the equation (4.1) for the Riemann function is equivalent to the hyperbolic canonical system [13]

$$(4.7) \quad U_x = f_1(x, y)U + f_2(x, y)V ,$$

$$(4.8) \quad V_y = g_1(x, y)U + g_2(x, y)V ,$$

where

$$(4.9) \quad f_1 \equiv -B = b ,$$

$$(4.10) \quad f_2 = 1 ,$$

$$(4.11) \quad g_1 \equiv AB - C + B_y = ab - c + a_x ,$$

$$(4.12) \quad g_2 \equiv -A = a .$$

For the system described by Eqs. (4.7) and (4.8) with boundary data (3.34) and (3.35) an existence and uniqueness theorem holds (see [13] for the Lipschitz conditions on boundary data), and we can therefore exploit the finite differences method to find approximate solutions for the Riemann function $R(x, y; \xi, \eta)$, and eventually $\chi(P)$ with the help of the integral representation (3.33).

5. CONCLUDING REMARKS

[14] and the Cauchy problem for hyperbolic equations with polynomial coefficients [15] have always been the object of intensive investigation in the mathematical literature. We have here considered the application of such issues to axisymmetric black hole collisions at the speed of light, relying on the work in [6, 7, 8, 9]. We have pointed out that, for the inhomogeneous equations (2.7) occurring in the perturbative analysis, the task of inverting the operator (2.8) can be accomplished with the help of the Riemann integral representation (3.33), after solving Eq. (4.1) for the Riemann function. One has then to solve a characteristic initial-value problem for a homogeneous hyperbolic equation in canonical form in two independent variables, for which we have developed formulae to be used for the numerical solution with the help of a finite differences scheme. For this purpose one studies the canonical system (cf. (4.7) and (4.8))

$$(5.1) \quad U_x = F(x, y, U, V) ,$$

$$(5.2) \quad V_y = G(x, y, U, V) ,$$

in the rectangle $\mathcal{R} \equiv \{x, y : x \in [x_0, x_0 + a], y \in [y_0, y_0 + b]\}$ with known values of U on the vertical side AD where $x = x_0$, and known values of V on the horizontal side AB where $y = y_0$. The segments AB and AD are then divided into m and n equal parts, respectively. On setting $a/m \equiv h$ and $b/n \equiv k$, the original differential equations become equations relating values of U and V at three intersection points of the resulting lattice, i.e.

$$(5.3) \quad \frac{U(P_{r,s+1}) - U(P_{rs})}{h} = F ,$$

$$(5.4) \quad \frac{V(P_{r+1,s}) - V(P_{rs})}{k} = G .$$

It is now convenient to set $U_{rs} \equiv U(P_{rs}), V_{rs} \equiv V(P_{rs})$, so that these equations read as

$$(5.5) \quad U_{r,s+1} = U_{rs} + hF(P_{rs}, U_{rs}, V_{rs}) ,$$

$$(5.6) \quad V_{r+1,s} = V_{rs} + kG(P_{rs}, U_{rs}, V_{rs}) .$$

Thus, if both U and V are known at P_{rs} , one can evaluate U at $P_{r,s+1}$ and V at $P_{r+1,s}$. The evaluation at subsequent intersection points of the lattice goes on along horizontal or vertical segments. In the former case, the resulting algorithm is

$$(5.7) \quad U_{rs} = U_{r0} + h \sum_{i=1}^{s-1} F(P_{ri}, U_{ri}, V_{ri}) ,$$

$$(5.8) \quad V_{rs} = V_{r-1,s} + kG(P_{r-1,s}, U_{r-1,s}, V_{r-1,s}) ,$$

while in the latter case one obtains the algorithm expressed by the equations

$$(5.9) \quad V_{rs} = V_{0s} + \sum_{i=1}^{r-1} G(P_{is}, U_{is}, V_{is}) ,$$

$$(5.10) \quad U_{rs} = U_{r,s-1} + hF(P_{r,s-1}, U_{r,s-1}, V_{r,s-1}) .$$

Stability of such solutions is closely linked with the geometry of the associated characteristics, and the criteria to be fulfilled are studied in section 13.2 of [16] (stability depends crucially on whether or not $h/k \leq 1$).

To sum up, one solves numerically Eq. (3.10) for $w = w(x, y) = w(x - y)$, from which one gets $t(x - y)$ with the help of (3.11), which is a fractional linear transformation. This yields a, b, c and \tilde{H} as functions of (x, y) according to (3.24)-(3.27), and hence A, B and C in the equation for the Riemann function are obtained according to (4.2)-(4.4), where derivatives with respect to x and y are evaluated numerically. Eventually, the system given by (4.7) and (4.8) is solved according to the finite-differences scheme of the present section, with

$$(5.11) \quad F = f_1U + f_2V = f_1R + f_2(R_x + BR) ,$$

$$(5.12) \quad G = g_1U + g_2V = g_1R + g_2(R_x + BR) .$$

Once the Riemann function $R = U$ is obtained with the desired accuracy, numerical evaluation of the integral (3.33) yields $\chi(P)$, and $\chi(q, r)$ is obtained upon using Eqs. (3.5) and (3.6) for the characteristic coordinates.

Our steps are conceptually desirable since they rely on well established techniques for the solution of hyperbolic equations in two independent variables [13, 16], and provide a viable alternative to the numerical analysis performed in [8], because all functions should be evaluated numerically. Our method is not obviously more powerful than the one used in [6, 7, 8, 9], but is well suited for a systematic and lengthy numerical analysis, while its analytic side provides an interesting alternative for the evaluation of Green functions both in black hole physics and in other problems where hyperbolic operators with variable coefficients might occur. This task remains very important because a strong production of gravitational radiation is mainly expected in the extreme events studied in [6, 7, 8, 9] and which motivated our paper. Any viable way of looking at mathematical and numerical aspects of the problem is therefore of physical interest for research planned in the years to come [17].

APPENDIX A. THE CHARACTERISTIC INITIAL-VALUE PROBLEM

In our expository article, we find it appropriate to include some background material, following, for example, the presentation in section IV of [1]. We therefore consider some four-dimensional region of space-time and choose in it a set of null hypersurfaces $u = \text{constant}$; the corresponding ray congruence with tangent vector $k_a = u_{,a}$ is assumed to have expansion $\rho \equiv (1/2)k^a_{;a} \neq 0$, which can always be arranged in a space-time patch, whereas outside of some patch the rays start to cross and hence our construction breaks down globally. On completing the k^a direction to a quasi-normal tetrad (k, m, t) , one finds the following split of the vacuum Einstein equations with Einstein tensor G_{ab} :

Main equations (6 equations)

$$(A.1) \quad k^a G_{ab} = 0, \quad G_{ab} t^a \bar{t}^b = 0,$$

trivial equation

$$(A.2) \quad G_{ab} t^a \bar{t}^b = 0,$$

and 3 supplementary conditions

$$(A.3) \quad G_{ab} m^a t^b = 0, \quad G_{ab} m^a m^b = 0,$$

where a single complex equation has been counted as two real equations. Remarkably, if the main equations hold everywhere, then the trivial equation holds everywhere and the supplementary conditions hold everywhere if they hold at one point on each ray. The fulfillment of the trivial equation is proved by writing, from the vacuum Einstein equations, that

$$(A.4) \quad G^a_{b;a} = 0,$$

and then exploiting the main equations (A.1) jointly with the split of $k_{a;b}$ as given in [1]:

$$(A.5) \quad k_{a;b} = z t_a \bar{t}_b + \sigma t_a t_b + \Omega t_a k_b + \zeta k_a t_b + \text{c.c.} + \xi k_a k_b.$$

Hence one gets

$$(A.6) \quad 0 = k^a G^b_{a;b} = -k_{a;b} G^{ab} = -2\rho G_{ab} t^a \bar{t}^b.$$

By hypothesis the expansion ρ does not vanish, so that the trivial equation is, indeed, identically satisfied. The fulfillment of (A.3) everywhere is proved along similar lines. Thus, one can again integrate the main equations (A.1) first and worry about the supplementary conditions (A.3) later.

Choose now the coordinate x^0 as the retarded time: $x^0 = u$. Let $r = x^1$ be a luminosity distance along the rays; let x^α (with $\alpha = 2, 3$) be any other pair of coordinates constant along the rays. The line element in these coordinates takes therefore the form (no confusion should arise with the β of section 1)

$$(A.7) \quad ds^2 = W du^2 + 2e^{2\beta} dudr - r^2 h_{\alpha\gamma} (dx^\alpha - U^\alpha du) (dx^\gamma - U^\gamma du),$$

where $W, \beta, h_{\alpha\gamma}, U^\alpha$ depend on the x^α coordinates. Since r is a luminosity distance, the determinant of $h_{\nu\mu}$ is independent of r . Bearing in mind that the luminosity distance is defined only up to a factor constant along each ray, one can demand without loss of generality that (here $\theta \equiv x^2, \phi \equiv x^3$)

$$(A.8) \quad 2h_{\mu\nu} dx^\mu dx^\nu = (e^{2\gamma} + e^{2\delta}) d\theta^2 + 4 \sinh(\gamma - \delta) d\theta d\phi \sin\theta + \sin^2\theta (e^{-2\gamma} + e^{-2\delta}) d\phi^2.$$

The metric corresponding to the line element (A.7) contains only six unknown functions of four variables, and our coordinate system is 'rigid' enough for our purposes [1].

One can either analyze the field in the neighbourhood of some point, or the field near infinity in an asymptotically flat space-time. Indeed, if in Minkowski space-time one uses a retarded time $u = t - r$ and spherical coordinates r, θ, ϕ one finds for the line element

$$(A.9) \quad ds^2 = du^2 + 2dudr - r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

hence one is led to require that, if asymptotic flatness holds,

$$(A.10) \quad \lim_{r \rightarrow \infty} W = 1, \quad \lim_{r \rightarrow \infty} (rU^\alpha) = \lim_{r \rightarrow \infty} \beta = \lim_{r \rightarrow \infty} \delta = \lim_{r \rightarrow \infty} \gamma = 0,$$

where all limits are taken as r approaches infinity with u, θ, ϕ fixed. The second requirement in (A.10), i.e. that all quantities of interest admit a power-series expansion in $1/r$, e.g.

$$(A.11) \quad (1-i) \frac{\delta + i\gamma}{2} = \frac{c(u, \theta, \phi)}{r} + \frac{d(u, \theta, \phi)}{r^2} + \dots,$$

is indeed restrictive. Such a requirement can be drastically weakened but not fully eliminated; moreover, it is closely related to an outgoing radiation condition of the Sommerfeld type. It should be stressed that all these requirements no longer hold when r becomes small to the extent that rays start to cross each other.

In the axially- and reflection-symmetric case considered by Bondi et al. [18], one has

$$(A.12) \quad \delta = \gamma, \quad U^3 = 0, \quad \frac{\partial g_{ab}}{\partial \phi} = 0,$$

and the ϕ -direction is a hypersurface-orthogonal Killing direction. The line element acquires the simpler form

$$(A.13) \quad ds^2 = \frac{V e^{2\beta}}{r} du^2 + 2e^{2\beta} dudr - r^2 [e^{2\gamma} (d\theta - U du)^2 + e^{-2\gamma} \sin^2 \theta d\phi^2],$$

where the peculiar form of the first coefficient is chosen to simplify the resulting calculations.

Interestingly, two main equations are found to be identically satisfied by virtue of axial symmetry, whereas the other four turn out to be linear combinations of [18]

$$(A.14) \quad R_{11} = -\frac{4}{r} \left(\beta_1 - \frac{1}{2} r \gamma_1^2 \right) = 0,$$

$$(A.15) \quad -2r^2 R_{12} = \left[r^4 e^{2(\gamma-\beta)} U_1 \right]_1 -$$

$$-2r^2 \left(-\gamma_{12} + 2\gamma_1 \gamma_2 - 2\gamma_1 \cot \theta + \beta_{12} - 2 \frac{\beta_2}{r} \right) = 0,$$

$$(A.16) \quad R_{22} e^{2(\beta-\gamma)} - r^2 R_3^3 e^{2\beta} = 2V_1 + \frac{1}{2} r^4 e^{2(\gamma-\beta)} U_1^2 -$$

$$-r^2 U_{12} - 4rU_2 - r^2 U_1 \cot \theta - 4rU \cot \theta +$$

$$+ 2e^{2(\beta-\gamma)} [\beta_{22} + \beta_2^2 - 1 - (3\gamma_2 - \beta_2) \cot \theta - \gamma_{22} + 2\gamma_2(\gamma_2 - \beta_2)] = 0,$$

$$(A.17) \quad -r^2 R_3^3 e^{2\beta} = 2r(r\gamma)_{01} + (1 - r\gamma_1)V_1 - (r\gamma_{11} + \gamma_1)V - r(1 - r\gamma_1)U_2 -$$

$$-r^2(\cot \theta - \gamma_2)U_1 + r(2r\gamma_{12} + 2\gamma_2 + r\gamma_1 \cot \theta - 3 \cot \theta)U +$$

$$+ e^{2(\beta-\gamma)} [-1 - (3\gamma_2 - 2\beta_2) \cot \theta - \gamma_{22} + 2\gamma_2(\gamma_2 - \beta_2)] = 0.$$

Equations (A.14)-(A.16) are called *hypersurface equations* because they contain no u derivatives, while Eq. (A.17) is called the *standard equation*.

Now if γ is given for one value of u , Eq. (A.14) and the boundary conditions (A.10) determine β uniquely. Next Eq. (A.15) and the boundary conditions determine U up to a function of integration $-6N(u, \theta)$ that can be added to $r^4 e^{2(\gamma-\beta)} U_1$. Equation (A.16) determines V up to the additive function $-2M(u, \theta)$; last, Eq. (A.17) determines γ_0 up to an additive function $c_0(u, \theta)/r$. One can then differentiate Eqs. (A.14)-(A.17) with respect to u and repeat the whole procedure. To sum up, given γ at one moment the main equations determine the future or past up to the three integration functions just mentioned. In the general case, the results are completely similar. One has to assign at one value of u the two functions γ and δ . The future is then determined up to five integration functions: a term

$-2M(u, \theta, \phi)/r$ to be added to W ; two functions $N^\alpha(u, \theta, \phi)$ which occur in the r^{-3} term for U ; and two ‘news functions’ $c_0(u, \theta, \phi)$, where the complex function c is given by Eq. (A.11).

As far as the supplementary conditions are concerned, the lemma just given makes it clear that they should only involve the functions M, N and c_0 , while a long calculation yields [1]

$$(A.18) \quad M_0 = -|c_0|^2 + \frac{1}{2}(\sin \theta)^{-1} \operatorname{Re} \left\{ \bar{\nabla} \left[\left(\frac{1}{\sin \theta} \right) \bar{\nabla} (c_0 \sin^2 \theta) \right] \right\},$$

$$(A.19) \quad 3(N^2 + i \sin \theta N^3) = -\nabla M - [4c \cot \theta + (\nabla c) + 3c \bar{\nabla}] \bar{c}_0,$$

where

$$\nabla \equiv \frac{\partial}{\partial \theta} + i \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}.$$

The desired M and N^α can be determined once that $c(u, \theta, \phi)$ and some initial values are given. In the axially symmetric case, Eqs. (A.18) and (A.19) take the simpler form [18]

$$(A.20) \quad M_0 = -c_0^2 + \frac{1}{2}(c_{22} + 3c_2 \cot \theta - 2c)_0,$$

$$(A.21) \quad -3N_0 = M_2 + 3cc_{02} + 4cc_0 \cot \theta + c_0 c_2.$$

The functions γ and δ given on the initial hypersurface $u = \text{constant}$, jointly with the two news functions c_0 given at $r = \infty$ describe the two transverse degrees of freedom. Moreover, one should specify M and N at the initial (or final) retarded time, and these three functions of two variables must be related to the longitudinal-timelike degrees of freedom of the gravitational field. In the characteristic value problem for general relativity, the independent data appear therefore in a very explicit form [1].

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