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On the singularities of the semi-Riemannian exponential map. Bifurcation of geodesics and light rays

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Abstract[1]. We discuss several notions of bifurcation of geodesics in semi-Riemannian manifolds, in relation with the singularities of the exponential map. In particular, we will consider the accumulation of images of a light source occurring at a conjugate instant along a null geodesic.

1. INTRODUCTION

Bifurcation theory is a very traditional topic in nonlinear analysis, with a huge number of applications in several areas of pure and applied mathematics. Recently, some authors have studied the singularities of the exponential map in a manifold endowed with a semi-Riemannian (i.e., non positive definite) metric tensor, using bifurcation theory. The results obtained aim at applications to General Relativity; for instance, bifurcation of light rays issuing from a light source describes in a satisfactory way the gravitational lensing phenomenon in the conjugate case, where the classical Morse theory does not apply. Similarly, bifurcation of solutions of the Lorentz force equation can be interpreted as accumulation of trajectories of massive charges under the action of the gravitational and the electromagnetic field in a general relativistic spacetime.

In this paper we will review very shortly some of the basic definitions and some results concerning the bifurcation of geodesics, and we will complement these results with a few observations. In the geodesic case, bifurcation (Definition 2) is a special kind of lack of local injectivity of the exponential map, and therefore it can occur only at conjugate instants. A sufficient condition for the occurrence of geodesic bifurcation is proven in [10] using the *spectral flow* of the family of index forms. Along Riemannian or Lorentzian causal geodesics, every conjugate instant determines a jump of the Morse index, hence it is a bifurcation instant, by classical results in variational bifurcation. More generally, using a suitable extension of the Morse index theorem to semi-Riemannian geometry (see [11]), one sees that a conjugate instant along a semi-Riemannian geodesic determines bifurcation if its contribution to the Maslov index of the geodesic is non null; moreover, *isolated* conjugate instants whose contribution to the Maslov index is non null determine a *continuous* bifurcating branch of geodesics (Proposition 4). Recall that for isolated conjugate instants, the contribution to the Maslov index is given by a sort of *algebraic multiplicity* of the conjugate instant (see for instance [12]). On the other hand, it is not clear whether non nullity of the algebraic multiplicity of a conjugate instant is necessary for bifurcation. We will introduce a weaker notion of geodesic bifurcation, by requiring that both initial and endpoints of the bifurcating branch of geodesics may vary along the given geodesic. Also in this case, we prove that weak bifurcation occurs only at conjugate instants (Proposition 6), by exploiting the properties of the exponential map (Lemma 1). It has been proven recently that

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a property very similar to weak bifurcation holds in the case of solutions of the Lorentz force equation with fixed charge-to-mass ratio, in a relativistic spacetime endowed with an exact magnetic two-form (see [5]). This result was obtained using a second order variational principle relating a conjugate instant along a solution of the Lorentz force equation having a fixed value of the charge-to-mass ratio with the conjugate instant along its *lightlike lift* in a suitable Kaluza–Klein model. Along Lorentzian lightlike geodesics, at each conjugate instant one has accumulation of lightlike geodesics issuing from any light source through the initial event (see [4]). The projection onto the base manifold of the bifurcating branch of lightlike geodesics gives a bifurcating branch of magnetic trajectories issuing from any observer through the initial event, and accumulating at the magnetic conjugate point. In particular, one can apply this result considering as the world-line of an observer the same magnetic trajectory, proving the occurrence of *weak* bifurcation at each magnetic conjugate instant.

We will conclude this short note with a few observations on the lightlike bifurcation, and we will show that, also in this case, accumulation of images can only occur at conjugate instants (Corollary 11). This is obtained by looking at the critical points of the restriction of the exponential map to the light cone subbundle of the tangent bundle (Proposition 10). In particular, we obtain the result that if accumulation of images from *some* light source occurs, then one has accumulation of images from *any* light source (Corollary 11). Finally, using the result in [3] in its Banach space formulation, we settle a question on the existence of a continuous bifurcating branch in the case of light rays (Proposition 13).

2. The exponential map

We will assume throughout the paper that (M, g) is a semi-Riemannian manifold; the symbol ∇ will denote the covariant derivative of the Levi–Civita connection of g. The classical textbooks [1] and [9] are standard references for all the basics of semi-Riemannian geometry.

Let us consider the tangent bundle TM of M, and let $\pi : TM \to M$ be the canonical projection. For $v \in TM$ and $p = \pi(v)$, the *vertical subspace* Ver_v of $T_v(TM)$ is the kernel of $\mathrm{d}\pi_v : T_v(TM) \to T_pM$; Ver_v can be identified to the tangent space to the fiber T_pM, and it is canonically isomorphic to this space. The connection ∇ determines a complementary subspace to Ver_v, called the *horizontal subspace* of $T_v(TM)$ and denoted by Hor_v. A vector $\xi \in T_v(TM)$ belongs to Hor_v if and only if $\xi = \Gamma'(0)$, where $\Gamma : \,]-\varepsilon, \varepsilon[\, \to TM$ is a smooth curve with $\Gamma(0) = v$, such that, setting $\gamma = \pi \circ \Gamma$, then Γ is parallel along γ.

The differential $\mathrm{d}\pi_v$ induces an isomorphism from Hor_v to T_pM; hence, we have:

$$T_v(TM) = \mathrm{Hor}_v \oplus \mathrm{Ver}_v \cong T_pM \oplus T_pM \, . \tag{1}$$

Denote by $\exp : \mathcal{A} \subset TM \to M$ the *exponential map* of (M, g), defined in an open neighborhood $\mathcal{A} = \mathcal{A}(M, g)$ of the zero section of TM; for $p \in M$, denote by $\mathcal{A}_p$ the intersection $\mathcal{A} \cap T_pM$, which is a non empty open star-shaped neighborhood of 0 in T_pM, and by $\exp_p$ the restriction of exp to $\mathcal{A}_p$, which is a smooth map with $\exp_p(0) = p$ and $\mathrm{d}\exp_p(0) : T_0(T_pM) \cong T_pM \to T_pM$ given by the identity. By definition, for $v \in \mathcal{A}$ and $p = \pi(v)$, the map $t \mapsto \exp_p(tv)$ is a geodesic in M. Given $p \in M$, $v \in \mathcal{A}_p$ and setting $q = \exp_p(v)$, q is said to be *conjugate to* p along the geodesic $[0, 1] \ni t \mapsto \exp_p(tv)$ if v is a critical point of $\exp_p$. Thus, by the inverse function theorem, if $q = \exp_p(v)$ is not conjugate to p, then $\exp_p$ gives a diffeomorphism between a neighborhood of v in T_pM and a neigborhood of q in M. Actually, we have the following stronger result:

Lemma 1. *Let $p \in M$ and $v \in \mathcal{A}_p$ be given, and set $q = \exp_p(v)$. If q is not conjugate to p along $[0, 1] \ni t \mapsto \exp_p(tv)$, then the map*

$$F = (\pi, \exp) : \mathcal{A} \subset TM \to M \times M$$

gives a diffeomorphism from a neighborhood of v in TM to a neighborhood of (p, q) in $M \times M$.

Proof. Under the assumptions, the differential $\mathrm{d}F_v : T_v(TM) \to T_pM \oplus T_qM$ is an isomorphism (see [9, Ch. 5, Lemma 6, p. 129]).

□

We want to study the occurrence of phenomena of lack of local injectivity of the exponential map around its singularities. By a classical result of Morse and Littauer (see [13]), the Riemannian exponential map is non locally injective around its critical points; the non positive definite case is more involved, due to the possible occurrence of *degenerate singularities* (see [6, 12]).

3. BIFURCATION OF GEODESICS

It was recently introduced the notion of *bifurcation* of geodesics (see [10]):

Definition 2. Let $\gamma : [0,1] \to M$ be a geodesic; set $p = \gamma(0)$ and $v_0 = \dot\gamma(0) \in T_pM$. An instant $t_0 \in\,]0,1[$ is said to be a *bifurcation instant along* γ if there exists a sequence $v_n \in T_pM$, and a sequence $(t_n)_{n\geq 1} \subset [0,1]$ such that:

1. $v_n \neq v_0$ for all n,
2. $\lim\limits_{n\to\infty} v_n = v_0$,
3. $\lim\limits_{n\to\infty} t_n = t_0$,
4. $\exp_p(t_nv_n) = \exp_p(t_nv_0) = \gamma(t_n)$ for all $n \geq 1$.

The instant t_0 is said to be a *branching instant along* γ if there exist continuous functions $I \ni s \mapsto t(s) \in [0,1]$ and $I \ni s \mapsto v(s) \in T_pM$ defined on a left or right neighborhood I of 0 in $\mathbb{R}$, such that $t(0) = t_0$, $v(0) = v_0$, $v(s) \neq v_0$ for all $s \in I \setminus \{0\}$, and with $\exp_p(t(s)v(s)) = \exp_p(t(s)v_0)$ for all $s \in I$.

A few remarks on the above definition are necessary. In first place, one observes immediately that if t_0 is a bifurcation instant, then $q = \gamma(t_0) = \exp_p(t_0v_0)$ is conjugate to $p = \gamma(0)$ along γ; namely, bifurcation at the instant t_0 implies that $\exp_p$ is not locally injective around t_0v_0, and thus t_0v_0 is a critical point of $\exp_p$.

It is proven in [10] that, conversely, every conjugate instant along γ whose contribution to the *Maslov index* of γ is non null is a bifurcation instant. Using a classical result in bifurcation theory due to Chow and Lauterbach ([3]), one can improve this result and obtain a characterization of branching points. We state the result of [3] in a form which is more suited to our purposes, using the notion of *spectral flow*; the spectral flow of a path of self-adjoint Fredholm operators through an isolated degeneracy is an algebraic count of the eigenvalues passing through the value zero at the degeneracy instant (see [10] for details).

Proposition 3. *Let H be a real separable Hilbert space, $U \subset H$ an open neighborhood of* 0 *and let $]a,b[\, \ni \lambda \mapsto F_\lambda$ be a smooth family of C^2 functionals $F_\lambda : U \to \mathbb{R}$. Assume the following:*

- 0 *is a critical point of F_λ for all $\lambda \in\,]a,b[$;*
- $\lambda \mapsto \mathrm{d}^2F_\lambda(0)$ *is a (continuous) path of Fredholm (self-adjoint) operators on H;*
- $\mathrm{d}^2F_\lambda(0)$ *is not injective if and only if $\lambda = \lambda_* \in\,]a,b[$;*
- *the spectral flow of the path $[\lambda_* - \varepsilon, \lambda_* + \varepsilon] \ni \lambda \mapsto \mathrm{d}^2F_\lambda(0)$ is non zero, for $\varepsilon > 0$ small enough.*

Then, there exist continuous functions $I \ni r \mapsto x(r) \in U$ and $I \ni r \mapsto \lambda(r) \in\,]a,b[$, where I is a left (or right) neighborhood of 0 *in* $\mathbb{R}$, *such that:*

- $\lambda(0) = \lambda_*$*;*
- $x(r) \neq 0$ *for* $r \neq 0$, *and* $x(0) = 0$*;*
- $\mathrm{d}F_{\lambda(r)}(x(r)) = 0$ *for all* $r \in I$.

The same result holds in the case that H is a separable Banach *space, provided that there exists a Hilbert space E and a bounded linear map $K : E \to H$ having dense image in H.*

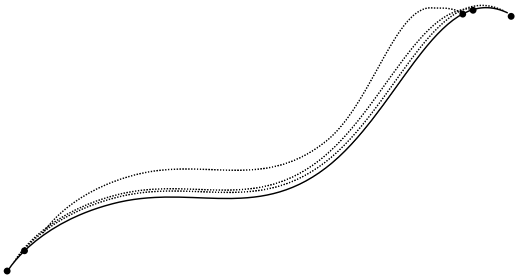

FIGURE 1. Weak bifurcation.

Proof. The existence of a continuous branch of critical points bifurcating at $\lambda = \lambda_*$ from $x = 0$ for a smooth family of functionals F_λ is proved in [3], under the assumptions that:

(a) $\mathrm{d}F_\lambda(0) = 0$ for all λ;
(b) for λ near λ_*, the Hessian $\mathrm{d}^2F_\lambda(0)$ is degenerate only at $\lambda = \lambda_*$;
(c) Ker $\big(\mathrm{d}^2F_{\lambda_*}(0)\big)$ is finite dimensional;
(d) 0 is an isolated point in the spectrum of $\mathrm{d}^2F_{\lambda_*}(0)$;
(e) the net number of eigenvalues of $\mathrm{d}^2F_\lambda(0)$ crossing the value 0 as λ passes through λ_* is non null.

Assumption (a) says that 0 is a critical point of F_λ for all λ. Assumption (b) guarantees that 0 is a degenerate critical point of F_λ only for $\lambda = \lambda_*$. Assumption (d) is equivalent to the fact that the image of $\mathrm{d}^2F_\lambda(0)$ is closed, so that (c) and (d) are equivalent to $\mathrm{d}^2F_{\lambda_*}(0)$ being Fredholm. Since the set of bounded self-adjoint Fredholm operators is open in the space of all bounded self-adjoint operators, and since the map $\lambda \mapsto \mathrm{d}^2F_\lambda(0)$ is continuous, from (c) and (d) it follows that $\mathrm{d}^2F_\lambda(0)$ is Fredholm for λ near 0. Finally, assumption (e) is equivalent to the non vanishing of the spectral flow of the path $[\lambda_* - \varepsilon, \lambda_* + \varepsilon] \ni \lambda \mapsto \mathrm{d}^2F_\lambda(0)$ for $\varepsilon > 0$ small enough.

As to the Banach space extension of the bifurcation result, see [3, Remark 4, pp. 52–53]. ☐

In view of Proposition 3, we can improve slightly the bifurcation result in [10] in the case of isolated conjugate instants.

Corollary 4. *Every* isolated *conjugate point along a semi-Riemannian geodesic γ whose contribution to the Maslov index of γ is non null is a branching instant along γ. In particular, every conjugate instant along a Riemannian or a causal Lorentzian geodesic is a branching instant.*

Examples of non bifurcating conjugate instants in general semi-Riemannian case are not known. In order to better understand the bifurcation phenomena of semi-Riemannian geodesics, we introduce a slightly weaker notion of bifurcation. Observe that, according to Definition 2, the bifurcating sequence of geodesics should have fixed initial point; we weaken this assumption by requiring that also the initial point may vary along γ:

Definition 5. Let $\varepsilon > 0$ and $\gamma : [-\varepsilon, 1] \to M$ be a geodesic; set $p = \gamma(0)$ and $v_0 = \dot\gamma(0)$. An instant $t_0 \in \,]0, 1[$ will be called a *weak bifurcation instant along* γ if there exist sequences $(r_n)_{n\geq1}, (t_n)_{n\geq1} \subset [-\varepsilon, 1]$ and $(v_n) \in TM$, with $v_n \in T_{\gamma(r_n)}M$ for all n, satisfying:

(1) $(\gamma(r_n), v_n) \neq (p, v_0)$ for all n,
(2) $\lim\limits_{n\to\infty} r_n = 0$, $\lim\limits_{n\to\infty} v_n = v_0$,
(3) $\lim\limits_{n\to\infty} t_n = t_0$,
(4) $\exp_{\gamma(r_n)}(t_nv_n) = \exp_p(t_nv_0)$ for all $n \geq 1$.

The definition of *weak branching instant* can be given similarly. Obviously, bifurcations instants are weak bifurcations instants; namely, it suffices to set $r_n \equiv 0$ in Definition 5 to obtain the definition of bifurcation instant. What is not totally obvious is that also weak bifurcation instants must be conjugate:

Lemma 6. *If* $t_0 \in]0,1[$ *is a weak bifurcation instant along* $\gamma : [-\varepsilon, 1] \to M$*, then* $\gamma(0)$ *and* $\gamma(t_0)$ *are conjugate along* γ*.*

Proof. The conditions (1)-(4) of Definition 5 imply that the map $F = (\pi, \exp)$ is not locally injective around (p, v_0), and the conclusion follows readily from Lemma 1.

□

Corollary 7. *Every weak bifurcation (resp., branching) instant along a Riemannian or a causal Lorentzian geodesic is a bifurcation (resp., branching) instant.*

Some remarks and open questions. The notion of weak bifurcation is motivated by a recent result in [5], where the authors study the problem of bifurcation for solutions of the Lorentz force equation having a *fixed* value of the charge-to-mass ratio constant. The main result in [5], obtained as a consequence of a second order variational principle, states that under the assumption that the magnetic field is *exact*, then every magnetic conjugate instant along a solution of the Lorentz force equation with charge q and mass m is a weak bifurcation instant by solutions having the same charge-to-mass ratio q/m. The notion of conjugacy introduced in [5] is given in terms of the linearized Lorentz force equation; it should be observed that in the case of an exact magnetic field, a different notion of magnetic conjugacy can be given in terms of the second variation of the magnetic action functional. In the latter case, conjugate instants would correspond to accumulation of solutions of the Lorentz force equation whose charge-to-mass ratio is not fixed; such conjugate instants may not even be isolated along each solution. Moreover, as observed in [7], the magnetic variational problem does not lead to the true Lorentz force equation, because it contains intrinsically a constraint that involves not only the charge-to-mass ratio q/m, but also the "proper length" of the solution.

It would be interesting to answer the following questions:

1. Do there exist weak bifurcation instants that are *not* bifurcation instants? (in case of a positive answer, one would have an example of a non bifurcating conjugate instant)
2. Is every conjugate instant a weak bifurcation instant?

4. BIFURCATION OF LIGHT RAYS

Let us now consider a (time oriented) Lorentzian manifold (M, g). Let us introduce some notation; for $p \in M$, let $\mathcal{L}_p$ denote the light cone at p:

$$\mathcal{L}_p = \{v \in T_pM : v \neq 0,\ g(v,v) = 0\}\ ;$$

in the time oriented case, one can also consider:

$$\mathcal{L}_p^+ = \{v \in \mathcal{L}_p : v \text{ future directed}\}\ .$$

For all p, $\mathcal{L}_p$ and $\mathcal{L}_p^+$ are smooth hypersurfaces of T_pM; $\mathcal{L}_p$ has two connected components, and $\mathcal{L}_p^+$ is one of them. Finally, given a smooth (C^1) curve $\gamma : I \to M$, set:

$$\mathcal{L}_\gamma = \bigcup_{s \in I} \mathcal{L}_{\gamma(s)}\ ,\ \ \mathcal{L}_\gamma^+ = \bigcup_{s \in I} \mathcal{L}_{\gamma(s)}^+\ .$$

In [4] it is given the following definition of bifurcation for lightlike geodesics in a (time oriented) Lorentzian manifold.

Definition 8. Let $\gamma : I \to M$ be a smooth timelike (future directed) curve defined on some open neighborhood $I \subset \mathbb{R}$ of 0, and let $z : [0,1] \to M$ be a (future directed) lightlike geodesic in M with $z(0) = \gamma(0)$ and $\dot{z}(0) = v_0 \in \mathcal{L}_{\gamma(0)}$. A point $z(t_0)$, $t_0 \in]0,1[$ is a point of *accumulation of images (from the past)* of the light source γ if there exists a sequence $(t_n)_n$

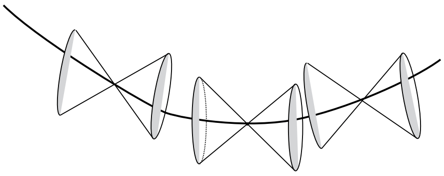

FIGURE 2. The manifold $\mathcal{L}_\gamma$.

in $]0,1[$ tending to t_0 as $n \to \infty$, and a sequence of pairwise distinct vectors $v_n \in \mathcal{L}_\gamma$ (resp., $v_n \in \mathcal{L}_\gamma^+$) with $\lim\limits_{n\to\infty} v_n = v_0$, such that $\exp(t_n v_n) = \exp(t_n v_0) = z(t_n)$ for all $n \geq 1$.

The point $z(t_0)$ is called a *point of accumulation of a continuum of images* (*from the past*) of γ if there exist a left or right neighborhood J of 0 in $\mathbb{R}$ and a continuous path $J \ni r \mapsto v_r \in \mathcal{L}_\gamma$ ($\mathcal{L}_\gamma^+$) of pairwise distinct vectors with $\lim\limits_{r\to 0} v_r = v_0$, a continuous path $J \ni r \mapsto t_r \in [0,1]$, with $\lim\limits_{r\to 0} t_r = t_0$, such that $\exp(t_r v_r) = \exp(t_r v_0) = z(t_r)$ for all $r \in J$.

The following result is proven in [4]:

Proposition 9. *If $z(t_0)$ is conjugate to $z(0)$ along z, then $z(t_0)$ is a point of accumulation of images of* any *light source γ through $z(0)$.*

We want to prove that the property of accumulation of images does in fact characterize conjugate points, i.e., that accumulation can only occur at conjugate points along null geodesics. For a precise statement of this fact, let us study the restriction of the exponential map to the manifolds $\mathcal{L}_\gamma$.

Proposition 10. *Let $\gamma : \,]-\varepsilon,\varepsilon[\, \to M$ be a smooth (C^1) curve; then $\mathcal{L}_\gamma$ is a smooth submanifold of TM. For $v \in \mathcal{L}_\gamma$ with $\pi(v) = \gamma(0) = p$, the tangent space $T_v(\mathcal{L}_\gamma)$ can be described using the decomposition* (1) *of $T_v(TM)$ as:*

$$T_v(\mathcal{L}_\gamma) = \\ = \big\{(w_1,w_2) \in \mathrm{Hor}_v \oplus \mathrm{Ver}_v \cong T_pM \oplus T_pM : w_1 \in \mathbb{R}\cdot\dot\gamma(0),\ w_2 \in v^\perp\big\}\ . \tag{2}$$

Assume that $\dot\gamma(0)$ is timelike; then a vector $v \in \mathcal{L}_p \cap \mathcal{A}_p$ is a critical point of the map $\exp : \mathcal{L}_\gamma \cap \mathcal{A} \to M$ *if and only if v is a critical point of* $\exp_p : \mathcal{A}_p \to M$.

Proof. $\mathcal{L}_\gamma$ is the inverse image of 0 of the map $\gamma^*(TM) \setminus \mathbf{0} \ni v \mapsto g(v,v) \in \mathbb{R}$, where $\gamma^*(TM)$ is the pull-back of TM by γ, which is a smooth vector bundle on $]-\varepsilon,\varepsilon[$, and $\mathbf{0}$ denotes the zero section. The proof of the first statement of the thesis follows easily from the observation that the map $v \mapsto g(v,v)$ has no critical points in $TM \setminus \mathbf{0}$; observe that $\dim(\mathcal{L}_\gamma) = \dim(M)$.

As to the tangent space $T_v(\mathcal{L}_\gamma)$, first observe that $T_v(\mathcal{L}_\gamma) \supset \{0\} \oplus v^\perp \subset \mathrm{Hor}_v \oplus \mathrm{Ver}_v$; namely, vectors of the form $(0,w_2)$, with $w_2 \in v^\perp$ are tangent vectors to curves in $\mathcal{L}_\gamma$ whose image remains in the light cone $\mathcal{L}_p$. Second, observe that parallel transport along γ carries light cones into light cones; hence, $\mathcal{L}_\gamma$ contains curves of the form $t \mapsto v(t)$, where v is a parallel vector field along γ. Tangent vectors to such curves are horizontal, and they are identified with $(\gamma'(t),0)$. In particular, $(\gamma'(0),0) \in T_v(\mathcal{L}_\gamma)$. The equality (2) follows easily from these two observations and from a dimension argument.

As to the statement on the critical points of the exponential map, let us recall that the differential at $v \in \mathcal{A}$ of the map $\exp : \mathcal{A} \to M$ is computed as follows: $\mathrm{d}\exp_v(w_1,w_2) = J(1) \in T_{\exp(v)}M$, where $(w_1,w_2) \in \mathrm{Hor}_v \oplus \mathrm{Ver}_v$ and J is the Jacobi field along the geodesic $[0,1] \ni t \mapsto \exp(tv) \in M$ satisfying $J(0) = w_1$ and $(\mathrm{D}/\mathrm{d}t)J(0) = w_2$. Assume that v is a regular point for $\exp_p : \mathcal{A}_p \to M$, i.e., that the map $\mathrm{d}\exp(v)$ gives an isomorphism between $\{0\} \oplus \mathrm{Ver}_v$ and $T_{\exp_v}M$; we claim that in this case the image of the subspace

$\{0\}\oplus v^\perp\subset T_v(\mathcal{L}_\gamma)$ is given by the subspace $\dot z(1)^\perp\subset T_{\exp(v)}M$, where z is the geodesic $z(t)=\exp_p(tv)$. Namely, using Gauss Lemma, for $w_2\in v^\perp$:

$$g\left(\dot z(1),\mathrm{d}\exp(v)(0,w_2)\right)=g\left(\mathrm{d}\exp_p(v)v,\mathrm{d}\exp_p(v)w_2\right)=g(v,w_2)=0\ ,$$

i.e., $\mathrm{d}\exp(v)(\{0\}\oplus v^\perp)\subset\dot z(1)^\perp$; by counting dimensions:

$$\mathrm{d}\exp(v)(\{0\}\oplus v^\perp)=\dot z(1)^\perp\ .$$

In order to prove that v is also a regular point for the restriction of exp to $\mathcal{L}_\gamma\cap\mathcal{A}$, i.e., that $\mathrm{d}\exp(v)(T_v(\mathcal{L}_\gamma))=T_{\exp(v)}M$, it suffices to show that $\mathrm{d}\exp(v)(\dot\gamma(0),0)$ does not belong to $\dot z(1)^\perp$. To this aim, let J be the Jacobi field along z such that $J(0)=\dot\gamma(0)$ and $(\mathrm{D}/\mathrm{d}t)J(0)=0$; recall that $g(J,\dot z)$ is an affine function, and, with the choice of such Jacobi field, it is indeed constant, because:

$$\frac{\mathrm{d}}{\mathrm{d}t}g(J,\dot z)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}g(J,\dot z)=g\left(\frac{\mathrm{D}}{\mathrm{d}t}J(0),\dot z(0)\right)=0\ .$$

Then:

$$\begin{aligned}g(\mathrm{d}\exp(v)(\dot\gamma(0),0),\dot z(1))=g(J(1),\dot z(1))=g(J(0),\dot z(0))=\\=g(\dot\gamma(0),\dot z(0))\neq 0\ ,\end{aligned}$$

where the last inequality follows from the fact that $\dot\gamma(0)$ is assumed timelike, while $\dot z(0)$ is lightlike.

Conversely, assume that v is a critical point of $\exp_p$; the computations we have just made above show that $\mathrm{d}\exp(v)(\{0\}\oplus v^\perp)\subset\dot z(1)^\perp$. Clearly, $\dot\gamma(0)\not\in v^\perp$; we claim that $\mathrm{d}\exp(v)(0,\dot\gamma(0))$ is not in $\dot z(1)^\perp$: if J is the Jacobi field along z satisfying $J(0)=0$ and $(\mathrm{D}/\mathrm{d}t)J(0)=\dot\gamma(0)$, then $\mathrm{d}\exp(v)(0,\dot\gamma(0))=J(1)$, and

$$\begin{aligned}g(\mathrm{d}\exp(v)(0,\dot\gamma(0)),\dot z(1))=g(J(1),\dot z(1))=\\=g(J(0),\dot z(0))+\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}g(J,\dot z)=g\left(\frac{\mathrm{D}}{\mathrm{d}t}\,J(0),\dot z(0)\right)=g(\dot\gamma(0),\dot z(0))\neq 0\ .\end{aligned}$$

From these observations, it follows that if $\mathrm{d}\exp_p(v)$ is not injective on T_pM, then it cannot be injective on $v^\perp$, hence $\mathrm{d}\exp(v)$ cannot be injective on $T_v(\mathcal{L}_\gamma)$. This concludes the proof.

□

Corollary 11. *Let (M,g) be a Lorentzian manifold and let $z:[0,1]\to M$ be a lightlike geodesic, with $\dot z(0)=v$. Given $t_0\in\,]0,1[$, the following statements are equivalent:*

(a) *$z(t_0)$ is a point of accumulation of images of* some *light source γ through $z(0)$;*
(b) *$z(t_0)$ is a point of accumulation of images of* any *light source through $z(0)$;*
(c) *$z(t_0)$ is conjugate to $z(0)$ along z;*
(d) *the exponential map $\exp:\mathcal{L}_\gamma\cap\mathcal{A}\to M$ is not locally injective around t_0v.*

Proof. Clearly, (b) $\Longrightarrow$ (a) $\Longrightarrow$ (d); by Proposition 10, (d) $\Longrightarrow$ (c), while Proposition 9 gives (c) $\Longrightarrow$ (b).

□

In analogy to the result of Proposition 4, also in the case of light rays one can prove that conjugate instants determine a *continuous* bifurcating branch of light rays issuing from any light source. The following elementary result will be needed for a formal proof of this fact:

Lemma 12. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and let $f:\mathcal{X}\to\mathcal{Y}$, $P:\mathcal{X}\to\mathcal{X}$ and $Q:\mathcal{Y}\to\mathcal{Y}$ be maps such that:*

- *$f(\mathcal{X})$ is dense in $\mathcal{Y}$;*
- *Q is continuous, and $Q^2=Q$;*
- *$f\circ P=Q\circ f$.*

Then, $f(P(\mathcal{X}))$ is dense in $Q(\mathcal{Y})$.

Proposition 13. *Every conjugate point along a lightlike geodesic* $z : [0,1] \to M$ *in a Lorentzian manifold* (M,g) *is a point of accumulation of a continuum of images issuing from any light source* γ *through* $z(0)$.

Proof. Also this result is obtained as an application of the bifurcation result of [3], in its Banach space formulation; observe that conjugate instants along lightlike Lorentzian geodesics are isolated. We recall from [4] that the bifurcation framework for light rays issuing from a fixed light source used a variational problem (Fermat principle for light rays) in a *Banach manifold* setup. Such manifold is modeled on a Banach space of the type:

$$\mathfrak{X}_{w,Z,\mathfrak{g}} = \big\{ X \in C^1([0,1],\mathbb{R}^n) \ : \ X(0) \in \mathbb{R}\cdot w,\ X(1) = 0,\ \mathfrak{g}(X',Z) \equiv 0 \big\} \ ,$$

where $w \in \mathbb{R}^n$ is a fixed vector, $\mathfrak{g}$ is a nondegenerate symmetric bilinear form on $\mathbb{R}^n$, $Z : [0,1] \to \mathbb{R}^n$ is a given smooth map with $\mathfrak{g}(Z(t),w) \neq 0$ for all $t \in [0,1]$. Here, $\mathfrak{X}_{w,Z,\mathfrak{g}}$ is assumed to be endowed with the Banach space structure inherited from $C^1([0,1],\mathbb{R}^n)$.

In order to prove our result, it will suffice to exhibit a Hilbert space that is mapped continuously and with dense image into the Banach space $\mathfrak{X}_{w,Z,\mathfrak{g}}$. To this aim, set:

$$\mathfrak{X}^{(2)}_{w,Z,\mathfrak{g}} = \big\{ X \in \mathfrak{X}_{w,Z,\mathfrak{g}} \ : \ X \text{ is of Sobolev class } H^2 \big\} \ ;$$

Clearly, $\mathfrak{X}^{(2)}_{w,Z,\mathfrak{g}}$ is a closed subspace of the Hilbert space $H^2([0,1],\mathbb{R}^n)$, and the inclusion $\mathfrak{i} : \mathfrak{X}^{(2)}_{w,Z,\mathfrak{g}} \to \mathfrak{X}_{w,Z,\mathfrak{g}}$ is continuous. In order to prove that $\mathfrak{i}$ has dense image, we argue as follows. Define:

$$\begin{aligned} \mathcal{X} = &\ \ \big\{ X \in H^2([0,1],\mathbb{R}^n) \ : \ X(0) \in \mathbb{R}\cdot w,\ X(1) = 0 \big\} \ , \\ \mathcal{Y} = &\ \ \big\{ X \in C^1([0,1],\mathbb{R}^n) \ : \ X(0) \in \mathbb{R}\cdot w,\ X(1) = 0 \big\} \ , \end{aligned}$$

and let $f : \mathcal{X} \to \mathcal{Y}$ be the inclusion map; clearly, f has dense image. For $X \in \mathcal{Y}$, let $\lambda_X : [0,1] \to \mathbb{R}$ be the map:

$$\lambda_X(t) = -\int_t^1 \frac{\mathfrak{g}(X',Z)}{\mathfrak{g}(w,Z)}\,\mathrm{d}r \ ;$$

observe that, by continuity, $g(w,Z)$ is bounded away from 0, and λ_X is a well defined map having the same regularity as X. Moreover, $\mathcal{X} \ni X \mapsto \lambda_X \in H^2([0,1],\mathbb{R})$ (resp., $\mathcal{Y} \ni X \mapsto \lambda_X \in C^1([0,1],\mathbb{R})$) is linear and bounded, so that we can define bounded linear maps $P : \mathcal{X} \to \mathcal{X}$ and $Q : \mathcal{Y} \to \mathcal{Y}$ by setting:

$$P(X) = X - \lambda_X \cdot w \ \ , \ \ Q(Y) = Y - \lambda_Y \cdot w \ .$$

By construction, $P(\mathcal{X}) = \mathfrak{X}^{(2)}_{w,Z,\mathfrak{g}}$, $Q(\mathcal{Y}) = \mathfrak{X}_{w,Z,\mathfrak{g}}$, $\mathfrak{i} \circ P = Q \circ \mathfrak{i}$; an immediate calculation shows that if $Y \in \mathfrak{X}_{w,Z,\mathfrak{g}}$, then $\lambda_Y \equiv 0$ and so $Q^2 = Q$. Applying Lemma 12, the inclusion of the Hilbert space $\mathfrak{X}^{(2)}_{w,Z,\mathfrak{g}}$ into the Banach space $\mathfrak{X}_{w,Z,\mathfrak{g}}$ has dense image, and this concludes the proof.

□

Using the result in [5], Proposition 13 gives also the continuity of the bifurcating branch of solutions of the Lorentz force equation with fixed charge-to-mass ratio.

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