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On the nonlocal nonlinear Schrödinger equation and its integrable regimes

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Abstract[1]. Nonlocal nonlinear optics is one of more important developments in modern optics. In the recent years, it was shown that nonocality plays a crucial rôle in the propagation of stable spatial solitons in three-dimensions. The problem of integrability of these kind of systems is completely open, and, apart few (very interesting) cases, the main theoretical results are obtained by numerical analysis. The main aim of this paper is to illustrate possible integrable regimes in the high frequency limit. Firstly, we review some known results concerning exact solutions in the limits of high and weak nonlocality for Kerr-type media. We perform the modulational instability analysis for a general form of the nonlocal nonlinear response. It suggests that the specific form of nonlinearity and nonlocal distribution does not affect importantly the stability property. Then, we approach the study of integrability for a general form of the nonlocal nonlinear response. We evaluate the high frequency limit of the nonlocal nonlinear Schrödinger equation and analyze the phase equations. Singular wavefronts and nonlocal perturbations are also discussed.

1. Introduction

The phenomena in classical electrodynamics are governed by the celebrated Maxwell equations (see e.g. [1, 2])

$$\nabla \wedge \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 4\pi \mathbf{J} \qquad \nabla \cdot \mathbf{D} = 4\pi\rho$$
$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \tag{1.1}$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$, x, y, z are the spatial coordinates and t is the time. Moreover, for convenience, we put the velocity of the light $c = 1$. The vectors $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, while the displacement vector $\mathbf{D}$ and the magnetic induction $\mathbf{H}$ contain the information about the response of the medium when an external electromagnetic field is applied. $\mathbf{D}$ and $\mathbf{H}$ are certain functions of $\mathbf{E}$ and $\mathbf{B}$ and they are specified by the so-called *material equations* which, formally, can be written down as follows

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H} \, . \tag{1.2}$$

The *electric permittivity* ε and the *magnetic permeability* μ are, in general, tensors depending on the coordinates and on the fields. Suitable assumptions about the medium as isotropy, homogeneity, symmetries, can induce a lot of simplifications on solving the Maxwell equations.

The *optics* is the branch of electrodynamics concerning with the study of the propagation of the electromagnetic waves through a *dielectric* medium, that is with no current ($\mathbf{J} = 0$)

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and no charges ($\rho = 0$). Under these assumptions, the Maxwell equations look like as follows

$$\nabla \wedge \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{D} = 0$$

(1.3)

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \, .$$

In the case in which ε and μ do not depend on the field, but only on the coordinates apart from the frequency, the Maxwell equations are linear and describe a very broad class of phenomena in physics [1, 2, 3].

The *nonlinear optics* treats the class of media such that ε and μ are depending on the fields. In this case the Maxwell equations are nonlinear. For sake of simplicity, in what follows we put $\mu = 1$ and focus our attention on the electric permittivity ε and on the electric component of the field.

A dielectric medium is said to be nonlinear if the dielectric function ε is depending on the electric field $\mathbf{E}$, that is

$$\varepsilon = \varepsilon(\mathbf{E}) \, .$$

At the moment, we omit to specify the dependence of ε on frequency (*dispersion law*) and the possible dependence on the coordinates (non homogeneous medium).

Usually, it is convenient to write the displacement vector $\mathbf{D}$ as the sum of different order of nonlinear contributions

$$\mathbf{D} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \mathbf{D}^{(3)} + \dots$$

where

(1.4) $$D_i^{(1)} = \varepsilon_{ij}^{(1)} E_j$$

(1.5) $$D_i^{(2)} = \varepsilon_{ijk}^{(2)} \, E_j E_k$$

(1.6) $$D_i^{(3)} = \varepsilon_{ijkl}^{(3)} \, E_j E_k E_l \, ,$$

$$\dots$$

and the sum over repeated indices is assumed. It can be seen that the *nth*-contribution to the electric permittivity $\varepsilon^{(n)}$ is a $n+1$-rank tensor [4].

Each nonlinear term $\mathbf{D}^{(n)}$ is responsible of different specific nonlinear effects. Moreover, particular properties of the medium, such as isotropy, homogeneity, symmetries etc., impose certain constraints on the $\varepsilon^{(n)}$'s. For instance, if the medium is invariant under inversions, the quadratic term $\mathbf{D}^{(2)}$ does not occur. Besides, the isotropy of the medium imposes notable simplifications about the explicit form of $\mathbf{D}^{(n)}$. For example, in the isotropic case, $\varepsilon_{ij}^{(1)}$ is a scalar quantity.

The so-called second harmonic generation has been the first observed nonlinear effect [5]. A monochromatic laser beam of frequency ω incident upon a crystal, produces a laser of frequency 2ω. In analog fashion, two beams of frequency ω_1 and ω_2 generate sum-frequency ($\omega_1 + \omega_2$) and difference-frequency ($|\omega_1 - \omega_2|$) radiations [4]. This phenomenon can be well explained starting from the Maxwell equations with quadratic nonlinearity, and considering the interaction of three resonant waves. This procedure leads to a set on nonlinear integrable partial differential equations (PDE's) for the amplitudes of the waves [2, 4, 6, 7].

First nonlinear contribution to the incoming radiation of a certain frequency ω is given by the cubic term $\mathbf{D}^{(3)}$. Even in this case, one can derive some physically meaningful models described by nonlinear integrable PDE's. Indeed, the Maxwell equations with a cubic nonlinearity in paraxial approximation and under the assumption of weak nonlinearity give rise to the nonlinear Schrödinger (NLS) equation for the envelop of a monochromatic wave. NLS equation describes some very important phenomena in nonlinear optics, such as solitons in $(1+1)$D, self-focusing and the wave collapse of a steady radiation in 3D, with important applications in optical fibers and waveguides.

In the following we discuss some facts showing the pivotal rôle of the theory of integrable systems for applications in nonlinear optics.

The paper is organized as follows. In section 2 we review some basic properties of the standard nonlinear Schrödinger (NLS) equation. In sections 3 we present the derivation of nonlocal nonlinear Schrödinger (NNLS) equation in its more general form and specify the assumptions to derive a more simple model. Section 4 is devoted to discussion of the modulational instability for NNLS. Some results concerning with the special limits of high and weak nonlocality are reviewed in the section 5. The integrability in high frequency limit and the discussion of some exact properties of the phase, singular wavefronts and optical vortices are the subject of the section 6. Finally, some concluding remarks close the paper.

2. The "standard" NLS equation

Let us consider a medium invariant under spatial inversions. In this case the quadratic nonlinearity does not occurs. Moreover, we assume the nonlinearity to be not too large, in such a way that higher than the third order nonlinear contributions to the displacement can be neglected. Hence, the displacement vector $\mathbf{D}$ can be written as follows

$$\mathbf{D} = \mathbf{D}^{(1)} + \mathbf{D}^{(3)} \ . \tag{2.1}$$

For an isotropic medium the general expressions (1.4) and (1.6) are reduced to

$$\mathbf{D}^{(1)} = \varepsilon_0 \mathbf{E} \tag{2.2}$$

$$\mathbf{D}^{(3)} = \alpha(\omega)|\mathbf{E}|^2\mathbf{E} + \beta(\omega)\mathbf{E}^2\mathbf{E} \ . \tag{2.3}$$

Nevertheless, in experimental situations, a broad variety of materials obey to the following more simple law

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \varepsilon_2|\mathbf{E}|^2\mathbf{E} \ . \tag{2.4}$$

Response of these media depends only on the intensity of the electric field. The law (2.4) is usually referred to as *optical Kerr effect*. Various mechanisms can be responsible of the Kerr effect (see e.g. [6]), such as the *orientational Kerr effect* due to the tendency for anisotropic molecules in a liquid to align themselves in a strong electric field, or, for instance, the *electrostriction* which is consisting into the compression of a dielectric material by an applied external field.

It's easy to check that if the magnetic permeability is $\mu = 1$ the Maxwell equations (1.3) imply the following second order system

$$\begin{aligned} &\nabla \wedge \nabla \wedge \mathbf{E} + \frac{\partial^2 \mathbf{D}}{\partial t^2} = 0 \\ &\nabla \cdot \mathbf{D} = 0 \ . \end{aligned} \tag{2.5}$$

Using the relation (2.4) and the identity

$$\nabla \wedge \nabla \wedge \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

it is straightforward to verify that the monochromatic wave

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \ , \tag{2.6}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{E}_0 = \text{const}$, satisfy the system (2.5) with the dispersion relation

$$k^2 = |\mathbf{k}|^2 = \varepsilon_0\omega^2 + \varepsilon_2|\mathbf{E}_0|\omega^2 \ . \tag{2.7}$$

Let us introduce slow variables by means the substitution

$$x \to \sigma x \ \ , \ \ \to \sigma y \ \ , \ \ z \to \sigma^2 z \ \ t \to \sigma t \ \ \text{where } \sigma << 1 \ ,$$

and consider the weak amplitude-modulations of a plane wave propagating along z-direction

$$\mathbf{E}_{full} = \sigma \mathbf{E}(x, y, z) e^{i(k_0 z - \omega t)/\sigma} \tag{2.8}$$

where $k_0 = \varepsilon_0\omega^2$. Using the (2.8) in (2.5) one gets, at the leading order on σ, the following equation

$$2i\sqrt{\varepsilon_0}\,\omega\,\frac{\partial \mathbf{E}}{\partial z} + \nabla_\perp^2 \mathbf{E} + \varepsilon_2\,\omega^2 |\mathbf{E}|^2 \mathbf{E} = 0\ , \tag{2.9}$$

where $\nabla_\perp = (\partial_x, \partial_y)$. The equation (2.9), was derived in two different forms by Kelley [8] and Talanov [9]. It is usually referred to as $(2+1)$D NLS equation. The notation $(2+1)$D is adopted even for the spatial NLS to stress the particular dependence on the variable z, which behaves as a time. It describes a stationary wave propagating along z-direction. The envelope is uniquely determined once the shape of the incoming beam $\mathbf{E} = \mathbf{E}(x, y, z = 0)$ is specified.

For a first analysis of the equation (2.9), we consider a linearly polarized field, for instance, along x-direction

$$\mathbf{E}(x, y, z) = \psi(x, y, z)\ \mathbf{e}_1\ , \tag{2.10}$$

where $\mathbf{e}_1$ is the unit vector parallel to x-axis. Thus, the equation (2.9) can be written down as follows

$$2i\sqrt{\varepsilon_0}\,\omega\psi_z + \psi_{xx} + \varepsilon_2\,\omega^2|\psi|^2\psi = 0\ . \tag{2.11}$$

In this paper we use equivalently the notation $\partial f/\partial x$ and f_x to indicate the partial derivative of the function f with respect to the variable x. We can set $\varepsilon_0 = 1$ without lost of generality. Given the plane wave solution of equation (2.11)

$$\psi = \psi_0 e^{i(\varepsilon_2\omega/2)|\psi_0|^2 z} \tag{2.12}$$

we analyze small amplitude perturbations parallel to the electric field of the form

$$\psi' = (\psi_0 + \delta\psi)e^{i(\varepsilon_2\omega/2)|\psi_0|^2 x}\ . \tag{2.13}$$

Writing down the equation (2.11) for the function ψ', one gets the following equation for $\delta\psi$

$$2i\omega\,\frac{\partial\delta\psi}{\partial z} + \frac{\partial^2\delta\psi}{\partial x^2} + \varepsilon_2\,\omega^2\psi_0^2(\delta\psi + \overline{\delta\psi}) = 0\ . \tag{2.14}$$

Looking for solutions of equation (2.14) of the form

$$\delta\psi = Ae^{i(\mathbf{q}\cdot\mathbf{r}+\gamma z)} + \bar{B}e^{-i(\mathbf{q}\cdot\mathbf{r}+\gamma z)}\ , \tag{2.15}$$

where $\mathbf{r} = (x, y)$ and the vector $\mathbf{q}$ lies on the xy-plane , one gets the condition

$$\gamma = \pm\frac{q}{2\omega}\,\sqrt{q^2 - 2\varepsilon_2\,\omega^2\psi_0^2}\ ,$$

where $q^2 = |\mathbf{q}|^2$. The case $\varepsilon_2 < 0$ is associated with the so-called defocusing NLS equation. The parameter γ is a real one and the plane waves are modulationally stable. In the opposite case $\varepsilon_2 > 0$, the plane waves are modulationally stable yet for $q^2 > 2\varepsilon_2\,\omega^2\psi_0^2$. If $q^2 < 2\varepsilon_2\,\omega^2\psi_0^2$, γ is imaginary and $\delta\psi$ increases exponentially along z. In this case, the plane wave (2.12) is modulationally unstable. This behavior is responsible of the so-called wave collapse phenomenon [10]. In particular, let us note that the condition $\varepsilon_2 > 0$ means that a light beam is subject to a couple of concurrent effects: the self-focusing and the diffraction. In particular, it is interesting to investigate possible solutions such that self-focusing and diffraction cancel out. More specifically, looking for solutions of the equation (2.11) of the form

$$\psi = F(x)e^{i\kappa z} \tag{2.16}$$

and integrating one obtains

$$F = \left(\frac{4\kappa}{\varepsilon_2\,\omega}\right)^{1/2}\operatorname{sech}(\sqrt{2\omega}\ \kappa\, x)\ . \tag{2.17}$$

It describes a beam carrying a flux of energy

$$W \sim F^2(0)\Delta$$

where Δ is the width of the beam

$$\Delta \sim (\omega\kappa)^{-1/2} \ . \tag{2.18}$$

The power of the beam increases if κ increases. As a consequence, due to the relation (2.18) the beam becomes narrower. This effect is usually called *self-channeling* or *self-trapping*, to distinguish it from the stronger self-focusing in $2+1$-dimensions.

The solution (2.17) is well known in the theory of integrable systems and represents the 1-*soliton* solution for the $(1+1)$D NLS equation [6, 11]. The so-called inverse scattering transform (IST) method provides us with a powerful approach to integrate it completely [6]. Here, we do not discuss details of the IST method which is the subject of several books (e.g. [6, 11, 12]). This method is based on the linearization of a nonlinear partial differential equation (PDE) by using its representation in terms of the *Lax pair*. Hence, it is possible to formulate the problem into the space of the so-called *spectral data*. They play the rôle of a complete set of action-angle variables allowing to integrate the nonlinear PDE.

The IST method for NLS equation has been discussed by Zakharov and Shabat in the paper [13] and it is based on the following Lax pair

$$\begin{aligned} \Phi_x &= U(\lambda; x, z)\Phi \ , \\ \Phi_z &= V(\lambda; x, z)\Phi \ , \end{aligned} \tag{2.19}$$

where

$$U = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

$$V = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix} \ .$$

The system (2.19) is over-determined and its compatibility condition is

$$U_z - V_x + [U, V] = 0 \ , \tag{2.20}$$

where $[\cdot, \cdot]$ is the usual commutator. Equation (2.20) can be written equivalently as follows

$$\begin{aligned} ir_z + r_{xx} + 2qr^2 &= 0 \\ iq_z - q_{xx} - 2rq^2 &= 0 \ . \end{aligned} \tag{2.21}$$

Choosing the reduction $r = s\bar{q}$, with $s = \pm 1$, the system (2.21) becomes

$$ir_z + r_{xx} + 2s|r|^2 r = 0 \ . \tag{2.22}$$

Equation (2.22) is just the $1 + 1D$ NLS equation and coincides with equation (2.11) up to the following rescaling of the variables

$$z \to \frac{\varepsilon_2\,\omega^2}{4\omega} x \ , \ \ x \to \frac{\omega}{2} \sqrt{2|\varepsilon_2|}\, x \ , \ \ r \to \psi \ , \ \ s \to \operatorname{sgn}(\varepsilon_2) \ .$$

In both cases $s = 1$ and $s = -1$, the $(1+1)$D NLS equation is completely integrable, but only for $s = 1$ there exist soliton solutions such that $|\psi| \to 0$ when $|x| \to \infty$. The soliton (2.17) is the simplest solution obtained using the IST method. More specifically, in this case, the general solution is given by the N-soliton solution, associated to the discrete spectrum of the linear problem (2.19), and a residual radiation associated to the continuous spectrum [6]. In the limit $z \to \infty$ the relevant contribution is given by N-solitons which appear as N straight lines of constant intensity in the xz-plane. This kind of solutions describes self-channeling light beams. Unfortunately, N-soliton solutions are unstable for transverse perturbations along y-direction [14, 15] (see also [6]). Recently, the problem of stable waveguide production in nonlinear media has attracted a renewed interest due to the observation that the nonlocality combined with the nonlinearity permits the propagation of stable light beam in 3D [16, 17, 18, 19, 20, 21, 22, 23, 24].

Incidentally, we mention that $(2+1)$D generalization of the local $(1+1)$D NLS equation (for a Kerr-like medium), which is amenable by IST method, is the Davey-Stewartson (DS) equation (see e.g. [12])

$$\begin{aligned} & iq_t + \frac{1}{2}(q_{xx} + \delta^2 q_{yy}) + |q|^2 q + q\phi = 0 \\ & \phi_{yy} - \delta^2 \phi_{xx} - (\delta^2 - 1)|q|^2_{xx} + (\delta^2 + 1)|q|^2_{yy} = 0 \ . \end{aligned} \tag{2.23}$$

It is discussed in several papers [25, 26, 27, 28].

Since 2+1-dimensional NLS equation is not completely integrable and then there is not soliton solutions, we do not expect formation of waveguides. Nevertheless, the analysis of the self-focusing phenomenon is interesting to establish the limit of applicability of NLS equation. More specifically, it is possible to derive a sufficient condition such that the light beam is, on average, focused or not (see [2]). Let us observe that given an electric field linearly polarized as in (2.10), the NLS equation (2.9) implies the following conservation law

$$\frac{\partial |\psi|^2}{\partial z} + \nabla_\perp \cdot \mathbf{J} = 0 \tag{2.24}$$

where

$$\mathbf{J} = -\frac{1}{2i\omega}(\psi \nabla_\perp \psi^* - \psi^* \nabla_\perp \psi) \ .$$

The continuity equation (2.24) implies that the following quantities

$$\mathcal{N} = \int_{-\infty}^{+\infty} |\psi|^2 \, dx \, dy \tag{2.25}$$

$$\mathcal{I} = \frac{1}{8\omega^2} \int_{-\infty}^{+\infty} \left(|\nabla_\perp \psi|^2 - \frac{\varepsilon_2 \omega^2}{2} \, |\psi|^4 \right) dx \, dy \ , \tag{2.26}$$

are preserved along z.

Defining the mean beam radius R as follows

$$R^2(z) = \frac{1}{\mathcal{N}} \int_{-\infty}^{+\infty} (x^2 + y^2)|\psi|^2 \, dx \, dy \ ,$$

the use of the equation (2.24) leads us to the equation [2]

$$\mathcal{N} \, \frac{\partial^2 R^2}{\partial z^2} = 4\mathcal{I} \ . \tag{2.27}$$

Integrating, one gets

$$R^2(z) = 2 \, \frac{\mathcal{I}}{\mathcal{N}}(z - z_0)^2 + R_0^2 \ . \tag{2.28}$$

Equation (2.27) is the analog of the virial theorem used by Zakharov to discuss collapse condition for the N-body problem [29]. The equation (2.28) says that the behavior of the beam is critically depending on the sign of the invariant $\mathcal{I}$. If $\mathcal{I} < 0$ there is a finite value of z for which R becomes zero and the beam is focused. In the opposite case, $\mathcal{I} > 0$ the beam is completely defocused.

We precise that this condition is a sufficient one. Indeed, often the singularity occurs much earlier than the predicted value. In general, the blowing up of solutions at finite z, indicate that higher order nonlinear effects should be taken into account [30].

3. The nonlocal NLS equation

In this section we illustrate a class of phenomenological models to describe paraxial light beams propagating in nonlocal nonlinear media. Let us consider a displacement vector of the form

$$\mathbf{D} = \sigma \mathbf{E} + \sigma^3 \mathbf{D}^{(3)} \; . \tag{3.1}$$

Writing the Maxwell equations in paraxial approximation for a light beam propagating along z-axis in the limit $\sigma \to 0$ one can derive the nonlocal nonlinear Schrödinger (NNLS) equation

$$2i\omega \, \frac{\partial \mathbf{E}}{\partial z} + \nabla_\perp^2 \mathbf{E} + \omega^2 \mathbf{D}^{(3)} = 0 \; , \tag{3.2}$$

where $\nabla_\perp = (\partial_x, \partial_y)$. For a general nonlocal nonlinear medium we can write

$$\mathbf{D}^{(3)} = \int_{\mathbb{R}^3} R(\mathbf{r}' - \mathbf{r}; a) \; N(I(\mathbf{r}'))\mathbf{E}(\mathbf{r}') \, d^3\mathbf{r}' \tag{3.3}$$

where $\mathbf{r} = (x, y, z)$. We used the notation $\mathbf{D}^{(3)}$ to recall that in the case of a local Kerr medium the relation (3.3) returns the cubic nonlinearity discussed above. The distribution $R(\mathbf{r}' - \mathbf{r}; a)$ characterizes the nonlocal response around the point $\mathbf{r}$ and a is the "width" parameter (in the following it will be assumed to be depending on the frequency ω). $N(I)$ is an arbitrary nonlinear response depending on the intensity of the electric field $I = |\mathbf{E}|^2$.

We note that due to the paraxial approximation the nonlocal response along z can be neglected. As illustrative example to explain this fact let us assume a Gaussian nonlocal response

$$R(\mathbf{r}', \mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \, e^{-|\mathbf{r}' - \mathbf{r}|^2} \tag{3.4}$$

where $|\mathbf{r}|^2 = x^2 + y^2 + z^2$. In paraxial approximation, substituting $z \to \sigma^{-1} z$

$$\lim_{\sigma \to 0} \frac{1}{\sigma (2\pi)^{3/2}} \; e^{(z'-z)^2/\sigma^2} \; e^{(x'-x)^2 + (y'-y)^2} = \frac{1}{2\pi} \; \delta(z' - z) \; e^{(x'-x)^2 + (y'-y)^2} \; , \tag{3.5}$$

this means that the response along the direction z becomes local. For simplicity we focus on the $1+1$-dimensional case

$$2i\omega \, \frac{\partial \mathbf{E}}{\partial z} + \frac{\mathbf{E}}{\partial x^2} + \omega^2 \mathbf{D}^{(3)} = 0 \tag{3.6}$$

where

$$\mathbf{D}^{(3)} = \int_{-\infty}^{+\infty} R(x - x'; a) \; N(I(x')) \; \mathbf{E}(x') \, dx' \; . \tag{3.7}$$

Let us define the width δR of the nonlocal distribution $R(x - x'; a)$ as the minimum such that

$$R(x - x'; a) \simeq 0 \;\; , \;\; \forall x' \notin [x - \delta R, x + \delta R] \; . \tag{3.8}$$

Of course, δR is depending on the width parameter a. Similarly, we can introduce the widths δE and δN of the electric field and the nonlinear response respectively. Suppose they verify the following conditions

$$\delta R \sim \delta N \;\; , \;\; \delta R << \delta E \; . \tag{3.9}$$

Moreover, we assume a nonlinear response of the form

$$N(I(x)) = \tilde{N}(X) \; , \tag{3.10}$$

where $X = \gamma \, x$ and $\gamma = 1/\delta N$. Expanding $\mathbf{E}(x')$ and $N(I(x'))$ in Taylor series around x, one gets the following approximation of the formula (3.7)

$$\mathbf{D}^{(3)} \simeq \left(\int_{-\infty}^{+\infty} R(x - x'; a) N(I(x')) \, dx' \right) \mathbf{E}(x) \; , \tag{3.11}$$

where we kept into account that due to the equation (3.10) higher orders of the expansion of $N(I(x))$ are not negligible. Note that the formula (3.11), in the case of nonlocal Kerr-type medium, leads to the nonlocal nonlinear Schrödinger equation discussed in [31].
For instance, given a bell-shape electric field $\mathbf{E} = \mathbf{E}_0 \ \exp[-x^2/2\sigma^2]$, a nonlinear response of the form $N(I) = I^\alpha = |\mathbf{E}_0|^{2\alpha} \ \exp[-(\gamma x)^2/2\sigma^2]$, where $\gamma = \sqrt{2\alpha}$, satisfies the condition (3.10). Nevertheless, it's easy to see that only the validity of relations (3.9) is sufficient to obtain the model (3.11). For instance, choosing $\mathbf{E} = \mathbf{E}_0/\cosh^2(x)$ and $N(I) = I^\alpha$, condition (3.9) is verified for α large enough.

4. Modulational instability

In this section we perform the analysis of the modulational instability for the nonlocal nonlinear response (3.3). For sake of simplicity, we consider a linearly polarized electric field in $(1+1)$D case

$$\mathbf{E}(x,z) = \psi(x,z)\hat{\mathbf{e}} \tag{4.1}$$

where $\hat{\mathbf{e}}$ is a constant unit vector lying on the xy-plane. The NNLS equation looks like as follows

$$2i\omega\psi_z + \psi_{xx} + s\omega^2 \int_{-\infty}^{+\infty} R(x'-x)\ N(I(x'))\ \psi(x',z)\, dx' = 0\ , \tag{4.2}$$

where $s = +1$ or $s = -1$ for the attractive or repulsive case respectively. The plane wave

$$\psi = r\ e^{ikx - i\beta z} \qquad \text{where} \qquad r = \text{const} \tag{4.3}$$

is a solution of the equation (4.2) if and only if the following dispersion relation is satisfied

$$\beta = \frac{k^2}{2\omega} - \frac{\omega}{2}\ N(r^2)\hat{R}(k) \tag{4.4}$$

where

$$\hat{R}(k) = \int_{-\infty}^{+\infty} R(x'-x)\ e^{ik(x'-x)}\, dx' \tag{4.5}$$

is the Fourier transform of the distribution $R(x'-x)$. Let us consider a small amplitude perturbation of the form

$$\psi(x,z) = (r + \epsilon a(x,z))\ e^{ikx - i\beta z} \ \text{ where } \epsilon << 1\ . \tag{4.6}$$

We should not confuse perturbating function $a(x,z)$ with the width parameter of the nonlocal distribution function $R(\mathbf{x}-\mathbf{x}')$ introduced above.
Substitution of the function (4.6) in (4.2) gives, at the leading order in ϵ, the following equation for $a(x,z)$

$$\begin{gathered}(2i\omega a_z + 2\omega\beta a + a_{xx} + 2ika_x - k^2 a)\ e^{ikx} + \\ + N(r^2) \int_{-\infty}^{+\infty} R(x'-x)\ a(x',z)\ e^{ikx'}\, dx' + \\ +2sr^2\ N'(r^2) \int_{-\infty}^{+\infty} R(x'-x)\text{Re}\{a(x',z)\}\ e^{ikx'}\, dx' = 0\ .\end{gathered} \tag{4.7}$$

Introducing the variable

$$\xi = x - c_g z \qquad \text{where} \qquad c_g = \frac{k}{\omega}$$

one gets

$$
\begin{aligned}
&(2i\omega a_z + 2\omega\beta a + a_{\xi\xi} + 2ika_\xi - k^2 a)\, e^{ik\xi} + \\
&+\omega^2 N(r^2) \int_{-\infty}^{+\infty} R(\xi' - \xi) a(\xi', z)\, e^{ik\xi'}\, d\xi' + \\
&+2s\omega^2 r^2 N'(r^2) \int_{-\infty}^{+\infty} R(\xi', \xi)\, u(\xi', z)\, e^{ik\xi'}\, d\xi' = 0\ .
\end{aligned} \tag{4.8}
$$

Recall that, by definition, the Fourier transform of the function $a(\xi, z)$ on the variable ξ is

$$
\hat{a}(\xi, z) = \int_{-\infty}^{\infty} a(\xi, z)\, e^{ik\xi}\, d\xi\ . \tag{4.9}
$$

It is convenient to write $a(\xi, z) = u(\xi, z) + iv(\xi, z)$ where u and v are real-valued. Being the Fourier transform a linear operator, it follows

$$
\hat{a}(k, z) = \hat{u}(k, z) + i\hat{v}(k, z)\ .
$$

Integrating the equation (4.8) with respect to ξ and using the convolution theorem

$$
\hat{R}(k)\hat{u}(k) = \int_{-\infty}^{+\infty} R(\xi' - \xi) u(\xi', z)\, e^{ik\xi'}\, d\xi' \tag{4.10}
$$

one obtains

$$
2i\omega\hat{a}_z - k^2\hat{a} + 2s\omega^2 r^2 N'(r^2)\hat{R}(k)\hat{u}(k) = 0\ . \tag{4.11}
$$

Separating in equation (4.11) the real and the imaginary parts, one gets the system

$$
\hat{X}_z = A\, \hat{X} \tag{4.12}
$$

where

$$
\hat{X} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}\ ,\quad A = \begin{pmatrix} 0 & \dfrac{k^2}{2\omega} \\ -\gamma & 0 \end{pmatrix} \tag{4.13}
$$

and

$$
\gamma = \frac{k^2}{2\omega} - \omega s r^2 N'(r^2)\hat{R}(k)\ . \tag{4.14}
$$

The eigenvalues of the matrix A are

$$
\lambda^2 = -\frac{k^2}{2\omega}\left(\frac{k^2}{2\omega} - \omega s r^2 N'(r^2)\hat{R}(k)\right)\ . \tag{4.15}
$$

The eigenvalues λ corresponds to the solutions of the form

$$
a(\xi, z) = a_0(\xi, z)\, e^{\lambda z}\ . \tag{4.16}
$$

If λ is purely imaginary the perturbation remains bounded, otherwise the perturbation gives rise to modulational instability. Let us assume the medium to be such that $N'(r^2) > 0$ (note that when $N'(r^2) < 0$ the analysis remains the same swapping the attractive with repulsive case). It is true, for instance, for Kerr-type and logarithmic saturable media. In the repulsive case $s = -1$, if the Fourier spectrum of the nonlocality distribution is positive definite, as it happens e.g. for Gaussian, Lorentzian distribution etc., one has $\lambda^2 < 0$, $\forall k \in \mathbb{R}$ and the plane wave is modulationally stable. In the attractive case $s = +1$ the plane wave is modulational unstable for the values of k such that

$$
\frac{k}{2\omega} - \omega r^2 N'(r^2)\hat{R}(k) < 0\ . \tag{4.17}
$$

If the response distribution is even ($R(x) = R(-x)$) and peaked at $x = 0$, the Fourier transform is also even with respect to k, that is $\hat{R}(k) = \hat{R}(-k)$ and peaked at $k = 0$. Expanding $\hat{R}$ in Taylor series for $k \to 0$, one gets

$$\hat{R}(k) = \hat{R}(0) + \frac{1}{2}\hat{R}''(0)k^2 + \dots \;, \tag{4.18}$$

where

$$\hat{R}''(0) = \left.\frac{d^2\hat{R}}{dk^2}(k)\right|_{k=0}$$

and we note that $\hat{R}''(0) < 0$. The condition of modulational instability (4.17) in the limit $k \to 0$ becomes

$$\left(\frac{1}{2\omega} - \frac{\omega^2}{2}\, r^2\hat{R}''(0)\right)\, k^2 < \omega r^2 N'(0)\hat{R}(0) \;. \tag{4.19}$$

We conclude that modulational unstability there always exists for k belonging to a small enough band centered around $k = 0$. In the case of nonlocal response functions which are not sign definite, as discussed in the paper [22], for attractive nonlinearities and for large enough nonlocality one observes the appearance of higher order instability bands. It is interesting to note that these results are similar to ones discussed in the paper [22]. Moreover, the modulational instabilty analysis is not considerably influenced by the specific analytical form of nonlinear and nonlocal responses.

5. Exact solutions

Let us focus on the model (3.11) for a Kerr-type medium $N(I) = I$. The nonlocal response function $R(\mathbf{x})$ is assumed to be symmetrically distributed around z-direction. Here, we review some exact solutions in the special limit of high and weak nonlocality.

5.1. **High nonlocality.** The aim of this section is to illustrate the model by Snyder and Mitchell [16]. It describes spatial solitons in a high nonlocal Kerr-type medium. Validity of their model was recently proven experimentally by Conti, Peccianto and Assanto for laser beams propagating in nematic liquid crystals [18].

As an example, consider a paraxial light beam travelling along z-direction through a high nonlocal medium. The assumption of high nonlocality means that the nonlocal response distribution function is large and the refractive index profile induced by the light beam is much more larger than the beam's one. For a system rotationally invariant around z-direction, putting $r = \sqrt{x^2 + y^2}$, the displacement vector can be written down as follows

$$\mathbf{D}^{(3)}(r) = n^2[I(r)]\;\mathbf{E}(r) \tag{5.1}$$

where $n^2[I(r)]$ denotes the following nonlocal operator acting on the intensity $I(r)$

$$n^2[I(r)] = \int_{-\infty}^{+\infty} R(r - r')\; I(r')\, dr' \;. \tag{5.2}$$

For sake of simplicity we assume a linearly polarized electric field of the form (4.1). In virtue of the high nonlocality, inside the domain of interest, where the electric field is not vanishing, one can expand the nonlocal response function in Taylor series

$$R(r - r') \simeq R(r) \simeq R(0) + \frac{1}{2}\, R_{rr}(0)\, r^2 + \dots \;. \tag{5.3}$$

Note that the linear term in r is missing because $R(r)$ has a maximum at $r = 0$. Thus, the refractive index, up to third order terms, is

$$n^2[I(r)] = n_0^2 - \alpha(P)r^2 \tag{5.4}$$

where

$$n_0^2 = R(0)\; P \;\;,\;\; \alpha(P) = \frac{1}{2}\, R_{rr}(0)\; P$$

and P is the power of the light beam

$$P = 2\pi \int_0^{+\infty} |\psi|^2 \, r \, dr \ . \tag{5.5}$$

Performing the gauge transformation

$$\psi \longrightarrow \psi \, e^{n_0^2 z}$$

the NNLS equation (3.6) becomes

$$2i\omega\psi_z + \nabla_\perp^2 \psi - \omega^2 r^2 \psi = 0 \ . \tag{5.6}$$

Equation (5.6) is well known and well studied in physics as the quantum harmonic oscillator equation. In particular, it admits Gaussian solutions [16]

$$I(r,z) = \frac{P}{\pi \, \rho^2(z)} \, e^{-r^2/\rho^2(z)} \tag{5.7}$$

where

$$\frac{\rho^2(z)}{\rho_0^2} = \cos^2(qz) + \frac{\alpha^2(P_c)}{\alpha^2(P)} \, \sin^2(qz) \ , \tag{5.8}$$

ρ_0 is the initial width of the beam, $q = \alpha(P)/n_0$, $P_c = 2/(\omega^2 R_{rr}(0)^2 \rho_0^4)$ and $q = R_{rr}(0)\sqrt{P}/(2n_0)$. The beam preserves the initial Gaussian shape but the width breath sinusoidally travelling along the direction z. For $P < P_c$ the diffraction dominates and initially expand, while for $P > P_c$ the beam initially contracts. In the particular case $P = P_c$ the beam's profile is preserved during its propagation along z and one gets the solitons.

Now, we illustrate the interaction of two Gaussian light beams of total power P such that the axis z lies midway between the beams. It can be shown that given a solution $\psi(\mathbf{r}, z)$, another solution is given by

$$\psi(\mathbf{r} - \mathbf{r}_0, z) \, e^{i \, \mathbf{u}\cdot\mathbf{x} + i\phi} \tag{5.9}$$

where

$$\mathbf{u} = \omega n_0 \, \frac{d\mathbf{r}_0}{dz} \quad , \quad \frac{d\phi}{dz} = \frac{\omega}{2n_0} \left(\alpha^2(P)\mathbf{r}_0^2 - n_0^2 \left(\frac{d\mathbf{r}_0}{dz} \right)^2 \right)$$

and $\mathbf{r}_0$ satisfies the Newton equation for the classical harmonic oscillator, known in the optics as paraxial ray equation

$$\frac{d^2\mathbf{r}_0}{dz^2} = -\left(\frac{\alpha}{n_0} \, \mathbf{r}_0 \right)^2 \ . \tag{5.10}$$

Last formula is not surprisingly since it is well known that mean position of quantum harmonic oscillator behaves classically according to Newton's equations. In the present context, independently on the shape of the beam, its center oscillates according to the formula

$$\mathbf{r}_0(z) = \mathbf{r}_0(0)\cos(qz) \ \ , \ \ \text{with} \quad q = \frac{\alpha(P)}{n_0} \ . \tag{5.11}$$

Two identical beams are described by the function

$$\Psi(\mathbf{r}, z) = \psi(\mathbf{r} - \mathbf{r}_0, z) \, e^{i\mathbf{u}\cdot\mathbf{r} + i\phi} \pm \psi(\mathbf{r} + \mathbf{r}_0, z) \, e^{-i\mathbf{u}\cdot\mathbf{r} + i\phi} \ . \tag{5.12}$$

In particular, Gaussian beams are obtained choosing

$$\psi(r,z) = \frac{\sqrt{P}}{\sqrt{\pi}\rho(z)} \, e^{-(r^2/2\rho^2) + i\varphi} \tag{5.13}$$

which is the solution associated with the intensity distribution (5.7).

The intensity distribution is

$$I = \frac{1}{A}(I_+ + I_- \pm 2\sqrt{I_+ I_-} \, \cos\theta) \tag{5.14}$$

where $I_\pm = I(\mathbf{r}\pm\mathbf{r}_0)$ and A is a normalization constant chosen in such a way that the power is P and $\theta = -2\mathbf{u}\cdot\mathbf{r}$.

5.2. **Weak nonlocality.** The limit of weak nonlocality means that the width of distribution R is small. Here, we consider an even nonlocal distribution $R(x) = R(-x)$. Following the paper [31] we illustrate how $1+1$-D bright soliton solutions can be calculated. A similar analysis can be done for dark solitons.

Due to weak nonlocality, it is reasonable to expand the intensity in Taylor series in such a way that

$$\int_{-\infty}^{+\infty} R(x'-x)I(x')\,dx' = I + R_2 I_{xx}\ , \tag{5.15}$$

where R_n is, by definition, the n-moment

$$R_n = \frac{1}{n!}\int_{-\infty}^{+\infty} R(x)\ x^n\,dx\ . \tag{5.16}$$

Note that, due to $R(x)$ is an even function, the 1-moment vanishes. Hence, the NNLS equation assumes the following form

$$2i\omega\psi_z + \psi_{xx} + s(I + R_2 I_{xx})\psi = 0\ . \tag{5.17}$$

Bright solitons are obtained in the attractive case ($s = +1$), while looking for solutions of the form

$$\psi(x,z) = u(x)\ e^{i\omega(\Gamma/2)z} \tag{5.18}$$

where the function $u(x)$ is assumed to be real-valued. Substituting the ansatz (5.18) in the equation (5.17) one obtains

$$u_{xx} + \omega^2(u^2-\Gamma)u + 2\gamma(u^2)_{xx} = 0\ . \tag{5.19}$$

Integrating, one gets

$$(1+4\gamma u^2)(u_x)^2 + \omega^2\left(\frac{u^2}{2} - \Gamma\right)u^2 = C \tag{5.20}$$

where C is the integration constant. If we look for solutions decaying exponentially, evaluating equation (5.19) $x\to\pm\infty$ one gets $C = 0$. Moreover, if we assume the maximum u_0 at $x = 0$ we have the following well known relation between the maximum and the propagation constant Γ

$$u_0^2 = 2\Gamma\ . \tag{5.21}$$

Thus, the equation (5.20) becomes

$$(u_x)^2 = \frac{\omega^2\,u^2}{2}\ \frac{u_0^2-u^2}{1+4\gamma u^2}\ . \tag{5.22}$$

Naming

$$v = \frac{u_0^2-u^2}{1+4\gamma u^2}$$

the integration of (5.22) gives the implicit solution

$$\pm x = \frac{\sqrt{2}}{\omega u_0}\left[\tanh^{-1}\left(\frac{v}{u_0}\right) + \sqrt{4\gamma}\,u_0\tan^{-1}\left(\sqrt{4\gamma}\,v\right)\right]\ . \tag{5.23}$$

Note that real-valued solutions exist if and only if $v^2 > 0$. This condition is always verified for $\gamma > 0$. In the case $\gamma < 0$, the peak intensity $I_0 = u_0^2$ is required to be no too large, that is

$$I_0 < \frac{1}{4|\gamma|}\ .$$

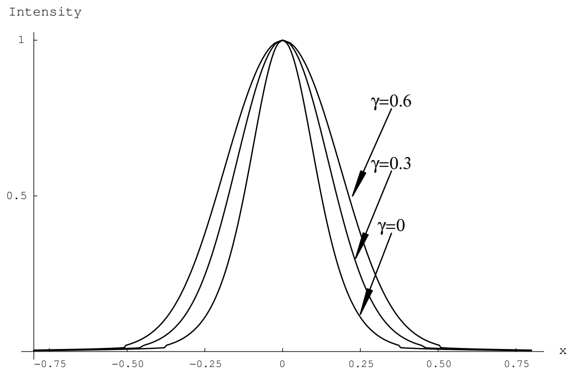

FIGURE 5.1. Intensity light beam profile for different values of the weak nonlocality parameter γ. We set $\omega = 10$ and $u_0 = 1$. Nonlocality increases the width of the bright soliton.

The solution (5.23) describe a bright soliton propagating in a weakly nonlocal Kerr-type medium. Note that in the local case $\gamma = 0$ it reduces to the single soliton

$$u(x) = u_0 \operatorname{sech}\left(\frac{\omega u_0}{\sqrt{2}}\, x\right) . \tag{5.24}$$

It coincides, up to a re-scale, with the 1-soliton discussed in the section 2.

As the figure 5.1 shows, the presence of nonlocality increases the width of the bright soliton. It is a consequence of the fact that the nonlocality smooths-out the nonlinear response.

It is important to highlight that the solution (5.23) is stable and can be proven using the stability criterion discussed by Laedke *et al.* [32]. For a single bright soliton, given the power of the beam

$$P = \int_{-\infty}^{+\infty} I(x)\, dx , \tag{5.25}$$

the soliton is stable if

$$\frac{\partial P}{\partial \Gamma} > 0 . \tag{5.26}$$

In this case, the power P can be calculated explicitly [31]

$$P = \sqrt{u_0^2} + \frac{1 + 4\gamma u_0^2}{\sqrt{4\gamma}}\, \tan^{-1}\left(\sqrt{4\gamma u_0^2}\right) . \tag{5.27}$$

One can see that for $\gamma > 0$, the condition (5.26) is satisfied and the soliton is stable. For $\gamma < 0$, the bright soliton exists if the intensity is small enough, that is $I < 1/(4|\gamma|)$, but it is unstable if $I > 0.7/(4|\gamma|)$.

6. Integrable high frequency regimes

In this section we are interested to discuss a model to describe paraxial light beams in nonlocal nonlinear media in the high frequency regime. In particular, we can interpret the nonlocal response as a resonance effect between the oscillating external electric field and the proper oscillations of the particles constituting the medium. It is reasonable to assume that the electric field oscillates quickly enough in such a way that the nonlocal response becomes negligible in the limit $\omega \to \infty$. In formulas, this means that

$$R(\mathbf{x} - \mathbf{x}'; a(\omega)) \underset{\omega\to\infty}{\longrightarrow} \delta(\mathbf{x} - \mathbf{x}') , \tag{6.1}$$

where $\delta(\mathbf{x})$ is the Dirac δ-function and the width parameter a depends in a suitable way on the frequency ω.
Representing the electric field in terms of the phase S^*

$$\mathbf{E} = \mathbf{E}_0 \, e^{i\omega S^*(x,y,z)} ,$$

the NNLS equation leads to the following equation

$$2S_z^* + (\nabla_\perp S^*)^2 = N_0(I_0(x,y,z)) , \tag{6.2}$$

where $\nabla_\perp = (\partial_x, \partial_y)$ and $N_0(I_0)$ is the high frequency limit of the intensity law

$$\lim_{\omega\to\infty} N(I) = N_0(I_0) .$$

In order to give a complete description of the system the phase equation (6.2) has to be considered together with the Poynting vector conservation law, i.e.

$$\nabla \cdot \mathbf{P} = 0 \tag{6.3}$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$, $\alpha << 1$ and the Poynting vector is

$$\mathbf{P} = I \, \frac{\nabla S^*}{|\nabla S^*|} . \tag{6.4}$$

Paraxial approximation of the equation (6.3) produces the following constraint on the xy-plane

$$\nabla_\perp I \cdot \nabla_\perp S^* + I \nabla_\perp^2 S^* = 0 . \tag{6.5}$$

Here, we focus on the class of solutions such that

$$S_z^* = \zeta_0 , \implies S^* = \zeta_0 z + S^*(x,y) , \tag{6.6}$$

where ζ_0 is a real constant. Then, the equations (6.2) and (6.5) assumes the form

$$S_x^2 + S_y^2 = u(I_0(x,y)) \tag{6.7a}$$

$$I_0(S_{xx} + S_{yy}) + I_{0x}S_x + I_{0y}S_y = 0 \tag{6.7b}$$

where $u(I_0) = \tilde{N}_0(I_0) - 2\zeta_0$. We refer to the functional dependence among u and I_0 as *intensity law*. It is determined by the specific physical properties of the medium. Note that once $u(I_0)$ is given, the system (6.7) is overdetermined and the consistency condition will be discussed in the next.

6.1. **The $(1+1)$D case.** Writing down the equations (6.2) and (6.7b) in the case in which the phase and the intensity do not depend, for instance, on y, one has

$$2S_z^* + (S_x^*)^2 = N_0(I_0) \tag{6.8a}$$

$$I_0 S_{xx}^* + I_{0x} S_x^* = 0 . \tag{6.8b}$$

Integrating once the equation (6.8b), the system (6.8) leads to

$$S_x^* = \frac{\gamma_0}{I_0} \tag{6.9a}$$

$$S_z^* = \frac{1}{2}\left(N_0(I_0) - \frac{\gamma_0^2}{I_0^2}\right) , \tag{6.9b}$$

where γ_0 is an arbitrary real constant. Imposing the that $S_{xz} = S_{zx}$ one gets

$$I_{0z} + \left(\frac{I_0^2 N_0'(I_0)}{2\gamma_0} + \frac{\gamma_0}{I_0}\right) I_{0x} = 0 , \tag{6.10}$$

where $N_0' := dN_0/dI_0$. General solution of the equation (6.10), calculated using the characteristics method [33], is provided by the following implicit relation

$$x - \left(\frac{I_0^2 N'(I_0)}{2\gamma_0} + \frac{\gamma_0}{I_0}\right) - \Phi(I_0) = 0 \tag{6.11}$$

where Φ is an arbitrary function of its argument. It is assigned by the initial profile of the intensity

$$I_0(x,0) = \Phi^{-1}(x) \ . \tag{6.12}$$

It is well known [34] that equations of type (6.10) exhibit breaking wave phenomena for finite z. Then, as shown in the figure 6.1, a smooth initial profile, such that $I_0 \to 0$ for $x \to \pm\infty$ is broken soon.

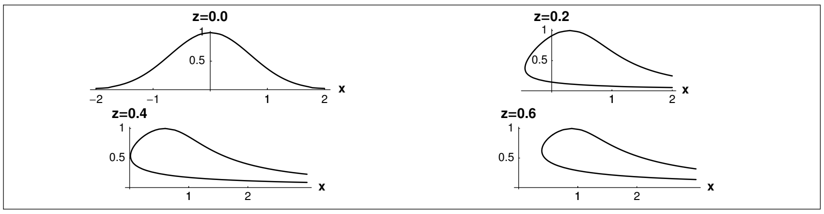

FIGURE 6.1. In $1+1$D a high frequency gaussian beam breaks becoming early multivalued.

6.2. **The $(2+1)$D case: elliptic intensity law.** Let us assume that the function $u(I_0)$ is invertible. For instance, in the case of *Kerr*-type media, where intensity law is of the form $u \propto I_0^\gamma$ ($\gamma > 0$), and for saturable media with logarithmic nonlinearity $u \propto \log(1 + I_0/I_{0t})$ (I_{0t} =const is the so-called threshold intensity), the invertibility condition is satisfied. Then, we can consider the function $I_0 = I_0(u)$. We referred it to as *inverse intensity law.* Calculating $\nabla_\perp I_0 = I_0' \nabla_\perp u$, where "prime" means the derivative with respect its argument u, and using equation (6.7a) in (6.7b), we obtain the following second order partial differential equation

$$AS_{xx} + BS_{yy} + 2CS_{xy} = 0 \ , \tag{6.13}$$

where by definition $\mathcal{I}_0 = \log I_0$ and

$$A = \mathcal{I}_0{}' S_x^2 + 2 \ , \quad B = \mathcal{I}_0{}' S_y^2 + 2 \ , \quad C = \mathcal{I}_0{}' S_x S_y \ . \tag{6.14}$$

The equations of the form (6.13) are well studied in mathematics (see e.g. [35] and references therein). Their properties are critically depending on the signature of the discriminant

$$\Delta = AB - C^2 \ . \tag{6.15}$$

In particular, one can distinguish three cases. Assuming $A > 0$, the equation (6.13) is said to be elliptic if $\Delta > 0$, parabolic if $\Delta = 0$ and hyperbolic if $\Delta < 0$.

In what follows, we will focus on the elliptic case motivated by the fact that a wide class physically relevant models for optical spatial solitons in Kerr-type or logarithmic saturable media satisfy the ellipticity condition $\Delta > 0$ uniformly.

We would like to stress that elliptic second order nonlinear equations of the form (6.13) possess several remarkable analytical and geometrical properties (see e.g. [35, 36, 37, 38, 39, 40]). We mention in particular the connection with the theory of quasiconformal mappings. In order to study the equation (6.13), let us introduce the complex variable $\lambda = x + iy$ and the complex gradient $w = S_x - iS_y$. Using these notations, the equation (6.13) looks as follows

$$aw_\lambda + bw_{\bar\lambda} + \bar a \bar w_{\bar\lambda} + \bar b \bar w_\lambda = 0 \ , \tag{6.16}$$

where

$$a = \frac{1}{2}(A - B + 2iC) \ , \quad b = \frac{1}{2}(A + B - 2iC) \ . \tag{6.17}$$

An interesting class of solutions of equation (6.16) is provided us by the solutions of, the so-called, nonlinear Beltrami equation

$$w_{\bar{\lambda}} = \mu(w, \bar{w}) w_{\lambda} \ , \ \ \mu = -\frac{a}{b} \ . \tag{6.18}$$

It's easy to verify that ellipticity condition $\Delta > 0$ implies $|\mu| < 1$. We highlight that the solutions of nonlinear Beltrami equation (6.18) possess a very remarkable geometrical meaning, being the so-called quasiconformal mapping of the plane with complex dilatation μ [39, 40]. Thus, the evolution of a light beam profile along the direction z can be described using suitable deformations of quasiconformal mappings.

More specifically, let us consider an input light beam profile such that $I_0 \neq 0$ inside a simply connected domain G and $I_0 = 0$ onto the smooth boundary Γ of G and outside it. Set $u_0 = u(I_0 = 0)$. Writing down the eikonal equation (6.7a) in terms of w and λ

$$w(\lambda)\bar{w}(\lambda) = 4u \ , \tag{6.19}$$

evaluating the equation (6.19) on the boundary Γ one has $|w| = 2\sqrt{u_0}$, that is Γ is mapped on the $2\sqrt{u_0}$-radius circle. Moreover, assuming w to be assigned in such a way that, for instance, $w(0) = 0$, where $0 \in G$, and the variation of the argument of the complex number w around Γ is

$$\underset{\Gamma}{\Delta} \arg w = 2\pi,$$

it can be proven that w is a homeomorphic mapping of domain G onto $2\sqrt{u_0}$-radius disk Γ' [36]. As a consequence, mapping w preserves the topology of domain G. Because of the assumption (6.6) the intensity profile does not depend on z, while the wavefront evolves according to the equation

$$\zeta_0 z + S(x, y) = \text{const} \ , \tag{6.20}$$

where $S(x, y)$ is a certain solution of the system (6.7).

We observe that the mapping $w = S_x - iS_y = (S_x, -S_y)$ can be also regarded as a two-dimensional vector field on the λ-plane associated with transverse components of the wavefront normal unit vector

$$\mathbf{n} = \frac{\nabla S^*}{|\nabla S^*|} = \left(\frac{S_x}{\sqrt{\zeta_0^2 + 4u}} \ , \ \frac{S_y}{\sqrt{\zeta_0^2 + 4u}} \ , \ \frac{1}{\sqrt{\zeta_0^2 + 4u}} \right) \ . \tag{6.21}$$

This means that the mapping w encodes information about light-rays distribution around direction z. For example, let us consider a mapping as in figure 6.2. Curves Γ and Γ' are oriented leaving domain on the right hand side. Under the assumptions mentioned above, there exists a homeomorphism w of domain G onto G' acting in such a way that $w(s_1, 0) = (-1, 0)$, $w(0, s_2) = (0, 1)$ and $w(-s_3, 0) = (1, 0)$. Consequently, the normal unit vector to wave-front on the boundary Γ is oriented in such a way that the light rays lie inside the rectangle circumscribing the domain G (see figure 6.2). Recall that y-component of w reverse y-component of $\mathbf{n}$. Then, this mapping describes a light beam 'trapped' around the direction z. Conversely, if $w(s_1, 0) = (1, 0)$, $w(0, s_2) = (0, -1)$ and $w(-s_3, 0) = (0, -1)$ radiation spreads transversely far from z-axis. Note that in both cases homeomorphism w is sense-reversing. If we consider a sense-preserving mapping such as, for instance, $w(s_1, 0) = (1, 0)$, $w(0, s_2) = (0, 1)$ and $w(-s_3, 0) = (-1, 0)$, the beam spreads along x-direction and tends to be trapped along y-axis (see figure 6.3). We emphasize, finally, that all of these observations can be generalized to the case of arbitrary n-connected domains.
Another class of solutions of equation (6.16) can be obtained solving the equation

$$b w_{\bar{\lambda}} + \bar{a} \bar{w}_{\bar{\lambda}} = 0 \ . \tag{6.22}$$

Let us introduce the reciprocal coordinates inverting the system

$$\begin{aligned} \lambda &= \lambda(w, \bar{w}) \\ \bar{\lambda} &= \bar{\lambda}(w, \bar{w}), \end{aligned} \tag{6.23}$$

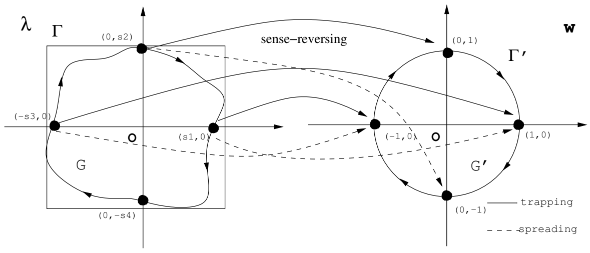

FIGURE 6.2. Sense-reversing mappings (solid line) describe beams which tend to be confined inside the rectangle or scattered (dashed line) outside it.

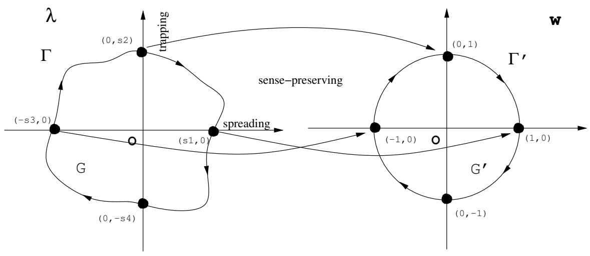

FIGURE 6.3. If the mapping is sense-preserving light beam spreads along x-direction and tends to be trapped along y-direction.

one converts equation (6.22) into the form of the *linear* Beltrami equation

$$\bar{\lambda}_{\bar{w}} = \nu(w, \bar{w})\bar{\lambda}_w \ , \tag{6.24}$$

where $\nu(w, \bar{w}) = -\bar{a}/b$ and, due to ellipticity, $|\nu| < 1$. The advantage of the use of reciprocal transformation (6.23) is that the Beltrami equation is linear and it can be solved explicitly for different choices of the intensity law. Moreover, it is well known that several theorems in analytic functions theory can be rigorously generalized to the quasi-analytic functions which are the solutions of the equation (6.24) (see e.g. Ref. [38]). In particular, we exploit the generalization of so-called Liouville theorem. If $\lambda = \lambda(w, \bar{w})$ is bounded on whole w-plane and satisfies the linear Beltrami equation (6.24) it can be shown (*Vekua's theorem*) that $\lambda(w, \bar{w}) \equiv$ constant. (See theorem 3.32 p. 213 in Ref. [38].) Of course, for the constant solution, mapping from w-to-λ-plane is singular and reciprocal transformation (6.23) is not defined. Then, any non-trivial solution of equation (6.24) must be singular somewhere on the complex plane and different type of singularities can occur, such as poles, essential singularities, singularities of the derivatives etc.

As illustrative example, we focus our attention on a solution $\bar{\lambda}(w, \bar{w})$ having a simple pole at $w = 0$. As discussed above, one can always consider a homeomorphism from $D' = \mathbb{C} \setminus D_\epsilon$ to D_R where D_ϵ is a disk of arbitrarily small radius ϵ on the w-plane and D_R is the R-radius disk on the λ-plane, mapping the boundary of D_ϵ on the boundary of D_R. Inverse mapping $w: \ D_R \to D'$, constructed in such a way, can be used to describe a beam "confined" around z-axis. Indeed, light rays crossing the boundary of D_R can be selfguided around the axis z with arbitrary accuracy.

Coming back to nonlinear Beltrami equation (6.18), we expect that for "mild enough" complex dilatations μ, Vekua's theorem still holds. In these cases, the only one bounded solution on whole λ-plane is $w =$ constant. In many physical situations, the intensity distribution on the λ-plane, at certain z, goes to zero for $\lambda \to \infty$, or equivalently, one can says that intensity vanishes outside a big enough R-radius disk D_R. Thus, outside D_R, refractive index assumes a constant value $u = u_0$ and the solutions of eikonal equation (6.7a) is

$$S = c_0 x + c_1 y + c_3 \ , \tag{6.25}$$

where c_0, c_1 and c_3 are constants and the condition $c_0^2 + c_1^2 = 4u_0$ holds. For a paraxial beam we have $c_0 = c_1 = 0$. By a consequence $w = S_x - iS_y = c_0 - ic_1 = 0$ and, of course it satisfies Beltrami equation in $\mathbb{C} \setminus D_R$. In virtue of Vekua's theorem, the only one bounded solution is $w \equiv 0$. Then, any non-trivial solution must be singular somewhere on the plane. In our example, wavefront is approximately plane for $\lambda \in \mathbb{C} \setminus D_R$ and possesses singularity inside D_R.

6.3. **Optical vortex.** As shown by Tricomi [41], the successive approximations method is a general approach to solve the linear Beltrami equation. In general, calculation of exact 'explicit' solutions can be very difficult and it is strongly influenced by the form of the complex dilatation. Incidentally, we note that solutions possessing cylindrical symmetry such that

$$S = S(r) \ \ , \ \ u = u(r) \ \ , \ \ r = \sqrt{x^2 + y^2} \ , \tag{6.26}$$

are not compatible with intensity law. It is straightforward to verify that equations (6.7) along with assumptions (6.26) imply that intensity depends explicitly on z-axis distance r

$$I_0 = \frac{1}{2cr\sqrt{u}} \ , \tag{6.27}$$

where c is an arbitrary constant.

Here, we approach the solution of the equation (6.13) in connection with the so-called non-parametric minimal surfaces. Non-parametric means that the surface is parametrized by the Cartesian coordinates: $(x, y, S(x, y))$. Hence, minimal surface equation is (see e.g. Ref. [42, 43])

$$\left(1 + S_y^2\right) S_{xx} + \left(1 + S_x^2\right) S_{yy} - 2 S_x S_y S_{xy} = 0 \ . \tag{6.28}$$

We restrict ourselves to the class of solutions of equation (6.13) which are also harmonic, that is

$$S_{xx} + S_{yy} = 0 \ . \tag{6.29}$$

Using condition (6.29) in equation (6.13), one gets the equation

$$S_x^2 S_{xx} + S_y^2 S_{yy} + 2 S_x S_y S_{xy} = 0 \ , \tag{6.30}$$

for any elliptic intensity law. It is straightforward to check that equations (6.30) and (6.28) coincide for harmonic solutions. In other words, a class of solutions of equation (6.13), whatever is the inverse intensity law $I_0 = I_0(u)$, is just given by the class of the harmonic minimal surfaces. It can be shown that the only non-trivial harmonic minimal surface in Cartesian coordinates is the helicoid [42]. It can be written as follows

$$S = K \arctan\left(\frac{\beta z + y}{x}\right) \ , \tag{6.31}$$

where K is an arbitrary constant. It is straightforward to check that function S in (6.31) satisfies equations (6.28), (6.29) and (6.30) simultaneously. Equation of corresponding wavefronts is

$$S^* \equiv z + K \arctan\left(\frac{\beta z + y}{x}\right) = \mathrm{const} \ , \tag{6.32}$$

where the constant K is the "pitch" of the helicoid. In particular the expression (6.32) describes the edge-screw dislocations discussed first time experimentally by Brynghdal in 1973 [44] and theoretically by Nye and Berry [45] in 1974. These class of phase defects has important phenomenological consequences connected with the *optical vortices* [46, 47, 48] (see also Refs. [50, 51] and references therein). If we assume, for simplicity, $\beta = 0$ one has the pure screw dislocations. The complex gradient associated with the helicoid

$$w = -i\,\frac{K}{\lambda} \tag{6.33}$$

is analytic and it has a simple pole singularity at $\lambda = 0$. Nevertheless, normal vector to the wavefront, whose components coincides (up to a sign) with real and imaginary parts of w, is not defined. Indeed, vector $w = (S_x, -S_y)$ has no limit as $(x, y) \to 0$. Figure 6.4 shows examples of helicoidal wavefronts (6.32) usually parametrized as follows

$$x = v\cos t \ , \ \ y = v\sin t \ , \ \ z = Kt \, . \tag{6.34}$$

One-started helicoid shown in figure 6.4a is obtained for $0 < v < +\infty$; two-started helicoids

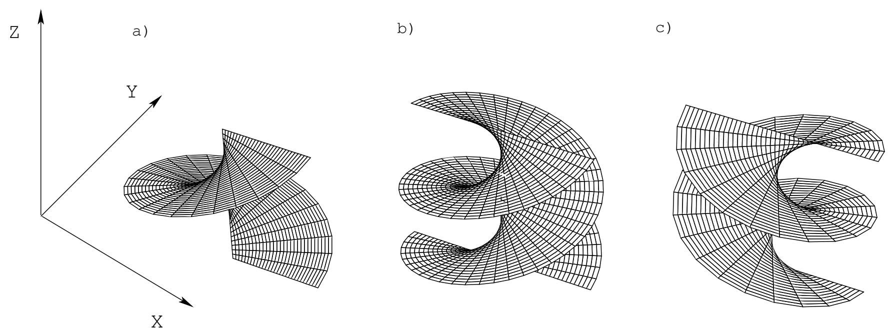

FIGURE 6.4. Helicoidal structures of wavefront around z-axis: a) one-start right-screw ($K = 1$); b) two-start right screw ($K = 1$); c) two-start left screw ($K = -1$).

shown in figure 6.4 b), c) are obtained for $-\infty < v < +\infty$. Refractive index

$$u = \frac{K^2}{4(x^2 + y^2)} \tag{6.35}$$

has cylindrical symmetry around z-axis and displays divergence at $\lambda = 0$. This means that around $\lambda = 0$ geometrical optics approximation fails and wave effects become relevant. In particular, necessary condition for the existence of singular wavefronts is that intensity vanishes where phase function is singular [45]. Indeed, in this region the interference phenomenon is no more negligible and can realize this condition.

6.4. **Nonlocal perturbations.** In the high frequency limit of the NNLS equation, the dependence of the refractive index n^2 on ω (the so-called *dispersion law*) plays a crucial rôle in the evaluation of wave corrections to geometric optics. According to the "classical" Debye's theory [52], the dispersion law assumes the form

$$n^2 = n_0^2 + \frac{\tilde{n}^2}{1 + 2i\omega} \ . \tag{6.36}$$

As usual, the geometric optics regime is provided us by the leading order in the high frequency expansion of NNLS. According to our description, the nonlocality is small for $\omega \to \infty$ and we expect it influences at most first wave corrections. Then, the evaluation of nonlocal effects in this regime requires the solutions of the *transport equations* for the coefficients of the ω^{-1}-expansion of the amplitude.

Nevertheless, in suitable physical situations, some simpler cases can be considered. In 1941 Cole and Cole [53] have shown that for a large family of liquid and solid polar media the Debye's law fails and it is modified by the following formula

$$n^2 = n_0^2 + \frac{\tilde{n}^2}{1 + (2i\omega)^{2\nu}} \quad , \quad \text{where} \quad 0 < \nu < \frac{1}{2} \, . \tag{6.37}$$

We stress that several media used in the experiments of nonlocal nonlinear optics such as liquid crystals obey the Cole-Cole law [54]. Following the paper [57], we illustrate shortly how it is possible to construct an integrable system of PDE's just using the Cole-Cole dispersion law.

Nonlocal perturbations can be described considering a nonlocal distribution given by the linear superposition of Dirac δ-function and its derivatives

$$R(\mathbf{r}-\mathbf{r}',\omega) = \sum_{l,m=0}^{\infty} R_{l,m}(\mathbf{r},\omega)\, \delta^{(l,m)}(\mathbf{r}-\mathbf{r}')\ , \tag{6.38}$$

where $\mathbf{r}=(x,y)$ the distributions $\delta^{(l,m)}$ are defined standardly as

$$\iint_{\mathbb{R}^2} f(\mathbf{r}')\, \delta^{l,m}(\mathbf{r}-\mathbf{r}')\, d\mathbf{r}' = (-1)^{l+m}\, \frac{\partial^{l+m} f(\mathbf{r})}{\partial x^l \partial y^m}\ , \tag{6.39}$$

where $f(\mathbf{r})$ is a trial-function. In this model the nonlocal nonlinear contribution to the displacement vector assumes the form

$$\mathbf{D}^{(3)} = n^2\mathbf{E} + \mathbf{F}_{nloc} \tag{6.40}$$

where

$$\mathbf{F}_{nloc} = \sum_{l=1}^{2} c_n\, \frac{\partial \mathbf{E}}{\partial x_l} + \sum_{l,m=1}^{2} c_{l,m}\, \frac{\partial^2 \mathbf{E}}{\partial x_l \partial x_m} + \dots\ . \tag{6.41}$$

We study a particular high frequency regime of NNLS equation where the phase of electric field varies slowly along z-direction in such a way that

$$z \to \omega^{2\nu} z\ ,\ \ \frac{\partial}{\partial z} \to \omega^{-2\nu}\frac{\partial}{\partial z}\ . \tag{6.42}$$

We also assume that nonlocal effects contribute, in the $\omega \to \infty$ limit, at most with $\omega^{-2\nu}$-order terms, i.e.

$$F_{nloc} = \omega^{-2\nu}\tilde{F}_{nloc} + O(\omega^{-4\nu})\ . \tag{6.43}$$

As usual, let us represent the electric field as follows

$$\mathbf{E} = \mathbf{E}_0(x,y,\omega^{2\nu})\ e^{i\omega S(x,y,\omega^{2\nu} z)}\ . \tag{6.44}$$

Expanding all functions with respect to small parameter $\omega^{-2\nu}$

$$\begin{aligned} \mathbf{E}_0(x,y,\omega^{2\nu} z) &= \mathbf{E}_0(x,y,z) + \omega^{-2\nu}\mathbf{E}_1(x,y,z) + \cdots \\ S(x,y,\omega^{2\nu} z) &= S(x,y,z) + \omega^{-2\nu} S_1(x,y,z) + \cdots \\ n^2(x,y,\omega^{2\nu} z) &= n_0^2(x,y,z) + \omega^{-2\nu} n_1^2(x,y,z) + \cdots\ , \end{aligned} \tag{6.45}$$

using the dispersion law (6.37) and the prescriptions (6.42), the leading orders in the NNLS equation (i.e. ω^2 and $\omega^{2-2\nu}$) provides us with the system

$$S_x^2 + S_y^2 = 4u\ , \tag{6.46a}$$

$$S_z = \varphi(S_x, S_y, x, y, z) \tag{6.46b}$$

where $4u = n_0^2$ and $\varphi = \left(n_1^2 + \tilde{F}_{nloc}\right)/2$. We stress that condition on the exponent $0 < \nu < 1/2$ is crucial to separate equation (6.46b) from the first equations containing the amplitude. Equation (6.46a) is the standard eikonal equation in two-dimensions. The function φ in equation (6.46b), in virtue of expression (6.41) is an N-degree polynomial in S_x and S_y and it describes an N-degree nonlocal response. Moreover, if we require that the equation (6.46b) possesses the inversion phase symmetry ($S \to -S$) just like the eikonal equation, polynomial degree of the function φ must be odd. We note also that the system of equations (6.46) has been derived first directly from the Maxwell equations [56].

As explicit examples we analyze the zero-degree (local) response and the first and third degree nonlocal responses. In local case we have

$$\varphi = \varphi(z) \ , \quad u = u(x,y) \ , \tag{6.47}$$

where φ and u are certain function of their arguments. For a first degree nonlocality we have

$$\varphi = \alpha_1 S_x + \alpha_2 S_y \ , \tag{6.48}$$

and u must satisfy the following linear equation

$$u_z = (\alpha_1 u)_x + (\alpha_2 u)_y \ , \tag{6.49}$$

where α_1 and α_2 are harmonic conjugate functions, that is they satisfy the Cauchy-Riemann conditions

$$\alpha_{1x} = \alpha_{2y} \quad \alpha_{1y} = -\alpha_{2x} \ .$$

Finally, for a third degree nonlocal response we have

$$\varphi = \frac{1}{4}\, S_x^3 - \frac{3}{4}\, S_x S_y^2 + V_1 S_x + V_2 S_y \tag{6.50}$$

and u satisfies the dispersionless Veselov-Novikov equation

$$\begin{aligned} u_z &= (V_1 u)_x + (V_2 u)_y \ , \\ V_{1x} - V_{2y} &= -3u_x \ , \\ V_{1y} + V_{2x} &= 3u_y \ . \end{aligned} \tag{6.51}$$

It is interesting to analyze the effect of the present class of nonlocal perturbations on the pure ($\beta = 0$) helicoidal wavefront (6.31). We will see what are the conditions such that the pure helicoid is preserved up to $\omega^{-2\nu}$-orders deformations. In this case, the phase (6.31) is defined up to an additive arbitrary function of z-variable $\psi(z)$, such that $\psi' = \varphi(z)$. Thus, using the phase expression

$$S = K \arctan\left(\frac{y}{x}\right) + \psi(z)$$

along with relation (6.35) in equations (6.48) and (6.49), we conclude that the helicoid is deformed for nonlocal harmonic data α_1 and α_2 satisfying simultaneously the following equations

$$(x^2+y^2)\alpha_{1x} = x\alpha_1 + y\alpha_2 - y\alpha_1 + x\alpha_2 = \varphi(z)(x^2+y^2) \ . \tag{6.52}$$

Trivial solutions $\alpha_1 = \alpha_2 = \varphi = 0$ corresponds to the local case discussed above. A simple non-trivial solution $\alpha_1 = -\gamma y$, $\alpha_2 = \gamma x$ and $\varphi = \gamma$, where γ is an arbitrary constant, provides us with the following wavefront (we remind that to return to "speed" variables one has to substitute $\psi(z) \to \psi(\omega^{-2\nu} z)$)

$$S^* \equiv z + \frac{K}{1+\omega^{-2\nu}\gamma}\ \arctan\left(\frac{y}{x}\right) = \text{const} \ . \tag{6.53}$$

For $\gamma = 0$ equation (6.53) coincides with the equation (6.32). As a consequence of nonlocal response the helicoid's pitch is compressed if $\gamma > 0$ and stretched if $\gamma < 0$.

6.5. **Intensity law and nonlocal perturbations.** The analysis of the compatibility between the class of nonlocal perturbations associated with the dVN hierarchy and the intensity conservation law (6.7b) shows that there are not nontrivial solutions for an arbitrary form of the intensity law. In particular, intensity law appears to be a quite restrictive constraint making the problem of compatible nonlocal responses rather non-trivial. Nevertheless, it is possible to see, by an explicit example, that there exists nontrivial intensity laws such that the system (6.46) and the intensity conservation law (6.7b) are compatible with the dVN hierarchy. Here, as illustrative example, we focus on the third degree of nonlocality associated with the so-called dVN equation.

In order to do that, we consider hydrodynamic type reductions of dVN equation. They have been found using symmetry constraint of the form [58]

$$\nabla_\perp^2 S = u_x \ .$$

It can be shown that dVN equation (6.51) is reduced to following hydrodynamic-type system

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ 2p_1 - 1 & 2p_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_x \tag{6.54}$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_z = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_x \tag{6.55}$$

where

$$A_{11} = 3p_1(p_1 - 1) \qquad A_{12} = 3p_2 \ ,$$

$$A_{21} = 3p_2(2p_1 - 1) \qquad A_{22} = 3p_1(p_1 - 1) + 6p_2^2$$

and $p_1 := S_x$, $p_2 := S_y$.

Looking for solutions such that $p_2 = p_2(p_1)$, p_1 and p_2 are given in implicit form in terms of the following algebraic system

$$x + G'y + H'z - \Phi(p_1) = 0 \tag{6.56}$$

$$\begin{aligned} p_2 &= \frac{1}{2}\left[q + \frac{2c - \log\left(q + \sqrt{1+q^2}\right)}{\sqrt{1+q^2}}\right] \\ q &= p_2 \pm \sqrt{p_2^2 + 2p_1 - 1} \end{aligned} \tag{6.57}$$

where c is an arbitrary constant, $G = p_2(p_1)$, $H = p_1^3 - (3/2)p_1^2 + (3/2)p_2^2(p_1)$ and 'prime' means the derivative with respect to p_1.

Differentiating eikonal equation (6.46a) with respect to x and taking into account that $u_x = \nabla_\perp^2 S$, one gets

$$\nabla_\perp\left(-\frac{S_x}{2}\right) \cdot \nabla_\perp S + \nabla_\perp^2 S = 0 \ . \tag{6.58}$$

Comparing equation (6.58) with intensity conservation equation (6.7b), which can be written equivalently as follows

$$\nabla_\perp S \cdot \nabla_\perp(\log I_0) + \nabla_\perp^2 S = 0 \tag{6.59}$$

one gets the following nice relation among intensity and p_1-component of the gradient

$$I_0 = C\, e^{-p_1/2} \ , \tag{6.60}$$

where C is an arbitrary real constant. Finally, eikonal equation provides us with intensity law

$$u(I_0) = \left(\log\frac{I_0}{C}\right)^2 + \frac{1}{4}\left[p_2\left(-2\log\frac{I_0}{C}\right)\right]^2 \ , \tag{6.61}$$

where last term in r.h.s. is given by algebraic relation (6.57).

7. Concluding remarks

In this paper we have reviewed and discussed some aspects of nonlocal nonlinear optics with particular attention to the problem of the integrability. As it was shown, nonlocality is crucial to generate stable soliton-like light beams in 3D. Indeed, the nonlocality smooth out the nonlinear response profile in such a way to compensate the otherwise prevalent self-focusing and consequent wave collapse in three dimensions. Moreover, it is well known, the 3D NLS equation is not integrable by the inverse scattering transform method unlike the $1+1$-dimensional one. Nevertheless, $(1+1)$D solitons of NLS do not possess "good" properties in three-dimensions. In particular, they are not stable for transverse perturbations [10]. We stress that even in $1+1$-dimensions nonlocality involves a new and very interesting phenomenology as, for instance, the attraction of dark solitons [59].

From the analytical point of view, the NNLS equation is quite different with respect to the local one and the extension of the standard methods to approach the NLS equation is highly nontrivial. As discussed in the section 5 there exists few cases which are treatable analytically. Nevertheless, a large part of results of phenomenological interest, concerning, for instance, solitons interaction, stability properties of optical vortices [23, 24] are obtained by means of numerical analysis sometimes combined with a variational approach.

The modulational instability analysis in the section 4 shows that stability is not strongly depending of the analytical form of nonlinear response once the refractive index derivative on the intensity is assumed to be positive (or negative) definite. On the other hand, several interesting phenomena seem to be not sensitive to the analytical detail of nonlocal (positive definite, symmetrically distributed around the axis z and exponentially decaying for $r \to \infty$) distribution.

It is a general philosophy that the integrability should be connected with some kind of "regular" phenomena. According to this idea, we could interpret the existence of stable soliton-like light beams as a clue of the existence of an "integrability" property, whatever it means.

Our interest in the high frequency limit is due to the observation that the equations for the phase are treatable for a general class of nonlinear responses and, as discussed in the sections 6.2 and 6.3, the analytical form of intensity law does not affect the possibility to have waveguides. Moreover, for $\omega \to \infty$, we expect a "naturally" small nonlocal response. As a consequence, in high frequency limit, one can simplify the analysis, separating nonlinearity and nonlocality and performing exact calculations. In particular, knowledge of exact solutions of the phase equations could have remarkable consequences. This solutions could suggest, for example, a new nontrivial ansatz to approach the study of NNLS equation using a variational method.

Finally, the study of the phase is useful since its singularities gives important information about vortex-type behaviours [45]. In particular the Beltrami equation together with the theory of generalized analytic functions seems to be a powerful tool to approach this problem opening the way to the analysis of a class of singular wavefronts never considered before.

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