Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 5(2006), pp. 153 - 171.

The interplay of the polar decomposition theorem and the Lorentz group

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Abstract[1]. In this paper we consider the well-known standard physical decomposition of Lorentz group in boost and spatial rotation from the point of view of bounded operator theory in Hilbert spaces. In this context we prove that the polar decomposition theorem of operators in (real) Hilbert spaces, when applied to real matrices of $SO(1,3)\uparrow$ produces just the standard physical decomposition of any matrix of the orthochronous proper Lorentz group $SO(1,3)\uparrow$. This result is interesting because the polar decomposition theorem is referred to a positive-defined scalar product while the Lorentz-group decomposition theorem deals with the indefinite Lorentz metric. A generalization to infinite dimensional spaces is also presented. It is finally shown that the polar decomposition of $SL(2,\mathbb{C})$ is preserved by the covering homomorphism of $SL(2,\mathbb{C})$ onto $SO(1,3)\uparrow$. In the first part the mathematical structure of Minkowski spacetime is reviewed in a pedagogical fashion.

1. MINKOWSKI SPACETIME

The main goal of this paper is to discuss the interplay of standard physical decomposition of Lorentz group in boost and spatial rotation and the so-called polar decomposition theorem for bounded operators in Hilbert spaces. The first part includes a pedagogical review on basic mathematical notions of Special Relativity theory suitable especially for mathematicians. Prerequisites are basic notions of differential geometry (including the notion of affine space) and Lie group theory.

1.1. General Minkowski spacetime structure.

As is well known, Poincaré group encodes all coordinate transformations between a pair of inertial frames moving in Minkowski spacetime. Let us recall some features of that extent from a mathematical point of view.

Definition 1. Minkowski spacetime $(\mathbb{M}^4, T^4, g)$ is a four-dimensional affine space $(\mathbb{M}^4, T^4)$, equipped with a non-degenerate but indefinite scalar product g, with signature $(-,+,+,+)$, in the real four-dimensional vector space of translations T^4. The following further definitions hold.
(a) The points of Minkowski spacetime are called *events.*
(b) The Cartesian coordinate systems x^0, x^1, x^2, x^3 induced from the affine structure by arbitrarily fixing any g-orthonormal basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in T^4 (with $g(\mathbf{e}_0, \mathbf{e}_0) = -1$, $g(\mathbf{e}_1, \mathbf{e}_1) = 1$, $g(\mathbf{e}_2, \mathbf{e}_2) = 1$, $g(\mathbf{e}_3, \mathbf{e}_3) = 1$) and any origin $O \in \mathbb{M}^4$, are called *Minkowskian coordinate frames.*

In practice, exploiting the affine structure and using standard notation for affine spaces, Minkowskian coordinates x^0, x^1, x^2, x^3 are defined in order that the map $\mathbb{M}^4 \ni p \mapsto (x^0(p), x^1(p), x^2(p), x^3(p)) \in \mathbb{R}^4$ satisfies

$$T^4 \ni p - O = \sum_{\mu=0}^{3} x^\mu(p)\mathbf{e}_i$$

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for every event $p \in \mathbb{M}^4$. The class of Minkowskian coordinate frames is in fact an C^∞ atlas for $\mathbb{M}^4$, which, as a consequence acquires the structure of a differentiable manifold. Every Minkowskian coordinate frame x^0, x^1, x^2, x^3 is a global coordinate system, i.e. the map $\mathbb{M}^4 \ni p \mapsto (x^0(p), x^1(p), x^2(p), x^3(p)) \in \mathbb{R}^4$ is a diffeomorphisms onto $\mathbb{R}^4$. Moreover Minkowskian charts induces, from the scalar producr g in T^4, the same smooth metric which turns out to be constant and diagonal in those coordinates:

$$-dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \; . \tag{1}$$

We conclude that $(\mathbb{M}^4, T^4, g)$ is in fact a *globally flat* semi-Riemannian manifold, whose metric has hyperbolic signature $(-,+,+,+)$. Such a metric is said to be of *Lorentzian type.* From now on the metric of Minkowski spacetime is indicated by g, since it is completely determined by the scalar product in T^4.

Physically speaking the events are the minimal space-temporal characterization of everything occurs in the universe. Modulo technicalities we shall examine shortly, Minkowskian coordinate frames individuate (not biunivocally) the the class of coordinate system of all *inertial observers*. Referring to the decomposition (1) of g, the coordinates x^1, x^2, x^3, are thought as "spatial coordinates", whereas the coordinate x^0 is a temporal coordinate. The metric g is used to perform measurements either in time and in space as we clarify in the rest of this section. If $X_p \neq 0$ is a vector in $T_p\mathbb{M}^4$, it may represent either infinitesimal temporal displacements if $g_p(X_p, X_p) \leq 0$ or infinitesimal spatial displacements if $g_p(X_p, X_p) > 0$. In both cases $|g_p(X_p, X_p)|$ has the physical meaning of the length (duration) of V_p. Actually the distinguishable case $g(X_p, X_p) = 0$ (but $X_p \neq 0$) deserves a particular comment. These vectors represent an infinitesimal part of the history of a light particle.

Definition 2. For every event $p \in \mathbb{M}^4$, $T_p\mathbb{M}^4 \setminus \{0\}$ is decomposed in three pairwisely disjoint subsets:

(i) the set of *spacelike* vectors which satisfy: $g_p(X_p, X_p) > 0$,
(ii) the set of *timelike* vectors which satisfy: $g_p(X_p, X_p) < 0$,
(iii) the set of *lightlike*, also called *null*, vectors which satisfy: $g_p(X_p, X_p) = 0$.

The following further definitions hold.
(a) The union of the sets $\{0\}$ and timelike and lightlike vectors is a closed cone, $\overline{V_p}$, called *closed light cone* at p. Its non-vanishing element are called *causal* vectors.
(b) The interior V_p of $\overline{V_p}$ is called *open light cone* at p.
(c) The boundary ∂V_p is called *light cone* at p.

1.2. **Time orientation.**

Definition 3. Two smooth timelike vector fields T, T' on $\mathbb{M}^4$ are said to *have the same time orientation* if $g(T_p, T'_p) < 0$ for every $p \in \mathbb{M}^4$.

To go on we notice the following fact. Using Minkowskian coordinates and referring to the base of $T_p\mathbb{M}^4$ associated with these coordinates, one sees that the open light cone is pictured as the set

$$V_p = \{(X^0, X^1, X^2, X^3) \in \mathbb{R}^4 \setminus \{0\} \;\mid\; (X^0)^2 > (X^1)^2 + (X^2)^2 + (X^3)^2\} \; .$$

Thus, in those Minkowskian coordinates, V_p is made of two disjoint halves

$$\begin{aligned} V_p^{(>)} &:= \{(X^0, X^1, X^2, X^3) \in V_p \;\mid\; X^0 > 0\} \; , \\ V_p^{(<)} &:= \{(X^0, X^1, X^2, X^3) \in V_p \;\mid\; X^0 < 0\} \; . \end{aligned} \tag{2}$$

We remark that, *a priori*, this decomposition depends on the used coordinate system.

Proposition 1. *The class $\mathcal{T}(\mathbb{M}^4)$ of smooth timelike vector fields on $\mathbb{M}^4$ satisfies following.*
(a) *$\mathcal{T}(\mathbb{M}^4)$ is not empty.*
(b) *Referring to a Minkowskian coordinate frame and decomposing every V_p into the two disjoint halves (2), if $T, T' \in \mathcal{T}(\mathbb{M}^4)$ it holds constantly in $p \in \mathbb{M}^4$, $g(T_p, T'_p) < 0$ or*

$g(T_p, T'_p) > 0$. The former happens if and only if both T_p, T'_p belong to the same half of V_p, the latter holds if and only if they belong to different halves.
(c) *"to have the same time orientation" is a equivalence relation in $\mathcal{T}(\mathbb{M}^4)$ and it admits two equivalence classes only.*

Proof. (a) $\mathcal{T}(\mathbb{M}^4)$ is not empty since it includes the vector field ∂_{x^0} associated to any Minkowskian coordinate frame on $\mathbb{M}^4$. Let us show (b). Consider a smooth timelike vector field S. Using Minkowskian coordinates x^0, x^1, x^2, x^3 and the associated orthonormal bases of elements $\mathbf{e}_{\alpha,p} = \partial_{x^\alpha}|_p \in T_p\mathbb{M}^4$, $\alpha = 0, 1, 2, 3$, one has:

$$(S^0_p)^2 > \sum_{i=1}^{3} (S^i_p)^2 \,. \tag{3}$$

Consider the two halves $V_p^{(>)}$ and $V_p^{(<)}$ said above. S^0_p cannot change it sign varying $p \in \mathbb{M}$ because it would imply that $S^0_p = 0$ somewhere and (3) would entail, in turn, that $S_p = 0$ which is not allowed. Therefore it holds $S_p \in V_p^{(>)}$ constantly in $p \in \mathbb{M}^4$, that is

$$S^0_p > \sqrt{\sum_{i=1}^{3} (S^i_p)^2} \,, \quad \text{for all } p \in \mathbb{M}^4 \,, \tag{4}$$

or $S_p \in V_p^{(<)}$ constantly in $p \in \mathbb{M}^4$, that is

$$S^0_p < \sqrt{\sum_{i=1}^{3} (S^i_p)^2} \,, \quad \text{for all } p \in \mathbb{M}^4 \,. \tag{5}$$

Now consider two timelike smooth vector fields T and T'. One has

$$g(T, T') = -T^0 T'^0 + \sum_{i=1}^{3} T'^i T^i \,. \tag{6}$$

On the other hand, by Cauchy-Schwartz inequality

$$\left| \sum_{i=1}^{3} T^i T'^i \right| \leq \sqrt{\sum_{i=1}^{3} (T^i_p)^2} \sqrt{\sum_{i=1}^{3} (T'^i_p)^2} \,.$$

Now, taking into account that it must hold either (4) or (5) for T and T' in place of S, we conclude from (6) that, constantly in $p \in \mathbb{M}^4$, $g(T_p, T'_p) < 0$ or $g(T_p, T'_p) > 0$ and the former happens if and only if both T, T' belong to the same half of V_p, whereas the latter arises if and only if they belong to different halves. The proof of (b) is concluded.
Using (b) result we can discuss the extent in a single tangent space fixing $p \in \mathbb{M}^4$ arbitrarily and we can prove (c), that "to have the same time orientation" is a equivalence relation. By definition of timelike vector $g(T_p, T_p) < 0$, so T has the same time orientation as T itself. If $g(T_p, T'_p) < 0$ then $g(T'_p, T_p) = g(T_p, T'_p) < 0$ so that the symmetric property is satisfied. Let us come to transitivity. Suppose that $g(T_p, T'_p) < 0$ so that T_p and T'_p belong to the same half of V_p, and $g(T'_p, S_p) < 0$ so that T'_p and S_p belong to the same half of V_p. We conclude that T_p and S_p belong to the same half of V_p and thus $g(T_p, S_p) < 0$. This proves transitivity.
To conclude, notice that, if T is a smooth timelike vector field on $\mathbb{M}^4$, T and $-T$ belong to different equivalence classes and if $g(T, S) > 0$ then $g(-T, S) < 0$ so that, every timelike smooth vector field belongs to the equivalence class of T or to the equivalence class of $-T$.

□

Corollary 1. *The decomposition of V_p into two disjoint halves does not depend on used Minkowskian coordinate frame. In other words, considering two Minkowskian coordinate* *frames* x^0, x^1, x^2, x^3 *and* $x^0_1, x^1_1, x^2_1, x^3_1$, $p \in \mathbb{M}^4$ *and and the associated decompositions* (2), $V_p = V_p^{(>)} \cup V_p^{(<)}$ *and* $V_{1,p}^{(>)} \cup V_{1,p}^{(<)}$*, one of the following exclusive cases must occur:*

$$(1)\;\; V_p^{(>)} = V_{1,p}^{(>)} \;\; \textit{and} \;\; V_p^{(<)} = V_{1,p}^{(<)} \;\; \textit{or} \;\; (2)\;\; V_p^{(>)} = V_{1,p}^{(<)} \;\; \textit{and} \;\; V_p^{(<)} = V_{1,p}^{(>)}\,.$$

Proof. By (b) of Proposition 1, $V_p^{(>)}$ and $V_{1,p}^{(>)}$ are respectively made of the restrictions to p of the vector fields in $\mathcal{T}(\mathbb{M}^4)$ which have the same time orientation as ∂_{x^0} and $\partial_{x^0_1}$ respectively. On the other hand, using (b) again, one finds that $\partial_{x^0_1}$ must belong to $V_p^{(>)}$ or it must belong to $V^{(<)}$. Suppose that the former is valid. In this case $\partial_{x^0_1}$ and $\partial_{x^0_1}$ have the same time orientation. By transitivity one conclude that $V_p^{(>)} = V_{1,p}^{(>)}$. Since $V_p^{(>)} \cup V_p^{(<)} = V_{1,p}^{(>)} \cup V_{1,p}^{(<)}$ and $V_p^{(>)} \cap V_p^{(<)} = V_{1,p}^{(>)} \cap V_{1,p}^{(<)} = \emptyset$ we conclude also that $V_p^{(<)} = V_{1,p}^{(<)}$. If $\partial_{x^0_1} \in V_p^{(<)}$, one concludes that ${V'}_p^{(>)} = V_{1,p}^{(>)}$ and ${V'}_p^{(<)} = V_{1,p}^{(<)}$ where ${V'}_p^{(>,<)}$ are referred to Minkowskian coordinates $-x^0, x^1, x^2, x^3$. Since ${V'}_p^{(>)} = V_p^{(<)}$ and ${V'}_p^{(<)} = V_p^{(>)}$, we conclude that $V_p^{(>)} = V_{1,p}^{(<)}$ and $V_p^{(<)} = V_{1,p}^{(>)}$.

Definition 4. A pair $(\mathbb{M}^4, \mathcal{O})$ where $\mathcal{O}$ is one of the two equivalence classes of the relation "to have the same time orientation" in the set of smooth timelike vector fields on $\mathbb{M}^4$, is called *time oriented* Minkowski spacetime, and $\mathcal{O}$ is called *time orientation* of $\mathbb{M}^4$.

There is an alternative way to fix a time orientation: form Proposition 1 and its corollary we also conclude that.

Proposition 2. *The assignment of a time orientation* $\mathcal{O}$ *is equivalent to smoothly select one of the two disjoint halves of* V_p*, for every* $p \in \mathbb{M}^4$*: that containing time vectors which are restriction to* p *of a vector field in* $\mathcal{O}$*.*

Definition 5. Considering a time oriented Minkowski spacetime $(\mathbb{M}^4, \mathcal{O})$ and an event $p \in \mathbb{M}^4$, the half open cone at p containing vectors which are restrictions to p of vector fields in $\mathcal{O}$ is denoted by V_p^+ and is called *future open light cone* at p. Its elements are said *future-directed timelike vectors* at p. The lightlike vectors of ∂V_p^+ are equally said to be *future-directed lightlike vectors* at p.

Remark 1. Henceforth we suppose to deal with a time oriented Minkowski spacetime and we indicate it by means of $\mathbb{M}^4$ simply.

1.3. **Curves as histories.** If $I \subset \mathbb{R}$ is an interval, a regular (at least C^1) curve $\gamma : I \to \mathbb{M}^4$ may represent the history of a particle of matter provided its tangent vector $\dot\gamma$ is causal and future directed. A particle of light, in particular, has a history representd by a curve with future-directed lightlike tangent vector.

Definition 6. A regular curve $\gamma : I \to \mathbb{M}^4$ is said to be *timelike*, *spacelike*, *causal*, *lightlike* if all of its tangent vectors $\dot\gamma_p$, $p \in \gamma$, are respectively timelike, spacelike, causal, lightlike.

A causal, timelike, lightlike curve γ with future oriented tangent vectors is said to be *future-directed* (resp. *causal*, *timelike*, *lightlike*) *curve.*

Definition 7. If $\gamma = \gamma(\xi)$, $\xi \in I$ with $I \subset \mathbb{R}$ any interval of $\mathbb{R}$, is any C^1 future-directed causal curve in $\mathbb{M}^4$ and $\xi_0 \in I$, the length function

$$\tau(\xi) := \int_{\xi_0}^{\xi} \sqrt{|g(\dot\gamma(l), \dot\gamma(l))|}\, dl \tag{7}$$

is called *proper time* of γ. Moreover, for future-directed timelike curves, the tangent vector obtained by re-parameterizing the curve with the proper time is called *four-speed* of the curve.

Notice that proper time is independent from the used parametrization of γ but it is defined up to the choice of the origin: the point on γ where $\tau = 0$.

From the point of view of physics, when one considers timelike curves, proper time is time measured by an ideal clock at rest with the particle whose history is γ.
For timelike curves proper time can be used as natural parameter to describe the history γ of the particle because, in that case, directly from (7), $d\tau/d\xi > 0$. Notice also that, if $\dot\gamma$ is a four-speed (7) implies that:

$$g(\dot\gamma, \dot\gamma) = -1 \,, \tag{8}$$

that is a four-speed is *unitary.*
Future-directed causal curves which are straight lines with respect to the affine structure of $\mathbb{M}^4$ (and so they are also complete geodesics for the metric g) are thought to be the histories of particles not subjected to forces. These are particles evolving with inertial motion.

1.4. **Minkowskian reference frames.** If x^0, x^1, x^2, x^3 are Minkowskian coordinates, the curves tangent to x^0 determine a constant (with respect to Levi-Civita connection associated with g) vector field ∂_{x^0} which is unitary, that is it satisfies $g(\partial_{x^0}, \partial_{x^0}) = -1$. A unitary constant timelike vector field can always be viewed as the tangent vector to the coordinate x^0 of a Minkowskian coordinate frame. There are anyway *several* Minkowskian coordinate frames for a unitary constant timelike vector field.

Definition 8. A constant timelike vector field $\mathcal{F}$ on $\mathbb{M}^4$ is said a (*Minkowskian*) *reference frame* provided it is future oriented and unitary, i.e. $g(\mathcal{F}, \mathcal{F}) = -1$. Furthermore the following definitions hold.
(a) Any Minkowskian coordinate frame such that $\partial_{x^0} = \mathcal{F}$, is said to be *co-moving with* (or equivalently *adapted to*) $\mathcal{F}$.
(b) If $\gamma = \gamma(\xi)$, where $\xi \in \mathbb{R}$, is an integral curve of $\mathcal{F}$ (and thus it is both a geodesic and a affine straight line), the length function

$$t_{\mathcal{F}}(\gamma(\xi)) := \int_0^\xi \sqrt{|g(\mathcal{F}_{\gamma(l)}, \mathcal{F}_{\gamma(l)})|}\; dl$$

defines a *time coordinate for* $\mathcal{F}$ (*along* γ).

For a fixed $\mathcal{F}$, an adapted Minkowskian coordinate frame can be obtained by fixing a three-dimensional affine plane Σ orthogonal to γ in O, and fixing an orthonormal basis in $T_p\mathbb{M}$ such that $\mathbf{e}_0 = \mathcal{F}$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ belong to Σ. Now one proves straightforwardly that the Minkowskian coordinate frame associated with O, $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is co-moving with $\mathcal{F}$ and, along γ, x^0 coincides with $t_{\mathcal{F}}$ up an additive constant. Notice that $t_{\mathcal{F}}$ defines in this way a *global temporal coordinate* associated with $\mathcal{F}$ since one may attribute a time coordinate $t_{\mathcal{F}} := x^0$ to every event of $\mathbb{M}^4$, not only to those of $\mathcal{F}$. By construction, starting from a different integral curve γ' of $\mathcal{F}$, one gets the same global temporal coordinate up to an additive constant. Moreover, if a time coordinate $t_{\mathcal{F}}$ is defined for $\mathcal{F}$, every three-dimensional affine plane orthogonal to $\mathcal{F}$ contains only points with constant value of that time coordinate.

Definition 9. Let $\mathcal{F}$ be a reference frame in $\mathbb{M}^4$ and t an associated time coordinate. Any three-dimensional affine plane $\Sigma^{(\mathcal{F})}_t$ orthogonal to $\mathcal{F}$, $t \in \mathbb{R}$ being the value of the time coordinate of the points in the plane, is called *rest space of* $\mathcal{F}$ *at time* t.

By construction every rest space $\Sigma^{(\mathcal{F})}_t$ of $\mathcal{F}$ is an embedded three-dimensional submanifold of $\mathbb{M}^4$ and the metric induced by g on $\Sigma^{(\mathcal{F})}_t$ is positive defined.
From the point of view of physics a Minkowski frame is nothing but an inertial reference frame, any time coordinate is time measured by ideal clocks and the metric induced by g on the rest frames is the mathematical tool corresponding to physical spatial measurements performed by rigid rulers.
The mathematical extent is sufficiently developed to allows one to define the notion of speed of a particle represented by a C^1 future-directed causal curve γ, with respect to a reference frame $\mathcal{F}$ at time t. The procedure is straightforward. Consider the event $e := \Sigma_t \cap \gamma$

where the curve intersect the rest frame $\Sigma_t^{(\mathcal{F})}$ (the reader should prove that each rest frame intersects γ exactly in a point). $T_e\mathbb{M}^4$ is decomposed to the orthogonal direct sum

$$T_e\mathbb{M}^4 = L(\mathcal{F}_e) \oplus T_e\Sigma_t^{(\mathcal{F})} \ , \tag{9}$$

$L(\mathcal{F}_e)$ being the linear space spanned by the vector $\mathcal{F}_e$. As a consequence $\dot{\gamma}_e$ turns out to be uniquely decomposed as

$$\dot{\gamma}_e = \delta t \mathcal{F}_e + \delta X \ , \tag{10}$$

where $\delta X \in T_e\Sigma_t^{(\mathcal{F})}$ and $\delta t \in \mathbb{R}$. The fact that $\dot{\gamma}_e$ is causal prevent δt to vanish (the reader should prove it), so that it makes sense to give the following definition of speed.

Definition 10. Let γ be a C^1 future-directed causal curve and $\mathcal{F}$ a reference frame in $\mathbb{M}^4$. The *speed of* γ *with respect to* $\mathcal{F}$ *at time* t is the vector $\mathbf{v}_{\gamma,t}^{(\mathcal{F})} \in T_{\Sigma_t^{(\mathcal{F})}\cap\gamma}\Sigma_t^{(\mathcal{F})}$, given by

$$\mathbf{v}_{\gamma,t}^{(\mathcal{F})} := \frac{\delta X}{\delta t} \ .$$

referring to (9) and (10) with $e := \Sigma_t \cap \gamma$.

Within our framework one has the following physically well-known result.

Proposition 3. *Consider a future-directed causal curve* γ *and fix a reference frame* $\mathcal{F}$*,*
(a) *The absolute value of* $\mathbf{v}_{\gamma,t}^{(\mathcal{F})}$ *is bounded up by* 1 *and that value is attained only at those events along* γ *where the tangent vector of* γ *is lightlike.*
(b) *If* γ *is timelike and* $\dot{\gamma}(t)$ *is the four-speed of* γ *at time* t *of* $\mathcal{F}$*:*

$$\dot{\gamma}(t) = \frac{1}{\sqrt{1-\mathbf{v}_{\gamma,t}^{(\mathcal{F})\,2}}}\,\mathcal{F} + \frac{\mathbf{v}_{\gamma,t}^{(\mathcal{F})}}{\sqrt{1-\mathbf{v}_{\gamma,t}^{(\mathcal{F})\,2}}} \ . \tag{11}$$

(c) *If* γ *intersects the rest spaces* $\Sigma_{t_1}^{(\mathcal{F})}$ *and* $\Sigma_{t_2}^{(\mathcal{F})}$ *with* $t_2 > t_1$ *at proper time* τ_1 *and* $\tau_2 > \tau_1$ *respectively, the corresponding intervals of time satisfy*

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1-\mathbf{v}_{\gamma,t}^{(\mathcal{F})\,2}}\,dt \tag{12}$$

and so $\Delta\tau < \Delta t$ *unless* $\mathbf{v}_{\gamma,t}^{(\mathcal{F})} = 0$ *in the considered interval of time and, in that case,* $\Delta\tau = \Delta t$*.*

Remark 2.
(1) The absolute value $||\mathbf{v}_{\gamma,t}^{(\mathcal{F})}||$ is referred to the scalar product induced by g in the rest spaces of $\mathcal{F}$. As previously said this is the physical metric tool which corresponds to perform measurements.
(2) The speed of light turns out to have the universal value 1 with respect to every reference frame. This value is usually denoted, changing the units of measure, by c.
(3) One recognizes in (c) the mathematical formulation of the celebrated relativistic phenomenon called *dilatation of time.*

Proof of Proposition 3. Using Definition 10, decomposition (10), the orthogonality of δX and $\mathcal{F}$ and the fact that $\mathcal{F}$ is unitary, one has

$$0 \geq g(\dot{\gamma},\dot{\gamma}) = -\delta^2 + g(\delta X, \delta X) = -\delta t^2 + ||\delta X||^2 \ .$$

The the sign = occurs only if $\dot{\gamma}$ is lightlike. This implies the thesis immediately. (b) follows immediately from (10) and the definition of $\mathbf{v}_{\gamma,t}^{(\mathcal{F})}$ if imposing $g(\dot{\gamma},\dot{\gamma}) = -1$. Finally (c) is a straightforward consequence of (b) since the the factor in front of $\mathcal{F}$ in (11) is $dt/d\tau$.

$\square$

2. LORENTZ AND POINCARÉ GROUPS

The following natural question arises: if $\mathcal{F}$ and $\mathcal{F}'$ are two Minkowskian reference frames equipped with co-moving Minkowskian coordinates x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 respectively, what about the most general transformation rule between these coordinates?
Since both coordinate frame are Minkowskian which, in turn, are Cartesian coordinate frames, the relation must be linear[2]:

$$x'^\mu = \Lambda^\mu{}_\nu(x^\nu + T^\nu)\ , \tag{13}$$

the requirement of non singularity of Jacobian determinant is obviously equivalent to non singularity of the matrix Λ of coefficient $\Lambda^\mu{}_\nu$. Finally, the requirement that in both coordinate system g must have the form (1), i.e.

$$g = \eta_{\alpha\beta} dx^\alpha \otimes dx^\beta = \eta_{\mu\nu}\, dx'^\mu \otimes dx'^\nu \tag{14}$$

where we have introduced the matrix $\eta = diag(-1, +1, +1, +1)$ of coefficients $\eta_{\alpha\beta}$, leads to the requirement

$$\Lambda^t \eta \Lambda = \eta\ . \tag{15}$$

Notice that this relation implies the non singularity of matrices Λ because, taking the determinant of both sides one gets:

$$(\det \Lambda)^2 \det \eta = \det \eta\ ,$$

where we exploited $\det(\Lambda^t\eta\Lambda) = \det(\Lambda^t)\det(\eta)\det(\Lambda) = \det(\Lambda)\det(\eta)\det(\Lambda)$. Since $\det \eta = -1$, it must be $\det \Lambda = \pm 1$. Proceeding backwardly one sees that if x'^0, x'^1, x'^2, x'^3 is a Minkowskian coordinate frame and (13) hold with Λ satisfying (15), then x^0, x^1, x^2, x^3 is a Minkowskian coordinate frame too. Summarizing one has the following straightforward result.

Proposition 4. *If x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 are Minkowskian coordinate frame on $\mathbb{M}^4$, the transformation rule between these coordinates has the form* (13) *where T^μ, $\mu = 0, 1, 2, 3$ are suitable reals and the matrix Λ satisfies* (15).
Conversely, if x'^0, x'^1, x'^2, x'^3 is a Minkowskian coordinate frame and (13) *hold for arbitrary real constants T^μ, $\mu = 0, 1, 2, 3$ and an arbitrary matrix Λ satisfying* (15)*, then x^0, x^1, x^2, x^3 is a another Minkowskian coordinate frame.*

2.1. **Lorentz group.** The result proved above allows one to introduce the celebrated Lorentz group. In the following $M(4, \mathbb{R})$ will denote the algebra of real 4×4 matrices and $GL(4, \mathbb{R})$ indicates the usual Lie group of real 4×4 matrices.

Proposition 5. *The set, called* Lorentz group,

$$O(1, 3) := \{\Lambda \in M(4, \mathbb{R}) \ \mid\ \Lambda^t \eta \Lambda = \eta\}\ , \tag{16}$$

is a Lie group which is a Lie subgroup of $GL(4, \mathbb{R})$.

Proof. From the general theory of Lie groups [1] we know that to show that $O(1, 3)$ is a Lie subgroup of the Lie group of $GL(2, \mathbb{R})$ it is sufficient to prove that the former is a topologically-closed algebraic subgroup of the latter. The fact that $O(1, 3)$ is a closed subset of $GL(2, \mathbb{R})$, where the latter is equipped with the topology (and the differentiable structure) induced by $\mathbb{R}^{16}$, is obvious from $\Lambda^t\eta\Lambda = \eta$ since the product of matrices and the transposition of a matrix are continuous operations. To show that $O(1, 3)$ is a subgroup of $GL(4, \mathbb{R})$ it is sufficient to prove that it is closed with respect to the multiplication of matrices, and this is trivial from (16) using the fact that $\eta\eta = I$, and that if $\Lambda \in O(1, 3)$ also $\Lambda^{-1} \in O(1, 3)$. The proof this last fact is consequence of (a), (b) and (c) in Proposition 6 whose proofs is entirely based on (16).

□

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[2] From now on we exploit the convention of summation over repeated indices.

The next technical proposition will allow us to introduce some physically relevant subgroups of $O(1,3)$.

Proposition 6. *The Lorentz group enjoys the following properties.*
(a) $\eta, -I, -\eta, \in O(1,3)$.
(b) $\Lambda \in O(1,3)$ *if and only if* $\Lambda^t \in O(1,3)$.
(c) *If* $\Lambda \in O(1,3)$ *then* $\Lambda^{-1} = \eta \Lambda^t \eta$.
(d) *If* $\Lambda \in O(1,3)$ *then* $\det \Lambda = \pm 1$. *In particular, if* $\Lambda, \Lambda' \in O(1,3)$ *and* $\det \Lambda = \det \Lambda = 1$ *then* $\det(\Lambda\Lambda') = 1$ *and* $\det \Lambda^{-1} = 1$.
(e) *If* $\Lambda \in O(1,3)$ *then* ${\Lambda^0}_0 \geq 1$ *or* ${\Lambda^0}_0 \leq -1$. *In particular, if* $\Lambda, \Lambda' \in O(1,3)$ *and* ${\Lambda^0}_0 \geq 1$ *and* ${\Lambda'^0}_0 \geq 1$ *then* ${(\Lambda\Lambda')^0}_0 \geq 1$ *and* ${(\Lambda^{-1})^0}_0 \geq 1$.

Proof. The proof of (a) is immediate from (16) also using $\eta\eta = I$. To prove (b) we start from $\Lambda^t \eta \Lambda = \eta$. Since Λ is not singular, Λ^{-1} exists and one has $\Lambda^t \eta \Lambda \Lambda^{-1} = \eta \Lambda^{-1}$, namely $\Lambda^t \eta = \eta \Lambda^{-1}$. Therefore, applying η on the right $\Lambda^t = \eta \Lambda^{-1} \eta$. Finally applying $\Lambda\eta$ on the left one gets

$$\Lambda \eta \Lambda^t = \eta \ ,$$

so $\Lambda^t \in O(1,3)$ if $\Lambda \in O(1,3)$.
To show (c) we notice that $\eta\Lambda^t\eta \in O(1,3)$ because this set is closed with respect to composition of matrices and $\eta, \Lambda^t \in O(1,3)$ for (a) and (b). Finally: $\Lambda(\eta\Lambda^t\eta) = (\lambda\eta\Lambda^t)\eta = \eta\eta = I$. Since every $\Lambda \in SO(1,3)$ is non singular as noticed below (15) we can conclude that $\eta\Lambda^t\eta = \Lambda^{-1}$ but also that $\Lambda^{-1} \in O(1,3)$ if $\Lambda \in O(1,3)$.
The first part of (d) has been proved previously. The remaining part is straightforward: $\det(\Lambda\Lambda') = (\det \Lambda) \cdot (\det \Lambda') = 1 \cdot 1 = 1$ and $\det(\Lambda^{-1}) = (\det \Lambda)^{-1} = (1)^{-1} = 1$.
Let us conclude the proof by demonstrating (e) whose proof is quite involved. The constraint $A^t \eta A = \eta$ gives :

$$({A^0}_0)^2 = 1 + \sum_{\alpha=1}^{3} {A^0}_\alpha {A^0}_\alpha \ , \tag{17}$$

so that ${A^0}_0 \geq 1$ or ${A^0}_0 \leq -1$ if $A \in O(1,3)$. This proves the first statement of (e) if $\Lambda = A$.
Let us pass to the second part. Suppose that $\Lambda, \Lambda' \in O(1,3)$ and ${\Lambda^0}_0 \geq 1$ and ${\Lambda'^0}_0 \geq 1$, we want to show that ${(\Lambda\Lambda')^0}_0 \geq 1$. Actually it is sufficient to show that ${(\Lambda\Lambda')^0}_0 > 0$ because of the first statement. We shall prove it.
We start from the identity

$${(\Lambda\Lambda')^0}_0 = {\Lambda^0}_0 {\Lambda'^0}_0 + \sum_{\alpha=1}^{3} {\Lambda^0}_\alpha {\Lambda'^\alpha}_0 \ .$$

It can be re-written down as

$${(\Lambda\Lambda')^0}_0 = {\Lambda^0}_0 {(\Lambda'^t)^0}_0 + \sum_{\alpha=1}^{3} {\Lambda^0}_\alpha {(\Lambda'^t)^0}_\alpha$$

Using Cauchy-Schwarz' inequality:

$$\left| \sum_{\alpha=1}^{3} {\Lambda^0}_\alpha {(\Lambda'^t)^0}_\alpha \right|^2 \leq \left(\sum_{\alpha=1}^{3} {\Lambda^0}_\alpha {\Lambda^0}_\alpha \right) \left(\sum_{\beta=1}^{3} {(\Lambda'^t)^0}_\beta {(\Lambda'^t)^0}_\beta \right) \ ,$$

so that

$$\left| {(\Lambda\Lambda')^0}_0 - {\Lambda^0}_0 {(\Lambda'^t)^0}_0 \right|^2 \leq \left(\sum_{\alpha=1}^{3} {\Lambda^0}_\alpha {\Lambda^0}_\alpha \right) \left(\sum_{\beta=1}^{3} {(\Lambda'^t)^0}_\beta {(\Lambda'^t)^0}_\beta \right) \ . \tag{18}$$

(17) implies, for $A \in O(1,3)$,:

$$\sum_{\alpha} {A^0}_\alpha {A^0}_\alpha < ({A^0}_0)^2 \ .$$

Exploiting that inequality in (18) for $A = \Lambda, \Lambda'^t$ (using the fact that $O(1,3)$ is closed with respect to transposition of matrices), we obtains:

$$\left|(\Lambda\Lambda')^0{}_0 - \Lambda^0{}_0(\Lambda'^t)^0{}_0\right|^2 < (\Lambda^0{}_0)^2((\Lambda'^t)^0{}_0)^2 \ . \tag{19}$$

Since $\Lambda^0{}_0 \geq 0$ and $(\Lambda'^t)^0{}_0 = \Lambda'^0{}_0 \geq 0$ by hypotheses, we have

$$\left|(\Lambda\Lambda')^0{}_0 - \Lambda^0{}_0\Lambda'^0{}_0\right| < \Lambda^0{}_0\Lambda'^0{}_0 \tag{20}$$

that is

$$\Lambda^0{}_0\Lambda'^0{}_0 - \Lambda^0{}_0\Lambda'^0{}_0 < (\Lambda\Lambda')^0{}_0 < \Lambda^0{}_0\Lambda'^0{}_0 + \Lambda^0{}_0\Lambda'^0{}_0$$

and thus

$$0 < (\Lambda\Lambda')^0{}_0 < 2\Lambda^0{}_0\Lambda'^0{}_0 \ .$$

In particular $(\Lambda\Lambda')^0{}_0 > 0$ which is that we wanted to prove.
To prove the last statement, notice that, if $\Lambda^0{}_0 \geq 1$, from (c), $(\Lambda^{-1})^0{}_0 = (\eta\Lambda^t\eta)^0{}_0 = \Lambda^0{}_0 \geq 1$. □

2.2. **Poincaré group.** Considering the complete transformation (13) we can introduce the celebrated Poincaré group, also called inhomogeneous Lorentz group.

Proposition 7. *The set, called* Poincaré group *or* inhomogeneous Lorentz group*,*

$$qIO(1,3) := O(1,3) \times T^4 \ , \tag{21}$$

is a group when equipped with the composition rule

$$(\Lambda, T) \circ (\Lambda', T') := (\Lambda\Lambda', T + \Lambda T') \ . \tag{22}$$

The proof of the theorem is immediate and it is left to the reader. We make only a few remarks.

Remark 3.
(1) The composition rule (22) is nothing but that obtained by composing the two transformations of Minkowskian coordinates:

$$x_1^\mu = \Lambda^\mu{}_\nu(x_2^\nu + T^\nu) \quad \text{and} \quad x_2^\nu = \Lambda'^\nu{}_\tau(x_3^\tau + T'^\tau)$$

obtaining

$$x_1^\mu = (\Lambda\Lambda')^\nu{}_\tau\,(x_3^\tau + (\Lambda T')^\tau) \ .$$

(2) It is possible to provide $IO(1,3)$ with the structure of Lie group which is also subgroup of $GL(5,\mathbb{R})$ and that includes $O(1,3)$ and T^4 as Lie subgroups. One start with the injective map

$$IO(1,3) \ni (\Lambda, T) \mapsto \left[\begin{array}{c|c} 1 & 0 \\ \hline T & \Lambda \end{array}\right] \in GL(5,\mathbb{R}) \ , \tag{23}$$

and he verifies that the map is in fact an injective group homomorphism. The matrix group in the right hand side of (23) define, in fact a closed subset of $\mathbb{R}^{25}$ and thus of $GL(5,\mathbb{R})$. As a consequence this matrix group is a Lie group which is a Lie subgroup of $GL(5,\mathbb{R})$.
(3) From a kinematic point of view it makes sense to define the speed of $\mathcal{F}$ with respect to $\mathcal{F}'$ when they are connected by Poincaré transformation (Λ, T). The speed turns out to be constant in space and time. Indeed, let $\mathcal{F}$ and $\mathcal{F}'$ be Minkowskian reference frames with associated co-moving Minkowskian coordinate frames x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 respectively and suppose that (13) hold. Let γ represent the history of a point *at rest* with respect to $\mathcal{F}$, that is, γ admits parametrization $x^i(\xi) = x_0^i$ constant for $i = 1, 2, 3$, $x^0 = x^0(\xi)$. The speed of γ with respect to $\mathcal{F}'$ does not depend on x_0^i and it is constant in $\mathcal{F}'$-time so that, indicating it by $\mathbf{v}_{\mathcal{F}}^{(\mathcal{F}')}$, their components in coordinates x'^1, x'^2, x'^3 turns out to be:

$$\mathbf{v}_{\mathcal{F}}^{(\mathcal{F}')i} = \frac{\Lambda^i{}_0}{\Lambda^0{}_0} \ .$$

The proof is immediate by applying the definition of speed of γ with respect to $\mathcal{F}'$.

2.3. **Orthochronous, proper subgroups and connected components of $O(1,3)$.** Consider two Minkowskian frames $\mathcal{F}$ and $\mathcal{F}'$ and let x^0, x^1, x^2, x^3, x'^0, x'^1, x'^2, x'^3 be two respectively co-moving Minkowski coordinate frames. We know that $\partial_{x^0} = \mathcal{F}$ and $\partial_{x'^0} = \mathcal{F}'$, finally we know that $\mathcal{F}$ and $\mathcal{F}'$ must have the same time-orientation they being future directed. We conclude that it must be $g(\partial_{x^0}, \partial_{x'^0}) < 0$. Only transformations (13) satisfying that constraint may make sense physically speaking. From (13) one has $\partial_{x^0} = \Lambda^\mu{}_0 \partial_{x'^\mu}$, so that the requirement $g(\partial_{x^0}, \partial_{x'^0}) < 0$ is equivalent to ${\Lambda_0}^0 > 0$ which is, in turn, equivalent to ${\Lambda_0}^0 > 1$ because of the first statement in (e) in Proposition 6. One expect that the Poincaré or Lorentz transformations fulfilling this constraint form a subgroup. Indeed this is the case and the group is called the *orthochronous subgroup*: it embodies all physically sensible transformations of coordinates between inertial frames in special relativity. The following proposition states that results introducing also two other relevant subgroups. The proof of the following proposition is immediate from (e) and (d) of Proposition 6.

Proposition 8. *The subsets of $IO(1,3)$ and $O(1,3)$ defined by*

$$\begin{aligned} IO(1,3)\!\uparrow &:= \{(\Lambda, T) \in IO(1,3) \;\mid\; \Lambda^0{}_0 \geq 1\}\,, \\ O(1,3)\!\uparrow &:= \{\Lambda \in O(1,3) \;\mid\; \Lambda^0{}_0 \geq 1\}\,, \end{aligned} \tag{24}$$

and called respectively orthochronous Poincaré group *and* orthochronous Lorentz group*, are (Lie) subgroups of $IO(1,3)$ and $O(1,3)$ respectively.*
The subsets of $IO(1,3)$ and $O(1,3)$ defined by

$$\begin{aligned} ISO(1,3) &:= \{(\Lambda, T) \in IO(1,3) \;\mid\; \det\Lambda = 1\}\,, \\ SO(1,3) &:= \{\Lambda \in O(1,3) \;\mid\; \det\Lambda = 1\}\,, \end{aligned} \tag{25}$$

and called respectively proper Poincaré group *and* proper Lorentz group*, are (Lie) subgroups of $IO(1,3)$ and $O(1,3)$ respectively.*

We remark that the condition $\Lambda^0{}_0 \geq 1$ can be replaced with the equivalent constraint $\Lambda^0{}_0 > 0$, whereas the condition $\det\Lambda = 1$ can be replaced with the equivalent constraint $\det\Lambda > 0$. Since the intersection of a pair of (Lie) groups is a (Lie) group, we can give the following final definition.

Definition 11. The (Lie) subgroups of $IO(1,3)$ and $O(1,3)$ defined by

$$ISO(1,3)\!\uparrow := IO(1,3)\!\uparrow \cap ISO(1,3) \qquad SO(1,3)\!\uparrow := O(1,3)\!\uparrow \cap SO(1,3) \tag{26}$$

are called respectively *orthochronous proper Poincaré group* and *orthochronous proper Lorentz group.*

To conclude this short landscape of properties of Lorentz group we state and partially prove (the complete proof needs a result we shall achieve later) the following proposition about connected components of Lorentz group. These are obtained by starting from $SO(1,3)\!\uparrow$ and transforming it under the left action of the elements of discrete subgroup of $O(1,3)$: $\{I, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}$ where $\mathcal{T} := \eta$ and $\mathcal{P} := -\eta$ (so that $\mathcal{PT} = \mathcal{TP} = -I$, $\mathcal{PP} = \mathcal{TT} = I$). In this context $\mathcal{T}$ is called *time reversal* operator - since it changes the time orientation of causal vectors - and $\mathcal{P}$ is also said to be the *(parity) inversion* operator - since it corresponds to the spatial inversion in the rest space.

Proposition 9. *$SO(1,3)$ admits four connected components which are respectively, with obvious notation, $SO(1,3)\!\uparrow$, $\mathcal{P}SO(1,3)\!\uparrow$, $\mathcal{T}SO(1,3)\!\uparrow$, $\mathcal{PT}SO(1,3)\!\uparrow$. Only the first is a subgroup.*

Proof. By construction: (1) if $\Lambda \in \mathcal{P}SO(1,3)\!\uparrow$, both $\det\Lambda = -1$ and $\Lambda^0{}_0 \geq 1$, (2) if $\Lambda \in \mathcal{T}SO(1,3)\!\uparrow$, both $\Lambda^0{}_0 \leq -1$ and $\det\Lambda = 1$, (3) if $\Lambda \in \mathcal{PT}SO(1,3)\!\uparrow$, both $\Lambda^0{}_0 \leq -1$ and $\det\Lambda = -1$.

Thus the last statement is an immediate consequence of the fact that, as I satisfies $\det I = 1$ and $(I)^0_0 = 1$, it cannot belong to the three sets by construction $\mathcal{P}SO(1,3)\!\uparrow$, $\mathcal{T}SO(1,3)\!\uparrow$, $\mathcal{PT}SO(1,3)\!\uparrow$. Assume that $SO(1,3)\!\uparrow$ is connected. We shall prove it later. Since the maps $O(1,3) \ni \Lambda \mapsto \mathcal{T}\Lambda$ and $O(1,3) \ni \Lambda \mapsto \mathcal{P}\Lambda$ are continuous, they transform connected sets to connected sets. As a consequence $\mathcal{P}SO(1,3)\!\uparrow$, $\mathcal{T}SO(1,3)\!\uparrow$, $\mathcal{PT}SO(1,3)\!\uparrow$ are connected sets. To conclude, it is sufficient to prove that the considered sets are pairwisely disconnected. To this end it is sufficient to exhibit continuous real-valued function defined on $O(1,3)$ which change their sign passing from a set to the other. By construction two functions are sufficient $O(1,3) \ni \Lambda \mapsto \det \Lambda$ and $O(1,3) \ni \Lambda \mapsto {\Lambda^0}_0$.

3. DECOMPOSITION OF LORENTZ GROUP

Consider a reference frame $\mathcal{F}$, what is the relation between two Minkowskian coordinate frames x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 both co-moving with $\mathcal{F}$? These transformations are called *internal* to $\mathcal{F}$. The answer is quite simple. The class of all internal Poincaré transformations is completely obtained by imposing the further constraint $\partial_{x^0} = \partial_{x'^0} (= \mathcal{F})$ on the equations (13) and assuming that $\Lambda \in O(1,3)\!\uparrow$.

Since $\partial_{x^0} = {\Lambda^\mu}_0 \partial_{x'^\mu}$, the constraint is equivalent to impose ${\Lambda^0}_0 = 1$ and ${\Lambda^i}_0 = 0$ for $i = 1,2,3$. Lorentz condition $\Lambda^t \eta \Lambda = \eta$ implies in particular that:

$$({\Lambda^0}_0)^2 = 1 + \sum_{i=1}^{3} {\Lambda^0}_i {\Lambda^0}_i \ ,$$

and thus, since ${\Lambda^0}_0 = 1$, we find that ${\Lambda^0}_i = 0$. Summing up, internal Lorentz transformations must have the form

$$\Omega_R = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right] . \tag{27}$$

By direct inspection one finds that, in this case, $\Omega^t \eta \Omega = \eta$ reduces to

$$R^t R = I \ . \tag{28}$$

This is nothing but the equation determining the orthogonal group $O(3)$. Conversely, starting from any matrix $R \in O(3)$ and thus satisfying (28), and defining Ω_R as in (27), it is immediate to verify that $\Omega_R \in O(1,3)\!\uparrow$ and

$$x'^\mu = {(\Omega_R)^\mu}_\nu (x^\nu + T^\nu) \ ,$$

with $T \in T^4$ fixed arbitrarily, is an internal Poincaré transformation. It is immediate to show also that Ω_R with form (27) belongs to $SO(1,3)\!\uparrow$ if and only if $R \in SO(3)$, the group of special rotations made of rotations of $O(3)$ with unitary determinant.

Remark 4. It is worthwhile noticing, from a kinematic point of view that, the speed of $\mathcal{F}$ with respect to $\mathcal{F}'$ seen in (3) in Remark 3 is invariant under changes of co-moving Minkowskian coordinates when the transformations of coordinates are internal to $\mathcal{F}$ and $\mathcal{F}'$. We leave the simple proof of this fact to the reader.

Definition 12. The Lorentz transformations Ω_R defined in (27) with $R \in O(3)$ are called *spatial rotations.* If $R \in SO(3)$, Ω_R is called *spatial proper rotations.*

From now on we focus on the Lorentz part of Poincaré group only and we extract the non internal part. This goal will be achieved after a preliminary analysis of the Lie algebra of $SO(1,3)\!\uparrow$ and the corresponding exponentiated operators.

3.1. **Spatial rotations and boosts: Lie algebra and Lie group analysis.** As $SO(1,3)\uparrow$ is a matrix Lie group subgroup of $GL(4,\mathbb{R})$, its Lie algebra can be obtained as a Lie algebra of matrices in $M(4,\mathbb{R})$ with the commutator $[\cdot,\cdot]$ given by the usual matrix commutator. As the maps $O(1,3) \ni \Lambda \mapsto \det \Lambda$ $O(1,3) \ni \Lambda \mapsto {\Lambda_0}^{0}$ are continuous and $detI = 1$ and ${(I)_0}^{0} > 0$, every $\Lambda \in O(1,3)$ sufficiently close to I must belong to $SO(1,3)\uparrow$ and *viceversa.* Hence the Lie algebra of $O(1,3)$, $o(1,3)$, coincides with that of its Lie subgroup $SO(1,3)\uparrow$ because it is completely determined by the behavior of the group in an arbitrarily small neighborhood of the identity.

Proposition 10. *The Lie algebra of* $SO(1,3)\uparrow$ *admits a vector basis made of the following* 6 *matrices called* boost generators K_1, K_2, K_3 *and* spatial rotation generators S_1, S_2, S_3*:*

$$K_1 = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] , \; K_2 = \left[\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] , \; K_3 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] . \tag{29}$$

$$S_i = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & T_i & \\ 0 & & & \end{array}\right] \qquad \textit{with}$$

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} , \; T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} , \; T_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{30}$$

These generators enjoy the following commutation relations, which, as a matter of facts, determines the structure tensor of $o(1,3)$*:*

$$[S_i, K_j] = \sum_{k=1}^{3} \epsilon_{ijk} K_k \;,\; [S_i, S_j] = \sum_{k=1}^{3} \epsilon_{ijk} S_k \;,\; [K_i, K_j] = -\sum_{k=1}^{3} \epsilon_{ijk} S_k \; . \tag{31}$$

Above ϵ_{ijk} *is the usual completely antisymmetric Ricci indicator with* $\epsilon_{123} = 1$.

Proof. If a matrix $N \in M(4,\mathbb{R})$ is in $o(1,3)$, the generated one-parameter subgroup $\{e^{uN}\}_{u\in\mathbb{R}}$ in $GL(4,\mathbb{R})$ satisfies $(e^{uN})^t \eta e^{uN} = \eta$ for u in a neighborhood of 0, that is $e^{uN^t} \eta e^{uN} = \eta$ for the same values of u. Taking the derivative at $t = 0$ one gets the necessary condition

$$N^t \eta + \eta N = 0 \; . \tag{32}$$

These equations are also sufficient. Indeed, from standard properties of the exponential map of matrices, one has

$$\frac{d}{du}\left(e^{uN^t} \eta e^{uN} - \eta\right) = e^{uN^t}\left(N^t \eta + \eta N\right) e^{uN} \; .$$

Thus, the validity of (32) implies that $e^{uN^t} \eta e^{uN} - \eta =$ constant. For $u = 0$ one recognizes that the constant is 0 and so $(e^{uN})^t \eta e^{uN} = \eta$ is valid (for every $u \in \mathbb{R}$). Eq. (32) supplies 10 linearly independent conditions so that it determines a subspace of $M(4,\mathbb{R})$ with dimension 6. The 6 matrices $S_i, K_j \in M(4.\mathbb{R})$ are linearly independent and satisfy (32), so they are a basis for $o(1,3)$. The relations (31) can be checked by direct inspection.

□

From now on $\mathbf{K}, \mathbf{S}, \mathbf{T}$ respectively denote the formal vector with components K_1, K_2, K_3, the formal vector with components S_1, S_2, S_3 and the formal vector with components T_1, T_2, T_3. $\mathbb{S}^2$ will indicate the sphere of three-dimensional unit vectors.

Generators S_1, S_2, S_3 individuates proper spatial rotations as stated in the following proposition.

Proposition 11. *The following facts about Proper rotations holds.*
(a) *Every proper spatial rotations has the form* $\Omega_R = e^{\theta\mathbf{n}\cdot\mathbf{S}}$ *- or equivalently* $R = e^{\theta\mathbf{n}\cdot\mathbf{T}}$ *for all* $R \in SO(3)$ *- with suitable* $\theta \in \mathbb{R}$ *and* $\mathbf{n} \in \mathbb{S}^2$ *depending on* R*.*
(b) *Every matrix* $e^{\theta\mathbf{n}\cdot\mathbf{S}}$ *with* $\theta \in \mathbb{R}$ *and* $\mathbf{n} \in \mathbb{S}^2$ *is a proper rotation* Ω_R*, and is associated with* $R = e^{\theta\mathbf{n}\cdot\mathbf{T}} \in SO(3)$*.*
(c) *The following equivalent identities hold true, for every* $U \in SO(3)$*,* $\mathbb{N} \in \mathbb{S}^2$*,* $\theta \in \mathbb{R}$*:*

$$\Omega_U e^{i\theta\mathbf{n}\cdot S}\Omega_U{}^t = e^{i\theta(U\mathbf{n})\cdot S} \;\; , \;\; U e^{i\theta\mathbf{n}\cdot T} U^t = e^{i\theta(U\mathbf{n})\cdot T} \; . \tag{33}$$

The latter holds, more generally, also if $R \in SL(3, \mathbb{C})$*.*

Proof. (a) and (c). Since, from the given definitions,

$$e^{i\theta\mathbf{n}\cdot S} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & e^{i\mathbf{n}\cdot\mathbf{T}} & \\ 0 & & & \end{array}\right] , \tag{34}$$

it is obvious that $\Omega_R = e^{\theta\mathbf{n}\cdot\mathbf{S}}$ are completely equivalent $R = e^{\theta\mathbf{n}\cdot\mathbf{T}}$, so we deal with the latter.
If $R \in SO(3)$, the induced operator in $\mathbb{R} + i\mathbb{R}$ is unitary and thus it admits a base of eigenvectors with eigenvalues λ_i with $|\lambda_i| = 1$, $i = 1, 2, 3$. As the characteristic polynomial of R is real, an eigenvalue must be real, the remaining pair of eigenvalues being either real or complex and conjugates. Since $\det R = \lambda_1\lambda_2\lambda_3 = 1$, 1 is one of the eigenvalues. If another eigenvalue coincides with 1 all three eigenvalues must do it and $R = I$. In this case every non-vanishing real vector is an eigenvector of R. Otherwise, the eigenspace of $\lambda = 1$ must be one-dimensional and thus, as R is real, it must contain a real eigenvector. We conclude that, in every case, R has a real normalized eigenvector $\mathbf{n}$ with eigenvalue 1. Consider an orthonormal basis $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3 := \mathbf{n}$, related with the initial one by means of $R' \in SO(3)$, and represent R in the new basis. Imposing the requirement that $\mathbf{n}_3$ is an eigenvector with eigenvalue 1 as well as that the represented transformation belong to $SO(3)$, one can easily prove that, in such a base, R is represented by the matrix $e^{\theta T_3}$ for some $\theta \in [0, 2\pi]$. In other words, coming back to the initial basis, $R = R' e^{\theta T_3} {R'}^t$ for some $R' \in SO(3)$. Now notice that $(T_i)_{jk} = -\epsilon_{ijk}$. This fact entails that $\sum_{i,j,k} U_{pi}U_{qj}U_{rk}\epsilon_{ijk} = \epsilon_{pqr}$ for all $U \in SL(3, \mathbb{C})$. That identity can be re-written as $\mathbf{n} \cdot U\mathbf{T}U^t = (U\mathbf{n}) \cdot \mathbf{T}$ for every $U \in SL(3, \mathbb{C})$. By consequence, if $U \in SO(3)$ in particular, it also holds $U e^{\theta\mathbf{n}\cdot\mathbf{T}} U^t = e^{\theta(U\mathbf{n})\cdot\mathbf{T}}$. (This proves the latter in (33), the former is a trivial consequence of the given definitions). Therefore, the identity found above for any $R \in SO(3)$, $R = R' e^{\theta T_3} {R'}^t$ with $R' \in SO(3)$, can equivalently be written as $R = e^{\theta\mathbf{n}\cdot\mathbf{T}}$ for some versor $\mathbf{n} = R'\mathbf{e}_3$.
(b) Finally, every matrix $e^{\theta\mathbf{n}\cdot\mathbf{T}}$ belongs to $SO(3)$ because $(e^{\theta\mathbf{n}\cdot\mathbf{T}})^t = e^{\theta\mathbf{n}\cdot\mathbf{T}^t} = e^{-\theta\mathbf{n}\cdot\mathbf{T}} = (e^{\theta\mathbf{n}\cdot\mathbf{T}})^{-1}$ and $\det e^{\theta\mathbf{n}\cdot\mathbf{T}} = e^{\theta\mathbf{n}\cdot\mathrm{tr}\,\mathbf{T}} = e^0 = 1$.

□

Remark 5. We leave to the reader the proof of the following facts. The correspondence between pairs $(\theta, \mathbf{n})$ and $SO(3)$ is one-to-one with the following exceptions: $\theta = 0$ individuates the trivial rotation I for all $\mathbf{n}$, $(\theta, \mathbf{n})$ and $(\theta + 2k\pi, \mathbf{n})$ with $k \in \mathbb{Z}$ individuate the same rotation and finally, the pairs $(\pi, \mathbf{n})$ and $(\pi, -\mathbf{n})$ individuates the same rotation. (Notice that the latter fact implies that $SO(3)$ is not simply connected.)

Definition 13. The elements of $SO(1,3)\!\uparrow$ with the form $\Lambda = e^{\chi\mathbf{m}\cdot\mathbf{K}}$, with $\chi \in \mathbb{R}$ and $\mathbf{m} \in \mathbb{S}^2$, are called *boosts* or *pure transformations.*

Let us investigate the basic properties of boosts. These are given by the following proposition.

Proposition 12. *The boost enjoy the following properties.*
(a) *All matrices* $\Lambda = e^{\chi\mathbf{m}\cdot\mathbf{K}}$ *with arbitrary* $\chi \in \mathbb{R}$ *and* $\mathbf{m} \in \mathbb{S}^2$ *belong to* $SO(1,3)\!\uparrow$ *and thus are boosts.*

(b) *For every pair* $\mathbf{m} \in \mathbb{S}^1$, $\chi \in \mathbb{R}$ *and every* $R \in SO(3)$ *one has*

$$\Omega_R e^{i\chi \mathbf{m}\cdot\mathbf{K}} \Omega_R{}^t = e^{i\chi (R\mathbf{m})\cdot\mathbf{K}} \,. \tag{35}$$

(c) *For* $\mathbf{n} \in \mathbb{S}^2$, *the explicit form of* $e^{i\chi \mathbf{n}\cdot\mathbf{K}}$ *reads:*

$$e^{i\chi \mathbf{n}\cdot\mathbf{K}} = \left[\begin{array}{c|c} \cosh\chi & (\sinh\chi)\mathbf{n}^t \\ \hline (\sinh\chi)\mathbf{n} & I - (1-\cosh\chi)\mathbf{n}\mathbf{n}^t \end{array}\right] , \tag{36}$$

(d) *Every boost is symmetric and* (*strictly*) *positive defined.*

Proof. (a) It has been proved in the proof of proposition 10, taking into account that $N := \chi\mathbf{m}\cdot\mathbf{K} \in o(1,3)$.
(b) Fix $\mathbf{n} \in \mathbb{S}^2$ and $j = 1,2,3$. Now, for $i = 1,2,3$ define the functions

$$f_j(\theta) := e^{i\theta\mathbf{n}\cdot S} K_j e^{i\theta\mathbf{n}\cdot S} \,, \quad g_j(\theta) := \sum_{k=1}^{3} \left(e^{i\theta\mathbf{n}\cdot T}\right)_{jk} K_k \,.$$

Taking the first derivative in θ and using both (31) and the explicit form of the matrices T_h, one finds that the smooth functions f_k and the smooth functions g_k satisfies the same system of differential equation of order 1 written in normal form: for $j = 1,2,3$,

$$\frac{df_j}{d\theta} = i\sum_{k=1}^{3} n_k \epsilon_{kjp} f_p(\theta) \,, \quad \frac{dg_j}{d\theta} = i\sum_{k=1}^{3} n_k \epsilon_{kjp} g_p(\theta) \,.$$

Since $f_j(0) = g_j(0)$ for $j = 1,2,3$, we conclude that these functions coincide for every $\theta \in \mathbb{R}$:

$$e^{i\theta\mathbf{n}\cdot S} K_j e^{i\theta\mathbf{n}\cdot S} = \sum_{k=1}^{3} \left(e^{i\theta\mathbf{n}\cdot T}\right)_{jk} K_k \,.$$

Now, by means of exponentiation we get (35) exploiting (c) of Proposition 11.
(c) First consider the case $\mathbf{n} = \mathbf{e}_3$. In this case directly from Taylor's expansion formula

$$e^{\chi\mathbf{e}_3\cdot\mathbf{K}} = \left[\begin{array}{c|ccc} \cosh\chi & 0 & 0 & \sinh\chi \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\chi & 0 & 0 & \cosh\chi \end{array}\right] = \tag{37}$$

$$= \left[\begin{array}{c|c} \cosh\chi & (\sinh\chi)\mathbf{e}_3^t \\ \hline (\sinh\chi)\mathbf{e}_3 & I - (1-\cosh\chi)\mathbf{e}_3\mathbf{e}_3^t \end{array}\right] .$$

If $\mathbf{n} \in \mathbb{S}^2$ there is $R \in SO(3)$ such that $\mathbf{n} = R\mathbf{e}_3$. Using this R in (b) with $\mathbf{m} = \mathbf{e}_3$ one gets (36) with $\mathbf{n}$.
(d) Symmetry is evident from (36). Using (b), it is sufficient to prove positivity if $\mathbf{m} = \mathbf{e}_3$. In this case strictly positivity can be checked by direct inspection using (37).

□

Recalling (3) in Remark 3, the kinematic meaning of parameters χ and $\mathbf{m}$ in boosts is clear from the following last proposition.

Proposition 13. *Let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be Minkowskian reference with associated co-moving Minkowskian coordinate frames* x^0, x^1, x^2, x^3 *and* x'^0, x'^1, x'^2, x'^3 *respectively and suppose that* (13) *hold with*

$$\Lambda = e^{i\chi\mathbf{m}\cdot\mathbf{K}} \,.$$

Let γ represent the history of a point at rest with respect to $\mathcal{F}$, that is, γ admits parametrization $x^i(\xi) = x^i_0$ constant for $i = 1, 2, 3$, $x^0 = x^0(\xi)$. The speed of γ with respect to $\mathcal{F}'$ does not depend on x^i_0 and it is constant in $\mathcal{F}'$-time so that, indicating it by $\mathbf{v}^{(\mathcal{F}')}_{\mathcal{F}}$, it holds

$$\mathbf{v}^{(\mathcal{F}')}_{\mathcal{F}} = (\tanh\chi)\mathbf{m} .$$

Proof. The proof follows immediately from (36) and Definition 10.

□

Remark 6.
(1) We leave to the reader the proof of the following facts. The correspondence between boosts and pairs $(\chi, \mathbf{n})$ is one-to-one with the following exceptions: $\chi = 0$ defines the trivial boost I not depending on $\mathbf{n}$, finally $(\chi, \mathbf{n})$ and $(-\chi, -\mathbf{n})$ define the same boost.
(2) Differently from spatial pure rotations, the class of boosts with different $\mathbf{n} \in \mathbb{S}^2$ does not for a subgroup of $SO(1,3)\!\uparrow$. The proof is immediate by direct inspection using (36) for $n = \mathbf{e}_1$ and $\mathbf{n} = \mathbf{e}_3$ and verifying that the product of these boosts cannot be represented as in (36). For a fixed $\mathbf{n}$, $\mathbb{R} \ni \chi \mapsto e^{i\chi\mathbf{n}\cdot\mathbf{K}}$ describes a one-parameter subgroup so that this class of boosts define a subgroup.
(3) Boosts along $\mathbf{n} := \mathbf{e}_3$ are know in the literature as "special Lorentz transformations" along z (an analogous name is given replacing $\mathbf{e}_3$ with $\mathbf{e}_1$ and $\mathbf{e}_2$).

3.2. **Decomposition theorem for $SO(1,3)\!\uparrow$.** In the previous subsection we have determined two classes of transformations in $SO(1,3)\!\uparrow$: spatial pure rotations and boosts. It is natural to wonder if those transformations encompass the whole group $SO(1,3)\!\uparrow$ in some sense. The answer is positive and is stated in a theorem of decomposition of $SO(1,3)\!\uparrow$ we go to state. The proof of the theorem will be given in the next section using a nonstandard approach based on the so-called polar decomposition theorem.

Theorem 1. *Take $\Lambda \in SO(1,3)\!\uparrow$, the following holds.*
(a) *There is exactly one boost P and exactly a spatial pure rotation U such that*

$$\Lambda = UP .$$

(b) *There is exactly one boost P' and exactly a spatial pure rotation U' such that*

$$\Lambda = P'U' .$$

(c) *It holds $U' = U$ and $P' = UPU^t$.*

As said above, the proof of the theorem will be given in the next section. Here we want emphasize a consequence of this theorem. It arises that $SO(1,3)\!\uparrow$ is arch-connected and thus connected. This is because, if $\Lambda \in SO(1,3)\!\uparrow$, Theorem 1 and Proposition 12 entails that, for some $\mathbf{n}, \mathbf{m} \in \mathbb{S}^2$ and $\theta, \in \chi \in \mathbb{R}$

$$\Lambda = e^{i\theta\mathbf{n}\cdot S} e^{i\chi\mathbf{m}\cdot K} .$$

It is simply proved that the curve

$$[0,1] \ni \lambda \mapsto e^{i\lambda\theta\mathbf{n}\cdot S} e^{i\lambda\chi\mathbf{m}\cdot K}$$

is continuous in the topology of $SO(1,3)\!\uparrow$ (actually it is C^∞). It starts from I and achieves Λ. Therefore $SO(1,3)\!\uparrow$ is arch-connected and thus connected.

4. POLAR DECOMPOSITION THEOREM OF $SO(1,3)\!\uparrow$

4.1. **The general polar decomposition theorem.** A *real* Hilbert space H is a vector space equipped with a symmetric scalar product $(\cdot|\cdot)$ and complete with respect to the induced norm topology. Henceforth we adopt the usual notation and definitions concerning adjoint, self-adjoint, unitary operators in Hilbert spaces (e.g, see [5]), using them either in complex or real Hilbert spaces H. Moreover $\mathcal{B}(H)$ denotes the space of all bounded operators $T : H \to H$. $T \in B(H)$ is said *positive* ($T \geq 0$) if $(u|Tu) \geq 0$ for all $u \in H$. The lemma and the subsequent theorem below straightforwardly extend the *polar decomposition* *theorem* (Theorem 12.35 in [5]) encompassing both the real and the complex case. The proofs are supplied in the appendix. (In complex Hilbert spaces bounded positive operators are self-adjoint [5], in that case the self-adjointness property can be omitted in the hypotheses and the thesis of the lemma and the theorem and elsewhere in this work.)

Lemma 1 (**Existence and uniqueness of (positive) square roots in Hilbert spaces).** *Let $T \in \mathcal{B}(H)$ be a self-adjoint positive operator where H is a, either real or complex, Hilbert space. There exists exactly one operator $\sqrt{T} \in \mathcal{B}(H)$ such that $\sqrt{T}^* = \sqrt{T} \geq 0$ and $\sqrt{T}^2 = T$. If T is bijective, $\sqrt{T}$ is so. $\sqrt{T}$ is said the* (positive) square root *of T.*

Proposition 14 (**Polar decomposition in either real or complex Hilbert spaces).** *If $T \in \mathcal{B}(H)$ is a bijective operator where H is a, either real or complex, Hilbert space:*
(a) *there is a unique decomposition $T = UP$, where U is unitary, and P is bounded, bijective, self-adjoint and positive. In particular $P = \sqrt{T^*T}$ and $U = T(\sqrt{T^*T})^{-1}$;*
(b) *there is a unique decomposition $T = P'U'$, where U' is unitary e and P' is bounded, bijective, self-adjoint and positive. In particular $U' = U$ and $P' = UPU^*$.*

4.2. **Polar decomposition and standard physical decomposition.** If $\Lambda \in SO(1,3)\uparrow$ it can be seen as a (bounded) operator in the finite-dimensional real Hilbert space $\mathbb{R}^4$. Therefore one may consider the polar decomposition $\Lambda = PU = U'P'$. $U = U'$ are now unitary operators, i.e. (real) matrices of $O(4)$ and P, P' are a self-adjoint positive operators, i.e. (real) symmetric positive matrices. *A priori* those decompositions could be physically meaningless because U and P, P' could not to belong to $SO(1,3)\uparrow$: the notions of symmetry, positiveness, orthogonal group $O(4)$ are refereed to the positive scalar product of $\mathbb{R}^4$ instead of the indefinite Lorentz scalar product. Nevertheless we shall show that the polar decompositions of $\Lambda \in SO(1,3)\uparrow$ are in fact physically meaningful. Indeed, they coincides with the known decompositions of Λ in spatial-rotation and boost parts as in Theorem 1. In part, the result can be generalized to infinite dimensional (real or complex) Hilbert spaces. As a subsequent issue, considering the universal covering of $SO(1,3)\uparrow$, $SL(2,\mathbb{C})$ [2, 3, 4], we show that the covering homomorphism $\Pi : SL(2,\mathbb{C}) \to SO(1,3)\uparrow$ preserves the polar decompositions of $SL(2,\mathbb{C})$ transforming them into the analogous decompositions in $SO(1,3)\uparrow$. Let us come to the main point by focusing attention on the real Hilbert space $H = \mathbb{R}^4$ endowed with the usual positive scalar product. In that case $\mathcal{B}(H) = M(4,\mathbb{R})$. Unitary operators are orthogonal matrices, i.e., elements of of $O(4)$ and, if $A \in \mathcal{B}(H)$ the adjoint A^* coincides with the transposed matrix A^t, therefore self-adjoint operators are symmetric matrices. We have the following theorem which proves Theorem 1 as an immediate consequence.

Theorem 2. *If $UP = P'U = \Lambda$ (with $P' = UU^t$) are polar decompositions of $\Lambda \in SO(1,3)\uparrow$:*
(a) *$P, P', U \in SO(1,3)\uparrow$, more precisely P, P' are boosts and U a spatial proper rotation;*
(b) *there are no other decompositions of Λ as a product of a Lorentz boost and a spatial proper rotation different from the two polar decompositions above.*

Proof. If $P \in M(4,\mathbb{R})$ we exploit the representation:

$$P = \left[\begin{array}{c|c} g & B^t \\ \hline C & A \end{array}\right], \tag{38}$$

where $g \in \mathbb{R}$, $B, C \in \mathbb{R}^3$ and $A \in M(3,\mathbb{R})$.
(a) We start by showing that $P, U \in O(1,3)$. As $P = P^t$, $\Lambda^t\eta\Lambda = \Lambda$ entails $PU^t\eta UP = \eta$. As $U^t = U^{-1}$ and $\eta^{-1} = \eta$, the obtained identity is equivalent to $P^{-1}U^t\eta UP^{-1} = \eta$ which, together with $PU^t\eta UP = \eta$, implies $P\eta P = P^{-1}\eta P^{-1}$, namely $\eta P^2\eta = P^{-2}$, where we have used $\eta = \eta^{-1}$ once again. Both sides are symmetric (notice that $\eta = \eta^t$) and positive by construction, by Lemma 1 they admit unique square roots which must coincide. The square root of P^{-2} is P^{-1} while the square root of $\eta P^2\eta$ is $\eta P\eta$ since $\eta P\eta$ is symmetric positive

and $\eta P\eta\eta P\eta = \eta PP\eta = \eta P^2\eta$. We conclude that $P^{-1} = \eta P\eta$ and thus $\eta = P\eta P$ because $\eta = \eta^{-1}$. Since $P = P^t$ we have found that $P \in O(1,3)$ and thus $U = \Lambda P^{-1} \in O(1,3)$. Let us prove that $P, U \in SO(1,3)\!\uparrow$. $\eta = P^t\eta P$ entails det $P = \pm 1$, on the other hand $P = P^t$ is positive and thus det $P \geq 0$ and $P^0{}_0 \geq 0$. As a consequence det $P = 1$ and $P^0{}_0 \geq 0$. We have found that $P \in SO(1,3)\!\uparrow$. Let us determine the form of P using (38). $P = P^t$, $P \geq 0$ and $P\eta P = \eta$ give rise to the following equations: $C = B$, $0 < g = \sqrt{1+B^2}$, $AB = gB$, $A = A^*$, $A \geq 0$ and $A^2 = I + BB^t$. Since $I + BB^t$ is positive, the solution of the last equation $A = \sqrt{A^2} = I + BB^t/(1+g) \geq 0$ is the unique solution by Lemma 1. We have found that a matrix $P \in O(1,3)$ with $P \geq 0$, $P = P^*$ must have the form

$$P = \left[\begin{array}{c|c} \cosh\chi & (\sinh\chi)\mathbf{n}^t \\ \hline (\sinh\chi)\mathbf{n} & I - (1-\cosh\chi)\mathbf{n}\mathbf{n}^t \end{array}\right] = e^{i\chi\mathbf{n}\cdot\mathbf{K}}\ , \tag{39}$$

where we have used the parameterization $B = (\sinh\chi)\mathbf{n}$, $\mathbf{n}$ being any versor in $\mathbb{R}^3$ and $\chi \in \mathbb{R}$. By (c) in proposition we have found that P is a boost. (The same proofs apply to P'.)
Let us pass to consider U. Since $\Lambda, P \in SO(1,3)\!\uparrow$, from $\Lambda P^{-1} = U$, we conclude that $U \in SO(1,3)\!\uparrow$. $U\eta = \eta(U^t)^{-1}$ (i.e. $U \in O(1,3)$) and $U^t = U^{-1}$ (i.e. $U \in O(4)$) entail that $U\eta = \eta U$ and thus the eigenspaces of η, E_λ (with eigenvalue λ), are invariant under the action of U. In those spaces U acts as an element of $O(\dim\,(E_\lambda))$ and the whole matrix U has a block-diagonal form. $E_{\lambda=-1}$ is generated by $\mathbf{e}_0$ and thus U reduces to $\pm I$ therein. The sign must be $+$ because of the requirement $U^0{}_0 > 0$. The eigenspace $E_{\lambda=1}$ is generated by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and therein U reduces to an element of $R \in O(3)$. Actually the requirement det $U = 1$ (together with $U^0{}_0 = 1$) implies that $R \in SO(3)$ and thus one has that $U = \Omega_R$ is a spatial pure rotation as well.
(b) Suppose that $\Omega B = \Lambda \in SO(1,3)\!\uparrow$ where B is a pure boost and Ω is a spatial proper rotation. $\Omega \in O(4)$ by construction, on the other hand, from (d) of Proposition 12: $B^t = B > 0$ Thus $\Omega B = \Lambda$ is a polar decomposition of Λ. Uniqueness (Proposition 14) implies that $\Omega = U$ and $B = P$. The other case is analogous.

□

The result can be partially generalized into the following theorem *dropping the hypotheses of finite dimensionality of the Hilbert space.* The proof is part of the proof of the statement (1) of Theorem 2 with $\mathbb{R}^4, \eta, \cdot^t$ replaced by $H, E, \cdot^*$ respectively.

Theorem 3. *Let H be a, either real or complex, Hilbert space and G_E the group of all of operators $\Lambda \in \mathcal{B}(H)$ such that $\Lambda^* E\Lambda = E$, for a fixed $E \in \mathcal{B}(H)$ which is not necessarily positive and satisfies $E = E^{-1} = E^*$. The polar decompositions of $\Lambda \in G_E$, $\Lambda = PU = UP'$ (where U is the unitary operator) are such that $P, P', U \in G_E$ and the eigenspaces of E are invariant for U.*

Notice that in the hypotheses above for E, $\sigma(E) \subset \{-1, +1\}$.

4.3. $SL(2,\mathbb{C})$**,** $SO(1,3)\!\uparrow$ **and polar decomposition.** Let us come to the last result. As is well known [1], the simply connected Lie group $SL(2,\mathbb{C})$ is the universal covering of $SO(1,3)\!\uparrow$ [2, 3, 4]. Hence there is a surjective Lie-group homomorphism $\Pi : SL(2,\mathbb{C}) \to SO(1,3)\!\uparrow$ which is a local Lie-group isomorphism about each $L \in SL(2,\mathbb{C})$.

Theorem 4. *Let $\boldsymbol{\sigma}$ denote the vector whose components are the well-known* Pauli's matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\ ,\quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\ ,\quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ . \tag{40}$$

If $L \in SL(2,\mathbb{C})$ and $L = PU = UP'$ are its polar decompositions:
(a) *$P, P', U \in SL(2,\mathbb{C})$, in particular $P = e^{\chi\mathbf{n}\cdot\boldsymbol{\sigma}/2}$, $U = e^{-\theta\mathbf{m}\cdot i\boldsymbol{\sigma}/2}$ for some $\mathbf{n}, \mathbf{m}$ versors in $\mathbb{R}^3$ and $\chi, \theta \in \mathbb{R}$.*

(b) *$\Pi(e^{\chi\mathbf{n}\cdot\boldsymbol{\sigma}/2}) = e^{\chi\mathbf{n}\cdot\mathbf{K}}$ and $\Pi(e^{-\theta\mathbf{m}\cdot i\boldsymbol{\sigma}/2}) = e^{-\theta\mathbf{m}\cdot\mathbf{S}}$ and thus Π maps the polar decompositions of any $L \in SL(2,\mathbb{C})$ into the corresponding polar decompositions of $\Pi(L) \in SO(1,3)\!\uparrow$.*

Proof. (a) We deal with the decomposition PU only the other case being analogous. As $0 \leq P = P^*$, P can be reduced in diagonal form with positive eigenvalues so that $\det P \geq 0$. As a consequence $1 = \det L = \det P \det U$ entails that $\det U > 0$. In turn, the condition $U^{-1} = U^*$ implies $|\det U|^2 = 1$ and thus $\det U = 1$. We have proved that $U \in SL(2,\mathbb{C})$ and also that and $P = LU^{-1} \in SL(2,\mathbb{C})$. From the spectral theorem (see theorem 12.37 in [5]) there are two bounded self-adjoint operators S, Q (i.e. Hermitian matrices of $M(2,\mathbb{C})$) such that $P = e^S$ and $U = e^{iQ}$. Since the matrices $\sigma_0 := I, \sigma_1, \sigma_2, \sigma_3$ are a basis of the real vector space of 2×2 Hermitian matrices, $S = aI + \chi\mathbf{n}\cdot\boldsymbol{\sigma}$ and $Q = bI + \theta\mathbf{m}\cdot\boldsymbol{\sigma}$ for some versors $\mathbf{n}, \mathbf{m} \in \mathbb{R}^3$ and reals a, b, χ, θ. Using $\det e^X = e^{\mathrm{tr}\, X}$ and the fact that Pauli matrices are traceless, the constraint $\det P = \det U = 1$ implies $a = b = 1$. This completes the proof of (a).
(b) By definition Π maps a one-parameter subgroup with initial tangent vector X into a one-parameter subgroup with initial tangent vector $d\Pi_I X$. Since $\Pi(L)^i{}_j = \mathrm{tr}\,\left(L\sigma_j L^*\sigma_i\right)/2$ where $i, j = 0, 1, 2, 3$ [3], it holds $d\Pi_I : -i\mathbf{n}\cdot\boldsymbol{\sigma}/2 \mapsto \mathbf{n}\cdot\mathbf{S}$ and similarly $d\Pi_I : \mathbf{n}\cdot\boldsymbol{\sigma}/2 \mapsto \mathbf{n}\cdot\mathbf{K}$ for $i = 1, 2, 3$. Hence the one parameters groups $\theta \mapsto e^{-i\theta\mathbf{m}\cdot\boldsymbol{\sigma}/2}$ and $\chi \mapsto e^{\chi\mathbf{n}\cdot\boldsymbol{\sigma}/2}$ are respectively mapped into $\theta \mapsto e^{\theta\mathbf{m}\cdot\mathbf{S}}$ and $\chi \mapsto e^{\chi\mathbf{n}\cdot\mathbf{K}}$.

□

5. Appendix A. Proofs of some propositions

If H is a real Hilbert space $H + iH$ denotes the complex Hilbert space obtained by defining on $H \times H$: (i) the product $(a + ib)(u + iv) := au - bv + i(bu + av)$ where $a + ib \in \mathbb{C}$ and we have defined $u + iv := (u, v) \in H \times H$, (ii) the sum of $u + iv$ and $x + iy$ in $H \times H$: $(u + iv) + (x + iy) := (u + x) + i(v + y)$, and (iii) the, anti-linear in the former entry, Hermitian scalar product $\langle u + iv | w + ix \rangle := (u|v) + (v|x) + i(u|x) - i(v|w)$. Let us introduce a pair of useful operators. The *complex conjugation* $J : u + iv \mapsto u - iv$ turns out to be an anti linear operator with $\langle J(u + iv) | J(w + ix) \rangle = \langle w + ix | u + iv \rangle$ and $JJ = I$. The unitary *flip operator* $C : u + iv \mapsto v - iu$ satisfies $C = C^* = C^{-1}$. A bounded operator $A : H + iH \to H + iH$ is said to be *real* if $JA = AJ$. It is simply prover that, (1) A is real if and only if there is a (uniquely determined) pair of bounded operators $A_j : H \to H$, $j = 1, 2$, such that $A(u + iv) = A_1 u + iA_2 v$ for all $u + iv \in H + iH$; (2) A is real and $AC = CA$, if and only if there is a (uniquely determined) bounded operator $A_0 : H \to H$, such that $A(u + iv) = A_0 u + iA_0 v$ for all $u + iv \in H + iH$.

Proof of Lemma 1. The proof in the complex case is that of Theorem 12.33 in [5]. Let us consider the case of a real Hilbert space H. If $T \in \mathcal{B}(H)$ is positive and self-adjoint, the operator on $H + iH$, $A : u + iv \mapsto Tu + iTv$ is bounded, positive and self-adjoint. By Theorem 12.33 in [5] there is only one $B \in \mathcal{B}(H + iH)$ with $0 \leq B (= B^*)$ and $B^2 = A$, that is the square root of A which we indicate by $\sqrt{A}$. Since A is bounded and commutes with both J and C, all projectors P_Ω ($\Omega \subset \sigma(A)$ being a Borel set) do so. In turn, every real Borel function of A does so, $\sqrt{A}$ in particular. We conclude that $\sqrt{A}$ is real with the form $\sqrt{A} : u + iv \mapsto Ru + iRv$. The operator $\sqrt{T} := R$ fulfills all of requirements it being bounded, self-adjoint and positive because $\sqrt{A}$ is so and $R^2 = T$ since $(\sqrt{A})^2 = A : u + iv \mapsto Tu + iTv$. If T is bijective, A is so by construction. Then, by Theorem 12.33 in [5], $\sqrt{A}$ turns out to be bijective and, in turn, R is bijective too by construction. Let us consider the uniqueness of the found square root. If R' is another bounded positive self-adjoint square root of T, $B : u + iv \to R'u + iR'v$ is a bounded self-adjoint positive square root of A and thus it must coincide with $\sqrt{A}$. This implies that $R = R'$.

□

Proof of Proposition 14. (a) Consider the bijective operator $T : H \to H$ where H is either real or complex. T^*T is bounded, self-adjoint, positive and bijective by construction. Define

$P := \sqrt{T^*T}$, which exists and is bounded, self-adjoint, positive and bijective by Lemma 1, and $U := TP^{-1}$. U is unitary because $U^*U = P^{-1}T^*TP^{-1} = P^{-1}P^2P^{-1} = I$, where we have used $P^* = P$. This proves that a polar decomposition of T exists because $UP = T$ by construction. Let us pass to prove the uniqueness of the decomposition. If $T = U_1P_1$ is a other polar decomposition, $T^*T = P_1U_1^*U_1P_1 = PU^*UP$. That is $P_1^2 = P^2$. Lemma 1 implies that $P = P_1$ and $U = T^{-1}P = T^{-1}P_1 = U_1$.
(b) $P' := UPU^*$ is bounded, self-adjoint, positive and bijective since U^* is unitary and $P'U' = UPU^*U = UP = T$. The uniqueness of the decomposition in (b) is equivalent to the uniqueness of the polar decomposition ${U'}^*{P'}^* = T^*$ of T^* which holds true by (a) replacing T by T^*.

□

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