

## Pseudoharmonic maps with potential

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**Abstract**<sup>1</sup>. We study pseudoharmonic maps with potential  $V$ , i.e. smooth maps  $\phi : M \rightarrow N$  of a compact strictly pseudoconvex CR manifold  $M$  into a Riemannian manifold  $N$  which are critical points of the functional  $E_V(\phi) = \frac{1}{2} \int_M \{ \text{trace}_{G_\theta}(\pi_H \phi^* h) - 2V \circ \phi \} \theta \wedge (d\theta)^n$ . We derive the first and second variation formulae for  $E_V$ . We show that any pseudoharmonic map with a concave  $C^2$  potential into a Riemannian manifold of nonpositive curvature is stable. Also, any nonconstant pseudoharmonic map  $\phi : M \rightarrow S^\nu$  with a strictly convex  $C^2$  potential is shown to be unstable.

### 1. INTRODUCTION

Let  $(N, h)$  be a  $\nu$ -dimensional Riemannian manifold and  $V : N \rightarrow \mathbb{R}$  a smooth function (a *potential* on  $N$ ). Let  $(M, T_{1,0}(M))$  be a compact strictly pseudoconvex CR manifold of CR dimension  $n$ ,  $\theta$  a contact form on  $M$ , and  $G_\theta$  the corresponding Levi form. Let  $\phi : M \rightarrow N$  be a smooth map and let us consider the functional

$$E_V(\phi) = \frac{1}{2} \int_M \{ \text{trace}_{G_\theta}(\pi_H \phi^* h) - 2V \circ \phi \} \theta \wedge (d\theta)^n.$$

Here  $\pi_H \phi^* h$  denotes the restriction of  $\phi^* h$  to the Levi distribution. A *pseudoharmonic map with potential  $V$*  is a critical point  $\phi$  of  $E_V$ . The Euler-Lagrange equations of the variational principle  $\delta E_V(\phi) = 0$  are

$$(1.1) \quad \tau(\phi) + (DV) \circ \phi = 0,$$

where  $\tau(\phi)$  is the *tension field* and  $DV$  the gradient of  $V$  (with respect to the Riemannian metric  $h$ ). Cf. our Theorem 3.1 in Section 3. Given a local orthonormal frame  $\{X_1, \dots, X_{2n}\}$  of the Levi distribution ( $G_\theta(X_a, X_b) = \delta_{ab}$ ), defined on an open set  $U \subseteq M$ , and a local coordinate system  $(\Omega, y^1, \dots, y^\nu)$  on  $N$  (such that  $\phi(U) \subseteq \Omega$ ), the equation (1.1) may be written as

$$(1.2) \quad \Delta_b \phi^i + \sum_{a=1}^{2n} (\Gamma_{jk}^i \circ \phi) X_a(\phi^j) X_a(\phi^k) + (DV)^i \circ \phi = 0, \quad 1 \leq i \leq \nu,$$

where  $\phi^i = y^i \circ \phi$ ,  $\Gamma_{jk}^i$  are the Christoffel symbols of  $h$ , and  $\Delta_b$  is the *sublaplacian* of  $(M, \theta)$ , a subelliptic operator of order  $1/2$  on  $M$ . Cf. [4] or Section 2 of this paper for a reminder of CR and pseudohermitian geometry. The case  $V = 0$  has

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been examined by E. Barletta et al., [2] (and the smooth critical points of  $E_0$  are referred to as *pseudoharmonic maps*). Given a smooth 2-parameter variation  $\Phi : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times M \rightarrow N$  of a pseudoharmonic map  $\phi$  ( $\Phi(0, 0, x) = \phi(x)$ ,  $x \in M$ ) with potential  $V$  we show (cf. our Theorem 6.1 in Section 6) that

$$(1.3) \quad \frac{\partial^2}{\partial s \partial t} \{E_V(\phi_{s,t})\}_{s=t=0} = I_\phi(\mathbb{T}, \mathbb{S})$$

where  $\phi_{s,t}(x) = \Phi(s, t, x)$ ,

$$\mathbb{T}_x = (d_{(0,0,x)}\Phi) \frac{\partial}{\partial t} \Big|_{(0,0,x)}, \quad \mathbb{S}_x = (d_{(0,0,x)}\Phi) \frac{\partial}{\partial s} \Big|_{(0,0,x)}, \quad x \in M,$$

and

$$(1.4) \quad I_\phi(A, B) = \int_M \sum_{a=1}^{2n} h^\phi(D_{X_a}^\phi A, D_{X_a}^\phi B) \omega - \int_M \{ \text{trace}_{G_\theta} (\pi_H R^D(A, (d\phi)\cdot, B, (d\phi)\cdot)) + (\text{Hess}V)^\phi(A, B) \} \omega$$

for any vector fields  $A, B$  on  $N$  along  $\phi$ . Here  $\omega$  is short for  $\theta \wedge (d\theta)^n$ . The meaning of the objects in (1.4) is explained in Section 6. The identity (1.3) is the *second variation formula* for the functional  $E_V$ . It leads naturally to the following concept of stability. Let  $\phi : M \rightarrow N$  be a pseudoharmonic map with potential  $V$ . We say  $\phi$  is *stable* if

$$I_\phi(A, A) \geq 0,$$

for any vector field  $A$  on  $N$  along  $\phi$  (otherwise  $\phi$  is *unstable*). As shown in [2], any pseudoharmonic map of a compact strictly pseudoconvex CR manifold into a Riemannian manifold of nonpositive sectional curvature is stable. The corresponding result for pseudoharmonic maps with potential involves an additional requirement on the potential  $V$ . We recall that  $V : N \rightarrow \mathbb{R}$  is *concave* (respectively *convex*) if for any geodesic  $\gamma(t)$  in  $N$  the function  $V(\gamma(t))$  is concave (respectively convex). We shall prove the following

**Theorem 1.1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold. Let  $N$  be a Riemannian manifold of nonpositive sectional curvature and  $V : N \rightarrow \mathbb{R}$  a concave function of class  $C^2$ . Then any pseudoharmonic map  $\phi : M \rightarrow N$  with potential  $V$  is stable.*

Again by a result in [2] pseudoharmonic maps into spheres are unstable. As another result of the present paper, we shall show that

**Theorem 1.2.** *Let  $M$  be a compact strictly pseudoconvex CR manifold. Let  $\phi : M \rightarrow S^\nu$  be a nonconstant pseudoharmonic map with a strictly convex potential  $V : S^\nu \rightarrow \mathbb{R}$  of class  $C^2$ . Then  $\phi$  is unstable.*

The paper is organized as follows. In section 2 we recall the notions and results of pseudohermitian geometry needed through the paper. Section 3 is devoted the first variation formula for the functional  $E_V$ . In section 4 and 5 we show that the equations (1.1) form a nonlinear system of subelliptic equations and compare the notion of a pseudoharmonic map (with potential) with that of a subelliptic harmonic map (with potential). The second variation formula for the functional  $E_V$  is derived in section 6 (by adapting a technique due to E. Mazet, [8]). Stability of pseudoharmonic maps with potential is discussed in section 7 where Theorems

1.1 and 1.2 are proved. In section 8 we emphasize on the interplay between wave maps (cf. e.g. [11]) with potential and pseudoharmonic maps with potential.

## 2. CR AND PSEUDOHERMITIAN GEOMETRY

Let  $M$  be a real  $(2n + 1)$ -dimensional  $C^\infty$  manifold. A *CR structure* on  $M$ , of *CR dimension*  $n$ , is a complex subbundle  $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ , of complex rank  $n$ , such that

$$T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0), \quad x \in M,$$

and

$$Z, W \in T_{1,0}(M) \implies [Z, W] \in T_{1,0}(M),$$

(the formal integrability property). Here  $T_{0,1}(M) = \overline{T_{1,0}(M)}$  and overbars denote complex conjugation. A pair  $(M, T_{1,0}(M))$  is a *CR manifold* (of CR dimension  $n$ ). The *Levi distribution* is

$$H(M) = \operatorname{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

The Levi distribution carries the complex structure

$$J : H(M) \rightarrow H(M), \quad J(Z + \bar{Z}) = i(Z - \bar{Z}), \quad Z \in T_{1,0}(M),$$

( $i = \sqrt{-1}$ ). Let  $H(M)^\perp \subset T^*(M)$  be the conormal bundle, i.e.

$$H(M)_x^\perp = \{\alpha \in T_x^*(M) : \operatorname{Ker}(\alpha) \supseteq H(M)_x\}, \quad x \in M.$$

A global nowhere vanishing smooth section  $\theta$  in  $H(M)^\perp$  is a *pseudohermitian structure* on  $M$ . Pseudohermitian structures are known to exist (cf. [12]) under the mere hypothesis that  $M$  is orientable. Given a pseudohermitian structure  $\theta$  on  $M$  the *Levi form* is

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

By the formal integrability property  $G_\theta(JX, JY) = G_\theta(X, Y)$  (and in particular  $G_\theta$  is symmetric). It is customary to consider also the complex bilinear form

$$L_\theta(Z, \bar{W}) = -i(d\theta)(Z, \bar{W}), \quad Z, W \in T_{1,0}(M).$$

This is referred to as the Levi form, as well. Note that  $L_\theta$  and (the extension by  $\mathbb{C}$ -linearity of)  $G_\theta$  coincide on  $T_{1,0}(M) \otimes T_{0,1}(M)$ . We say  $(M, T_{1,0}(M))$  is *nondegenerate* (respectively *strictly pseudoconvex*) if  $G_\theta$  is nondegenerate (respectively positive definite) for some  $\theta$ . If  $\hat{\theta}$  is another pseudohermitian structure then  $\hat{\theta} = f\theta$  for some  $C^\infty$  function  $f : M \rightarrow \mathbb{R} \setminus \{0\}$  (as  $H(M)^\perp$  is a real line bundle). Hence  $d\hat{\theta} = df \wedge \theta + f d\theta$  yields  $G_{\hat{\theta}} = f G_\theta$ , that is nondegeneracy is a *CR invariant* property (i.e. invariant under a transformation  $\theta \mapsto f\theta$  as above). Each pseudohermitian structure  $\theta$  on a nondegenerate CR manifold  $M$  is a contact form on  $M$ , i.e.  $\theta \wedge (d\theta)^n$  is a volume form on  $M$ .

Let  $M$  be a nondegenerate CR manifold and  $\theta$  a contact form. There is (cf. [12]) a unique nowhere zero globally defined tangent vector field  $T$  on  $M$ , transverse to  $H(M)$ , such that

$$\theta(T) = 1, \quad (d\theta)(T, X) = 0, \quad X \in T(M),$$

( $T$  is the *characteristic direction* of  $d\theta$ ). Once a contact form is fixed it is customary to extend  $J$  to an endomorphism of  $T(M)$  by requiring that  $J(T) = 0$ . Then

$J^2 = -I + \theta \otimes T$  (where  $I$  is the identity). As  $T(M) = H(M) \oplus \mathbb{R}T$  we may consider the semi-Riemannian metric  $g_\theta$  on  $M$  given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

(the *Webster metric* of  $(M, \theta)$ ) for any  $X, Y \in H(M)$ . If  $M$  is strictly pseudoconvex and  $\theta$  is chosen in such a way that  $G_\theta$  is positive definite then  $g_\theta$  is a Riemannian metric on  $M$ . One of the main tools in pseudohermitian geometry is a canonical linear connection on  $M$ , similar to both the Levi-Civita connection of a Riemannian manifold and the Chern connection in Hermitian geometry, associated to a fixed contact form  $\theta$  on a nondegenerate CR manifold  $M$ . Let  $\nabla$  be a linear connection on  $M$ . Let  $T_\nabla$  be the torsion tensor field of  $\nabla$ . We say  $T_\nabla$  is *pure* if

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \bar{W}) = 2iL_\theta(Z, \bar{W})T,$$

for any  $Z, W \in T_{1,0}(M)$ , and

$$\tau \circ J + J \circ \tau = 0,$$

where  $\tau(X) = T_\nabla(T, X)$ ,  $X \in T(M)$ . By a result of N. Tanaka, [10], and S. Webster, [12], for any contact form  $\theta$  on  $M$  there is a unique linear connection  $\nabla$  on  $M$  (the *Tanaka-Webster connection* of  $(M, \theta)$ ) such that 1)  $H(M)$  is parallel with respect to  $\nabla$ , 2)  $\nabla J = 0$ ,  $\nabla g_\theta = 0$ , 3)  $T_\nabla$  is pure.

Let  $\phi : M \rightarrow N$  be a smooth map of a strictly pseudoconvex CR manifold  $M$  into a Riemannian manifold  $(N, h)$ . We shall need the bundle  $\phi^{-1}TN \rightarrow M$ , i.e. the pullback of  $T(N) \rightarrow N$  by  $\phi$ . Smooth cross sections in  $\phi^{-1}TN \rightarrow M$  are referred to as tangent vector fields on  $N$  along  $\phi$ . Any tangent vector field  $Y : \Omega \subseteq N \rightarrow T(N)$  on  $N$  admits a *natural lift*  $Y^\phi : \phi^{-1}(\Omega) \rightarrow \phi^{-1}TN$  given by  $Y^\phi(x) = Y_{\phi(x)}$ , for any  $x \in \phi^{-1}(\Omega)$ . Given a local coordinate system  $(\Omega, y^i)$  on  $N$  the natural lifts  $\{Y_i : 1 \leq i \leq \nu\}$  of the local vector fields  $\{\partial/\partial y^i : 1 \leq i \leq \nu\}$  form a local frame in  $\phi^{-1}TN$  defined on  $\phi^{-1}(\Omega)$ . With any vector field  $X$  on  $M$  we associate a cross section  $\phi_*X$  in  $\phi^{-1}TN \rightarrow M$  given by

$$(\phi_*X)_x = (d_x\phi)X_x, \quad x \in M.$$

The Riemannian metric  $h$  on  $N$  induces a Riemannian bundle metric  $h^\phi$  in  $\phi^{-1}TN \rightarrow M$ . Also, the Levi-Civita connection  $D$  of  $(N, h)$  induces a connection  $D^\phi$  in the vector bundle  $\phi^{-1}TN \rightarrow M$  which is easiest to describe locally.

Let  $(U, x^1, \dots, x^{2n+1})$  be a local coordinate system on  $M$  such that  $\phi(U) \subseteq \Omega$ . Then  $D^\phi$  is locally given by

$$D_{\partial/\partial x^A}^\phi Y_j = \frac{\partial \phi^i}{\partial x^A} (\Gamma_{ij}^k \circ \phi) Y_k$$

where  $\Gamma_{jk}^i$  are the Christoffel symbols of the second kind of  $h$ . Let  $\theta$  be a fixed contact form on  $M$  such that  $G_\theta$  is positive definite. Let  $\nabla$  be the Tanaka-Webster connection of  $(M, \theta)$ . The *second fundamental form* of  $\phi$  is

$$\beta_\phi(X, Y) = D_X^\phi \phi_* Y - \phi_* \nabla_X Y,$$

for any  $X, Y \in T(M)$ . The notion is due to R. Petit, [9]. The *tension field* of  $\phi$  is

$$\tau(\phi) = \text{trace}_{G_\theta}(\pi_H \beta_\phi).$$

Here  $\pi_H \beta_\phi$  is the restriction of  $\beta_\phi$  to  $H(M) \otimes H(M)$ . Clearly  $\tau(\phi)$  is a vector field on  $N$  along  $\phi$ . A  $C^\infty$  map  $\phi : M \rightarrow N$  is *pseudoharmonic* if  $\tau(\phi) = 0$ . The notion is well defined for any nondegenerate (not necessarily compact) CR

manifold  $M$ . In the theory of harmonic maps between Riemannian manifolds the second fundamental form of a smooth map  $\phi : M \rightarrow N$  is built in terms of the Levi-Civita connections of the given Riemannian metrics on  $M$  and  $N$  (cf. e.g. [1], p. 69). R. Petit replaced (cf. *op. cit.*) the Levi-Civita connection of the Riemannian metric on the source manifold with the Tanaka-Webster connection. While this may seem rather formal, it turned out (cf. [2]) that  $\tau(\phi) = 0$  is the Euler-Lagrange equation of the variational principle  $\delta E_0(\phi) = 0$ .

### 3. THE FIRST VARIATION FORMULA

Let  $M$  be a compact strictly pseudoconvex CR manifold and  $\phi : M \rightarrow N$  a smooth map of  $M$  into a Riemannian manifold  $N$ . Let  $\Phi$  be a smooth 1-parameter variation of  $\phi$ , i.e. a  $C^\infty$  map  $\Phi : (-\epsilon, \epsilon) \times M \rightarrow N$  ( $\epsilon > 0$ ) such that  $\Phi(0, x) = \phi(x)$ , for any  $x \in M$ . We set  $\phi_t(x) = \Phi(t, x)$  and  $\tilde{M} = (-\epsilon, \epsilon) \times M$ . Let  $\beta_t : M \rightarrow \tilde{M}$  be given by  $\beta_t(x) = (t, x)$ . Given a vector field  $X \in \mathcal{X}(M)$  we consider the vector field  $\tilde{X} \in \mathcal{X}(\tilde{M})$  given by

$$\tilde{X}_{(t,x)} = (d_x \beta_t) X_x, \quad (t, x) \in \tilde{M}.$$

Let  $\mathbb{T} \in \Gamma^\infty(\phi^{-1}TN)$  be the infinitesimal variation induced by  $\Phi$  i.e.

$$\mathbb{T}_x = (d_{(0,x)} \Phi) \left. \frac{\partial}{\partial t} \right|_{(0,x)}, \quad x \in M.$$

$\phi$  is a critical point of  $E_V$  if

$$\frac{d}{dt} \{E_V(\phi_t)\}_{t=0} = 0$$

for any smooth 1-parameter variation  $\Phi$  of  $\phi$ . We establish the following

**Theorem 3.1.** *Let  $\phi : M \rightarrow N$  be a smooth map of a compact strictly pseudoconvex CR manifold into a Riemannian manifold  $N$ . Let  $\Phi : \tilde{M} \rightarrow N$  be a smooth 1-parameter variation of  $\phi$  and  $\mathbb{T}$  the induced infinitesimal variation. Let  $V : N \rightarrow \mathbb{R}$  be a  $C^\infty$  function. Then*

$$(3.1) \quad \frac{d}{dt} \{E(\phi_t)\}_{t=0} = - \int_M h^\phi(\mathbb{T}, \tau(\phi) + (DV) \circ \phi) \omega.$$

Consequently  $\phi$  is a critical point of  $E_V$  if and only if

$$\tau(\phi) + (DV) \circ \phi = 0.$$

(3.1) is the *first variation formula* for the functional  $E_V$ . *Proof of Theorem 3.1.* Let  $h^\Phi$  be the Riemannian bundle metric in  $\Phi^{-1}TN \rightarrow \tilde{M}$  induced by  $h$ . Note that  $\phi_t = \Phi \circ \beta_t$ . Hence, for any  $x \in U$  we have

$$\begin{aligned} \text{trace}_{G_\theta}(\pi_H \phi_t^* h)_x &= \sum_{a=1}^{2n} (\phi_t^* h)(X_a, X_a)_x = \\ &= \sum_a h_{\Phi(t,x)}((d_x \phi_t) X_{a,x}, (d_x \phi_t) X_{a,x}) = \sum_a h^\Phi(\Phi_* \tilde{X}_a, \Phi_* \tilde{X}_a)_{(t,x)} \end{aligned}$$

so that for any  $|t| < \epsilon$

$$(3.2) \quad E(\phi_t) = \frac{1}{2} \int_M \left\{ \sum_a h^\Phi(\Phi_* \tilde{X}_a, \Phi_* \tilde{X}_a) - 2V \circ \Phi \right\} \circ \beta_t \omega.$$

On the other hand, for any  $X \in \mathcal{X}(M)$  (by  $D^\Phi h^\Phi = 0$ )

$$\begin{aligned} & \frac{\partial}{\partial t} \left( h^\Phi(\Phi_* \tilde{X}, \Phi_* \tilde{X}) \right) = 2h^\Phi(D_{\partial/\partial t}^\Phi \Phi_* \tilde{X}, \Phi_* \tilde{X}) = \\ & \text{(by } D_{\tilde{X}}^\Phi \Phi_* \tilde{Y} = D_{\tilde{Y}}^\Phi \Phi_* \tilde{X} + \Phi_*[\tilde{X}, \tilde{Y}] \text{ and } [\partial/\partial t, \tilde{X}] = 0) \\ & = 2h^\Phi(D_{\tilde{X}}^\Phi \Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X}) = 2\{\tilde{X}(h^\Phi(\Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X})) - h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}}^\Phi \Phi_* \tilde{X})\}. \end{aligned}$$

At this point we may take the derivative of (3.2) at  $t = 0$ . Let  $\alpha_x : (-\epsilon, \epsilon) \rightarrow \tilde{M}$  be given by  $\alpha_x(t) = (t, x)$ . If  $d/dt$  is thought of as a tangent vector field on  $(-\epsilon, \epsilon)$  then

$$\frac{\partial}{\partial t} \Big|_{(t,x)} = (d_t \alpha_x) \frac{d}{dt} \Big|_t$$

hence, for any  $f \in C^\infty(\tilde{M})$  and any  $x \in M$ , the derivative of the function  $t \mapsto f(\beta_t(x))$  is  $(\partial f/\partial t)(0, x)$ . Therefore

$$\begin{aligned} & \frac{d}{dt} \{E_V(\phi_t)\}_{t=0} = \\ & = \int_M \sum_a \{ \tilde{X}_a(h^\Phi(\Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X}_a)) - h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a) \} \circ \beta_0 - \\ & \quad - \int_M \frac{\partial(V \circ \Phi)}{\partial t} \circ \beta_0. \end{aligned}$$

Note that  $\tilde{X}(f) \circ \beta_0 = X(f \circ \beta_0)$ , for any  $f \in C^\infty(\tilde{M})$ . On the other hand one checks easily that

$$h^\Phi(\Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X}) \circ \beta_0 = h^\phi(\mathbb{T}, \phi_* X), \quad (D_{\tilde{X}}^\Phi \Phi_* \tilde{X}) \circ \beta_0 = D_X^\phi \phi_* X,$$

hence

$$\begin{aligned} (3.3) \quad & \frac{d}{dt} \{E_V(\phi_t)\}_{t=0} = \\ & = \int_M \sum_a \{ X_a(h^\phi(\mathbb{T}, \phi_* X_a)) - h^\phi(\mathbb{T}, D_{X_a}^\phi \phi_* X_a) \} - \int_M h^\phi(\mathbb{T}, (DV) \circ \phi) \end{aligned}$$

because of

$$\begin{aligned} \frac{\partial(V \circ \Phi)}{\partial t}(0, x) &= \frac{\partial V}{\partial y^i}(\phi(x)) \frac{\partial \Phi^i}{\partial t}(0, x) = \mathbb{T}_x(V) = \\ &= h_{\phi(x)}(\mathbb{T}_x, (DV)_{\phi(x)}) = h^\phi(\mathbb{T}, (DV) \circ \phi)_x \end{aligned}$$

where  $DV$  is the gradient of  $V$  with respect to  $h$ . Let  $Y_\Phi \in H(M)$  be defined by

$$G_\theta(Y_\Phi, X) = h^\phi(\mathbb{T}, \phi_* X)$$

for any  $X \in H(M)$ . Let  $\text{div}(Y_\Phi)$  be the divergence of  $Y_\Phi$  with respect to the volume form  $\omega = \theta \wedge (d\theta)^n$ . Since  $\nabla \omega = 0$  (and  $H(M)$  is parallel with respect to  $\nabla$ )

$$\begin{aligned} \text{div}(Y_\Phi) &= \text{trace}\{X \in H(M) \mapsto \nabla_X Y_\Phi\} = \\ &= \sum_a g_\theta(\nabla_{X_a} Y_\Phi, X_a) = \quad (\text{as } \nabla g_\theta = 0) \\ &= \sum_a \{ X_a(g_\theta(Y_\Phi, X_a)) - g_\theta(Y_\Phi, \nabla_{X_a} X_a) \} = \\ &= \sum_a \{ X_a(h^\phi(\mathbb{T}, \phi_* X_a)) - h^\phi(\mathbb{T}, \phi_* \nabla_{X_a} X_a) \} \end{aligned}$$

that is

$$(3.4) \quad \sum_a X_a(h^\phi(\mathbb{T}, \phi_* X_a)) = \operatorname{div}(Y_\Phi) + \sum_a h^\phi(\mathbb{T}, \phi_* \nabla_{X_a} X_a).$$

By (3.3)-(3.4) and Green's lemma we obtain

$$\begin{aligned} & \frac{d}{dt} \{E_V(\phi_t)\}_{t=0} = \\ & = - \int_M h^\phi(\mathbb{T}, \sum_a \{D_{X_a}^\phi \phi_* X_a - \phi_* \nabla_{X_a} X_a\} + (DV) \circ \phi) \omega = \\ & = - \int_M h^\phi(\mathbb{T}, \tau(\phi) + (DV) \circ \phi) \omega \end{aligned}$$

and (3.1) is proved. To prove the second statement in Theorem 3.1 let  $\phi : M \rightarrow N$  be a harmonic map with potential  $V$ . Let  $x_0 \in M$  and  $(\Omega, \varphi = (y^1, \dots, y^\nu))$  a local coordinate system on  $N$  such that  $\phi(x_0) \in \Omega$ . Let us set  $U = \phi^{-1}(\Omega) \subseteq M$ . Then  $\tau(\phi) = \tau(\phi)^i Y_i$  on  $U$  and we consider the functions

$$F^i : (-\epsilon, \epsilon) \times U \rightarrow \mathbb{R},$$

$$F^i(t, x) = \phi^i(x) + t[\tau(\phi)_x^i + (DV)_x^i],$$

where  $\phi^i = y^i \circ \phi$ . Let us consider a test function  $\eta \in C_0^\infty(M)$  such that  $0 \leq \eta \leq 1$  on  $M$ ,  $\operatorname{supp}(\eta) \subset U$ , and  $\eta(x_0) \neq 0$ . We may assume w.l.o.g. that  $\varphi(\Omega) = \mathbb{R}^\nu$  and consider the smooth 1-parameter variation  $\Phi : M \rightarrow N$  of  $\phi$  given by

$$\Phi(t, x) = \begin{cases} \varphi^{-1}(\Phi^i(t, x)e_i) & \text{if } x \in U, \\ \phi(x) & \text{if } x \in M \setminus U, \end{cases}$$

$$\Phi^i(t, x) = \eta(x)F^i(t, x) + (1 - \eta(x))\phi^i(x), \quad x \in U.$$

Here  $\{e_1, \dots, e_\nu\} \subset \mathbb{R}^\nu$  is the canonical linear basis. The infinitesimal variation induced by  $\Phi$  is

$$\begin{aligned} \mathbb{T}_x &= (d_{(0,x)}\Phi) \frac{\partial}{\partial t} \Big|_{(0,x)} = \frac{\partial \Phi^i}{\partial t}(0, x) Y_{i,x} = \\ &= \begin{cases} \eta(x)(\tau(\phi)_x^i + (DV)_{\phi(x)}^i) Y_{i,x} & \text{if } x \in U, \\ 0 & \text{if } x \in M \setminus U, \end{cases} \end{aligned}$$

so that

$$0 = \int_M h^\phi(\mathbb{T}, \tau(\phi) + (DV) \circ \phi) \omega = \int_U \eta \|\tau(\phi) + (DV) \circ \phi\|^2 \omega$$

and then  $\eta \|\tau(\phi) + (DV) \circ \phi\|^2 = 0$  on  $U$  and in particular in  $x_0$ . Q.e.d.

#### 4. THE SUBLAPLACIAN

Let  $M$  be a strictly pseudoconvex CR manifold, of CR dimension  $n$ , and  $\theta$  a contact form on  $M$  such that  $G_\theta$  is positive definite. The *sublaplacian* is the differential operator given by

$$\Delta_b u = \operatorname{div}(\nabla^H u), \quad u \in C^2(M),$$

where  $\nabla^H u = \pi_H \nabla u$  and  $\nabla u$  is the gradient of  $u$  with respect to the Webster metric, i.e.  $g_\theta(\nabla u, X) = X(u)$ , for any  $X \in \mathcal{X}(M)$ . Also  $\pi_H X$  is the projection of  $X$  on  $H(M)$  with respect to the decomposition  $T(M) = H(M) \oplus \mathbb{R}T$ . Let  $(U, x^1, \dots, x^{2n+1})$  be a local coordinate system on  $M$  and  $\{X_a\}$  a local orthonormal

$(g_\theta(X_a, X_b) = \delta_{ab})$  frame of  $H(M)$ , defined on the open set  $U \subseteq M$ . We set  $X_a = b_a^i \partial / \partial x^i$  and  $a^{ij} = \sum_{a=1}^{2n} b_a^i b_a^j$ . A calculation shows that

$$\Delta_b u = \frac{\partial}{\partial x^i} (a^{ij} \frac{\partial u}{\partial x^j}) + c^i \frac{\partial u}{\partial x^i}, \quad c^i = -\frac{\partial a^{ij}}{\partial x^j} - a^{jk} \Gamma_{jk}^i.$$

$a^{ij}$  is symmetric and positive semi-definite, yet not definite, in general. Hence  $\Delta_b$  is a degenerate elliptic operator. Yet  $\Delta_b$  is a formally selfadjoint second order operator such that for any  $x \in M$  there is an open neighborhood  $U \subseteq M$  of  $x$  and a constant  $C > 0$  such that

$$\|u\|_{1/2}^2 \leq C(|(\Delta_b u, u)| + \|u\|^2)$$

for any  $u \in C_0^\infty(U)$ . Here  $(u, v) = \int_M u \bar{v} \theta \wedge (d\theta)^n$  is the ordinary  $L^2$  inner product,  $\|u\| = (u, u)^{1/2}$ , and  $\|u\|_\epsilon$  is the Sobolev norm of order  $\epsilon$ . That is,  $\Delta_b$  is a subelliptic operator of order  $1/2$ . Consequently  $\Delta_b$  is hypoelliptic and for any  $x \in M$  there is an open neighborhood  $U \subseteq M$  such that the following *a priori* estimates are satisfied

$$\|u\|_{s+1}^2 \leq C_s (\|\Delta_b u\|_s^2 + \|u\|^2), \quad s \geq 0,$$

for any  $u \in C_0^\infty(U)$ .

Let  $\phi : M \rightarrow N$  be a smooth map of  $M$  into a Riemannian manifold. Let  $(\Omega, y^i)$  a local coordinate system on  $N$ , such that  $\phi(U) \subseteq \Omega$ . Then the tension  $\tau(\phi)$  of  $\phi$  is locally given by

$$(4.1) \quad \tau(\phi) = \{\Delta_b \phi^i + \sum_{a=1}^{2n} (\Gamma_{jk}^i \circ \phi) X_a(\phi^j) X_a(\phi^k)\} Y_i.$$

where  $\phi^i = y^i \circ \phi$ . The proof is elementary. (4.1) shows that (1.1) may be locally written in the form (1.2).

## 5. SUBELLIPTIC HARMONIC MAPS WITH POTENTIAL

Let  $X_1, \dots, X_p$  be a set of  $C^\infty$  vector fields on an open set  $U \subseteq \mathbb{R}^m$  satisfying the *Hörmander condition*, i.e. the  $X_a$ 's together with their commutators up to a fixed length  $r$  span the tangent space at each  $x \in U$ . If  $(x^1, \dots, x^m)$  are the Cartesian coordinates on  $\mathbb{R}^m$  we set  $X_a = b_a^i(x) \partial / \partial x^i$ . Let  $X_a^*$  be the formal adjoint of  $X_a$ , i.e.

$$X_a^* f = -\frac{\partial}{\partial x^i} (b_a^i f), \quad f \in C_0^1(U).$$

Let  $Hu = -\sum_{a=1}^p X_a^* X_a u$  be the *Hörmander operator*. Let  $(N, h)$  be a Riemannian manifold and  $V : N \rightarrow \mathbb{R}$  a  $C^\infty$  function. Assume that  $N$  may be covered by a single coordinate chart  $\psi : N \rightarrow \mathbb{R}^\nu$ ,  $\psi = (y^1, \dots, y^\nu)$ . A smooth solution  $\phi : U \rightarrow N$  to the nonlinear subelliptic system

$$(5.1) \quad H_V \phi^i \equiv H \phi^i + \sum_{a=1}^p X_a(\phi^j) X_a(\phi^k) (\Gamma_{jk}^i \circ \phi) + (DV) \circ \phi = 0,$$

for  $1 \leq i \leq \nu$  (where  $\phi^i = y^i \circ \phi$  and  $DV$  is the gradient of  $V$  with respect to  $h$ ) is a *subelliptic harmonic map with potential  $V$* . The case  $V = 0$  was investigated by J. Jost et al., [6]. Let  $W_X^{1,2}(U)$  be the space of all  $u \in L^2(U)$  such that for any  $a \in \{1, \dots, p\}$  there is  $u_a \in L^2(U)$  with

$$\int_U u_a \varphi \, dx = \int_U u X_a^* \varphi \, dx$$

for any  $\varphi \in C_0^\infty(U)$ . Given  $u \in W_X^{1,2}(U)$  the functions  $u_a$  are uniquely determined a.e. in  $U$  and commonly denoted by  $X_a u$ . As well known (cf. e.g. [13], p. 149)  $W_X^{1,2}(U)$  is a separable Hilbert space with the inner product

$$(u, v)_{W_X^{1,2}(U)} = \int_U u \bar{v} \, dx + \int_U \sum_{a=1}^p (X_a u)(X_a \bar{v}) \, dx.$$

Let  $W_X^{1,2}(U, N)$  be the space of all maps  $\phi : U \rightarrow N$  such that  $\phi^i \in W_X^{1,2}(U)$ , for  $1 \leq i \leq \nu$ . A *weak solution* to (5.1) is a map  $\phi \in W_X^{1,2}(U, N)$  such that

$$\begin{aligned} \sum_{a=1}^p \int_U \{X_a(\phi^i)X_a(\varphi) + (\Gamma_{jk}^i \circ \phi)X_a(\phi^j)X_a(\phi^k)\varphi\} \, dx + \\ + \int_U [(DV)^i \circ \phi]\varphi \, dx = 0, \end{aligned}$$

for any  $\varphi \in C_0^\infty(U)$ . Let  $\Omega \subset \mathbb{R}^m$  be a domain such that  $\bar{\Omega} \subset U$ . Given  $f \in C^0(\bar{\Omega}, N)$  the *Dirichlet problem* for the subelliptic harmonic maps with potential is to solve

$$(5.2) \quad H_V \phi = 0 \text{ in } \Omega, \quad \phi = f \text{ on } \partial\Omega,$$

for  $\phi \in W_X^{1,2}(\Omega, N)$ . The Dirichlet problem with  $V = 0$  was solved in [6]. The problem (5.2) is open. Following Z-R. Zhou, [14], one may slightly generalize the notion of a subelliptic harmonic map with potential, as shown below. Let  $\gamma_{ab}(x)$  be a symmetric, positive definite matrix of functions on  $U$  and let us set  $\gamma = \det(\gamma_{ab})$  and  $(\gamma^{ab}) = (\gamma_{ab})^{-1}$ . Then a smooth map  $\phi : U \rightarrow N$  is *subelliptic harmonic with potential*  $V$  if

$$(5.3) \quad \frac{1}{\sqrt{\gamma}} X_b^* (\sqrt{\gamma} \gamma^{ab} X_a \phi^i) + \gamma^{ab} (\Gamma_{jk}^i \circ \phi) X_a(\phi^j) X_b(\phi^k) + (DV)^i \circ \phi = 0,$$

for  $1 \leq i \leq \nu$ . Clearly, the two notions coincide when  $\gamma_{ab} = \delta_{ab}$ . Let  $\Omega$  be a domain such that  $\bar{\Omega} \subset U$ . The problem whether smooth solutions to (5.3) with the same boundary values on  $\partial\Omega$  coincide (the counterpart of Theorem 5.1 in [14], p. 589, for subelliptic harmonic maps with potential) is open. The formal analogy between subelliptic harmonic maps (with potential) and the local manifestation, with respect to a local coordinate system  $(U, x^i)$  on  $M$ , of a pseudoharmonic map  $\phi : M \rightarrow N$  (with potential) becomes clear by observing that

$$\Delta_b u = - \sum_{a=1}^{2n} X_a^* X_a u$$

where  $\{X_a\}$  is a local orthonormal frame of  $H(M)$  on  $U$  and  $X_a^*$  is the formal adjoint of  $X_a$  with respect to the inner product  $(u, v) = \int_U u \bar{v} \theta \wedge (d\theta)^n$ .

## 6. THE SECOND VARIATION FORMULA

Let  $M$  be a compact strictly pseudoconvex CR manifold and  $\phi : M \rightarrow N$  a pseudoharmonic map with potential  $V : N \rightarrow \mathbb{R}$ . Let us consider a smooth 2-parameter variation  $\Phi : \tilde{M} = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times M \rightarrow N$  of  $\phi$  ( $\Phi(0, 0, x) = \phi(x)$ ,  $x \in M$ ). We consider the injection

$$\beta_{s,t} : M \rightarrow \tilde{M}, \quad \beta_{s,t}(x) = (s, t, x), \quad (s, t, x) \in \tilde{M},$$

and set  $\phi_{s,t} = \Phi \circ \beta_{s,t}$ . For any  $X \in \mathcal{X}(M)$  we consider  $\tilde{X} \in \mathcal{X}(\tilde{M})$  given by

$$\tilde{X}_{(s,t,x)} = (d_x \beta_{s,t}) X_x.$$

Also, with any vector field  $A \in \mathcal{X}(\tilde{M})$  we associate the vector field along  $\Phi$  given by

$$(\Phi_* A)_{(s,t,x)} = (d_{(s,t,x)} \Phi) A_{(s,t,x)}.$$

We shall need the vectors fields along  $\phi$

$$\mathbb{S}_x = (d_{(0,0,x)} \Phi) \frac{\partial}{\partial s} \Big|_{(0,0,x)}, \quad \mathbb{T}_x = (d_{(0,0,x)} \Phi) \frac{\partial}{\partial t} \Big|_{(0,0,x)}.$$

We establish the following

**Theorem 6.1.** *Let  $\phi : M \rightarrow N$  a pseudoharmonic map with potential  $V$ , from a compact strictly pseudoconvex CR manifold  $M$  into a Riemannian manifold  $N$ . Let  $\Phi : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times M \rightarrow N$  be a smooth 2-parameter variation of  $\phi$  and  $\mathbb{S}, \mathbb{T} \in \Gamma^\infty(\phi^{-1}TN)$  the induced vector fields on  $N$  along  $\phi$ . Then*

$$(6.1) \quad \frac{\partial^2}{\partial s \partial t} \{E_V(\phi_{s,t})\}_{s=t=0} = \int_M \sum_a h^\phi(D_{X_a}^\phi \mathbb{T}, D_{X_a}^\phi \mathbb{S}) \omega - \\ - \int_M \{ \text{trace}_{G_\theta}(\pi_H R^D(\mathbb{T}, (d\phi)\cdot, \mathbb{S}, (d\phi)\cdot)) + (\text{Hess } V)^\phi(\mathbb{T}, \mathbb{S}) \} \omega.$$

Here  $R^D$  is the Riemann-Christoffel tensor field of  $(N, h)$ . Also we set  $(\text{Hess } V)^\phi(A, B)_x = (\text{Hess } V)_{\phi(x)}(A_x, B_x)$  for any vector fields  $A, B$  on  $N$  along  $\phi$  and any  $x \in M$ .

The identity (6.1) is the *second variation formula* for the functional  $E_V$ . Let  $x \in M$  and  $\{X_a\}$  be a local orthonormal ( $G_\theta(X_a, X_b) = \delta_{ab}$ ) frame of  $H(M)$ , defined on an open set  $U$  such that  $x \in U$ . Then  $\sum_a h^\phi(D_{X_a}^\phi \mathbb{T}, D_{X_a}^\phi \mathbb{S})_x$  doesn't depend upon the choice of orthonormal frame at  $x$ . The Hessian  $\text{Hess } V$  is given by

$$(\text{Hess } V)(X, Y) = (D_X(dV))Y = X(Y(V)) - (D_X Y)V$$

for any  $X, Y \in \mathcal{X}(N)$ . *Proof of Theorem 6.1.* For any  $|s| < \epsilon$ ,  $|t| < \epsilon$

$$\text{trace}_{G_\theta}(\pi_H \phi_{s,t}^* h) = \sum_a h^\phi(\Phi_* \tilde{X}_a, \Phi_* \tilde{X}_a) \circ \beta_{s,t}$$

hence

$$\frac{\partial}{\partial t} (\text{trace}_{G_\theta}(\pi_H \phi_{s,t}^* h)) = \sum_a \left\{ \frac{\partial}{\partial t} (h^\Phi(\Phi_* \tilde{X}_a, \Phi_* \tilde{X}_a)) \right\} \circ \beta_{s,t} = \\ = 2 \sum_a \{ h^\Phi(D_{\partial/\partial t}^\Phi \Phi_* \tilde{X}_a, \Phi_* \tilde{X}_a) \} \circ \beta_{s,t} = \\ = 2 \sum_a \{ h^\Phi(D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial t} + \Phi_* [\frac{\partial}{\partial t}, \tilde{X}_a], \Phi_* \tilde{X}_a) \} \circ \beta_{s,t} = \\ = 2 \sum_a \{ \tilde{X}_a (h^\Phi(\Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X}_a)) - h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a) \} \circ \beta_{s,t}.$$

For any fixed  $|s| < \epsilon$ ,  $|t| < \epsilon$  we consider the vector field  $Y_{s,t} \in H(M)$  given by

$$G_\theta(Y_{s,t}, X) = h^\Phi(\Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X}) \circ \beta_{s,t}$$

for any  $X \in H(M)$ . A calculation shows that

$$\tilde{X}_a(h^\Phi(\Phi_* \frac{\partial}{\partial t}, \Phi_* \tilde{X}_a)) \circ \beta_{s,t} = X_a(G_\theta(Y_{s,t}, X_a)).$$

Hence

$$\begin{aligned} & \frac{\partial}{\partial t} (\text{trace}_{G_\theta}(\pi_H \phi_{s,t}^*)) = \\ & = 2 \sum_a \{X_a(G_\theta(Y_{s,t}, X_a)) - h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a) \circ \beta_{s,t}\} = \\ & = 2 \text{div}(Y_{s,t}) + 2 \sum_a \{G_\theta(Y_{s,t}, \nabla_{X_a} X_a) - h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a) \circ \beta_{s,t}\} = \\ & = 2 \text{div}(Y_{s,t}) - 2 \sum_a h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a}) \circ \beta_{s,t} \end{aligned}$$

and then (by Green's lemma)

$$(6.2) \quad \begin{aligned} & \frac{\partial}{\partial t} E_V(\phi_{s,t}) = - \int_M \left\{ \frac{\partial(V \circ \Phi)}{\partial t} \circ \beta_{s,t} + \right. \\ & \left. + \sum_a h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a}) \circ \beta_{s,t} \right\} \omega. \end{aligned}$$

Moreover

$$\begin{aligned} & \frac{\partial}{\partial s} (h^\Phi(\Phi_* \frac{\partial}{\partial t}, \sum_a (D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a})) = \\ & = h^\Phi(D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t}, \sum_a (D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a})) + \\ & + h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\partial/\partial s}^\Phi \sum_a (D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a})). \end{aligned}$$

A calculation shows that

$$(D_{\tilde{X}}^\Phi \Phi_* \tilde{Y}) \circ \beta_{0,0} = D_X^\phi \phi_* Y,$$

for any  $X, Y \in \mathcal{X}(M)$ , hence

$$(6.3) \quad \begin{aligned} & \sum_a \left\{ \frac{\partial}{\partial s} (h^\Phi(\Phi_* \frac{\partial}{\partial t}, \sum_a (D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a})) \right\} \circ \beta_{0,0} = \\ & = h^\phi((D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t}) \circ \beta_{0,0}, \tau(\phi)) + \\ & + \sum_a h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\partial/\partial s}^\Phi D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - D_{\partial/\partial s}^\Phi \Phi_* \widetilde{\nabla_{X_a} X_a}) \circ \beta_{0,0}. \end{aligned}$$

Let  $R^\Phi$  be the curvature tensor field of  $D^\Phi$ . Then (as  $[\partial/\partial s, \tilde{X}_a] = 0$ )

$$\begin{aligned} & h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\partial/\partial s}^\Phi D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - D_{\partial/\partial s}^\Phi \Phi_* \widetilde{\nabla_{X_a} X_a}) = \\ & = h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi D_{\partial/\partial s}^\Phi \Phi_* \tilde{X}_a) + \\ & + h^\Phi(\Phi_* \frac{\partial}{\partial t}, R^\Phi(\frac{\partial}{\partial s}, \tilde{X}_a) \Phi_* \tilde{X}_a - D_{\partial/\partial s}^\Phi \Phi_* \widetilde{\nabla_{X_a} X_a}) = \\ & = \tilde{X}_a(h^\Phi(\Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s})) - h^\Phi(D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s}) + \end{aligned}$$

$$+h^\Phi(\Phi_* \frac{\partial}{\partial t}, R^\Phi(\frac{\partial}{\partial s}, \tilde{X}_a)\Phi_* \tilde{X}_a - D_{\partial/\partial s}^\Phi \Phi_* \widetilde{\nabla_{X_a} X_a}).$$

Yet  $\phi$  is pseudoharmonic with potential  $V$ , i.e.  $\tau(\phi) = -(DV) \circ \phi$ , and then (6.3) becomes

$$(6.4) \quad \begin{aligned} & \sum_a \left\{ \frac{\partial}{\partial s} \left( h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, \sum_a (D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a}) \right) \right) \right\} \circ \beta_{0,0} = \\ & = -h^\phi \left( (D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t}) \circ \beta_{0,0}, (DV) \circ \phi \right) + \\ & + \sum_a \left\{ \tilde{X}_a \left( h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s} \right) \right) - h^\Phi \left( D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s} \right) \right\} \circ \beta_{0,0} + \\ & + \sum_a h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, R^\Phi \left( \frac{\partial}{\partial s}, \tilde{X}_a \right) \Phi_* \tilde{X}_a - D_{\partial/\partial s}^\Phi \Phi_* \widetilde{\nabla_{X_a} X_a} \right) \circ \beta_{0,0}. \end{aligned}$$

At this point we consider the vector field  $Y_\Phi \in H(M)$  determined by

$$G_\theta(Y_\Phi, X) = h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}}^\Phi \Phi_* \frac{\partial}{\partial s} \right),$$

for any  $X \in H(M)$ . A calculation shows that

$$X_a(G_\theta(Y_\Phi, X_a)) = \{ \tilde{X}_a \left( h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s} \right) \right) \} \circ \beta_{0,0}$$

so that (by  $\nabla g_\theta = 0$ )

$$\begin{aligned} & \sum_a \left\{ \tilde{X}_a \left( h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s} \right) \right) \right\} \circ \beta_{0,0} = \\ & = \sum_a \{ g_\theta(\nabla_{X_a} Y_\Phi, X_a) + g_\theta(Y_\Phi, \nabla_{X_a} X_a) \} = \\ & = \operatorname{div}(Y_\Phi) + \sum_a h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, D_{\widetilde{\nabla_{X_a} X_a}}^\Phi \Phi_* \frac{\partial}{\partial s} \right) \circ \beta_{0,0} = \end{aligned}$$

(as  $[\partial/\partial s, \widetilde{\nabla_{X_a} X_a}] = 0$ )

$$= \operatorname{div}(Y_\Phi) + \sum_a h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, D_{\partial/\partial s}^\Phi \Phi_* \widetilde{\nabla_{X_a} X_a} \right)$$

so that (6.4) becomes

$$(6.5) \quad \begin{aligned} & \sum_a \left\{ \frac{\partial}{\partial s} \left( h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, \sum_a (D_{\tilde{X}_a}^\Phi \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a}) \right) \right) \right\} \circ \beta_{0,0} = \\ & = -h^\phi \left( (D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t}) \circ \beta_{0,0}, (DV) \circ \phi \right) + \operatorname{div}(Y_\Phi) + \\ & + \sum_a \left\{ h^\Phi \left( \Phi_* \frac{\partial}{\partial t}, R^\Phi \left( \frac{\partial}{\partial s}, \tilde{X}_a \right) \Phi_* \tilde{X}_a \right) - h^\Phi \left( D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s} \right) \right\} \circ \beta_{0,0} \end{aligned}$$

so that (by differentiating (6.2) with respect to  $s$  and by Green's lemma)

$$\begin{aligned} & \frac{\partial^2}{\partial s \partial t} \{ E_V(\phi_{s,t}) \}_{s=t=0} = \\ & = \int_M \left\{ h^\phi \left( (D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t}) \circ \beta_{0,0}, (DV) \circ \phi \right) - \frac{\partial^2 (V \circ \Phi)}{\partial s \partial t} \circ \beta_{0,0} \right\} + \end{aligned}$$

$$\begin{aligned}
& + \int_M \sum_a h^\Phi(D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial t}, D_{\tilde{X}_a}^\Phi \Phi_* \frac{\partial}{\partial s}) \circ \beta_{0,0} - \\
& - \int_M \sum_a h^\Phi(\Phi_* \frac{\partial}{\partial t}, R^\Phi(\frac{\partial}{\partial s}, \tilde{X}_a) \Phi_* \tilde{X}_a) \circ \beta_{0,0}.
\end{aligned}$$

Note that

$$D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t} = \left\{ \frac{\partial^2 \Phi^i}{\partial s \partial t} + \frac{\partial \Phi^j}{\partial s} \frac{\partial \Phi^k}{\partial t} (\Gamma_{jk}^i \circ \Phi) \right\} Y_i$$

where  $Y_i$  are the natural lifts of the  $\partial/\partial y^i$  to  $\Phi^{-1}TN \rightarrow \tilde{M}$ . Then

$$\begin{aligned}
& \{h^\phi((D_{\partial/\partial s}^\Phi \Phi_* \frac{\partial}{\partial t}) \circ \beta_{0,0}, (DV) \circ \phi) - \frac{\partial^2(V \circ \Phi)}{\partial s \partial t} \circ \beta_{0,0}\}_x = \\
& = -(\text{Hess } V)_{\phi(x)}(\mathbb{T}_x, \mathbb{S}_x),
\end{aligned}$$

for any  $x \in M$ . In the end, the identities

$$\begin{aligned}
(D_{\tilde{X}}^\Phi \Phi_* \frac{\partial}{\partial s})_{(0,0,x)} &= (D_X^\phi \mathbb{S})_x, \quad (D_{\tilde{X}}^\Phi \Phi_* \frac{\partial}{\partial t})_{(0,0,x)} = (D_X^\phi \mathbb{T})_x, \\
(R^\Phi(\frac{\partial}{\partial s}, \tilde{X}) \Phi_* \tilde{Y})_{(0,0,x)} &= R_{\phi(x)}^D(\mathbb{S}_x, (\phi_* X)_x)(\phi_* Y)_x,
\end{aligned}$$

lead to (6.1). Q.e.d.

## 7. STABILITY

Let  $\phi : M \rightarrow N$  be a pseudoharmonic map with potential  $V$ , from a compact strictly pseudoconvex CR manifold  $M$  into a Riemannian manifold  $N$ . For any  $A, B \in \Gamma^\infty(\phi^{-1}TN)$  we set

$$\begin{aligned}
I_\phi(A, B) &= \int_M \sum_{a=1}^{2n} h^\phi(D_{X_a}^\phi A, D_{X_a}^\phi B) \omega - \\
& - \int_M \{ \text{trace}_{G_\theta}(\pi_H R^D(A, (d\phi)\cdot, B, (d\phi)\cdot)) + (\text{Hess } V)^\phi(A, B) \} \omega
\end{aligned}$$

and

$$\|D^\phi A\|_x^2 = \sum_a h^\phi(D_{X_a}^\phi A, D_{X_a}^\phi A)_x, \quad x \in U,$$

so that

$$\begin{aligned}
(7.1) \quad I_\phi(A, A) &= \int_M \{ \|D^\phi A\|^2 - (\text{Hess } V)^\phi(A, A) \} \omega - \\
& - \int_M \text{trace}_{G_\theta}(\pi_H R^D(A, (d\phi)\cdot, A, (d\phi)\cdot)) \omega.
\end{aligned}$$

*Proof of Theorem 1.1.* Let  $x \in M$  and  $\{X_a\}$  an orthonormal frame of  $H(M)$ , defined on a neighborhood  $U$  of  $x$ . For any vector field  $A$  on  $N$  along  $\phi$  let  $\sigma_a(A) \subset T_{\phi(x)}(N)$  the subspace spanned by  $\{A_x, (d_x \phi)X_{a,x}\}$  and let us set  $J_A = \{a \in \{1, \dots, 2n\} : \dim_{\mathbb{R}} \sigma_a(A) = 2\}$ . Then

$$\begin{aligned}
& \text{trace}_{G_\theta}(\pi_H R^D(A, (d\phi)\cdot, A, (d\phi)\cdot))_x = \\
& = \sum_a R_{\phi(x)}^D(A_x, (d_x \phi)X_{a,x}, A_x, (d_x \phi)X_{a,x}) = \\
& = \sum_{a \in J_A} (\|A_x\|^2 \|(d_x \phi)X_{a,x}\|^2 - h_{\phi(x)}(A_x, (d_x \phi)X_{a,x})) k(\sigma_a(A)) \leq 0,
\end{aligned}$$

where  $k : G_2(N) \rightarrow \mathbb{R}$  is the sectional curvature of  $(N, h)$ . Moreover, let  $v = A_x \in T_{\phi(x)}(N)$  and let  $\gamma : (-\epsilon, \epsilon) \rightarrow N$  the geodesic of initial data  $(\phi(x), v)$ . Then

$$(\text{Hess } V)^\phi(A, A)_x = (\text{Hess } V)_{\phi(x)}(v, v) = \frac{d^2(V \circ \gamma)}{dt^2}(0) \leq 0$$

and (7.1) implies that  $\phi$  is stable. Q.e.d.

Let us look now at the stability of pseudoharmonic maps  $\phi : M \rightarrow S^\nu$  with a potential  $V : S^\nu \rightarrow \mathbb{R}$ . Here  $j : S^\nu \subset \mathbb{R}^{\nu+1}$  is the standard unit sphere. Let  $\bar{h}$  be the Euclidean metric on  $\mathbb{R}^{\nu+1}$  and  $\bar{D}$  the corresponding Levi-Civita connection. We set  $h = j^*\bar{h}$ . Let  $\{V_r : 1 \leq r \leq \nu + 1\}$  be an orthonormal  $(\bar{h}(V_r, V_s) = \delta_{rs})$  frame of  $\mathbb{R}^{\nu+1}$  such that  $\bar{D}V_r = 0$ ,  $1 \leq r \leq \nu + 1$ . Let  $\tan_p : T_p(\mathbb{R}^{\nu+1}) \rightarrow T_p(S^\nu)$  and  $\text{nor}_p : T_p(\mathbb{R}^{\nu+1}) \rightarrow T(S^\nu)_p^\perp$  be the projections associated with the direct sum decomposition

$$T_p(\mathbb{R}^{\nu+1}) = T_p(S^\nu) \oplus T(S^\nu)_p^\perp, \quad p \in S^\nu.$$

Let us set  $W_r = \tan(V_r)$ . The proof of Theorem 1.2 is to show that  $\sum_{r=1}^{\nu+1} I_\phi(W_r^\phi, W_r^\phi) < 0$ , where  $W_r^\phi \in \Gamma^\infty(\phi^{-1}TS^\nu)$  is the natural lift of  $W_r$ . If  $W_r = f_r^j \partial/\partial y^j$  then  $W_r^\phi = (f_r^j \circ \phi)Y_j$  and the identity

$$D_{\partial/\partial y^i} W_r = \left( \frac{\partial f_r^k}{\partial y^i} + f_r^j \Gamma_{ij}^k \right) \frac{\partial}{\partial y^k}$$

leads to

$$(7.2) \quad D_{X_a}^\phi W_r^\phi = X_a(\phi^i)(D_{\partial/\partial y^i} W_r)^\phi.$$

Here  $\{X_a\}$  is a local orthonormal frame of  $H(M)$ . Let  $D$  and  $B$  be the induced connection on  $S^\nu$  and the second fundamental form of  $j$ , respectively. By the Gauss equation

$$\bar{D}_{\partial/\partial y^i} W_r = D_{\partial/\partial y^i} W_r + B\left(\frac{\partial}{\partial y^i}, W_r\right).$$

Let us set  $\xi_r = \text{nor}(V_r)$ . Let  $A_{\xi_r}$  and  $D^\perp$  be the Weingarten operator associated with the normal section  $\xi_r$  and the normal connection, respectively. Then, by the Weingarten equation

$$\bar{D}_{\partial/\partial y^i} W_r = \bar{D}_{\partial/\partial y^i}(V_r - \xi_r) = -\bar{D}_{\partial/\partial y^i} \xi_r = A_{\xi_r} \frac{\partial}{\partial y^i} - D_{\partial/\partial y^i}^\perp \xi_r$$

hence, by identifying the tangential components

$$(7.3) \quad D_{\partial/\partial y^i} W_r = A_{\xi_r} \frac{\partial}{\partial y^i}.$$

By (7.1)-(7.2) we obtain

$$D_{X_a}^\phi W_r^\phi = X_a(\phi^i)(A_{\xi_r} \frac{\partial}{\partial y^i})^\phi$$

where from

$$\begin{aligned} \|D^\phi W_r^\phi\|^2 &= \sum_a h^\phi(D_{X_a}^\phi W_r^\phi, D_{X_a}^\phi W_r^\phi) = \\ &= \sum_a X_a(\phi^i) X_a(\phi^j) h(A_{\xi_r} \frac{\partial}{\partial y^i}, A_{\xi_r} \frac{\partial}{\partial y^j}) \circ \phi. \end{aligned}$$

Given  $x \in M$  we set for simplicity  $v_{a,r} = X_a(\phi^i)_x(A_{\xi_r}\partial/\partial y^i)_{\phi(x)} \in T_{\phi(x)}(S^\nu)$ . Then (as  $V_s$  are orthonormal)

$$\begin{aligned} \|D^\phi W_r^\phi\|_x^2 &= \sum_a \|v_{a,r}\|^2 = \sum_a \sum_{s=1}^{\nu+1} \bar{h}_{\phi(x)}(v_{a,r}, V_{s,\phi(x)})^2 = \\ &= \sum_{a,s} [X_a(\phi^i)_x h(A_{\xi_r} \frac{\partial}{\partial y^i}, W_s)_{\phi(x)}]^2. \end{aligned}$$

Let us recall the identity

$$h(A_\xi X, Y) = \bar{h}(B(X, Y), \xi),$$

for any  $X, Y \in \mathcal{X}(S^\nu)$ ,  $\xi \in \Gamma^\infty(T(S^\nu)^\perp)$ . Then

$$\|D^\phi W_r^\phi\|_x^2 = \sum_{a,s} [X_a(\phi^i)_x \bar{h}(B(\frac{\partial}{\partial y^i}, W_s), \xi_r)_{\phi(x)}]^2.$$

Next, as the sphere  $S^\nu$  is totally umbilical in  $\mathbb{R}^{\nu+1}$ , i.e.  $B = h \otimes H$  (where  $H$  is the mean curvature vector of  $j$ )

$$\|D^\phi W_r^\phi\|_x^2 = \sum_{a,s} [X_a(\phi^i)_x h(\frac{\partial}{\partial y^i}, W_s)_{\phi(x)} \bar{h}(H, \xi_r)_{\phi(x)}]^2 =$$

(by  $\phi_* X_a = X_a(\phi^i) Y_i$ )

$$= \sum_{a,s} h^\phi(\phi_* X_a, W_s^\phi)_x^2 \bar{h}(H, \xi_r)_{\phi(x)}^2.$$

Note that

$$\begin{aligned} \sum_s h^\phi(\phi_* X_a, W_s^\phi)_x^2 &= \sum_s \bar{h}_{\phi(x)}((d_x \phi) X_{a,x}, V_{s,\phi(x)})^2 = \\ &= h_{\phi(x)}((d_x \phi) X_{a,x}, (d_x \phi) X_{a,x}) \end{aligned}$$

so that

$$(7.4) \quad \sum_{a,s} h^\phi(\phi_* X_a, W_s^\phi)_x^2 = \text{trace}_{G_\theta}(\pi_H \phi^* h).$$

Let  $\xi = u^r \partial/\partial u^r$ , where  $(u^1, \dots, u^{\nu+1})$  are the Cartesian coordinates on  $\mathbb{R}^{\nu+1}$ . The restriction of  $\xi$  to  $S^\nu$  is a unit normal vector on  $S^\nu$ . Then  $\bar{h}(H, \xi_r)^2 = \|\xi_r\|^2 \|H\|^2$  and we may conclude that (by (7.4))

$$(7.5) \quad \|D^\phi W_r^\phi\|^2 = (\|\xi_r\| \circ \phi)^2 \|H\|^2 \text{trace}_{G_\theta}(\pi_H \phi^* h).$$

(as  $S^\nu$  has constant mean curvature). Note that  $\sum_r \|\xi_r\|^2 = 1$ . Then (by (7.5))

$$(7.6) \quad \sum_{r=1}^{\nu+1} \|D^\phi W_r^\phi\|^2 = \|H\|^2 \text{trace}_{G_\theta}(\pi_H \phi^* h).$$

Let us recall that the curvature  $R^D$  of any totally umbilical submanifold  $N$  of a Riemannian manifold  $\bar{N}$  of constant curvature  $\bar{k}$  is given by  $R^D(X, Y)Z = (\bar{k} + \|H\|^2)\{h(Y, Z)X - h(X, Z)Y\}$ , for any  $X, Y, Z \in T(N)$ , i.e.  $N$  is a space form itself, of constant curvature  $k + \|H\|^2$  (the mean curvature  $\|H\|$  is constant as a consequence of the Codazzi equation). In the case at hand  $\bar{k} = 0$  hence (by (7.4))

$$\sum_r \text{trace}_{G_\theta}(\pi_H R^D(W_r, (d\phi) \cdot, W_r, (d\phi) \cdot)) =$$

$$\begin{aligned}
&= \sum_{r,a} h(R^D(W_r, (d\phi)X_a)(d\phi)X_a, W_r) = \\
&= \|H\|^2 \sum_{r,a} \{h^\phi(\phi_*X_a, \phi_*X_a)(\|W_r\| \circ \phi)^2 - h^\phi(W_r^\phi, \phi_*X_a)^2\} = \\
&= \|H\|^2 \text{trace}_{G_\theta}(\pi_H \phi^* h) \left[ \sum_r (\|W_r\| \circ \phi)^2 - 1 \right]
\end{aligned}$$

and  $\sum_r \|W_r\|^2 = \nu$  yields

$$\sum_r \text{trace}_{G_\theta}(\pi_H R^D(W_r, (d\phi) \cdot, W_r, (d\phi) \cdot)) = (\nu - 1) \|H\|^2 \text{trace}_{G_\theta}(\pi_H \phi^* h).$$

Summing up

$$\begin{aligned}
\sum_{r=1}^{\nu+1} I_\phi(W_r^\phi, W_r^\phi) &= -2(\nu - 2) \|H\|^2 E_0(\phi) - \\
&- \sum_{r=1}^{\nu+1} \int_M (\text{Hess } V)^\phi(W_r^\phi, W_r^\phi) \omega < 0.
\end{aligned}$$

Q.e.d.

## 8. WAVE MAPS WITH POTENTIAL

Let  $M$  be a compact strictly pseudoconvex CR manifold and  $\theta$  a contact form with  $G_\theta$  positive definite. Let  $S^1 \rightarrow C(M) \xrightarrow{\pi} M$  be the canonical circle bundle (cf. e.g. [2], p. 728). By a result of J.M. Lee, [7],  $C(M)$  carries a Lorentz metric  $F_\theta$  (the Fefferman metric of  $(M, \theta)$ ) such that  $\hat{\theta} = e^u \theta$  yields  $F_{\hat{\theta}} = e^{u \circ \pi} F_\theta$  (so that the restricted conformal class  $[F_\theta] = \{e^{u \circ \pi} F_\theta : u \in C^\infty(M)\}$  is a CR invariant). Let  $T$  be the characteristic direction of  $d\theta$  and  $S$  the tangent to the  $S^1$ -action on  $C(M)$ .  $S^1 \rightarrow C(M) \rightarrow M$  carries a natural connection 1-form  $\sigma$  (cf. (2.5) in [2], p. 728). Let  $T^\uparrow$  be the horizontal lift of  $T$  with respect to  $\sigma$ . Then  $T^\uparrow - S$  is timelike, hence  $(C(M), F_\theta)$  is time oriented by  $T^\uparrow - S$ , i.e.  $(C(M), F_\theta)$  is a (compact) space-time (see [3], p. 17).

A smooth map  $\Phi : C(M) \rightarrow N$  into a Riemannian manifold  $N$  is a *wave map with potential*  $V : N \rightarrow \mathbb{R}$  if

$$(8.1) \quad \square \Phi^i + \sum_{p,q=1}^{2n+2} F^{pq} (\Gamma_{jk}^i \circ \Phi) \frac{\partial \Phi^j}{\partial u^p} \frac{\partial \Phi^k}{\partial u^q} + (DV)^i \circ \Phi = 0,$$

for  $1 \leq i \leq \nu$ . Cf. [11], p. 3. Here  $\square$  is the wave operator (the Laplace-Beltrami operator of  $(C(M), F_\theta)$ ). Also  $F_{pq} = F_\theta(\partial_p, \partial_q)$ ,  $(F^{pq}) = (F_{pq})^{-1}$ , and  $\partial_p = \partial/\partial u^p$ , where  $(\pi^{-1}(U), u^p)$  are the natural local coordinates induced on  $C(M)$  by  $(U, x^A)$  (i.e.  $u^A = x^A \circ \pi$ ,  $1 \leq A \leq 2n+1$ , and  $u^{2n+2}$  is a fibre coordinate).

**Theorem 8.1.** *A  $S^1$ -invariant map  $\Phi : C(M) \rightarrow N$  is a wave map with potential  $V : N \rightarrow \mathbb{R}$  if and only if the corresponding base map  $\phi : M \rightarrow N$  is pseudoharmonic with potential  $V$ .*

*Proof.* (8.1) are the Euler-Lagrange equations of the variational principle  $\delta \mathbb{E}_V(\Phi) = 0$ , where

$$\mathbb{E}_V(\Phi) = \frac{1}{2} \int_{C(M)} \{\text{trace}_{F_\theta}(\Phi^* h) - 2V \circ \Phi\} d \text{vol}(F_\theta).$$

Let  $\Phi : C(M) \rightarrow N$  be a smooth  $S^1$ -invariant map and  $\phi : M \rightarrow N$  the corresponding base map. Then integration along the fibre (cf. e.g. [5], p. 300) leads to

$$\mathbb{E}_V(\Phi) = 2\pi E_V(\phi)$$

hence the necessity. The sufficiency follows from the proof of Theorem 2.1 in [2] (cf. (2.9) in *op. cit.*, p. 730).

#### REFERENCES

- [1] P. Baird & J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monographs, N.S., Ed. by E.B. Davies & P.M. Neumann, Clarendon Press, Oxford, 2003.
- [2] E. Barletta, S. Dragomir & H. Urakawa, *Pseudoharmonic maps from a nondegenerate CR manifold into a Riemannian manifold*, Indiana University Mathematics Journal, (2)50(2001), 719-746.
- [3] J. K. Beem & P. E. Ehrlich, *Global Lorentzian geometry*, M. Dekker, Inc., New York - Basel, 1981.
- [4] S. Dragomir, *A survey of pseudohermitian geometry*, The Proceedings of the Workshop on Differential Geometry and Topology, Palermo (Italy), June 3-9, 1996, in *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, Serie II, 49(1997), 101-112.
- [5] W. Greub, S. Halperin & R. Vanstone, *Connections, curvature, and cohomology*, Vol. I, Academic Press, New York - London, 1972.
- [6] J. Jost & C.-J. Xu, *Subelliptic harmonic maps*, Trans. of A.M.S., (11)350(1998), 4633-4649.
- [7] J. M. Lee, *The Fefferman metric and pseudohermitian invariants*, Trans. A.M.S., (1)296(1986), 411-429.
- [8] E. Mazet, *La formule de la variation seconde de l'énergie au voisinage d'une application harmonique*, J. Diff. Geom., 8(1973), 279-296.
- [9] R. Petit, *Harmonic maps and strictly pseudoconvex CR manifolds*, Comm. Anal. Geom., (2)41(2002), 575-610.
- [10] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Kinokuniya Book Store Co., Ltd., Kyoto, 1975.
- [11] D. Tataru, *The wave maps equation*, Bull. Amer. Math. Soc., N. S., (2)41(2004), 185-204.
- [12] S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Diff. Geometry, 13(1978), 25-41.
- [13] C.-J. Xu, *Subelliptic variational problems*, Bull. Soc. Math. France, 118(1990), 147-169.
- [14] Z. R. Zhou, *Uniqueness of subelliptic harmonic maps*, Annals of Global Analysis and Geometry, 17(1999), 581-594.