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A strict maximum principle for singular fully nonlinear operators

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Abstract[1]. In this paper we prove a strict maximum principle for the equation $F(\nabla u, D^2u) = \beta(u)$ with F a class of fully nonlinear operators modelled on the p-Laplacian but not in divergence form and β an optimal increasing function. Clearly the solutions considered are viscosity solutions. The conditions on β are linked with the homogeneity of the operator F.

1. Introduction

Let Ω be some domain of $\mathbb{R}^N$ and β be a nondecreasing continuous function $\mathbb{R} \to \mathbb{R}$ such that $\beta(0) = 0$. We prove here a strict maximum principle for fully nonlinear equations such as

$$-F(\nabla u, D^2u) + \beta(u) = 0 \text{ in } \Omega, \tag{1.1}$$

where F is assumed to be some continuous function on $(\mathbb{R}^N)^* \times \mathbf{S}_N$, $\mathbf{S}_N$ denoting the space of symmetric matrices on $\mathbb{R}^N$; moreover, F satisfies the following conditions:

(1) $F(p, 0) = 0$, $\forall p \in (\mathbb{R}^N)^*$.
(2) $\exists \alpha > -1$, $\lambda, \Lambda \in (\mathbb{R}^+)^2$, $\forall (p, M, N) \in \mathbb{R}^N \times \mathbf{S}_N \times \mathbf{S}_N^+$.

$$\lambda |p|^\alpha trN \le F(p, M + N) - F(p, M) \le \Lambda |p|^\alpha trN$$

Example 1: One example we have in mind is the q-Laplacian:

$$F(p, M) = |p|^{q-2} trM + (q-2)|p|^{q-4}(Mp, p)$$

which satisfies 2 with $\alpha = q - 2$.
Example 2: Other examples of operators satisfying the above assumptions

$$F(p, M) = |p|^\alpha \mathcal{M}^+_{\lambda\Lambda}(M) \text{ and } F(p, M) = |p|^\alpha \mathcal{M}^-_{\lambda\Lambda}(M)$$

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where $\mathcal{M}^+_{\lambda\Lambda}(M) = \Lambda\sum_{e_i>0} e_i + \lambda\sum_{e_i<0} e_i$ and $\mathcal{M}^-_{\lambda\Lambda}(M) = \lambda\sum_{e_i>0} e_i + \Lambda\sum_{e_i<0} e_i$, with $e_1, ..., e_N$ the eigenvalues of M, are the so-called Pucci operators (see Caffarelli-Cabrè [2]).

In [1], the first authors proved a comparison result for operators satisfying 1 and 2. They even consider operators depending explicitely on x.

The aim of this paper is to give some necessary and sufficient conditions on β in order that the following strong maximum principle hold true: if u is not identically 0 in Ω and satisfies:

$$-F(\nabla u, D^2u) + \beta(u) \geq 0 \text{ in } \Omega \text{ with } u \geq 0 \text{ on } \partial\Omega$$

then $u > 0$ in Ω.

This issue when F is the q-Laplacian Δ_q has been tackled out by Vazquez in [11], though in a quite different setting : In this paper, the inequation $-\Delta_q u + \beta(u) \geq 0$ is taken in the distributional sense and the function u is assumed to belong to $W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$; one then has an underlying regularity result which states that as soon as $\Delta_q u \in L^k_{loc}(\Omega)$ with $k > qN/(q-1)$ then $u \in C^{1+\alpha}$ with $\alpha \in (0,1)$ (see [5] and [10]).

Of course, we cannot consider weak solutions of (1.1) since F is not in the divergence form, we shall instead define viscosity solutions adapted to our setting: this framework allows the solutions of (1.1) to be merely continuous functions. (for a more detailed description see [3], [7], [8]...).

As in the works of Birindelli, Demengel [1] or Juutinen, Lindqvist, Manfredi [9], we must take into account the singularity of F: the test functions are not allowed to have a vanishing gradient since the operator F is not defined at $p = 0$ (when $\alpha < 0$).

Definition 1. *Let Ω be an open set in $\mathbb{R}^N$, then $v \in \mathcal{C}(\Omega)$ is called a viscosity super-solution of $F = g(x, .)$ if for all $x_0 \in \Omega$,*
-Either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $g(x, c) \geq 0$
-Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum on x_0 and $D\varphi(x_0) \neq 0$, one has

$$F(D\varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, v(x_0)). \tag{1.2}$$

Of course u is a viscosity sub-solution if for all $x_0 \in \Omega$,
-Either there exists a ball $B(x_0, \delta)$, $\delta > 0$ on which $u = cte = c$ and $g(x, c) \leq 0$,
-Or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $u - \varphi$ has a local maximum on x_0 and $D\varphi(x_0) \neq 0$, one has

$$F(D\varphi(x_0), D^2\varphi(x_0)) \geq g(x_0, u(x_0)). \tag{1.3}$$

Definition 2. *We shall say that v is a strict super-solution (respectively u is a strict sub-solution) if there exists $\epsilon > 0$ such that for all $x_0 \in \Omega$, either there exists an open ball $B(x_0, \delta)$, $\delta > 0$ in Ω on which $v = cte = c$ and $g(x, c) \geq \epsilon$, or $\forall \varphi \in \mathcal{C}^2(\Omega)$, such that $v - \varphi$ has a local minimum on x_0 and $D\varphi(x_0) \neq 0$, one has*

$$F(D\varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, v(x_0)) - \epsilon\ .$$

(respectively either $u = cte$ on a ball $B(x_0, \delta)$ and $g(x, cte) \leq -\epsilon$, or in (1.3), one has $F(x_0, D\varphi(x_0), D^2\varphi(x_0)) \geq g(x_0, u(x_0)) + \epsilon$).

We now state our main result:

Theorem 1. *Let Ω be some domain of $\mathbb{R}^N$ and F satisfying 1. and 2. Let $\beta : \mathbb{R} \to \mathbb{R}$ be some nondecreasing function satisfying the additional property: either there exists $t > 0$ such that $\beta(t) = 0$, or*

$$\int_0^\delta (j(s))^{-1/(\alpha+2)} ds = \infty, \ (\delta > 0) \tag{1.4}$$

where j is the primitive of β vanishing at 0.

Let $u \in C(\bar{\Omega})$ be a supersolution of

$$-F(\nabla u, D^2 u) + \beta(u) \geq 0 \ \text{in } \Omega$$

such that $u \geq 0$ on $\partial\Omega$.

Then either $u \equiv 0$ in Ω, either $u > 0$ in Ω.

Condition (1.4) says that β must not grow rapidly near $u = 0$. For instance, it is satisfied as soon as $\beta(u) \leq Cu^{\alpha+1}$ for some $C > 0$.

In a last section, we prove that the sufficient condition (1.4) given in Theorem 1 is optimal in some sense:

Theorem 2. *Assume that F satisfies 1, 2, 3, and that β is a continuous nondecreasing function, $\beta(0) = 0$; we assume that $\beta(t) > 0$ for all $t > 0$ and*

$$\int_0^\delta (j(s))^{-1/(\alpha+2)} ds < \infty \quad (\delta > 0)$$

where $j(s) = \int_0^s \beta(t)dt$.

Then for any strict subdomain Ω of $\mathbb{R}^N$, there exist a nonnegative supersolution v of (1.1) in Ω and $x_0 \in \Omega$ such that $v(x_0) = 0$ but v is not identically zero in Ω.

Clearly a key ingredient to prove strict maximum principle is the existence of a comparison theorem see e.g. [6] and [11]. In our setting, we shall use the following result which is proved in [1]:

Theorem 3. *Suppose that F satisfies condition 1, and 2. Let $u \in C(\bar{\Omega})$ be a subsolution of (1.1) and $v \in C(\bar{\Omega})$ be a strict supersolution of (1.1) such that $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

2. Proof of Theorem 1

First step: Let M be a symmetric matrix and $M = M^+ - M^-$ be a minimal decomposition of M into the difference of two nonnegative matrices. Condition 2 implies:

$$\lambda |p|^\alpha tr(M^+) \leq F(p, M) - F(p, -M^-) \leq \Lambda |p|^\alpha tr(M^+), \tag{2.1}$$

$$\lambda |p|^\alpha tr(M^-) \leq F(p, 0) - F(p, -M^-) \leq \Lambda |p|^\alpha tr(M^-). \tag{2.2}$$

Substracting (2.2) from (2.1) and using condition 1, one gets:

$$|p|^\alpha (\lambda tr(M^+) - \Lambda tr(M^-)) \leq F(p, M) \ \text{ for all } p \in (\mathbb{R}^N)^*, M \in \mathbf{S}_N$$

Setting $H(p, M) = |p|^\alpha (\lambda tr(M^+) - \Lambda tr(M^-))$, one observes that u is a viscosity subsolution of:

$$-H(\nabla u, D^2 u) + \beta(u) \geq 0 \ \text{ in } \Omega \tag{2.3}$$

Second step: Suppose that u vanishes somewhere in Ω but is not identically zero: the sets $N = \{x \in \Omega; u(x) = 0\}$ and $P = \{x \in \Omega; u(x) > 0\}$ are not empty. Using the classical procedure in Hopf's construction (see [6]), one can then pick $x_1 \in P$ such that $d(x_1, N) < \frac{2}{3} d(x_1, \partial\Omega)$. Setting $R = d(x_1, N)$ and $B_r = B_r(x_1)$, one has:

$$\bar{B}_{3R/2} \subset \Omega \text{ and } \exists x_0 \in \partial B_R,\ u(x_0) = 0\ .$$

We now consider the annulus

$$G = \left\{ x \in \Omega; \frac{R}{2} < |x - x_1| < \frac{3R}{2} \right\} \subset \Omega$$

and we set

$$m = \inf \left\{ u(x); |x - x_1| = \frac{R}{2} \right\} > 0\ .$$

The idea is to construct a radial strict subsolution v of $-H(\nabla v, D^2 v) + \beta(v) = 0$ and use the comparison principle to get a contradiction.

Lemma 1. *Let a, m be some positive constants and $\gamma : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function such that $\gamma(0) = 0$. Assume that j, the primitive of γ vanishing at 0, satisfies*

$$\int_0^\delta (j(s))^{-1/(\alpha+2)} ds = \infty,\ (\delta > 0)\ .$$

Then there exists a unique C^2 solution $v = v(r)$ to the problem

$$\text{(2.4)} \qquad \begin{cases} (\alpha+1)v''|v'|^\alpha &= av'|v'|^\alpha + \gamma(v)\ ,\ 0 < r < R \\ v(0) = 0, v(R) &= m. \end{cases}$$

and $0 < v < m$ on $]0, R[$, $v' > 0$, $v'' \geq 0$ on $[0, R]$.

The proof of Lemma 1 is postponed at the end of this section.

Let v be the solution of (2.4) with

$$a > \frac{2\Lambda(N-1)(\alpha+1)}{3\lambda R} \quad \text{and } \gamma(u) = \frac{(\alpha+1)\beta(u)}{\lambda}\ .$$

Let $\bar{v}(r) = v(\frac{3}{2}R - r)$ and

$$\tilde{v}(x) = \bar{v}(r) - v\left(\frac{R}{2}\right), \quad \text{where } r = |x - x_1|, x \in G.$$

Let us recall that the eigenvalues of $D^2\psi$ where $\psi(x) = \phi(|x|)$ is a radial function, are $\phi''(r)$ of multiplicity 1 and $\dfrac{\phi'(r)}{r}$ of multiplicity $(N-1)$ see [4]. Using the definition of the Pucci operators we obtain :

$$\begin{aligned} H(\nabla\tilde{v}, D^2\tilde{v}) &= |\bar{v}'(r)|^\alpha \left(\lambda\bar{v}''(r) - \frac{\Lambda(N-1)}{r}\bar{v}'(r) \right) = \\ &= \frac{\lambda}{\alpha+1}\left((\alpha+1)\bar{v}'(r)^\alpha \bar{v}''(r) - a\bar{v}'(r)^{\alpha+1} \right) + \\ &+ \left(\frac{\lambda a}{\alpha+1} - \frac{\Lambda(N-1)}{r} \right) \bar{v}'(r)^{\alpha+1} = \\ &= \beta\left(\bar{v}(r)\right) + \left(\frac{\lambda a}{\alpha+1} - \frac{\Lambda(N-1)}{r} \right) \bar{v}'(r)^{\alpha+1}\ . \end{aligned}$$

a has been chosen in order that the second term in the RHS is positive and hence using the fact that β is nondecreasing, one gets:

$$H(\nabla\tilde{v}(x), D^2\tilde{v}(x)) > \beta\left(v\left(\frac{3R}{2} - r\right)\right) + \epsilon \geq \beta(\tilde{v}(x)) + \epsilon \text{ in } G\,. \tag{2.5}$$

In other words, $\tilde{v}$ is a strict subsolution of $-H + \beta = 0$ in G.

We now want to prove that $u \geq \tilde{v}$ in G. One has

$$\forall x \text{ such that } |x - x_1| = R/2,\ \tilde{v}(x) \leq v(R) = m \leq u(x)\ ,$$

$$\forall x \text{ such that} |x - x_1| = 3R/2,\ \ \tilde{v}(3) = v(0) - v(R/2) \leq 0 \leq u(x)\ ,$$

which means that $\tilde{v} \leq u$ on ∂G. Coming back to (2.3) and (2.5), and applying Theorem 3 to the operator H, one deduces $u \geq \tilde{v}$ in G.

Finally, since $\nabla\tilde{v}(x_0) \neq 0$ then $\tilde{v}$ is a test function for u at x_0 in the sense of Definition 1. Thus, as $\tilde{v}(x_0) = 0 = u(x_0)$, the following inequality holds true

$$-H(\nabla\tilde{v}(x_0), D^2\tilde{v}(x_0)) + \beta(\tilde{v}(x_0)) \geq 0\ .$$

This contradicts (2.5). Hence, u cannot be 0 inside Ω.

Proof of Lemma 1.

- Existence: Let us consider the following minimization problem

$$\inf_{v\in W^{1,\alpha+2}(0,R);v(0)=0,v(R)=m}\left\{\int_0^R \left(\frac{1}{\alpha+2}|v'(r)|^{\alpha+2} + j(v(r))\right) e^{-ar}dr\right\}\ .$$

  There exists some $v \in W^{1,\alpha+2}(0,R)\cap C([0,R])$ which realizes this infimum. It satisfies the Euler equation:

$$(\alpha+1)|v'|^{\alpha}v'' = a|v'|^{\alpha}v' + \gamma(v)\ . \tag{2.6}$$

- Uniqueness: suppose that v_1 and v_2 satisfy (2.4); then, multiplying (2.6) by e^{-ar}, one has

$$\left(e^{-ar}v_i^{'}|v_i^{'}|^{\alpha}\right)^{'} = e^{-ar}\gamma(v_i),\ 0 < r < R,\ i = 1, 2\ .$$

  Substracting the two equations, multiplying by $v_1 - v_2$ and integrating, one deduces

$$\int_0^R \left(e^{-ar}(v_1^{'}|v_1^{'}|^{\alpha} - v_2^{'}|v_2^{'}|^{\alpha})\right)^{'}(v_1 - v_2)dr = \int_0^R e^{-ar}(\gamma(v_1) - \gamma(v_2))(v_1 - v_2)dr$$

  which gives after an integration by part:

$$-\int_0^R e^{-ar}\underbrace{(v_1^{'}|v_1^{'}|^{\alpha} - v_2^{'}|v_2^{'}|^{\alpha})(v_1^{'} - v_2^{'})}_{\geq 0}dr = \int_0^R e^{-ar}\underbrace{(\gamma(v_1) - \gamma(v_2))(v_1 - v_2)}_{\geq 0}dr\ .$$

  One then gets that both sides are equal to zero, which implies $v_1' = v_2'$ in $[0, R]$; since they have the same value at 0 and R, one deduces that $v_1 = v_2$ on $[0, R]$.

- $v \geq 0$: As we have seen, the solution v of (2.4) satisfies:

$$\left(e^{-ar}v^{'}|v^{'}|^{\alpha}\right)^{'} = e^{-ar}\gamma(v),\ 0 < r < R$$

  Multiplying by v^- and integrating by parts over $[0, R]$, one gets:

$$\int_0^R e^{-ar}|(v^-)'|^{\alpha+2} = -\int_0^R e^{-ar}\gamma(v^-)$$

which implies that $(v^-)' \equiv 0$ on $]0,R[$; since v^- is zero at 0 and R, one deduces that $v^- \equiv 0$ on $[0,R]$.

- $v' \geq 0$: v satisfies $\left(e^{-ar}v'|v'|^\alpha\right)' = e^{-ar}\gamma(v) \geq 0$ which implies that $r \mapsto e^{-ar}v'(r)|v'(r)|^\alpha$ is nondecreasing. Moreover, since $v \geq 0$ and $v(0)=0$, one has $v'(0) \geq 0$. One deduces that $v'(r) \geq 0$, for all $r \in [0;R]$.
- $v'(0) > 0$ and $v(r) > 0$ for all $0 < r < R$: Indeed, by contradiction let $r_0 \in [0;R[$ be the largest r for which $v(r) = 0$. One has

$$\int_{r_0}^{R} \frac{v'(s)ds}{j(v(s))^{1/(\alpha+2)}} = \int_{v(r_0)}^{v(R)} \frac{ds}{j(s)^{1/(\alpha+2)}} = \int_0^m \frac{ds}{j(s)^{1/(\alpha+2)}} = \infty$$

Now, setting $w = (v')^{\alpha+2}$, one computes

$$\begin{aligned} j(v)' = \gamma(v)v' &= \left((\alpha+1)v''(v')^\alpha - a(v')^{\alpha+1}\right)v' = \\ &= (\alpha+1)v''(v')^{\alpha+1} - a(v')^{\alpha+2} = \\ &= \frac{\alpha+1}{\alpha+2}w' - aw \end{aligned}$$

and then

$$e^{-\frac{\alpha+2}{\alpha+1}ar}\left[j(v(r))\right]' = \frac{\alpha+1}{\alpha+2}\left(e^{-\frac{\alpha+2}{\alpha+1}ar}w(r)\right)'$$

Suppose that $v'(r_0) = 0$; integrating the last equality between r_0 and $r > r_0$, one obtains

$$e^{-\frac{\alpha+2}{\alpha+1}ar_0}j(v(r)) \geq \frac{\alpha+1}{\alpha+2}e^{-\frac{\alpha+2}{\alpha+1}aR}w(r)$$

and then

$$\int_{r_0}^{R}\frac{v'(r)dr}{j(v(r))^{1/(\alpha+2)}} \leq \left(\frac{\alpha+2}{\alpha+1}\right)^{1/(\alpha+2)} e^{\frac{1}{\alpha+1}a(R-r_0)}(R-r_0) < \infty$$

which yields a contradiction. Hence $v'(r_0) > 0$ and, from the definition of r_0, this implies $r_0 = 0$. It follows that $v'(0) > 0$ and $v(r) > 0$ for all $0 < r < R$.
- Coming back to the equation (2.6), we obtain finally that v is C^2 and $v'' \geq 0$.

3. Necessity of condition (1.4)

This last section is devoted to the proof of Theorem 2. Let us first notice that, coming back to (2.1) and (2.2), and using condition 1., one has for all $p \in (\mathbb{R}^N)^*$ and $M \in \mathbf{S}_N$:

$$F(p,M) \leq |p|^\alpha(\Lambda tr(M^+) - \lambda tr(M^-)) \leq \Lambda|p|^\alpha tr(M^+) \tag{3.1}$$

Let us then choose $a \in]0,\delta[$ and define $I(a) = K\int_0^a j(s)^{-1/(\alpha+2)}ds$ where $K = \left(\frac{\Lambda}{\alpha+2}\right)^{1/(\alpha+2)}$.

One notes that $I(a) \to 0$ when $a \to 0$. We then define $u(x)$ as being the unique $y \in [0,a]$ such that

$$x = K\int_y^a j(s)^{-1/(\alpha+2)}ds \ \text{ for all } \ x \in [0, I(a)]\,.$$

The uniqueness of y comes easily from the fact that $j(s) > 0$ for $s > 0$.

One easily checks that u is nonincreasing on $[0, I(a)]$ and satisfies

$$\begin{cases} \Lambda u''(x)(-u'(x))^\alpha &= \beta(u(x)) \text{ on }]0, I(a)[\\ u(0) = a, u(I(a)) &= 0. \end{cases}$$

We then set $u \equiv 0$ on $[I(a); +\infty[$ so that u is C^1 on $]0, \infty[$ and C^2 on $]0, I(a)[\bigcup]I(a), \infty[$.

Suppose now that $N \geq 2$ and let Ω be some subdomain of $\mathbb{R}^N$, $\Omega \neq \mathbb{R}^N$. Let $x_0 \in \partial\Omega$ and $a > 0$ small enough so that $\Omega \backslash \overline{B}(x_0, I(a)) \neq \emptyset$.

We define

$$v(x) = u(|x - x_0|) \text{ for } x \in B(x_0, I(a)) \cap \Omega$$

and

$$v \equiv 0 \text{ in } \Omega \backslash \overline{B}(x_0, I(a)) \ .$$

Using (3.1) and the fact that the only positive eigenvalue of $D^2 v(x)$ is $u''(r)$ with multiplicity one, we get

$$\begin{aligned} F(\nabla v(x), D^2 v(x)) &\leq \Lambda(-u'(r))^\alpha u''(r) = \\ &= \beta(u(r)) = \beta(v(x)) \end{aligned}$$

for all $x \in B(x_0, I(a)) \cap \Omega$.

Take now $x \in \Omega \backslash B(x_0, I(a))$. Since $v \equiv 0$ in $\Omega \backslash B(x_0, I(a))$ and $\beta(0) = 0$ then

$$-F(\nabla v, D^2 v) + \beta(v) \geq 0 \text{ in } \Omega$$

in the sense of definition 1.

Nevertheless, v is not positive in Ω nor identically 0 in Ω.

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