

Perimeter, Mumford-Shah functionals, and doubling metric measures

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Abstract¹. In this paper we study the notion of perimeter and the existence of minima for Mumford-Shah type functionals associated with doubling metric measures or strong A_∞ weights in the sense of David & Semmes.

1. INTRODUCTION

In the Euclidean space \mathbb{R}^n , denote by $B^{\text{Euc}}(x, r) = \{y \in \mathbb{R}^n; |x - y| < r\}$ the Euclidean ball centered at x of radius $r > 0$, and let μ be a positive doubling measure (i.e. we assume there exists a constant $c > 0$ such that for all balls $B^{\text{Euc}} = B^{\text{Euc}}(x, r)$ in \mathbb{R}^n we have $\mu(2B^{\text{Euc}}) \leq c\mu(B^{\text{Euc}})$, where $2B^{\text{Euc}} = B^{\text{Euc}}(x, 2r)$). Following Semmes [18], we define a canonical quasi-metric D for $x, y \in \mathbb{R}^n$ as follows:

$$(1.1) \quad D(x, y) := \{\mu(B^{\text{Euc}}(x, |x - y|)) + \mu(B^{\text{Euc}}(y, |x - y|))\}^{1/n}.$$

Definition 1.1. We say that μ is a doubling metric measure if there exists a constant $C > 0$ and a metric $d(x, y)$ on \mathbb{R}^n such that

$$(1.2) \quad C^{-1}d(x, y) \leq D(x, y) \leq Cd(x, y).$$

Let us state now the definition of *strong- A_∞ weight*, as it is given in [8]. For sake of simplicity and since we shall deal only with this class of weights, let us suppose that the weight ω is a continuous function that is bounded on all of \mathbb{R}^n . Then the measure distance associated with ω is defined by

$$(1.3) \quad \delta(x, y) := \left\{ \int_{B_{x,y}^{\text{Euc}}} w(u) du \right\}^{1/n},$$

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1991 *Mathematics Subject Classification.* 49Q20, 28A12.

Keywords and phrases. Strong A_∞ -weights, Mumford-Shah functionals, weighted BV spaces.

Investigation supported by University of Bologna, funds for selected research topics and by GNAMPA of INdAM, Italy, under the project *Analysis in metric spaces and subelliptic equation*, by University of Bologna, funds for selected research topics, and by MIUR, Italy.

It is a pleasure to thank here Luigi Ambrosio for his invaluable and friendly help. In fact, our paper is largely inspired by his advice and his comments that lead us to to simplify our original proofs, to get rid of several ambiguities of the original version and to clarify the role of the assumptions.

where $B_{x,y}^{\text{Euc}}$ denotes the Euclidean ball with diameter $|x - y|$ containing x and y .

On the other hand, given a rectifiable arc $\gamma(t)$, $t \in [0, 1]$, we can define its ω -length by the Riemannian formula

$$l_\omega(\gamma) = \int_0^1 \omega(\gamma(t))^{1/n} |d\gamma(t)|,$$

where $|d\gamma(t)|$ denotes arclength measure on γ . The geodesic distance $d_\omega(x, y)$ is the infimum of the lengths of all curves joining x to y .

Definition 1.2 (David & Semmes [8]). *A weight function $\omega \in A_\infty$ is said to be a strong- A_∞ weight if there exists $C > 0$ such that*

$$(1.4) \quad C^{-1} d_\omega(x, y) \leq \delta(x, y) \leq C d_\omega(x, y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

If $x \in \mathbb{R}^n$ and $r > 0$ are given, then a metric ball $B(x, r)$ with respect to the metric d_ω is the set $B(x, r) := \{y \in \mathbb{R}^n : d_\omega(x, y) \leq r\}$. In [18], Semmes proves the following

Theorem 1.3. *A positive doubling measure μ satisfies (1.2) if and only if $\mu = \omega dx$, where ω is a strong- A_∞ weight and dx stands for Lebesgue measure.*

We refer to [8], [18], [6], [7] for further discussions and examples.

In the present paper we introduce a notion of perimeter associated with a strongly A_∞ weight ω following the general scheme of the perimeter theory in doubling metric spaces ([2], Miranda [17] and [5]), and we recall some evidences of the intrinsic nature of this notion. Then, based on this notion, we develop a theory of Mumford–Shah functionals associated with weighted measures.

More precisely, in this paper we prove the existence of minima for functionals of the form

$$(1.5) \quad MS(u) + \int_\Omega |u - g|^2 dx,$$

where $g \in L^\infty(\Omega)$, and

$$(1.6) \quad MS(u) = \int_\Omega |\nabla u|^2 \omega^{1-\frac{2}{n}}(x) dx + \int_{S_u^\omega} \omega^{1-\frac{1}{n}} d\mathcal{H}^{n-1},$$

where, for $s \geq 0$, \mathcal{H}^s is the s -dimensional Euclidean Hausdorff measure, and the set S_u^ω is the set of ω -approximated discontinuities of u in Ω (see Definition 3.6 below). As in the unweighted setting, it is crucial to show that the set S_u^ω is $(n-1)$ -countably rectifiable in the usual sense. Denoting by \mathcal{H}_ω^s the s -dimensional Hausdorff measure built from d_ω , the functional in (1.6) can be better written in the intrinsic form

$$(1.7) \quad MS(u) = \int_\Omega |\nabla u|^2 \omega^{1-\frac{2}{n}}(x) dx + \mathcal{H}_\omega^{n-1}(S_u^\omega(\Omega)).$$

The crucial points of our proof consist of i) a good and consistent definition of $SBV(\Omega; \omega^{1-\frac{1}{n}})$ space associated with ω as a subclass of the related $BV(\Omega; \omega^{1-\frac{1}{n}})$ space we previously studied in [6] in the setting of Ambrosio's and Miranda's spaces; ii) closure and compactness theorems for minimizing sequences in these spaces.

We notice that the form of the functional in (1.6) even though it shows clearly our aim, i.e. that when minimizing the functional we want somehow to neglect the regions where ω is very small, nevertheless is somehow misleading, since it can be written for every non negative continuous weight function, regardless of it being a strong A_∞ weight. However, the problem is not correctly formulated for

general continuous weights. In fact, in order to formulate in a correct way our variational problem, we need a structure of BV space associated with the weight, and hence - in order to apply Ambrosio's and Miranda's theory [3], [17] - we must associate with the weight ω a "good" metric space, i.e. a doubling metric space where a $L^1 - L^1$ Poincaré inequality holds. Strong A_∞ weights precisely provide such a "good" metric (the metric d_ω we mentioned above), thanks to the Poincaré inequality proved by David & Semmes [8].

If we drop the assumption on ω , then minimizing sequences for the functional (1.5) (that can be written for any continuous weight ω) are not compact in $L^1(\Omega, \omega^{1-\frac{1}{n}} d\mathcal{L}^n)$, i.e. natural imbedding theorems fail to hold. It is important to stress that, if we take ω to be an arbitrary doubling weight, minimizing sequences still admit converging subsequences to a function u in $L^1_{\text{loc}}(\Omega \setminus F)$, but this does not yield any result of some interest, since the crucial point is that the "variation" of u given by (1.6) is by no means intrinsic without the connection given by Poincaré inequality (Theorem 2.8). Now, the lack of imbedding in $L^1(\Omega, \omega^{1-\frac{1}{n}})$ shows precisely that Poincaré inequality does not hold, and thus, u is a function for which the functional (1.6) has no intrinsic (or metric) meaning.

In fact, the notion of strong- A_∞ weight is strictly related to that of perimeter: among all doubling measures in the Euclidean space, a doubling metric measure enjoys a crucial peculiar property: an intrinsic relative isoperimetric inequality (see formula (2.14) below).

Full proofs and more detailed discussion on the role of the assumptions can be found in [7] (see also [6]). Recently, similar problems have been studied in the setting of metric doubling spaces in [5].

2. STRONG- A_∞ WEIGHTS AND PERIMETER

Since A_∞ weights yield doubling measures, it is easy to prove the following fact.

Proposition 2.1. *If K is a compact set, then there exists $\varepsilon > 0$ and constants $c_K, C_K > 0$ such that for $x, y \in K$*

$$c_K |x - y|^{1/\varepsilon} \leq d_\omega(x, y) \leq C_K |x - y|.$$

The following result is proved in [18], (B.4.16) and (B.4.13).

Theorem 2.2. *There exist $a, A > 0$ such that for any $x \in \mathbb{R}^n$, $r > 0$ we have*

$$(2.8) \quad ar^n \leq \omega(B(x, r)) \leq Ar^n.$$

Moreover, there exists $c > 0$ such that for any $r > 0$ there exists $R = R(r)$ such that

$$(2.9) \quad B^{\text{Euc}}(x, cR) \subseteq B(x, r) \subseteq B^{\text{Euc}}(x, R)$$

for any $x \in \mathbb{R}^n$.

Since in addition a Poincaré-type inequality for strong- A_∞ weights on d_ω -balls holds (see Theorem 2.8 below), we can define the weighted BV space associated with a strong- A_∞ weight ω as it is done in [17], [2] and [3], i.e. the space of bounded variation functions with respect to the measure $\omega^{1-\frac{1}{n}}(x) dx$ (which is doubling since $\omega \in A_\infty$) and to the distance d_ω , by means of a relaxation argument.

Definition 2.3. Let A be an open subset of \mathbb{R}^n and let ω be a strong- A_∞ weight function. We shall say that a function $u \in L^1(A; \omega^{1-\frac{1}{n}})$ belongs to $BV(A; \omega^{1-\frac{1}{n}})$ if there exists a sequence $(u_h)_{h \in \mathbb{N}} \subset \text{Lip}_{\text{loc}}(A)$ converging to u in $L^1_{\text{loc}}(A; \omega^{1-\frac{1}{n}})$ and satisfying

$$\limsup_{h \rightarrow +\infty} \int_A |\nabla u_h| \omega^{1-\frac{1}{n}}(x) dx < \infty.$$

Moreover, we put

$$\|Du\|_{\omega^{1-\frac{1}{n}}}(A) := \inf \left\{ \liminf_{h \rightarrow +\infty} \int_A |\nabla u_h| \omega^{1-\frac{1}{n}}(x) dx : (u_h) \subset \text{Lip}_{\text{loc}}(A), u_h \xrightarrow{L^1_{\text{loc}}(A; \omega^{1-\frac{1}{n}})} u \right\},$$

and

$$\|u\|_{BV(A; \omega^{1-\frac{1}{n}})} = \|u\|_{L^1(A; \omega^{1-\frac{1}{n}})} + \|Du\|_{\omega^{1-\frac{1}{n}}}(A).$$

We shall say that $u \in BV_{\text{loc}}(\mathbb{R}^n; \omega^{1-\frac{1}{n}})$ if $u \in BV(A; \omega^{1-\frac{1}{n}})$ for any bounded open set A in \mathbb{R}^n .

The following crucial imbedding theorem is proved in [1] and [17].

Theorem 2.4. Let Ω be an open set in \mathbb{R}^n . Then $BV(\Omega; \omega^{1-\frac{1}{n}})$ is compactly imbedded in $L^1_{\text{loc}}(\Omega; \omega^{1-\frac{1}{n}})$.

We denote by $\|Du\|$ the variational measure of a function u in the classical $BV(\Omega)$ space (see e.g. [4], [10], [11], [19]).

We can give now the notion of the intrinsic perimeter associated with a strongly A_∞ -weight.

Definition 2.5. Let Ω be an open set in \mathbb{R}^n . A Borel set E is said to have finite ω -perimeter in Ω if $\chi_E \in BV(\Omega; \omega^{1-\frac{1}{n}})$ and we write $\|D\chi_E\|_{\omega^{1-\frac{1}{n}}}(\Omega) = \|\partial E\|_{\omega^{1-\frac{1}{n}}}(\Omega)$.

Let us denote by $\mathcal{B}(\Omega)$ the Borel σ -algebra of an open set $\Omega \subset \mathbb{R}^n$. We recall the following result proved in [17], Theorem 3.4 (see also [1], Theorem 3.3).

Theorem 2.6. Let $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$. Then the set function

$$A \longrightarrow \|Du\|_{\omega^{1-\frac{1}{n}}}(A)$$

is the restriction to the open subset of Ω of a finite Borel measure $\|Du\|_{\omega^{1-\frac{1}{n}}}(\cdot)$ in Ω defined by

$$\|Du\|_{\omega^{1-\frac{1}{n}}}(B) := \inf \{ \|Du\|_{\omega^{1-\frac{1}{n}}}(A) : A \supset B, A \text{ open} \},$$

for all $B \in \mathcal{B}(\Omega)$.

Remark 2.7. Since in \mathbb{R}^n any open set is σ -compact (i.e. it is the countable union of compact sets), then without loss of generality we may assume $\|Du\|_{\omega^{1-\frac{1}{n}}}$ is a Radon measure.

We notice the strong A_∞ assumption cannot be dropped since it yields Poincaré inequality and hence the intrinsic character of the total variation. This argument can be made now explicit in the following statement whose proof can be obtained starting from [8] using Theorem 7 in [6] and the arguments of [14], [13], [15].

Theorem 2.8. *There exists $C_P > 0$ such that, if $x \in \mathbb{R}^n$ and $r > 0$ and we set $B := B(x, r)$, then, for $u \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$,*

$$(2.10) \quad \left(\int_B |u - u_B|^{n/(n-1)} \omega(x) dx \right)^{(n-1)/n} \leq C_P r \int_B |\nabla u| \omega^{1-\frac{1}{n}}(x) dx .$$

Hence

$$(2.11) \quad \left(\int_B |u - u_B|^{n/(n-1)} \omega(x) dx \right)^{(n-1)/n} \leq C_P \|Du\|_{\omega^{1-\frac{1}{n}}}(B)$$

and for any $x \in B$

$$(2.12) \quad |u(x) - u_B| \leq C'_P \int_B d_\omega(x, y)^{1-n} d\|Du\|_{\omega^{1-\frac{1}{n}}}(y) .$$

Finally, both (2.11) and (2.12) are equivalent to (2.10).

The notion of ω -perimeter introduced in Definition 2.5 can be shown to be “natural” by proving it satisfies some “natural” properties: first of all a representation formula in terms of the Euclidean perimeter: ([6], Theorem 7):

Theorem 2.9. *If $E \subseteq \Omega$ is a Caccioppoli set, i.e. $\chi_E \in BV(\Omega)$, then*

$$(2.13) \quad \|\partial E\|_{\omega^{1-\frac{1}{n}}}(A) = \int_A \omega^{1-\frac{1}{n}} d\|\partial E\| .$$

From Theorems 2.8 and 2.9 we can derive the following relative isoperimetric inequality.

Theorem 2.10. *If $E \subseteq \mathbb{R}^n$ is a Caccioppoli set, and $B = B(x, r)$ a metric ball, then*

$$(2.14) \quad \min \left\{ \int_{E \cap B} \omega(x) dx, \int_{B \setminus E} \omega(x) dx \right\} \leq C \left(\int_B \omega^{1-\frac{1}{n}} d\|\partial E\| \right)^{\frac{n}{n-1}} .$$

A third “natural” property is provided by the the following “Minkowski content” formula (see [6], Proposition 2).

Theorem 2.11. *Let $E \subseteq \mathbb{R}^n$ be a compact set such that ∂E is a smooth manifold. If $r > 0$ define a tubular neighborhood*

$$I_r = I_r(\partial E) = \{x \in \mathbb{R}^n; d_\omega(x, \partial E) < r\} .$$

Moreover, if $\Omega \subseteq \mathbb{R}^n$ is an open set, put

$$M^+(\partial E)(\Omega) = \limsup_{r \rightarrow 0^+} \frac{\omega(I_r \cap \Omega)}{2r} , \quad M^-(\partial E)(\Omega) = \liminf_{r \rightarrow 0^+} \frac{\omega(I_r \cap \Omega)}{2r} .$$

Then, if $\mathcal{H}_{n-1}(\partial E \cap \partial \Omega) = 0$,

$$(2.15) \quad M^+(\partial E)(\Omega) = M^-(\partial E)(\Omega) = \|\partial E\|_{\omega^{1-\frac{1}{n}}}(\Omega) .$$

Finally, also the following variational property of the ω perimeter supports our definition ([6], Theorem 3):

Theorem 2.12. *Let Ω be a bounded open set in \mathbb{R}^n with Lipschitz boundary. For any $\varepsilon > 0$ let F_ε be defined by*

$$(2.16) \quad F_\varepsilon(u) = \begin{cases} \int_{\Omega} \left(\varepsilon |Du|^2 \omega^{1-\frac{2}{n}} + \frac{W(u)}{\varepsilon} \omega \right) dx & \text{if } u \in S^{1,2}(\Omega; \omega^{1-\frac{2}{n}}) \\ +\infty & \text{otherwise,} \end{cases}$$

where $W(u) = u^2(1-u)^2$ and

$$S^{1,2}(\Omega; \omega^{1-\frac{2}{n}}) := \left\{ u \in L^2(\Omega; \omega^{1-\frac{2}{n}}) \cap W_{\text{loc}}^{1,1}(\Omega) : |Du| \in L^2(\Omega; \omega^{1-\frac{2}{n}}) \right\}.$$

Set now

$$(2.17) \quad F(u) = \begin{cases} 2\sigma \|\partial E\|_{\omega^{1-\frac{1}{n}}}(\Omega) & \text{if } u = \chi_E \in BV(\Omega; \omega^{1-\frac{1}{n}}) \\ +\infty & \text{otherwise,} \end{cases}$$

with $\sigma = \int_0^1 \sqrt{W(t)} dt$. Then the functionals F_ε Γ -converge to F in $L^1(\Omega; \omega^{1-\frac{1}{n}})$ as $\varepsilon \rightarrow 0$.

3. MUMFORD-SHAH FUNCTIONALS AND SBV SPACE

The proofs of the results of this Section are contained in [7].

The notion of ω -perimeter has its natural counterpart in the notion of ω -Hausdorff measure. More precisely, for any $k > 0$ we denote by $\mathcal{H}_\omega^k := \lim_{\delta \rightarrow 0} \mathcal{H}_{\omega,\delta}^k$ the k -dimensional Hausdorff measure associated with d_ω , where $\mathcal{H}_{\omega,\delta}^k$ is defined according to Carathéodory's construction ([11], 2.10.2) relative to the distance d_ω .

We can prove that the $(n-1)$ -dimensional weighted Hausdorff measure \mathcal{H}_ω^{n-1} is absolutely continuous with respect to \mathcal{H}^{n-1} , the usual Euclidean $(n-1)$ -dimensional Hausdorff measure, i.e.

Proposition 3.1. $\mathcal{H}_\omega^{n-1} \ll \mathcal{H}^{n-1}$.

Throughout all the remainder of this paper we assume the following hypothesis is satisfied.

Hypothesis 1. *The strong- A_∞ weight ω satisfies*

$$\mathcal{H}^{n-1}(F) := \mathcal{H}^{n-1}(\{x : \omega(x) = 0\}) < \infty.$$

Then we have

Proposition 3.2. *Let $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$, and suppose Hypothesis 1 holds. Then*

$$(3.18) \quad \mathcal{H}_\omega^{n-1}(F) = 0,$$

and

$$(3.19) \quad \|Du\|_{\omega^{1-\frac{1}{n}}}(F) = 0.$$

If $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$ then $u \in BV_{\text{loc}}(\Omega \setminus F)$, and it is well known that $\|Du\|$ is a Radon measure in $\Omega \setminus F$ (see [10], Section 5.1). Thus we have

Proposition 3.3. *If $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$, then*

$$(3.20) \quad \|Du\|_{\omega^{1-\frac{1}{n}}}(A) = \int_{A \setminus F} \omega^{1-\frac{1}{n}} d\|Du\|.$$

for all open set $A \subset \Omega$.

Remark 3.4. *The right hand side of identity (3.20) can be written for any continuous weight function ω , even when it does not belong to strong A_∞ (in fact, even the A_∞ condition is unnecessary). Thus one might erroneously think that the subsequent theory can be carried on in spite of ω being a strong A_∞ weight. In fact, to show that this conjecture is false, it is easy to produce examples of weight functions ω not belonging to strong A_∞ for which the subset of $L^1(\Omega; \omega^{1-\frac{1}{n}}) \cap BV_{\text{loc}}(\Omega \setminus F)$ of functions u such that*

$$\int_{\Omega} |u| \omega^{1-\frac{1}{n}} d\mathcal{L}^n + \int_{\Omega \setminus F} \omega^{1-\frac{1}{n}} d\|Du\| \leq 1$$

is not compact in $L^1_{\text{loc}}(\Omega; \omega^{1-\frac{1}{n}})$. In other words, this means that the measure defined in Ω by the right and side of (3.20) is not linked to u by Poincaré inequality (that would imply the compactness), i.e. it differs from its intrinsic variation.

For instance, let $\Omega \subset \mathbb{R}^2$ be a bounded neighborhood of the origin. The weight function could be defined in the following way: denote by $\lambda : [0, \infty) \rightarrow [0, \infty)$ the continuous piecewise linear function such that $\lambda\left(\frac{1}{k}\right) = 0$, $\lambda\left(\frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right)\right) = \frac{1}{k} - \frac{1}{k+1}$ for $k \in \mathbb{N}$. Now we set

$$\omega(x) = |x|^6 + \lambda(|x|),$$

and

$$u_k(x) = \begin{cases} k^4 & \text{if } 1/(k+1) < |x| \leq 1/k \\ 0 & \text{otherwise.} \end{cases}$$

An elementary computation shows that

$$(3.21) \quad \int_{\Omega} |u_k| \omega^{1-\frac{1}{n}} d\mathcal{L}^n \approx 1,$$

and

$$\int_{\Omega \setminus F} \omega^{1-\frac{1}{n}} d\|Du_k\| \approx k^4 \int_{S_{u_k}} \omega^{1/2} d\mathcal{H}^1 \approx 1.$$

Suppose by contradiction $\{u_k, k \in \mathbb{N}\}$ is compact in $L^1_{\text{loc}}(\Omega; \omega^{1-\frac{1}{n}})$; then, without loss of generality, we could assume $u_k \rightarrow u$ in $L^1_{\text{loc}}(\Omega; \omega^{1-\frac{1}{n}})$ as $k \rightarrow \infty$. But $u_k \rightarrow 0$ a.e. in \mathbb{R}^2 , and hence $u \equiv 0$, contradicting (3.21).

Notice that, using the arguments of [12], Section 6, we can show that $\omega \in A_\infty$, since both λ and $x \rightarrow |x|^6$ satisfy a reverse Hölder inequality. But we stress explicitly that ω is not a strong A_∞ weight, since it does not satisfy David & Semmes' weighted isoperimetric inequality (2.14) for instance on sets of the form $\{x : 1/(k+1) < |x| < 1/k\}$.

We recall now some results for unweighted BV functions ([1], [10], [11]). Let A be an open set and let $f \in BV_{\text{loc}}(A)$. If $x \in A$, we set

$$\begin{aligned} f_+(x) &= \inf\{t \in [-\infty, +\infty] : \{y \in A : f(y) > t\} \text{ has zero density at } x\}, \\ f_-(x) &= \sup\{t \in [-\infty, +\infty] : \{y \in A : f(y) < t\} \text{ has zero density at } x\}. \end{aligned}$$

It is known that $f_+(x), f_-(x) \in \mathbb{R}$ for \mathcal{H}^{n-1} -a.e. $x \in A$.

The jump set S_f of f in A is defined as

$$(3.22) \quad S_f := \{x \in A : f_-(x) < f_+(x)\}.$$

The set S_f is \mathcal{H}^{n-1} -measurable and $(n-1)$ -countably rectifiable ([4], Proposition 3.78).

If x does not belong to S_f , the approximate limit $\tilde{f}(x)$ of f at x is the common value of $f_+(x)$ and $f_-(x)$. Let $f \in L^1_{\text{loc}}(A)$ and $x \in A \setminus S_f$. We say that f is approximated differentiable at x if there exists a vector $L \in \mathbb{R}^n$ such that

$$(3.23) \quad \lim_{\rho \downarrow 0} \int_{B^{\text{Euc}}(x, \rho)} \frac{|f(y) - \tilde{f}(x) - \langle L, (y-x) \rangle|}{\rho} dy = 0,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^n .

If f is approximately differentiable at x then L is uniquely determined by (3.23) and it is called the approximated gradient of f at x and it is denoted by ∇f .

We recall also the definition of special function of bounded variation as it was introduced in [9] in the unweighted setting. If A is an open set and $f \in BV_{\text{loc}}(A)$ and we denote by Df its distributional derivative identified with a vector-valued Radon measure, then Radon-Nikodym theorem implies that $Df = D^a f + D^s f$, with $D^a f$ absolutely continuous with respect to Lebesgue measure \mathcal{L}^n , and $D^s f$ singular with respect to \mathcal{L}^n . In addition (see e.g. [10], Section 6.1.1), $D^a f = \nabla f \mathcal{L}^n$. In turn, the singular part can be decomposed in two more parts. More precisely, $D^s f = D^j f + D^c f$, where $D^j f$ – called the jump part of the derivative – is the restriction of $D^s f$ to the jump set S_f , and $D^c f$ – the Cantor part of the derivative – is the restriction of $D^s f$ to $\Omega \setminus S_f$.

Moreover, there exists a Borel map $\nu_f : S_f \rightarrow \mathbf{S}^{n-1}$ such that $\nu_{E_t} = \nu_f$ for \mathcal{H}^{n-1} -a.e. $x \in S_{\chi_{E_t}} \cap S_f$ for all t such that $\chi_{E_t} \in BV(A)$, where $E_t = \{f > t\}$ (see [1], formula (1.5)).

Then the following decomposition formula holds ([1], formula (1.7)).

Theorem 3.5. *Let A be an open set, and let $f \in BV_{\text{loc}}(A)$ be given. Since $f \in L^1_{\text{loc}}(A)$, its distributional derivative $Df \in \mathcal{D}'(A)$ is well defined and is identified with a vector-valued Radon measure. Then*

$$(3.24) \quad Df = \nabla f \mathcal{L}^n \llcorner A + (f_+ - f_-) \nu_f \mathcal{H}^{n-1} \llcorner S_f + D^c f,$$

and $\nabla f \in L^1_{\text{loc}}(A)$.

Then the space $SBV(A)$ is defined as the set of functions $f \in BV(A)$ such that $D^c f = 0$.

Going back to the space $BV(\Omega; \omega^{1-\frac{1}{n}})$, we want to introduce the corresponding notations of jump and Cantor parts. Let us start by the definition of the set of ω -approximated discontinuities of a function $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$.

Definition 3.6. *Let Ω be an open set and let $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$. We define the set of ω -approximated discontinuities of u in Ω as*

$$(3.25) \quad S_u^\omega := \bigcup_{k \in \mathbb{N}} S_{u|\Omega_k}$$

where $\Omega_k = \{x \in \mathbb{R}^n : \omega(x) > \frac{1}{k}\}$, and $S_{u|\Omega_k}$ is defined as in (3.22), since $u|_{\Omega_k} \in BV(\Omega_k)$. If $A \subset \Omega$, we write $S_u^\omega(A)$ for $S_u^\omega \cap A$.

If $x \in S_u^\omega$, by definition there exists $k \in \mathbb{N}$ such that $x \in S_{u|\Omega_k}$, and we set

$$(3.26) \quad u_\pm(x) := (u|_{\Omega_k})_\pm(x).$$

This definition is well posed since, if $k < h$, then

$$S_{u|\Omega_k} = S_{u|\Omega_h} \cap \Omega_k$$

and

$$(u|\Omega_k)_\pm(x) = (u|\Omega_h)_\pm(x) \quad \text{for any } x \in S_{u|\Omega_k},$$

because of local character of the definition of approximated limits.

Notice if $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$, then the set S_u^ω is \mathcal{H}^{n-1} -measurable and $(n-1)$ -countably rectifiable, because of the corresponding properties of $S_{u|\Omega_k}$ for any $k \in \mathbb{N}$.

Definition 3.7. Let Ω be an open subset of \mathbb{R}^n and let ω be a strong- A_∞ weight function satisfying Hypothesis 1. We define the space $SBV(\Omega; \omega^{1-\frac{1}{n}})$ of weighted special BV-functions by setting

$$SBV(\Omega; \omega^{1-\frac{1}{n}}) := \left\{ u \in BV(\Omega; \omega^{1-\frac{1}{n}}) : u|_{\Omega_k} \in SBV(\Omega_k) \forall k \in \mathbb{N} \right\}.$$

In other words,

$$(3.27) \quad SBV(\Omega; \omega^{1-\frac{1}{n}}) = BV(\Omega; \omega^{1-\frac{1}{n}}) \cap SBV_{\text{loc}}(\Omega \setminus F).$$

Hence we can apply the Euclidean decomposition (3.24) to $Du|_{\Omega \setminus F} \in \mathcal{D}'(\Omega \setminus F)$ and we get

$$(3.28) \quad Du = \nabla u \mathcal{L}^n \llcorner (\Omega \setminus F) + (u_+ - u_-) \nu_u \mathcal{H}^{n-1} \llcorner S_u.$$

Our aim here is to prove the existence of minima for the weighted Mumford-Shah functional

$$MS(u) + \int_{\Omega} |u - g|^2 dx,$$

where $g \in L^\infty(\Omega)$, and

$$(3.29) \quad MS(u) = \int_{\Omega} |\nabla u|^2 \omega^{1-\frac{2}{n}}(x) dx + \mathcal{H}_\omega^{n-1}(S_u^\omega(\Omega)), \quad u \in SBV(\Omega; \omega^{1-\frac{1}{n}}).$$

From (3.28) it follows in particular that, if $u \in BV(\Omega; \omega^{1-\frac{1}{n}})$, then ∇u is well defined for \mathcal{L}^n -a.e. $x \in \Omega \setminus F$ and it makes sense to write $\int_{\Omega} |\nabla u|^2 \omega^{1-2/n}(x) dx$ since $|F| = 0$. Then we have completely shown that formula (3.29) makes sense.

We show now that $MS(u)$ can be written also in the form

$$(3.30) \quad MS(u) = \int_{\Omega} |\nabla u|^2 \omega^{1-\frac{2}{n}}(x) dx + \int_{S_u^\omega} \omega^{1-\frac{1}{n}} d\mathcal{H}^{n-1}.$$

To this end, the following representation formula provides the key tool.

Theorem 3.8. Let S be a \mathcal{H}^{n-1} -measurable and $(n-1)$ -countably rectifiable set. Then

$$(3.31) \quad \mathcal{H}_\omega^{n-1} \llcorner S = \omega^{1-\frac{1}{n}} \mathcal{H}^{n-1} \llcorner S.$$

Notice both measures in (3.31) are Borel regular (see e.g. [16], Theorem 1.9).

Let $g \in L^\infty(\Omega)$. We want to obtain the existence of minimizers of $MS(u) + \int_{\Omega} (u-g)^2 dx$ in $SBV(\Omega; \omega^{1-\frac{1}{n}})$. As in [1], [4], this can be obtained by direct methods of Calculus of Variations once we have proved both a w^* -lower semicontinuity and a w^* -compactness result in $SBV(\Omega; \omega^{1-\frac{1}{n}})$. Our main result is contained in the following theorem.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $(u_h)_{h \in \mathbb{N}}$ be a sequence in $SBV(\Omega; \omega^{1-\frac{1}{n}})$ such that*

$$(3.32) \quad \|u_h\|_\infty + \int_\Omega |\nabla u_h|^2 \omega^{1-\frac{2}{n}}(x) dx + \mathcal{H}_\omega^{n-1}(S_{u_h}^\omega(\Omega)) < C ,$$

with the constant C independent of h . Then there exists a subsequence (u_{h_s}) weakly converging in $BV(\Omega; \omega^{1-\frac{1}{n}})$ to a function u in $SBV(\Omega; \omega^{1-\frac{1}{n}})$. In addition*

$$(3.33) \quad \int_\Omega |\nabla u|^2 \omega^{1-\frac{2}{n}}(x) dx \leq \liminf_{s \rightarrow \infty} \int_\Omega |\nabla u_{h_s}|^2 \omega^{1-\frac{2}{n}}(x) dx ,$$

and

$$(3.34) \quad \mathcal{H}_\omega^{n-1}(S_u^\omega(\Omega)) \leq \liminf_{s \rightarrow \infty} \mathcal{H}_\omega^{n-1}(S_{u_{h_s}}^\omega(\Omega)) .$$

The proof of Theorem 3.9 relies on the following result that has been kindly pointed out to the authors by Luigi Ambrosio. It provides a “local” form of Ambrosio’s compactness and semicontinuity theorem in SBV and can be derived from its “global” counterpart (Theorem 2.1 in [1], see also [4], Theorem 4.7).

Theorem 3.10. *Let $A \subset \mathbb{R}^n$ be an open set, and let $(u_h)_{h \in \mathbb{N}}$ be a sequence in $SBV_{\text{loc}}(A)$ such that*

$$(3.35) \quad \|u_h\|_{L^\infty(B)} + \int_B |\nabla u_h|^2 dx + \mathcal{H}^{n-1}(S_{u_h}) \leq C(B) < +\infty$$

for any open set $B \subset\subset A$. Then

- i) *Then there exists a subsequence (u_{h_s}) converging in $L^1_{\text{loc}}(A)$ to a function u as $s \rightarrow \infty$;*
- ii) *$u \in SBV_{\text{loc}}(A)$ and $\nabla u_{h_s} \rightarrow \nabla u$ weakly in $(L^2_{\text{loc}}(A))^n$;*
- iii) *the sequence of Radon measures $(\mathcal{H}^{n-1} \llcorner S_{u_{h_s}})_{s \in \mathbb{N}}$ weakly converges to a Radon measure $\mu \geq \mathcal{H}^{n-1} \llcorner S_u$.*

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