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# Regularity of positive solutions of $p$-Laplace equations on manifolds and its applications

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**Abstract**[1]. We consider the Dirichlet problem for positive solutions of the equation $-\Delta_p(u) = f(u)$ in a compact smooth manifold $\mathcal{M}$ with boundary, where $f$ is locally Lipschitz continuous and $p > 2$, and prove some regularity results for weak $C^1(\overline{\mathcal{M}})$ solutions. In particular, when $f(s) > 0$ for $s > 0$, we prove summability properties of $1/|\nabla u|$, and Sobolev and Poincaré type inequalities in weighted Sobolev spaces with weight $|\nabla u|^{p-2}$ under some assumptions on the manifold (namely, for homogenous manifolds). The point of view of considering $|\nabla u|^{p-2}$ as a weight is very useful when studying qualitative properties of a fixed solution. In particular, using these new regularity results we can prove a weak comparison principle for solutions. Our results generalize some of those obtained very recently in [9] for the Euclidean case.

## 1. Introduction and statement of the results

Let $(\mathcal{M}, g)$ be a $C^\infty$ $n$-dimensional compact Riemann manifold with boundary. We study the following equation

$$(1.1) \qquad \begin{cases} -\Delta_p u = f(u) & \text{in } \mathcal{M} \\ u \geq 0 & \text{in } \mathcal{M} \\ u = 0 & \text{on } \partial\mathcal{M} , \end{cases}$$

where $\Delta_p$ is the $p$-Laplace Beltrami operator. In local coordinates $(\Omega, x)$ it can be written as

$$(1.2) \qquad -\Delta_p u := \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_i} (\sqrt{\det g(x)} g^{ij}(x) |\nabla u|^{p-2} \frac{\partial u}{\partial x_j}) ,$$

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where $g(x) = (g_{ij}(x))$ is the metric matrix, $g^{-1}(x) = (g^{ij}(x))$ is the inverse matrix of $g(x)$ and

$$|\nabla u|^2 := \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} .$$

Consider a function $f : [0, +\infty) \to \mathbb{R}$ satisfying

**(A1)** $f$ is continuous on $[0, +\infty)$;
**(A2)** $f$ is locally Lipschitz continuous in $(0, +\infty)$;
**(A3)** $f(0) = 0$ and $f(t) > 0, \ \forall t > 0$.

We further assume that

**(B1)** $u \in C^1(\overline{\mathcal{M}})$ is a solution of (1.1).

It is well known that, since the $p$-Laplace operator is singular or degenerate elliptic, solutions of (1.1) belong generally to the class $C^{1,\tau}$ with $\tau < 1$ (see [17, 10]), and solve (1.1) only in the weak sense. Moreover there are no general comparison theorems for the solutions as in the case where $p = 2$.

In a recent paper [9], Sciunzi and the second author consider this equation in Euclidean spaces. They prove some regularity properties for positive solutions of (1.1), such as summability properties of $1/|\nabla u|$, and Sobolev and Poincaré type inequalities in weighted Sobolev spaces with weight $|\nabla u|^{p-2}$.

Using these regularity results, a weak comparison theorem for solutions of differential inequalities involving the $p$-Laplace operator is proved.

Inspired by [9], in this work we generalize such results to a certain class of manifolds - namely, we consider manifolds for which there exist $n$ linearly independent killing vector fields at each point - therefore, all homogenous manifolds are included. In particular, our results hold for the manifolds with constant curvature such as $\mathbb{R}^n$ (euclidean spaces), $\mathbb{S}^n$(spheres), $\mathbb{H}^n$ (hyperbolic spaces) and product manifolds of them.

In a forthcoming paper, the authors will combine all these results, together with the Alexandrov-Serrin moving plane method ([1] and [16]), to prove some directional monotonicity (and symmetry) results (as in [2],[3], [4], [6],[7],[8], [9], [11] and [16]) for solutions of (1.1) with $p > 2$ and $f$ positive.

In section 2 we prove in particular that $|\nabla u|^{p-2} X u \in W^{1,2}_{\mathrm{loc}}(\mathcal{M})$ for any regular vector field on $\mathcal{M}$ and then we exploit this result to study the linearized operator $L_u$(see section 2 for the precise statement) associated to problem (1.1). In particular, we prove that if $\varphi \in W^{1,2}(\mathcal{M})$ has compact support then

$$L_u(Xu, \varphi) \equiv$$

$$\int_{\mathcal{M}} (|\nabla u|^{p-2} \langle \nabla X u, \nabla \psi \rangle + (p-2)|\nabla u|^{p-4} \langle \nabla X u, \nabla u \rangle \langle \nabla u, \nabla \psi \rangle) \,\mathrm{dvol} -$$

$$- \int_{\mathcal{M}} f'(u) X u \psi \,\mathrm{dvol}$$

is well defined and the following equation holds

$$L_u(Xu, \varphi) = 0 \quad \forall \, \varphi \in W^{1,2}(\mathcal{M}) \,, \ \ \mathrm{supp}\ (\varphi) \subset \mathcal{M} \,, \tag{1.3}$$

where $X$ is a killing vector field on $\mathcal{M}$, dvol is the volume element, i.e. dvol $:= \sqrt{\det g(x)} dx_1 \wedge \ldots \wedge dx_n$, $\langle \cdot, \cdot \rangle$ is the inner product associated to the metric $g$ and

$\nabla v$ is the gradient of $v$. In local coordinates we have

$$\langle \nabla v, \nabla \omega_\varepsilon \rangle = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial \omega_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} \ .$$

The proofs of our regularity results will be based both on equation (1.1) and equation (1.3). Let us state some of these results in the following

**Theorem 1.** *Let* $u \in C^1(\mathcal{M})$ *be a weak solution of* (1.1) *and* $X$ *be a Killing vector field. Assume* (*A1*) *and* (*A2*) *are true and that* $p \geq 2$. *Then, for any* $E \subset\subset \mathcal{M}$ *and for any* $0 < \beta < 1$, *there exists a constant* $C$ *such that*

$$\sup_{x \in \mathcal{M}} \int_{E \setminus \{Xu=0\}} \frac{|\nabla u|^{p-2}}{(d(x,y))^\gamma |Xu|^\beta} |\nabla X u|^2 \, \mathrm{dvol}_y < C \ , \tag{1.4}$$

*where*

$\gamma < n-2$ *if* $n \geq 3$ *and* $\gamma = 0$ *if* $n = 2$;
$C = C(\beta, \gamma, E)$ *depends only on* $\beta, \gamma$ *and* $E$;
$d(x,y)$ *denotes the Riemann distance between* $x$ *and* $y$.

*Moreover, assume* $\forall x \in \mathcal{M}$, *there exist* $n$ *linearly independent Killing vector fields at* $x$. *Then, for any* $E \subset\subset \mathcal{M}$ *and any* $0 < \beta < 1$,

$$\sup_{x \in \mathcal{M}} \int_{E \setminus \mathcal{Z}} \frac{|\nabla u|^{p-2-\beta}}{(d(x,y))^\gamma} |D^2 u| \, \mathrm{dvol}_y < C \ , \tag{1.5}$$

*where* $\mathcal{Z} := \{x \in \mathcal{M} \ : \ |\nabla u| = 0\}$ *is the set of critical points of the solution. Furthermore, assuming also* (*A3*), *we have that, for every* $0 < m < 1$,

$$\sup_{x \in \mathcal{M}} \int_{\mathcal{M}} \frac{1}{|\nabla u|^{(p-1)m} (d(x,y))^\gamma} \, \mathrm{dvol}_y < C \ , \tag{1.6}$$

The lack of regularity of the solutions of (1.1) is one of the greatest difficulties in the applications.

Here, as in [9, 14, 18], if $\rho \in L^1(\mathcal{M})$, we define the space $W^{1,q}_\rho(\mathcal{M})$ as the completion of $C^1(\overline{\mathcal{M}})$ (or $C^\infty(\overline{\mathcal{M}})$) under the norm

$$\|v\|_{W^{1,q}_\rho} = \|v\|_{L^q(\mathcal{M})} + \|\nabla v\|_{L^q_\rho(\mathcal{M})} \tag{1.7}$$

and $\|\nabla v\|^q_{L^q_\rho(\mathcal{M})} = \int_{\mathcal{M}} |\nabla v|^q \rho \, \mathrm{dvol}$. In this way, we will prove that if $f(s) > 0$ for $s > 0$ and $u$ is a solution of (1.1) with $p \geq 2$, considering the weight $\rho = |\nabla u|^{p-2}$, for every $q \geq 2$ and $v \in W^{1,q}_{0,\rho}(\Omega)$ a weighted Poincaré inequality holds , i.e.

$$\|v\|_{L^q(\mathcal{M})} \leq C(\,\mathrm{vol}(\mathcal{M})) \|\nabla v\|_{L^q_\rho(\mathcal{M})} \tag{1.8}$$

where $C(\,\mathrm{vol}(\mathcal{M})) \to 0$ if $\mathrm{vol}(\mathcal{M}) \to 0$ and $\mathrm{vol}(\cdot)$ is the volume.
In [14, 18] equation (1.8) is proved by assuming that

$$\rho \in L^1(\mathcal{M}) \ , \quad \frac{1}{\rho} \in L^t(\mathcal{M}) \tag{1.9}$$

with $t > n/q$ and $q > 1+1/t$. In order to avoid this restriction in the applications, in section 3, following the ideas in [9], we will prove that a weighted Poincaré inequality in the space $W^{1,q}_{0,\rho}(\mathcal{M})$ can be obtained using classical potential estimates (similar to those in [14, 18]) and assuming that we have the following estimate for the weight $\rho$

$$\sup_{x\in\mathcal{M}}\int_{\mathcal{M}}\frac{1}{\rho(y)d(x,y)^{\gamma_0}}\ \mathrm{dvol}_y\le C$$

for some $\gamma_0\in(n-2,n)$. So, using the regularity results in Theorem 1, together with these abstract results, we can prove the following Poincaré type inequality for solutions of (1.1).

**Theorem 2.** *Assume that for each $x\in\mathcal{M}$, there exist $n$ linearly independent Killing vector fields at $x$, and (A1) to (A3), (B1) and $p\ge 2$. Then, if we consider $\rho=|\nabla u|^{p-2}$ we get, for every $q\ge 2$*

$$\|v\|_{L^q(\mathcal{M})}\le C(\mathrm{vol}(\mathcal{M}))\|\nabla v\|_{L^q_\rho(\mathcal{M})}\quad \textit{for every } v\in W^{1,q}_{0,\rho}(\mathcal{M}) \tag{1.10}$$

*where $C(\mathrm{vol}(\mathcal{M}))\to 0$ if $\mathrm{vol}(\mathcal{M})\to 0$.*
*In particular (1.10) holds for every $v\in W^{1,2}_{0,\rho}(\mathcal{M})$.*

We then use the weighted Poincaré type inequality obtained in Theorem 2 to prove the following

**Theorem 3** (Weak Comparison Principle in small domains)**.** *Suppose that either*
*$1<p<2$ and $u,v\in W^{1,\infty}(\mathcal{M})$, or*
*$p\ge 2$ and $u,v\in W^{1,p}(\mathcal{M})\cap L^\infty(\mathcal{M})$*
*and that either $\rho\equiv|\nabla u|^{p-2}$ or $\rho\equiv|\nabla v|^{p-2}$ satisfy the following condition*

$$\sup_{x\in\mathcal{M}}\int_{\mathcal{M}}\frac{1}{\rho(y)d(x,y)^{\gamma_0}}\ \mathrm{dvol}_y\le C$$

*for some $\gamma_0\in(n-2,n)$.*
*Suppose that $u$, $v$ weakly solve*

$$-\triangle_p u+g(x,u)-\Lambda u\le-\triangle_p v+g(x,v)-\Lambda v \ \textit{in } \mathcal{M} \tag{1.11}$$

*where $\Lambda\ge 0$ and $g\in C(\overline{\mathcal{M}}\times\mathbb{R})$ is such that for every $x\in\mathcal{M}$, $g(x,s)$ is nondecreasing for $|s|\le\max\{\|u\|_{L^\infty},\|v\|_{L^\infty}\}$.*
*Let $\mathcal{M}'\subseteq\mathcal{M}$ be open and suppose $u\le v$ on $\partial\mathcal{M}'$, then there exists $\delta>0$ such that, if $\mathrm{vol}(\mathcal{M}')\le\delta$, then $u\le v$ in $\mathcal{M}'$. If $\Lambda=0$ the thesis is true for every $\mathcal{M}'\subseteq\mathcal{M}$.*
*In particular the result holds if either $u$ or $v$ is a $C^1(\overline{\mathcal{M}})$ weak solution of (1.1) and (A1) to (A3) are satisfied, and if $\forall x\in\mathcal{M}$, there exist $n$ linearly independent Killing vector fields at $x$.*

The point of view of considering $\rho=|\nabla u|^{p-2}$ as a weight and working in the weighted Sobolev space $W^{1,2}_{0,\rho}(\Omega)$, which is a Hilbert space, is particularly useful when studying qualitative properties of a solution of (1.1).

The paper is organized as follows. In Section 2 we prove Theorem 1 and some related regularity results. In Section 3 we state sufficient conditions to obtain general weighted Sobolev and Poincaré inequalities and then we exploit them together with Theorem 1 to prove Theorem 2. Finally we take advantage of the weighted Poincaré inequality obtained and prove Theorem 3.

## 2. Regularity results

In this section we prove all the statements of Theorem 1 and some other related results.
Let us first recall a particular version of the Strong Maximum Principle and of Hopf's Lemma.

**Lemma 2.1.** *(Hopf) Let $B(a,r) \subset\subset \mathcal{M}$ with radius $r$ smaller than the injectivity radius. Let $\omega \in C^1(B(a,r)) \cap C^0(\overline{B(a,r)})$ be a non negative function such that*

$$\begin{cases} -\Delta_p \omega \geq 0 & \text{in } B(a,r) \\ \omega \geq 0 & \text{in } B(a,r) \ . \end{cases} \tag{2.1}$$

*Assume $\omega(x_0) = 0$ for some $x_0 \in \partial B(a,r)$. Then,*

$$\frac{\partial \omega}{\partial \nu}(x_0) < 0 \ ,$$

*where $\nu$ is the outward unit normal vector on $\partial B(a,r)$.*

*Proof.* We use expression (1.2). The lemma then follows directly from the work of Pucci, Serrin and Zhou [15] (see also [19]). In fact, the barrier function they construct is also valid in our case.

□

This result implies that, under assumptions (A1)-(A3) and (B1), $u$ is positive in $\mathcal{M}$ and $|\nabla u| > 0$ on $\partial\mathcal{M}$. We denote

$$\mathcal{Z} := \{x \in \mathcal{M} \ : \ |\nabla u| = 0\} \ , \tag{2.2}$$

the set of critical points. Hopf's lemma implies that $\mathcal{Z} \subset\subset \mathcal{M}$. Thus, classical regularity theory implies that $u$ is $C^2$ in $\mathcal{M} \setminus \mathcal{Z}$.

**Lemma 2.2.** *Let $X$ be a $C^\infty$ vector field such that $X(x) \neq 0$, $\forall x \in \mathcal{M}$ and assume $p \geq 2$. We have*

$$|\nabla u|^{p-2} X u \in H^1_{\mathrm{loc}}(\mathcal{M}) \ .$$

*Proof.* For simplicity we suppose $X = \dfrac{\partial}{\partial x_1}$ and we solve the auxiliary problem

$$\begin{cases} \mathcal{L}_\varepsilon \omega_\varepsilon = f(u) & \text{in } B(a,r) \\ \omega_\varepsilon = u & \text{on } \partial B(a,r) \ . \end{cases} \tag{2.3}$$

for any small $\varepsilon > 0$, where

$$\mathcal{L}_\varepsilon \omega_\varepsilon := -\sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_i}\left(\sqrt{\det g(x)} g^{ij}(x) |\varepsilon^2 + |\nabla \omega_\varepsilon|^2|^{\frac{p-2}{2}} \frac{\partial \omega_\varepsilon}{\partial x_j}\right)$$

This equation corresponds to the energy functional (defined for all $\omega \in W^{1,p}_u(B(a,r))$)

$$E_\varepsilon(\omega) := \frac{1}{p}\int_{B(a,r)} |\varepsilon^2 + |\nabla \omega|^2|^{\frac{p-2}{2}} \,\mathrm{dvol} - \int_{B(a,r)} f(u)\omega \,\mathrm{dvol} \ ,$$

where dvol is the volume element, i.e dvol $:= \sqrt{\det g(x)} dx_1 \wedge \ldots \wedge dx_n$ and

$$W^{1,p}_u(B(a,r)) := \{\omega \in W^{1,p}(B(a,r)) \ : \ \omega = u \text{ on } \partial B(a,r)\} \ . \tag{2.4}$$

Recall that $f(u) \in C^1(B(a,r))$. Therefore, standard elliptic theory yields that (2.3) has a unique solution $\omega_\varepsilon$. Moreover, $\omega_\varepsilon$ is $C^2$. We remark that

$$E_\varepsilon(\omega_\varepsilon) := \min_{\omega \in W_u^{1,p}} E_\varepsilon(\omega) \tag{2.5}$$

depends continuously on $\varepsilon$. This implies that $(\omega_\varepsilon)$ is bounded in $W^{1,p}(B(a,r))$.

Differentiating equation (2.3), we obtain, for all $\ell = 1, \dots, n$,

$$\frac{\partial}{\partial x_\ell}(\sqrt{\det g(x)}\mathcal{L}_\varepsilon \omega_\varepsilon) = \frac{\partial}{\partial x_\ell}(\sqrt{\det g(x)} f(u)) \text{ in } B(a,r)\ . \tag{2.6}$$

The corresponding variational equality is

$$\int_{B(a,r)} \sum_{i,j=1}^n \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_\ell}(\sqrt{\det g(x)}\ g^{ij}(x)) \cdot \tag{2.7}$$

$$\cdot(\varepsilon^2 + |\nabla\omega_\varepsilon|^2)^{\frac{p-2}{2}} \frac{\partial \omega_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} \,\mathrm{dvol} + \int_{B(a,r)} \langle \nabla v, \nabla \frac{\partial \omega_\varepsilon}{\partial x_\ell}\rangle (\varepsilon^2 + |\nabla\omega_\varepsilon|^2)^{\frac{p-2}{2}} \,\mathrm{dvol} +$$

$$+(p-2)\int_{B(a,r)} \langle \nabla v, \nabla \omega_\varepsilon\rangle (\varepsilon^2 + |\nabla\omega_\varepsilon|^2)^{\frac{p-4}{2}} \langle \nabla\omega_\varepsilon, D_{\frac{\partial}{\partial x_\ell}} \nabla\omega_\varepsilon\rangle \,\mathrm{dvol} =$$

$$= \int_{B(a,r)} \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_\ell}(\sqrt{\det g(x)} f(u)) v \,\mathrm{dvol}\ ,$$

where $\nabla v$ is the gradient of $v$, $\langle \cdot, \cdot \rangle$ is the inner product associated to the metric $g$, $D$ is the connection associated to $g$, and $v \in C_0^\infty(\mathcal{M})$. In local coordinates we have

$$\langle \nabla v, \nabla \omega_\varepsilon \rangle = \sum_{i,j=1}^n g^{ij}(x) \frac{\partial \omega_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j}\ ,$$

$$\langle \nabla \omega_\varepsilon, D_{\frac{\partial}{\partial x_\ell}} \nabla\omega_\varepsilon \rangle = \sum_{i,j=1}^n \left(\frac{1}{2}\frac{\partial g^{ij}}{\partial x_\ell} \frac{\partial \omega_\varepsilon}{\partial x_i} \frac{\partial \omega_\varepsilon}{\partial x_j} + g^{ij}(x) \frac{\partial \omega_\varepsilon}{\partial x_i} \frac{\partial^2 \omega_\varepsilon}{\partial x_\ell \partial x_j}\right)\ .$$

Let $\varphi \in C_0^\infty(B(a,r))$ be a cut-off function such that $\varphi \geq 0$ and $\varphi|_{B(a,r/2)} \equiv 1$. Taking $\varphi \dfrac{\partial \omega_\varepsilon}{\partial x_\ell}$ as test function in (2.7), we obtain

$$\int_{B(a,r)} \varphi |\nabla(\frac{\partial \omega_\varepsilon}{\partial x_\ell})|^2 (\varepsilon^2 + |\nabla\omega_\varepsilon|^2)^{\frac{p-2}{2}} \,\mathrm{dvol} \leq C \int_{B(a,r)} (1 + |\nabla\omega_\varepsilon|^p + |\frac{\partial\omega_\varepsilon}{\partial x_\ell}|) \,\mathrm{dvol}\ .$$

Thus, $(\varepsilon^2 + |\nabla\omega_\varepsilon|^2)^{\frac{p-2}{4}} \dfrac{\partial \omega_\varepsilon}{\partial x_\ell}$ is bounded in $H^1(B(a,r/2))$. We recall $\omega_\varepsilon$ is bounded in $W^{1,p}(B(a,r))$. Modulo a subsequence, we can suppose that

$$\begin{array}{lll} \omega_{\varepsilon_n} \rightharpoonup v & \text{weakly in } W^{1,p}(B(a,r)) \\ \omega_{\varepsilon_n} \to v & \text{strongly in } L^p(B(a,r))\ . \end{array} \tag{2.8}$$

By lower semi-continuity, it follows that

$$\int_{B(a,r)} |\nabla v|^p \,\mathrm{dvol} \leq \liminf_{\varepsilon_n \to 0} \int_{B(a,r)} |\nabla \omega_{\varepsilon_n}|^p \,\mathrm{dvol}\ ,$$

and

$$\lim_{\varepsilon_n \to 0} \int_{B(a,r)} f(u)\omega_{\varepsilon_n} \,\mathrm{dvol} = \int_{B(a,r)} f(u) v \,\mathrm{dvol}\ .$$

Hence, $v$ is a minimizer of the energy functional $E_0$ and thus $v \equiv u$ by uniqueness of the minimizer. Moreover,

$$\lim_{\varepsilon_n \to 0} \int_{B(a,r)} |\nabla \omega_{\varepsilon_n}|^p \,\mathrm{dvol} = \int_{B(a,r)} |\nabla u|^p \,\mathrm{dvol} .$$

Consequently, $\omega_\varepsilon$ converges strongly to $u$ in $W^{1,p}(B(a,r))$. This yields that $(\varepsilon^2 + |\nabla \omega_\varepsilon|^2)^{\frac{p-2}{4}} \dfrac{\partial \omega_\varepsilon}{\partial x_\ell} \to |\nabla u|^{\frac{p-2}{2}} \dfrac{\partial u}{\partial x_\ell}$ strongly in $L^2$.

We recall that $(\varepsilon^2 + |\nabla \omega_\varepsilon|^2)^{\frac{p-2}{4}} \dfrac{\partial \omega_\varepsilon}{\partial x_\ell}$ is bounded in $H^1_{\mathrm{loc}}$, and thus $|\nabla u|^{\frac{p-2}{2}} \dfrac{\partial u}{\partial x_\ell} \in H^1_{\mathrm{loc}}$. Moreover, we can write

$$|\nabla u|^{p-2} \frac{\partial u}{\partial x_\ell} = F_\ell(x, |\nabla u|^{\frac{p-2}{2}} \frac{\partial u}{\partial x_1}, \ldots, |\nabla u|^{\frac{p-2}{2}} \frac{\partial u}{\partial x_n}) ,$$

where $F : \mathcal{M} \times \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ map. Therefore, using the boundedness of $|\nabla u|$ it follows that $|\nabla u|^{p-2} \dfrac{\partial u}{\partial x_\ell} \in H^1_{\mathrm{loc}}$.

□

**Remark 2.1.** We remark that, on the one hand, in $\mathcal{M} \setminus \mathcal{Z}$, $u$ is $C^2$ and

$$\frac{\partial}{\partial x_j}(|\nabla u|^{p-2} \frac{\partial u}{\partial x_\ell}) \equiv |\nabla u|^{p-2} \frac{\partial^2 u}{\partial x_j \partial x_\ell} + (p-2)|\nabla u|^{p-4} \langle \nabla u \,, D_{\frac{\partial}{\partial x_j}} \nabla u \rangle \frac{\partial u}{\partial x_\ell} . \tag{2.9}$$

On the other hand, on $\mathcal{Z}$ we have $|\nabla u|^{p-2} \dfrac{\partial u}{\partial x_\ell} \equiv 0$. It follows from Stampacchia's theorem that

$$\frac{\partial}{\partial x_j}(|\nabla u|^{p-2} \frac{\partial u}{\partial x_\ell}) = 0 \text{ a.e. in } \mathcal{Z} . \tag{2.10}$$

Denoting

$$\overline{v} := \begin{cases} v \text{ in } \mathcal{M} \setminus \mathcal{Z}, \\ 0 \text{ in } \mathcal{Z}. \end{cases}$$

we have

$$\frac{\partial}{\partial x_j}(|\nabla u|^{p-2} \frac{\partial u}{\partial x_\ell}) = |\nabla u|^{p-2} \overline{\frac{\partial^2 u}{\partial x_j \partial x_\ell}} + (p-2)\overline{|\nabla u|^{p-4} \langle \nabla u \,, D_{\frac{\partial}{\partial x_j}} \nabla u \rangle} \frac{\partial u}{\partial x_\ell} .$$

Since $|\nabla u|^{p-1} \in H^1_{\mathrm{loc}}$, it follows that

$$(p-1)\overline{|\nabla u|^{p-3} \langle \nabla u \,, D_{\frac{\partial}{\partial x_j}} \nabla u \rangle} \in L^2_{\mathrm{loc}} .$$

Therefore $(p-2)\overline{|\nabla u|^{p-4} \langle \nabla u \,, D_{\frac{\partial}{\partial x_j}} \nabla u \rangle} \dfrac{\partial u}{\partial x_\ell} \in L^2_{\mathrm{loc}}(\mathcal{M})$, which yields

**Corollary 2.1.** $|\nabla u|^{p-2} \overline{\dfrac{\partial^2 u}{\partial x_j \partial x_\ell}} \in L^2_{\mathrm{loc}}(\mathcal{M})$.

Before proceeding, we recall that a Killing field $X$ is a vector field such that $L_X g = 0$, where $L_X$ is the Lie derivative along $X$.

**Lemma 2.3.** *Under the above assumptions, let $X$ be a Killing vector field. Then, for any $\psi \in H^1_0(\mathcal{M})$, we have*

$$\int_{\mathcal{M}} (|\nabla u|^{p-2} \langle \nabla X u, \nabla \psi \rangle + (p-2)|\nabla u|^{p-4} \langle \nabla X u, \nabla u \rangle \langle \nabla u, \nabla \psi \rangle) \,\mathrm{dvol} = \tag{2.11}$$

$$= \int_{\mathcal{M}} f'(u) X u \psi \, \mathrm{dvol} \, .$$

*Proof.* We start by considering the case where $\psi \in C_0^\infty(\mathcal{M})$. Let $\theta_t$ be the local one parameter group associated to $X$. We set

$$\psi_h := \psi \circ \theta_{-h} \, .$$

On the one hand, taking, respectively, $\psi$ and $\psi_h$ as test functions in (1.1), we obtain

$$\int_{\mathcal{M}} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle \, \mathrm{dvol} = \int_{\mathcal{M}} f(u) \psi \, \mathrm{dvol} \, ,$$

$$\int_{\mathcal{M}} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi_h \rangle \, \mathrm{dvol} = \int_{\mathcal{M}} f(u) \psi_h \, \mathrm{dvol} \, .$$

On the other hand, since $X$ is a Killing field, we have

$$\int_{\mathcal{M}} |\nabla (u \circ \theta_h)|^{p-2} \langle \nabla (u \circ \theta_h), \nabla \psi \rangle \theta_h^* (\mathrm{dvol}) =$$

$$= \int_{\mathcal{M}} |(\nabla u) \circ \theta_h|^{p-2} \langle (\nabla u) \circ \theta_h, \nabla \psi_h \circ \theta_h \rangle \theta_h^* (\mathrm{dvol}) \, .$$

Therefore, we obtain

$$\int_{\mathcal{M}} \frac{|\nabla (u \circ \theta_h)|^{p-2} \langle \nabla (u \circ \theta_h), \nabla \psi \rangle \theta_h^* (\mathrm{dvol}) - |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle \, \mathrm{dvol}}{h} = \tag{2.12}$$

$$= \int_{\mathcal{M}} \frac{f(u \circ \theta_h) \psi \theta_h^* (\, \mathrm{dvol}) - f(u) \psi \, \mathrm{dvol}}{h} \, .$$

We have

$$\lim_{h \to 0} \frac{f(u \circ \theta_h) \theta_h^* (\, \mathrm{dvol}) - f(u) \, \mathrm{dvol}}{h} = L_X (f(u) \, \mathrm{dvol}) = f'(u) X u \, \mathrm{dvol}$$

$$\lim_{h \to 0} \frac{\langle \nabla (u \circ \theta_h), \nabla \psi \rangle - \langle \nabla u, \nabla \psi \rangle}{h} = \lim_{h \to 0} \frac{[(\theta_h^*)(du) - du](\nabla \psi)}{h} =$$

$$= (L_X (du))(\nabla \psi) = D^2 u \langle X, \nabla \psi \rangle + du (D_{\nabla \psi} X) =$$

$$= D^2 u \langle \nabla \psi, X \rangle + du (D_{\nabla \psi} X) = (D_{\nabla \psi} du)(X) + du (D_{\nabla \psi} X) =$$

$$= D_{\nabla \psi} (du(X)) = \nabla \psi (X u) = \langle \nabla (X u), \nabla \psi \rangle \, .$$

Thus, we obtain (2.11) by letting $h \to 0$ in (2.12). The general case then follows by standard density arguments.

□

**Theorem 4.** *Let $u \in C^1(\mathcal{M})$ be a weak solution of* (1.1) *and $X$ be a Killing vector field. Assume* (*A1*) *and* (*A2*) *are true and that $p \geq 2$. Then, for any $E \subset\subset \mathcal{M}$ and for any $0 < \beta < 1$, there exists a constant $C$ such that*

$$\sup_{x \in \mathcal{M}} \int_{E \setminus \{X u = 0\}} \frac{|\nabla u|^{p-2}}{(d(x,y))^\gamma |X u|^\beta} |\nabla X u|^2 \, \mathrm{dvol}_y < C \, , \tag{2.13}$$

*where*

*$\gamma < n - 2$ if $n \geq 3$ and $\gamma = 0$ if $n = 2$;*
*$C = C(\beta, \gamma, E)$ depends only on $\beta, \gamma$ and $E$;*
*$d(x,y)$ denotes the Riemann distance between $x$ and $y$.*

*Proof.* Without loss of generality, it is sufficient to prove that for any measurale set $E \subset\subset \mathcal{M}$ we have

$$\sup_{x\in E} \int_{E\setminus\{Xu=0\}} \frac{|\nabla u|^{p-2}|\nabla Xu|^2}{(d(x,y))^\gamma |Xu|^\beta} \,\mathrm{dvol}_y = K(\beta,\gamma,E) < +\infty \;. \tag{2.14}$$

To see this is so, assume (2.14) and let $0 < \delta < 1/2\ d(E, \partial\mathcal{M})$. Defining $E_\delta := \{x \in \mathcal{M} \ : \ d(x, E) \le \delta\}$ and considering the cases $x \in E_\delta$ and $x \in \mathcal{M} \setminus E_\delta$, respectively, it follows that

$$\sup_{x\in \mathcal{M}} \int_{E\setminus\{Xu=0\}} \frac{|\nabla u|^{p-2}|\nabla Xu|^2}{(d(x,y))^\gamma |Xu|^\beta} \,\mathrm{dvol}_y \le K(\beta,\gamma,E_\delta) + \frac{1}{\delta^\gamma} K(\beta,0,E) \;,$$

and thus we obtain (2.13).

To prove (2.14), we start by considering a cut-off function $\varphi \in C_0^\infty(\mathcal{M})$ such that $\varphi|_{E_\delta} \equiv 1$, and defining, for any $\varepsilon > 0$,

$$G_\varepsilon(s) := \begin{cases} 0 & \text{if } |s| \le \varepsilon, \\ \dfrac{2s-2\varepsilon}{|s|^\beta} & \text{if } \varepsilon \le |s| \le 2\varepsilon, \\ \dfrac{s}{|s|^\beta} & \text{if } |s| > 2\varepsilon \;. \end{cases}$$

We discuss the two cases:
**Case 1:** assume $x \in E \cap \mathcal{Z}$.

Let $\psi_{\varepsilon,x}(y) := \dfrac{G_\varepsilon(Xu(y))\varphi(y)}{(d(x,y))^\gamma}$. Recall that $Xu \in C^1(\mathcal{M} \setminus \mathcal{Z})$. We have $\psi_{\varepsilon,x} \in H_0^1(\mathcal{M})$ and , using it as test function in (2.11) we obtain

$$\begin{aligned} \int_{\mathcal{M}} \frac{|\nabla u|^{p-2}\varphi\, G'_\varepsilon(Xu)}{(d(x,y))^\gamma} [|\nabla(Xu)|^2 + (p-2)|\nabla u|^{-2}\langle \nabla Xu, \nabla u\rangle\langle \nabla u, \nabla Xu\rangle] \,\mathrm{dvol} + \\ + \int_{\mathcal{M}} |\nabla u|^{p-2}\varphi\, G_\varepsilon(Xu)\langle \nabla Xu, \nabla(\frac{1}{(d(x,y))^\gamma})\rangle \,\mathrm{dvol} + \\ +(p-2)\int_{\mathcal{M}} |\nabla u|^{p-2}\varphi\, G_\varepsilon(Xu)|\nabla u|^{-2}\langle \nabla Xu, \nabla u\rangle\langle \nabla u, \nabla(\frac{1}{(d(x,y))^\gamma})\rangle \,\mathrm{dvol} + \\ + \int_{\mathcal{M}\setminus E_\delta} \frac{|\nabla u|^{p-2} G_\varepsilon(Xu)}{(d(x,y))^\gamma} [\langle \nabla Xu, \nabla\varphi\rangle + (p-2)|\nabla u|^{-2}\langle \nabla Xu, \nabla u\rangle\langle \nabla u, \nabla\varphi\rangle] \mathrm{dvol} = \\ = \int_{\mathcal{M}} \frac{f'(u)(Xu)\varphi G_\varepsilon(Xu)}{(d(x,y))^\gamma} \,\mathrm{dvol} \end{aligned}$$

Recalling that $f'(u), Xu \in L^\infty$, $|\nabla d(x,y)| = 1$ and $G'_\varepsilon \ge 0$, we obtain

$$\begin{aligned} \int_{\mathcal{M}} \frac{|\nabla u|^{p-2}|\nabla(Xu)|^2\varphi G'_\varepsilon(Xu)}{(d(x,y))^\gamma} \,\mathrm{dvol} \le \\ \le C\gamma \int_{\mathcal{M}} \frac{|\nabla u|^{p-2}|\nabla(Xu)|\varphi|G_\varepsilon(Xu)|}{(d(x,y))^{\gamma+1}} \,\mathrm{dvol} + \\ +C \int_{\mathcal{M}\setminus E_\delta} \frac{|\nabla u|^{p-2}|\nabla(Xu)||\nabla\varphi||G_\varepsilon(Xu)|}{(d(x,y))^\gamma} \,\mathrm{dvol} + C\int_{\mathcal{M}} \frac{\varphi}{(d(x,y))^\gamma} \,\mathrm{dvol} \;. \end{aligned}$$

We note that, for $x \in E$, we have that

$$\sup_{y \in \mathcal{M} \setminus E_\delta} \frac{1}{(d(x,y))^\gamma} \leq \frac{1}{\delta^\gamma} ,$$

and that $|\nabla u|^{p-2} |\nabla (Xu)| \in L^2_{\text{loc}}(\mathcal{M})$. Therefore, we have

$$\int_{\mathcal{M} \setminus E_\delta} \frac{|\nabla u|^{p-2} |\nabla (Xu)| |\nabla \varphi| |G_\varepsilon(Xu)|}{(d(x,y))^\gamma} \, \text{dvol} \leq C_1 ,$$

$$\int_{\mathcal{M}} \frac{\varphi}{(d(x,y))^\gamma} \, \text{dvol} \leq C_1 ,$$

where $C_1$ is independent of $x$. Consequently,

$$\int_{\mathcal{M}} \frac{|\nabla u|^{p-2} |\nabla (Xu)|^2 \varphi G'_\varepsilon(Xu)}{(d(x,y))^\gamma} \, \text{dvol} \leq C \int_{\mathcal{M}} \frac{|\nabla u|^{p-2} |\nabla (Xu)| \varphi |G_\varepsilon(Xu)|}{(d(x,y))^{\gamma+1}} \, \text{dvol} + C$$

$$\leq C + \frac{4C}{\sigma} \int_{\mathcal{M}} \frac{|\nabla u|^{p-2} \varphi |G_\varepsilon(Xu)|^2}{(d(x,y))^{\gamma+2} G'_\varepsilon(Xu)} \, \text{dvol} + \sigma \int_{\mathcal{M}} \frac{|\nabla u|^{p-2} |\nabla (Xu)|^2 \varphi G'_\varepsilon(Xu)}{(d(x,y))^\gamma} \, \text{dvol}$$

for some small $\sigma > 0$ (here we use the inequality $ab \leq \sigma a^2 + (4/\sigma) b^2$). Finally, since

$$\frac{G^2(s)}{G'(s)} \leq \frac{1}{1-\beta} s G(s) ,$$

we deduce

$$\int_{\mathcal{M}} \frac{(1-\sigma) |\nabla u|^{p-2} |\nabla (Xu)|^2 \varphi G'_\varepsilon(Xu)}{(d(x,y))^\gamma} \, \text{dvol} \leq C + \int_{\mathcal{M}} \frac{\varphi}{(d(x,y))^{\gamma+2}} \, \text{dvol} \leq C .$$

Letting $\varepsilon \to 0$ we obtain

$$\int_{\mathcal{M} \setminus \{Xu=0\}} \frac{|\nabla u|^{p-2} |\nabla (Xu)|^2 \varphi (1-\beta)(1-\sigma)}{(d(x,y))^\gamma |Xu|^\beta} \, \text{dvol}_y \leq C .$$

**Case 2:** assume $x \in E \setminus \mathcal{Z}$.

We consider a positive cut-off function $\varphi_{\varepsilon,x} \in C_0^\infty(\mathcal{M})$ such that

$\varphi_{\varepsilon,x} \equiv 0$ in $B(x,\varepsilon)$,
$\varphi_{\varepsilon,x} \equiv 1$ in $E_\delta \setminus B(x,2\varepsilon)$,
$|\nabla \varphi_{\varepsilon,x}| \leq C/\varepsilon$ in $B(x,2\varepsilon) \setminus B(x,\varepsilon)$ and
$|\nabla \varphi_{\varepsilon,x}| \leq C$ in $\mathcal{M} \setminus B(x,2\varepsilon)$.

Taking $\psi_{\varepsilon,x}(y) := \dfrac{G_\varepsilon(Xu(y)) \varphi_{\varepsilon,x}(y)}{(d(x,y))^\gamma}$ as text function in (2.11), by estimates analog to those done in case 1, we deduce

$$\int_{\mathcal{M}} \frac{(1-\sigma) |\nabla u|^{p-2} |\nabla (Xu)|^2 \varphi_{\varepsilon,x} G'_\varepsilon(Xu)}{(d(x,y))^\gamma} \, \text{dvol} \leq$$

$$\leq C + C \int_{B(x,2\varepsilon) \setminus B(x,\varepsilon)} \frac{|\nabla u|^{p-2} |\nabla (Xu)| |\nabla \varphi_{\varepsilon,x}| |G_\varepsilon(Xu)|}{(d(x,y))^\gamma} \, \text{dvol} .$$

We recall that $x \notin \mathcal{Z}$, and thus, $\forall y \in B(x,2\varepsilon) \setminus B(x,\varepsilon)$,

$$\frac{|\nabla u|^{p-2}|\nabla(Xu)||\nabla\varphi_{\varepsilon,x}||G_\varepsilon(Xu)|}{(d(x,y))^\gamma} \le \frac{C(x)}{\varepsilon^{\gamma+1}}.$$

Therefore,

$$\int_{\mathcal{M}} \frac{(1-\sigma)|\nabla u|^{p-2}|\nabla(Xu)|^2\varphi_{\varepsilon,x}G'_\varepsilon(Xu)}{(d(x,y))^\gamma}\,\mathrm{dvol} \le C + C(x)\varepsilon^{n-\gamma-1}\ .$$

Letting $\varepsilon \to 0$ we obtain

$$\int_{E\setminus\{Xu=0\}} (1-\beta)(1-\sigma)\frac{|\nabla u|^{p-2}|\nabla(Xu)|^2}{(d(x,y))^\gamma|Xu|^\beta}\,\mathrm{dvol}_y \le C$$

where $C$ is independent of $x$. This concludes the proof of Theorem 4.

□

From now on, we add a geometrical assumption on $\mathcal{M}$:

**(B2)** $\forall x \in \mathcal{M}$, there exist $n$ linearly independent Killing vector fields at $x$.

**Corollary 2.2.** *Under the same assumptions as in Theorem 4 plus assumption (B2), for any $E \subset\subset \mathcal{M}$ and any $0<\beta<1$,*

$$\sup_{x\in\mathcal{M}} \int_{E\setminus\mathcal{Z}} \frac{|\nabla u|^{p-2-\beta}}{(d(x,y))^\gamma}|D^2u|\,\mathrm{dvol}_y < C\ , \tag{2.15}$$

*where $C = C(\beta,\gamma,E)$ depends only on $\beta,\gamma$ and $E$;*

*Proof.* This result is a direct consequence of Theorem 4 in this setting.

□

**Theorem 5.** *Assume (A1) to (A3), (B1), (B2) and $p \ge 2$. Then, for any given $0<m<1$ and $0 \le \gamma < n-2$ (if $n \ge 3$, for $n = 2$ we should take $\gamma = 0$), we have*

$$\sup_{x\in\mathcal{M}} \int_{\mathcal{M}} \frac{\mathrm{dvol}_y}{(|\nabla u|^{p-1})^m(d(x,y))^\gamma} \le C\ . \tag{2.16}$$

*Moreover,* $\mathrm{vol}(\mathcal{Z}) = 0$.

*Proof.* Using Hopf's Lemma (Lemma 2.1), since $f \ge 0$, we see that $\exists E \subset\subset \mathcal{M}$ such that $\mathcal{Z} \subset\subset E \subset\subset \mathcal{M}$ . Moreover, $u(x) > 0,\ \forall x \in \mathcal{M}$. Therefore,

$$\int_{\mathcal{M}\setminus E} \frac{\mathrm{dvol}_y}{(|\nabla u|^{p-1})^m(d(x,y))^\gamma} \le \frac{1}{\min\limits_{\mathcal{M}\setminus E}|\nabla u|^{(p-1)m}} \int_{\mathcal{M}\setminus E} \frac{\mathrm{dvol}_y}{(d(x,y))^\gamma} \le C\ .$$

Thus, it suffices to prove

$$\sup_{x\in E} \int_{\mathcal{M}} \frac{\mathrm{dvol}_y}{(|\nabla u|^{p-1})^m(d(x,y))^\gamma} \le C\ . \tag{2.17}$$

As in the proof of the previous Theorem, we need to consider only the points $x \in E$. Let $\varphi_{\varepsilon,x}$ be the cut-off function in Theorem 4 and define

$$\psi_{\varepsilon,x} := \frac{1}{(|\nabla u|^{p-1}+\varepsilon)^m}\ \frac{\varphi_{\varepsilon,x}(y)}{(d(x,y))^\gamma}\ .$$

We have that $\psi_{\varepsilon,x} \in H_0^1(\mathcal{M})$ since $|\nabla u|^{p-1} \in H^1_{\text{loc}}(\mathcal{M})$ and $|\nabla u| \in L^\infty(\mathcal{M})$. $E \subset\subset \mathcal{M}$ and $f(s) > 0\ \forall s > 0$, hence, $\sup_{y\in E} f(u(y)) > \dfrac{1}{C_1}$, for some $C_1 > 0$. Thus,

$$
\begin{aligned}
&\int_E \psi_{\varepsilon,x}\,\text{dvol} \le C_1 \int_E \psi_{\varepsilon,x} f(u)\,\text{dvol} \le C_1 \int_{\mathcal{M}} \psi_{\varepsilon,x} f(u)\,\text{dvol} = \\
&= C_1 \int_{\mathcal{M}} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi_{\varepsilon,x}\rangle\,\text{dvol} \le \\
&\le C \int_{\mathcal{M}\setminus E_\delta} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^m} \frac{|\nabla \varphi_{\varepsilon,x}|}{(d(x,y))^\gamma}\,\text{dvol} + \\
&+ C \int_{\mathcal{M}} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^m} \frac{\varphi_{\varepsilon,x}}{(d(x,y))^{\gamma+1}}\,\text{dvol} + \\
&+ C \int_{B(x,2\varepsilon)\setminus B(x,\varepsilon)} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^m} \frac{|\nabla \varphi_{\varepsilon,x}|}{(d(x,y))^\gamma}\,\text{dvol} + \\
&+ C \int_{\mathcal{M}} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^{m+1}} \frac{|\nabla(|\nabla u|^{p-1})| \varphi_{\varepsilon,x}}{(d(x,y))^\gamma}\,\text{dvol}\,.
\end{aligned} \tag{2.18}
$$

Since $u \in C^1(\overline{\mathcal{M}})$ , it follows that

$$
\int_{\mathcal{M}\setminus E_\delta} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^m} \frac{|\nabla \varphi_{\varepsilon,x}|}{(d(x,y))^\gamma}\,\text{dvol} \le C \int_{\mathcal{M}\setminus E_\delta} \frac{\text{dvol}}{(d(x,y))^\gamma} \le C\,, \tag{2.19}
$$

$$
\int_{\mathcal{M}} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^m} \frac{\varphi_{\varepsilon,x}}{(d(x,y))^{\gamma+1}}\,\text{dvol} \le C \int_{\mathcal{M}} \frac{\text{dvol}}{(d(x,y))^{\gamma+1}} \le C\,, \tag{2.20}
$$

$$
\int_{B(x,2\varepsilon)\setminus B(x,\varepsilon)} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^m} \frac{|\nabla \varphi_{\varepsilon,x}|}{(d(x,y))^\gamma}\,\text{dvol} \le C\varepsilon^{n-\gamma-1} \le C\,. \tag{2.21}
$$

On the other hand, $u \in C^2(\overline{\mathcal{M}} \setminus \mathcal{Z})$ , hence,

$$
\begin{aligned}
&\int_{\mathcal{M}} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^{m+1}} \frac{|\nabla(|\nabla u|^{p-1})| \varphi_{\varepsilon,x}}{(d(x,y))^\gamma}\,\text{dvol} \le \\
&\le \int_{\mathcal{M}\setminus E} \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^{m+1}} \frac{|\nabla(|\nabla u|^{p-1})| \varphi_{\varepsilon,x}}{(d(x,y))^\gamma}\,\text{dvol} + \\
&+ \int_E \frac{|\nabla u|^{p-1}}{(|\nabla u|^{p-1}+\varepsilon)^{m+1}} \frac{|\nabla(|\nabla u|^{p-1})| \varphi_{\varepsilon,x}}{(d(x,y))^\gamma}\,\text{dvol} \le \\
&\le C \int_{\mathcal{M}\setminus E} \frac{\text{d}vol}{(d(x,y))^\gamma} + C \int_E \frac{|\nabla u|^{p-2} |D^2 u|^2 \varphi_{\varepsilon,x}}{(|\nabla u|^{p-1}+\varepsilon)^m (d(x,y))^\gamma}\,\text{dvol} \le \\
&\le C + C \left( \int_E \frac{|\nabla u|^{2(p-2)} |D^2 u|^2 \varphi_{\varepsilon,x}}{(|\nabla u|^{p-1}+\varepsilon)^m (d(x,y))^\gamma}\,\text{dvol} \right)^{\frac12} \times \\
&\times \left( \int_E \frac{\varphi_{\varepsilon,x}}{(|\nabla u|^{p-1}+\varepsilon)^m (d(x,y))^\gamma}\,\text{dvol} \right)^{\frac12}\,.
\end{aligned} \tag{2.22}
$$

Combining (2.18) to (2.22) and using Theorem 4, we have

$$
\int_E \psi_{\varepsilon,x}\,\text{dvol} \le C\,.
$$

Letting $\varepsilon \to 0$ it follows that

$$\int_E \frac{\mathrm{dvol}}{|\nabla u|^{m(p-1)}(d(x,y))^{\gamma}} \leq C \ , \tag{2.23}$$

where $C$ is independent of $x$. In particular, we obtain $\mathrm{vol}(\mathcal{Z}) = 0$.

□

As a consequence of Theorems 4 and 5, we have the following result.

**Corollary 2.3.** *Under the above assumptions (as in Theorem 5), for every* $\gamma_0 \in (n-2, n-2+2/(p-1))$,

$$\sup_{x\in\mathcal{M}} \int_{\mathcal{M}} \frac{\mathrm{dvol}_y}{|\nabla u|^{p-2}(d(x,y))^{\gamma_0}} \leq C \ . \tag{2.24}$$

*Moreover, for any* $1 \leq m < (p-1)/(p-2)$, *we have*

$$D^2 u \in L^m_{\mathrm{loc}}(\mathcal{M}) \tag{2.25}$$

*Proof.* We have that for any $\gamma_1 \in (0, n)$

$$\sup_{x\in\mathcal{M}} \int_{\mathcal{M}} \frac{\mathrm{dvol}_y}{(d(x,y))^{\gamma_1}} \leq C \ . \tag{2.26}$$

Hence, writing

$$|\nabla u|^{p-2}(d(x,y))^{\gamma_0} = \left[|\nabla u|^{p-2}(d(x,y))^{(n-2)\frac{p-2}{p-1}}\right] \cdot$$
$$\cdot \left[(d(x,y))^{(n-2)\frac{1}{p-1}}(d(x,y))^{\gamma_0-(n-2)}\right]$$

equation (2.24) follows from (2.16) and the Hölder inequality with exponents $r = \alpha(p-1)/(p-2)$, $s = \beta(p-1)$, with $0 < \alpha < 1 < \beta$ and $\alpha$, $\beta$ sufficiently close to 1. Therefore, using corollary 2.1 and Theorem 5, we prove (2.25).

□

## 3. Weighted Poincaré type inequality and weak comparison principle

In this section we prove a weighted Poincaré type inequality, and then we use it to prove a weak comparison principle in small domains. Our method is inspired from the work in [9]. They prove an integrability condition for the weighted function $\rho$, more precisely,

$$\sup_{x\in\mathcal{M}} \int_{\mathcal{M}} \frac{\mathrm{dvol}_y}{(\rho(y))^t(d(x,y))^{\gamma_0}} \leq C \ .$$

for some $\gamma_0 \in (0, n-2)$ and $t > 1$. Based on this, under suitable assumptions, they establish the weighted Sobolev inequalities. Here, we use a somewhat different approach for this technical point - we concentrate on the fixed $t = 1$ case and allow only the parameter $\gamma_0 \in (n-2, n)$ to change (thanks to Corollary 2.3) in order to obtain such inequalities.

Let us start by collecting in the next Lemma some known results about the potential of a function. If $f \in C^0(\overline{\mathcal{M}})$, and $0 < \alpha < n$ then the potential of order $\alpha$ generated by $f$ is defined by

$$U_\alpha[f](x) = \int_{\mathcal{M}} f(y) d(x,y)^{\alpha-n}\, \mathrm{dvol}_y.$$

**Lemma 3.1.** *1) If $1 < a < n/\alpha$, denote by $b$ the number defined by $\frac{1}{b} = \frac{1}{a} - \frac{\alpha}{n}$. Then, the map*

$$\begin{array}{llll} U_\alpha : & L^a(\mathcal{M}) & \to & L^b(\mathcal{M}) \\ & f & \mapsto & U_\alpha[f] \end{array}$$

*is continuous. Moreover, there exists a constant $C = C(n, \alpha, a, \mathcal{M}) > 0$ such that*

$$\|U_\alpha[f]\|_{L^b(\mathcal{M})} \leq C\|f\|_{L^a(\mathcal{M})} \ . \tag{3.1}$$

*2) If $a = n/\alpha$, then the map $U_\alpha$ is continuous from $L^a(\mathcal{M})$ into $L^q(\mathcal{M})$ for any $1 \leq q < \infty$. More precisely, there is a constant $C = C(n, \alpha, a, q, \mathcal{M}) > 0$, such that*

$$\|U_\alpha[f]\|_{L^q(M)} \leq C\|f\|_{L^a(\mathcal{M})} \ . \tag{3.2}$$

*3) If $a > n/\alpha$, then the map $U_\alpha$ is continuous from $L^a(\mathcal{M})$ into $L^\infty(\mathcal{M})$, that is, there is a constant $C = C(n, \alpha, a, \mathcal{M}) > 0$, such that*

$$\|U_\alpha[f]\|_{L^\infty(\mathcal{M})} \leq C\|f\|_{L^a(\mathcal{M})} \ . \tag{3.3}$$

We begin by proving general Sobolev and Poincaré type inequalities, using potential estimates as in [14, 18].

**Theorem 6.** *Assume $\rho \in L^1(\mathcal{M})$ is a positive weight function such that*

$$\int_{\mathcal{M}} \frac{1}{\rho(y) d(x,y)^\gamma} \,\mathrm{dvol}_y \leq C, \quad \forall x \in \mathcal{M} \ , \tag{3.4}$$

*where $0 \leq \gamma < n$ and $C$ does not depend on $x$.*
*Assume also that $q \geq 2$ satisfies $q > n - \gamma$. Then,*
*1) If $q < 2n - \gamma$ then there exists a constant $c_0 = c_0(n, q, \rho, \mathcal{M}, \gamma)$ such that the following weighted Sobolev inequality holds for any $u \in W^{1,q}_{0,\rho}(\mathcal{M})$:*

$$\|u\|_{L^{q^*}} \leq c_0 \|\nabla u\|_{L^q_\rho(\mathcal{M})}, \qquad \forall u \in W^{1,q}_{0,\rho}(\mathcal{M}) \tag{3.5}$$

*where $q^*$ is defined by*

$$\frac{1}{q^*} = \frac{2}{q} - \frac{1}{n} - \frac{\gamma}{nq}.$$

*2) If $q = 2n - \gamma$ then for any $r \geq 1$ there exists a constant $c_r$ such that for every $u \in W^{1,q}_{0,\rho}(\mathcal{M})$ we have*

$$\|u\|_{L^r(\mathcal{M})} \leq c_r \|\nabla u\|_{L^q_\rho(\mathcal{M})} \ , \tag{3.6}$$

*3) If $q > 2n - \gamma$ then there exists a constant $c_\infty$ such that for every $u \in W^{1,q}_{0,\rho}(\mathcal{M})$ we have*

$$\|u\|_{L^\infty(\mathcal{M})} \leq c_\infty \|\nabla u\|_{L^q_\rho(\mathcal{M})} \ , \tag{3.7}$$

*Proof.* By density arguments we may suppose $u \in C^\infty_0(\mathcal{M})$. Let $G(x, y)$ denote Green's function in $\mathcal{M}$. For every $x \in \mathcal{M}$, we have

$$u(x) = -\int_{\mathcal{M}} G(x,y) \triangle_g u(y) \,\mathrm{dvol}_y \ , \tag{3.8}$$

where $\triangle_g$ is the Laplace-Beltrami operator. As $u$ has compact support, we deduce

$$u(x) = \int_{\mathcal{M}} \langle \nabla_y G(x,y), \nabla_y y \rangle \,\mathrm{dvol}_y \ . \tag{3.9}$$

Recall that the following estimate holds

$$|\nabla_y G(x,y)| \leq C d(x,y)^{1-n} \ , \tag{3.10}$$

where $C$ is a constant independent of $x$ and $y$. Thus, we get

(3.11)

$$|u(x)| \leq C\int_{\mathcal{M}} \frac{|\nabla u(y)|}{d(x,y)^{n-1}}\,\mathrm{dvol}_y = C\int_{\mathcal{M}} \frac{|\nabla u(y)|\rho(y)^{\frac{1}{q}}}{d(x,y)^{n-1-\frac{\gamma}{q}}}\frac{1}{\rho(y)^{\frac{1}{q}}d(x,y)^{\frac{\gamma}{q}}}\,\mathrm{dvol}_y \leq$$

$$\leq C\left(\int_{\mathcal{M}} \frac{1}{\rho(y)d(x,y)^{\gamma}}\,\mathrm{dvol}_y\right)^{\frac{1}{q}} \left\|\frac{|\nabla u|\rho(y)^{\frac{1}{q}}}{d(x,y)^{n-1-\frac{\gamma}{q}}}\right\|_{L^{q'}(\mathcal{M})},$$

where $\dfrac{1}{q'}+\dfrac{1}{q}=1$. We set

$$f(y) = (|\nabla u(y)|\rho(y)^{\frac{1}{q}})^{q'}\ .$$

We remark that $n-1-\gamma/q > 0$ and set $\alpha = n-(n-1-\gamma/q)q'$. Recall $q > n-\gamma$, so that $\alpha > 0$. On the other hand, it follows from (3.11)

$$|u(x)|^{q'} \leq C|U_\alpha[(|\nabla u|\rho^{\frac{1}{q}})^{q'}]|.$$

We see that $f \in L^{\frac{q}{q'}}(\mathcal{M})$, where $q/q' \geq 1$. Let us consider first the case $q < 2n-\gamma$. In this case, we have $q/q' < n/\alpha$. Using Lemma 3.1, we obtain

$$\||u(x)|^{q'}\|_{L^b(\mathcal{M})} \leq C\|f\|_{L^{\frac{q}{q'}}(\mathcal{M})} = C\|\nabla u\|^{q'}_{L^q_\rho(\mathcal{M})}.$$

where $\dfrac{1}{b} = \dfrac{q'}{q} - \dfrac{\alpha}{n}$. This yields

$$\|u\|_{L^{bq'}(\mathcal{M})} \leq C\|\nabla u\|_{L^q_\rho(\mathcal{M})}.$$

Therefore, part 1) is proved. If $q = 2n-\gamma$, we have $q/q' = n/\alpha$. Using again Lemma 3.1, we have, for any $r \geq 1$,

$$\||u(x)|^{q'}\|_{L^r(\mathcal{M})} \leq C\|\nabla u\|^{q'}_{L^q_\rho(\mathcal{M})}.$$

Hence, we obtain result 2). Part 3) also follows from Lemma 3.1.

□

We will now apply this result to the case $\rho = |\nabla u|^{p-2}$, $p \geq 2$ and $u$ is a weak solution of (1.1).

**Theorem 7.** *Assume (A1) to (A3), (B1), (B2) and* $p \geq 2$. *Then, if we consider* $\rho = |\nabla u|^{p-2}$ *we get, for every* $q \geq 2$

$$\|v\|_{L^q(\mathcal{M})} \leq C(\,\mathrm{vol}(\mathcal{M}))\|\nabla v\|_{L^q_\rho(\mathcal{M})} \quad \text{for every } v \in W^{1,q}_{0,\rho}(\mathcal{M}) \tag{3.12}$$

*where* $C(\,\mathrm{vol}(\mathcal{M})) \to 0$ *if* $\mathrm{vol}(\mathcal{M}) \to 0$.
*In particular* (3.12) *holds for every* $v \in W^{1,2}_{0,\rho}(\Omega)$.

*Proof.* Since $u \in C^1(\overline{\mathcal{M}})$ and $p \geq 2$, obviously $\rho = |Du|^{p-2} \in L^1(\mathcal{M})$. By Corollary 2.3 we have

$$\sup_{x\in\mathcal{M}}\int_{\mathcal{M}} \frac{1}{\rho(y)d(x,y)^{\gamma_0}}\,\mathrm{dvol}_y \leq C$$

for some $\gamma_0 \in (n-2,n)$. We see $q \geq 2 > n-\gamma_0$. By Theorem 6, there exists some $q_1 > q$ such that

$$\|v\|_{L^{q_1}(\mathcal{M})} \leq C\|\nabla v\|_{L^q_\rho(\mathcal{M})}.$$

Consequently, we deduce

$$\|v\|_{L^q(\mathcal{M})} \le (\,\mathrm{vol}(\mathcal{M}))^{\frac{1}{q}-\frac{1}{q_1}} \|v\|_{L^{q_1}(\mathcal{M})} \le C(\,\mathrm{vol}(\mathcal{M}))^{\frac{1}{q}-\frac{1}{q_1}} \|\nabla v\|_{L^q_\rho(\mathcal{M})}$$

Thus, we conclude the proof.

□

Note that usually the case $q = 2$, which gives a Hilbert space $W^{1,2}_{0,\rho}(\mathcal{M})$, is considered. The previous inequality allows us to prove the following

**Theorem 8** (Weak Comparison Principle in small domains)**.** *Suppose that either*

$1 < p < 2$ *and* $u, v \in W^{1,\infty}(\mathcal{M})$*, or*
$p \ge 2$ *and* $u, v \in W^{1,p}(\mathcal{M}) \cap L^\infty(\mathcal{M})$

*and that either* $\rho \equiv |\nabla u|^{p-2}$ *or* $\rho \equiv |\nabla v|^{p-2}$ *satisfy the following condition*

$$\sup_{x\in\mathcal{M}} \int_{\mathcal{M}} \frac{1}{\rho(y)d(x,y)^{\gamma_0}}\, \mathrm{dvol}_y \le C$$

*for some* $\gamma_0 \in (n-2, n)$*.*

*Suppose that* $u$*,* $v$ *weakly solve*

$$-\triangle_p u + g(x,u) - \Lambda u \le -\triangle_p v + g(x,v) - \Lambda v \text{ in } \mathcal{M} \tag{3.13}$$

*where* $\Lambda \ge 0$ *and* $g \in C(\overline{\mathcal{M}} \times \mathbb{R})$ *is such that for every* $x \in \mathcal{M}$*,* $g(x,s)$ *is nondecreasing for* $|s| \le \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$*.*
*Let* $\mathcal{M}' \subseteq \mathcal{M}$ *be open and suppose* $u \le v$ *on* $\partial\mathcal{M}'$*, then there exists* $\delta > 0$ *such that, if* $\mathrm{vol}(\mathcal{M}') \le \delta$*, then* $u \le v$ *in* $\mathcal{M}'$*. If* $\Lambda = 0$ *the thesis is true for every* $\mathcal{M}' \subseteq \mathcal{M}$*. In particular the result holds if either* $u$ *or* $v$ *is a* $C^1(\overline{\mathcal{M}})$ *weak solution of* (1.1) *and (A1) to (A3) and (B2) are satisfied.*

*Proof.* In the case $1 < p < 2$, the weak comparison principle for small domains, in the Euclidean setting, was established in [5]. Since we can write the operators in divergence form, the same approach shows that the corresponding result in the general manifold setting is also valid.

Therefore, from now on, we may assume $p \ge 2$. Consider, in $\mathcal{M}'$, the function $(u-v)^+$. It is bounded, it vanishes on $\partial\mathcal{M}'$ and it belongs to $W^{1,p}_0(\mathcal{M})$, so that it belongs to $W^{1,2}_{0,\rho}(\mathcal{M}') \cap L^\infty(\mathcal{M}')$ and can be used as test function in (3.13), yielding

$$\begin{aligned} &\int_{[u\ge v]} \langle |\nabla u|^{p-2} Du - |\nabla v|^{p-2}\nabla v, \nabla u - \nabla v\rangle\, \mathrm{dvol} + \\ &+ \int_{[u\ge v]} [g(x,u) - g(x,v)](u-v)\, \mathrm{dvol} - \Lambda \int_{[u\ge v]} (u-v)^2\, \mathrm{dvol} \le 0\,. \end{aligned} \tag{3.14}$$

where $[u \ge v] = \{x \in \mathcal{M}' : u(x) \ge v(x)\}$. Moreover $g(x,u) \le g(x,v)$ if $u \ge v$, so that

$$\int_{[u\ge v]} \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla u - \nabla v\rangle\, \mathrm{dvol} \le \Lambda \int_{[u\ge v]} (u-v)^2\, \mathrm{dvol}\,. \tag{3.15}$$

By standard estimates (see e.g. [5] Lemma 2.1), we obtain the following inequality

$$\int_{\mathcal{M}'} (|\nabla u|^{p-2} + |\nabla v|^{p-2}) |\nabla(u-v)^+|^2\, \mathrm{dvol} \le C_p\, \Lambda \int_{\mathcal{M}'} [(u-v)^+]^2\, \mathrm{dvol}\,, \tag{3.16}$$

where $C_p$ depends on $p$, so that

$$\int_{\mathcal{M}'} |\nabla(u-v)^+|^2 \rho\, \mathrm{dvol} \le C_p\, \Lambda \int_{\mathcal{M}'} [(u-v)^+]^2\, \mathrm{dvol}\,, \tag{3.17}$$

where we can take $\rho \equiv |Du|^{p-2}$ or $\rho \equiv |Dv|^{p-2}$. Using Poincarè's inequality with weight (Theorem 7), we get

$$\int_{\mathcal{M}'} [(u-v)^+]^2 \text{ dvol} \leq C_1 \Lambda C(\text{ vol}(\mathcal{M}')) \int_{\mathcal{M}'} [(u-v)^+]^2 \text{ dvol} . \tag{3.18}$$

A contradiction occurs if $C_1 \Lambda C(\text{ vol}(\mathcal{M}')) < 1$, unless $(u-v)^+ = 0$ in $\mathcal{M}'$, i.e. $u \leq v$ in $\mathcal{M}'$. (Let us recall that the integral in the last inequality defines a norm). If $\Lambda = 0$, the same arguments prove the result for every $\mathcal{M}' \subseteq \mathcal{M}$.

□

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