Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 3(2004), pp. 65 - 75.

Harnack inequality for nonlinear Dirichlet forms

by Marco Biroli and Paola Gioia Vernole

Abstract[1]. We define the notion of Dirichlet functionals and forms in the nonlinear p-homogeneous case. We give the chain rule for the energy of functional or of form. Finally we consider the Riemannian case and, in this case we prove an Harnack inequality on balls for the harmonics relative to the form.

1. Introduction

The first goal in this paper is an extension of notion and fundamental properties of strongly local (regular) Dirichlet form to the nonlinear case. For the notion of Dirichlet form we refer to the book of Fukushima-Oshima-Takeda, [15]. In [15] a purely analytical proof of fundamental properties of Dirichlet form is given, this type of proof firstly appeared in [21]; we recall also the papers [4], [5], where an analytical investigation of the properties of the harmonics relative to a strongly local "Riemannian" Dirichlet forms ia carried on. From Beurling-Deny representation formula, [2] a Dirichlet form is represented as the sum of a strongly local part, of a "killing" part and of a global part. The Beurling-Deny representation theorem is the fundamental tool allowing to prove that same properties of Dirichlet forms (in particular the Markov property) hold again for energy measures in the strong local (regular) case. Using the above mentioned properties of energy measure it can be proved that for the energy measure of a strongly local (regular) Dirichlet form a chain rule and a Leibnitz rule hold; those properties are the starting point for an investigation of local regularity of harmonics relative to a strongly local (regular) Dirichlet form, see in particular [4], [5]. The Beurling-Deny representation theorem is proved using Riesz theorem on representation of measures, which is an essentially linear tool, then it seems that proving a nonlinear version of this result is difficult.

Previous work on a possible extension of the notion of Dirichlet form to the nonlinear case has been given by Benilan-Picard, [1], and Cipriani-Grillo, [12] [13]. In particular in [1] the relations between maximum principle and Markov property are investigated generalizing to nonlinear monotone case previous results obtained in [2] and [17] in the linear case. In [13] a notion of nonlinear Dirichlet form is given

---

[1] Authors' address: M. Biroli, Politecnico di Milano, Dipartimento di Matematica "F. Brioschi", Via Bonardi 9, 20133 Milano, Italy; e-mail: `marbir@mate.polimi.it` .

Research supported by MIUR Research Project 2002012589

P. G. Vernole, Universit degli Studi di Roma "La Sapienza", Dipartimento di Matematica "G. Castelnuovo", Piazzale Aldo Moro, 2, 00185 Roma; e-mail: `vernole@mat.uniroma1.it` .

Keywords. Dirichlet forms, Harnack inequality, Regularity of harmonic functions.
Subject Classification. 31C25, 31C05, 35B65.

and the relations with a class of nonlinear semigroups (the order preserving contractions semigroups with a cyclically monotone generator) are investigated. The above papers deal with the general global case and are interested in the properties of the corresponding nonlinear semigroup; then the existence of an energy measure is not ensured and there is no proof of chain or Leibnitz rule for the energy measure, when such a measure exists.The first paper concerning local forms was [19], where a suitable chain rule for the energy measure connected with the form is assumed and Sobolev-Morrey inequalities are proved as a consequence of a Poincaré inequality. In [7], [8], [9], [10]some nonlinear forms on fractals are given in explicit form and it is proved that the assumptions in [19] hold (see also the more recent papers [23], [16] on the p-Laplacian on the Sierpinski gasket).

In this paper we are interested to the notion of nonlinear strongly local Dirichlet forms and we give our assumptions (in particular the Markov property) directly on the energy measure of the form, whose existence is assumed. We are able to prove in this framework (by purely analytical methods in the line of [21]) suitable Leibnitz and chain rule. We define also the notion of Riemannian functionals and forms generalizing the notion relative to linear case [4], [5]). In the Riemannian case we investigate the local regularity of the harmonics relative to Riemannian forms and in particular we give a proof of an Harnack type inequality for positive harmonics (we observe that the chain rule proved here is the same assumed in [19] and similar to the one proved in the linear case in [4], [5]).

We finally observe that our framework contains the case of subelliptic p-Laplacian, $p > 1$, related to vector fields X_i, $i = 1, ..., m$, which satisfy an Hörmander condition or to Grushin vector fields, considered on $\mathbb{R}^N$ endowed with the Lebesgue measure or considered on $\mathbb{R}^N$ endowed with a measure having a density in the intrinsic A_p class as well as the p-Laplacians on fractals considered in [7], [8], [10] (see also [23], [16], where the Authors give a construction of a p-Laplacian on the Sierpinski gasket and investigate the Hölder continuity of harmonics). Finally we recall that an Harnack inequality for positive harmonics relative to vector fields X_i, $i = 1, ..., m$, which satisfy an Hörmander condition considered on $\mathbb{R}^N$ endowed with the Lebesgue measure has been proved in [11].

2. NONLINEAR DIRICHLET FUNCTIONALS AND FORMS

We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that supp$[m]= X$. Let $\Phi : L^p(X, m) \to [0, +\infty]$, $1 < p$, be a l.s.c. convex functional with domain D, i.e. $D = \{v;\ \Phi(v) < +\infty\}$, with $\Phi(0) = 0$. We assume that D is dense in $L^p(X, m)$ and that the following conditions hold:

(H_1) D is a dense linear subspace of $L^p(X, m)$,which can be endowed with a norm $||.||_D$; moreover D has a structure of uniformly convex Banach space with respect to the norm $||.||_D$ and the following estimate holds

$$c_1||v||_D^p \leq \Phi_1(v) = \Phi(v) + \int_X |v|^p dm \leq c_2||v||_D^p$$

for every $v \in D$, where $c_1,\ c_2$ are positive constants.

(H_2) We denote by D_0 the closure of $D \cap C_0(X)$ in D (with respect to the norm $||.||_D$ and we assume that $D \cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on X, moreover we assume that Φ_1 is locally uniformly convex on D_0, i.e. if we have

$\lim_{n\to 0}\Phi_1(\frac{u_n+u}{2}) = \Phi_1(u)$ and $\lim_{n\to 0} u_n = u$ weakly in D_0 then $\lim_{n\to 0} u_n = u$ in D_0.

Remark 2.1. We observe that, since Φ is convex, Φ is l.s.c. also with respect to the weak topology of $L^p(X,m)$. We remark that the assumption (H_1) substantially does not allow us to deal with the case $p = 1$ or with sublinear functionals. Moreover from the assumption (H_1) it follows that Φ is continuous on D for the norm $||.||_D$, [22] Ch.1 Sec.2 p. 20, then from (H_2) the restriction of Φ to D_0 coincides with the relaxation of Φ defined on $D \cap C_0(X)$.

The assumptions $(H_1)(H_2)$ have a global character; now we will define a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$. Let Φ satisfy (H_1) and (H_2); we say that Φ is a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$ if the following conditions hold:

(H_3) Φ has the following representation on D_0 $\Phi(u) = \int_X \alpha(u)(dx)$ where α is a non-negative bounded Radon measure depending on $u \in D_0$, which does not charge sets of zero capacity. We say that $\alpha(u)$ is the energy (measure) of our functional. The energy $\alpha(u)$ (of our functional) is convex with respect to u in D_0, i.e. let $u, v \in D_0$ and $t \in [0,1]$ then $\alpha(tu + (1-t)v) \leq t\alpha(u) + (1-t)\alpha(v)$, in the measures and is homogeneous of degree $p > 1$, i.e. $\alpha(tu) = |t|^p\alpha(u)$, $\forall u \in D_0$, $\forall t \in \mathbb{R}$. Moreover the following closure property holds: if $u_n \to u$ in D and $\alpha(u_n)$ converges to χ in the measures then $\chi \geq \alpha(u)$.

(H_4) α is of strongly local type, i.e. if $u, v \in D$ and $u - v = \mathit{constant}$ on an open set A we have $\alpha(u) = \alpha(v)$ on A.

(H_5) α is of Markov type, i.e. let $\beta \in C^1(\mathbb{R})$ such that $\beta'(t) \leq 1$ and $\beta(0) = 0$ and $u \in D \cap C_0(X)$ then $\beta(u) \in D \cap C_0(X)$ and $\alpha(\beta(u)) \leq \alpha(u)$ in the sense of measures.

We recall that as proved in [6] for a strongly local Dirichlet functional it is possible to define a domain relative to an open set Ω in X, denoted by $D_0(\Omega)$, ($D_0(\Omega)$ is the closure of $C_0(\Omega)$ into D_0) and also the local domain relative to Ω, denoted by $D_{loc}(\Omega)$ ($D_{loc}(\Omega)$ is the set of the functions in $L^P_{loc}(\Omega,m)$ such that for every relatively compact open set ω with closure in Ω there exists $v \in D_0(\Omega)$ such that $u = v$ a.e. on ω). We observe that the extension to X by 0 of a function in $D_0(\Omega)$ is in D_0 We recall also that in [6] has been proved that a notion of Choquet capacity relative to an open set Ω can be defined and that a function in $D_0(\Omega)$ can has a quasi-everywhere quasi-continuous representant for the capacity relative to Ω.

In [6] a chain rule is proved for strongly local Dirichlet functionals

Theorem 2.1. *Let $u \in D_0 \cap L^\infty(\Omega,m)$ and $\beta \in C^1(\mathbb{R})$ with $\beta(0) = 0$; then*

$$\alpha(\beta(u)) = |\beta'(u)|^p\alpha(u).$$

The same result holds if β' is bounded and $u \in D_0$.

Theorem 2.2. *Let $u \in D_0$ and $\beta_1(t) = inf(t,M)$, $\beta_2(t) = sup(t,-M)$, $\beta_3 = inf(sup(t,-M_1),M_2)$, $M, M_1, M_2 \geq 0$; then $\beta_i(u) \in D_0$ and*

$$\alpha(\beta_1(u)) = \mathbf{1}_{\{u<M\}}\alpha(v)$$

$$\alpha(\beta_2(u)) = \mathbf{1}_{\{u>-M\}}\alpha(v)$$

$$\alpha(\beta_3(u)) = \mathbf{1}_{\{-M_1<u<M_2\}}\alpha(v)$$

where $\mathbf{1}_E$ *denotes the characteristic function of the set* E *(which is defined up to sets of capacity zero).*

Remark 2.2. We observe that the results in Theorem 2.1 and 2.2 have meaning since the function u is defined up to sets of zero capacity.

Let $\Phi(u) = \int_X \alpha(u)(dx)$ be a strongly local Dirichlet functional with domain D_0. Assume that for every $u, v \in D_0$ we have

$$\lim_{t\to 0} \frac{\alpha(u+tv)-\alpha(u)}{t} = \mu(u,v)$$

in the weak* topology of the measures uniformly for u, v in a compact set of D_0, where $\mu(u,v)$ is defined on $D_0 \times D_0$ and is linear in v. We say that $\Psi(u,v) = \int_X \mu(u,v)(dx)$ is a strongly local Dirichlet form.

Remark 2.3. From the above assumptions it follows that if u_n, $n = 1, 2, ...$, converges to u in D_0 then

$$\lim_{n\to+\infty} \mu(u_n, v) = \mu(u,v)$$

in the weak* topology of the measures.

Theorem 2.3. *The following properties hold:*
(1) $\mu(\lambda u, v) = |\lambda|^{p-2}\lambda\mu(u,v)$ *for* $\lambda \neq 0$ *real and* $\mu(0,v) = 0$ *(here and in the following we assume* $|t|^{p-2}t = sign(t)|t|^{p-1}$*)*
(2) the measure $\mu(u,v)$ *does not charge sets of zero capacity*
(3) If we have $u_1 - u_2 = constant$ *on the open set* A, $u_1, u_2 \in D_0$, *then* $\mu(u_1, v) = \mu(u_2, v)$ *on* A *for every* $v \in D_0$
(4) If we have $v_1 - v_2 = constant$ *on an open set* A, $v_1, v_2 \in D_0$, *then* $\mu(u, v_1) = \mu(u, v_2)$ *on* A *for every* $u \in D_0$
(5) $p\alpha(u) = \mu(u,u)$
(6) $|\mu(u,v)| \leq 2^{p-1}a^{-p}\alpha(u) + 2^{p-1}a^{p(p-1)}\alpha(v)$ *for every* $a > 0$ *where* $u, v \in D_0$.

For strongly local also a suitable chain rule has been proved in [6]:

Theorem 2.4. *Let* $u \in D_0 \cap L^\infty(X,m)$,$v \in D_0$ *and* $\beta \in C^1(R)$ *with* $\beta(0) = 0$. *Then we have* $\beta(u), \beta(v) \in D_0$ *and*

$$\mu(\beta(u), v) = |\beta'(u)|^{p-2}\beta'(u)\mu(u,v)$$

$$\mu(u, \beta(v)) = \beta'(v)\mu(u,v)$$

The same result hold if β' *is bounded and* $u, v \in D_0$.

A Leibnitz rule does not hold in general with respect to the first argument of the form, but has been proved in [6] with respect to the second argument of the form:

Theorem 2.5. *Let* $u \in D_0$ *and* $v, w \in D_0 \cap L^\infty(X,m)$, *then* $vw \in D_0$ *and the following Leibnitz rule holds*

$$\mu(u, vw) = v\mu(u,w) + w\mu(u,v)$$

We also have suitable chain rules with respect to the truncate functions, [6]:

Theorem 2.6. *Let $u, v \in D_0$ and define $\beta_1(t) = inf(t, M)$, $\beta_2(t) = sup(t, -M)$, $\beta_3 = inf(sup(t, -M_1), M_2)$, $M, M_1, M_2 \geq 0$; then $\beta_i(u)$ and $\beta_i(v)$, $i = 1, 2, 3$, are in D_0 and*

$$\begin{aligned} \mu(\beta_1(u), v) &= \mathbf{1}_{\{u<M\}}\mu(u, v) \\ \mu(u, \beta_1(v)) &= \mathbf{1}_{\{v<M\}}\mu(u, v) \\ \mu(\beta_2(u), v) &= \mathbf{1}_{\{u>-M\}}\mu(u, v) \\ \mu(u, \beta_2(v)) &= \mathbf{1}_{\{v>-M\}}\mu(u, v) \\ \mu(\beta_3(u), v) &= \mathbf{1}_{\{-M_1<u<M_2\}}\mu(u, v) \\ \mu(u, \beta_3(v)) &= \mathbf{1}_{\{-M_1<v<M_2\}}\mu(u, v) \end{aligned}$$

where $\mathbf{1}_E$ denotes the characteristic function of the set E (which is defined up to sets of capacity zero).

From property (6) in Theorem 2.3 we obtain:

Proposition 2.7. *Let $f \in L^p(X, \alpha(u))$ and $g \in L^p(X, \alpha(u))$; then*

$$|fg||\mu(u, v)| \leq \epsilon|f|^p\alpha(u) + C_\epsilon|g|^p\alpha(v)$$

$$|fg||\mu(u, v)| \leq C_\epsilon|f|^p\alpha(u) + \epsilon|g|^p\alpha(v)$$

where $\epsilon > 0$ is arbitrary.

Proof. If f,g are functions which are measurable with respect to $\alpha(u)$ and $alpha(v)$ with a finite number of values the result follows easily from Theorem 2.3 with a suitable choice of a. In the general case we have that f (g) can be approximated in $L^p(X, alpha(u))$ $(L^p(X, alpha(u)))$ by a sequence f_n (g_n) of functions with a finite number of values. It follows that $f_n g_n$ converges to fg in $L^1(X, |\mu(u, v))$. The above inequalities hold for f_n, g_n for every n; then passing to the limit as $n \to +\infty$ we obtain the result.

□

As we have observed no Leibnitz formula is available for the first argument in the form or for the energy of the functional. In the following result we give a Leibnitz type inequality, which is enough from the technical point of view for the use of many methods of proof:

Proposition 2.8. *Let u,v be in$D_0 \cap L^\infty(X, m)$; then uv is again in $D_0 \cap L^\infty(X, m)$ and*

$$\alpha(uv) \leq C(|u|^p\alpha(v) + |v|^p\alpha(u))$$

in the measures, where C is a constant that does not depend on u,v.

Proof. From Theorem 2.1 we have that u^2 and v^2 are in $D_0 \cap L^\infty(X, m)$; then uv is in $D_0 \cap L^\infty(X, m)$. From Theorems 2.3, 2.5 we have

$$\alpha(uv) = \frac{1}{p}\ \mu(uv, uv) = \frac{1}{p}\ (u\mu(uv, v) + v\mu(uv, u))$$

By Proposition 2.7 we obtain

$$\alpha(uv) \leq \frac{1}{p}\ (\frac{1}{4}\alpha(uv) + C|u|^p\alpha(v) + \frac{1}{4}\ \alpha(uv) + C|v|^p\alpha(u)$$

and the result follows.

□

3. THE RIEMANNIAN CASE AND THE HARNACK INEQUALITY

We assume
(A_1) $D_0 \cap C_0(X)$ is separating ,i.e. for every $x, y \in X$, $x \neq y$, there exists a function φ in $D_0 \cap C_0(X)$ with $\alpha(\varphi, \varphi) \leq m$ such that $\varphi(x) \neq \varphi(y)$. In this case we can define a *distance* function related to Φ by

$$d(x,y) = sup\{\varphi(x) - \varphi(y); \quad \varphi \in D[a] \cap C_0(X) \ \text{ with } \ \alpha(\varphi,\varphi) \leq m\}$$

and we assume that the topology defined on X by d is equivalent to the initial one. We denote by $B(x,r)$ the ball for d of radius r and center x.
(A_2) a *duplication property*, that is

$$m(B(y,s)) \geq c_0(\frac{s}{t})^{\nu} m(B(y,t))$$

$0 < s < t$, with a constant c_0 does not depend on y in a given compact set.

Proposition 3.1. *We have that $d(x,.) \in D_{loc}(X)$ and $\alpha(d) \leq m$.*

Proof. To prove the result is enough to prove that for every $y \in X$ there exists a ball $B(y,r)$ such that $d(x,.)$ is in $D_{loc}[a, B(y,r)]$. We can cover $B(y,r)$ by a finite number of balls $B(y_i, n^{-1})$, i=1,...,m_n. By the same methods of Lemma 3.2 of [5] we can construct a function ϕ_n such that $\alpha(\phi_n, \phi_n) \leq m$, $\phi_n(x) = 0$, and $\phi_n(y) \geq d(x,.) - n^{-1}$. Let now ϕ be the supremum with respect to n of ϕ_n we have $\phi = d(x,.)$ on $B(y,r)$ and $\alpha(\phi,\phi) \leq m$, then $d(x,.) \in D_{loc}(B(y,r))$.

☐

As a consequence of Proposition 3.1 we have:

Proposition 3.2. *Consider the intrinsic balls $B(x,r)$ and $B(x,\rho)$, $\rho < R$. There exists a function ϕ (the cut-off function between the two balls) such that $\phi = 1$ on $B(x,\rho)$, $\phi = 0$ in $X - B(x,R)$ and*

$$\alpha(\phi) \leq \frac{C}{(r-\rho)^p} m$$

in the measures

Proof. Consider a function $\varphi(t)$ decreasing in $C^1(\mathbb{R}_+)$ such that $\varphi(t) = 1$ for $0 \geq t \leq \rho$, $\varphi(t) = 0$ if $t \geq r$, $0 \geq \varphi \leq 1$, $\varphi'(t) \leq \frac{C}{(r-\rho)}$. We obtain the result choosing $\phi(y) = \varphi(d(x,y))$. We also observe that, due to locality property $\alpha(\phi) = 0$ on $B(x,\rho)$ and on $X - B(x,r)$.

☐

We assume also:
(A_3) a *Poincaré* inequality holds of the type

$$\int_{B(x,r)} |f - f_{x,r}|^p m(dy) \leq c_1 r^p \int_{B(x,\kappa r)} \alpha(f) m(dy) \qquad (P)$$

for all f in $D_{loc}(B(x,\kappa r)$, $r \leq R_0$ and x varies in a relatively compact open set, where $\kappa \geq 1$ is a suitable fixed constant,$f_{y,r}$ is the average of f on $B(y,r)$ and c_1 does not depend on x in the given compact set. A Dirichlet functional satisfying (A_1)-(A_3) is called a *Riemannian* Dirichlet functional. A Dirichlet form relative to a Riemannian Dirichlet functional is called a Riemannian Dirichlet form.

We observe that a Riemannian Dirichlet functional satisfies the assumption in [19]; then Sobolev type inequalities hold:

Proposition 3.3. *Let $\nu > p$ and f be in $D_{loc}(B(X.\kappa r))$. Then we have*

$$(\frac{1}{m(B(x,r))}\int_{B(x,r)}|f-f_{x,r}|^s m(dy))^{\frac{1}{s}} \leq \qquad (S)$$

$$\leq c_2\, r\, (\frac{1}{m(B(x,\kappa r))}\int_{B(x,\kappa r)}\alpha(f)m(dy))^{\frac{1}{p}}$$

where $s = \frac{\nu p}{\nu - p}$, $r \leq R_0$ and x varies in a relatively compact open set.

We observe that we can always assume $\nu > p$; then in the following we deal only with the case $\nu > p$.

From (S) it follows easily that if $f \in D_0(B(x,r))$ we have

$$(\frac{1}{m(B(x,r))}\int_{B(x,r)}|f|^s m(dy))^{\frac{1}{s}} \leq c_3 r(\frac{1}{m(B(x,r))}\int_{B(x,r)}\alpha(f)m(dx))^{\frac{1}{p}} \qquad (S_1)$$

Using the cut off function η between the balls $B(x,\rho)$ and $B(x,r)$, $\frac{1}{2}r \geq \rho < r$, applying (S_1) to ηf where $f \in D_{loc}(B(x,2r)$, we obtain

$$(\frac{1}{m(B(x,\rho))}\int_{B(x,\rho)}|f|^s m(dy))^{\frac{1}{s}} \leq \qquad (S_2)$$

$$\leq c_3(r^p\frac{1}{m(B(x,r))}\int_{B(x,r)}\alpha(f)m(dx) + \frac{C}{(r-\rho)^p m(B(x,r))})^{\frac{1}{p}}\int_{B(x,r)}|f|pm(dy)$$

(in the proof of (S_2) we use also an approximation of f by bounded function in $D_{loc}(B(x,2r))$.

Let now Ω be a relatively compact open set and μ a Riemannian Dirichlet form:

Definition 3.1. *A function $u \in D_{loc}(\Omega)$ is subharmonic (with respect to a fixed Riemannian Dirichlet form μ if*

$$\int_\Omega \mu(u,\phi)(dx) \leq 0$$

for every nonnegative $\phi \in D_0(\Omega)$ with support contained in Ω. A function $u \in D_{loc}(\Omega)$ is superharmonic (with respect to a fixed Riemannian Dirichlet form μ) if

$$\int_\Omega \mu(u,\phi)(dx) \geq 0$$

for every nonnegative $\phi \in D_0(\Omega)$ with support contained in Ω. A function $u \in D_{loc}(\Omega)$ is harmonic (with respect to a fixed Riemannian Dirichlet form μ) if

$$\int_\Omega \mu(u,\phi)(dx) = 0$$

for every nonnegative $\phi \in D_0(\Omega)$ with support contained in Ω.

We observe that a function $u \in D_{loc}(\Omega)$ is harmonic iff is both subharmonic and super harmonic and superharmonic. Using the test function $\phi = \eta^p u$, where η is the cut-off function between the balls $B(x,\rho)$ and $B(x,r)$ we obtain easily a Caccioppoli type inequality for subharmonics:

Proposition 3.4. *Let $u \in D_{loc}(\Omega)$ be subharmonic in Ω, $B(x,r) \subset \Omega$ and $\rho < r$; we have*

$$\int_{B(x,\rho)} \alpha(u)(dy) \le \frac{C}{(r-\rho)^p} \int_{B(x,r)} u^p m(dy)$$

Proposition 3.4 and (S_2) allow us to use a Moser iteration technique,[20] and prove the following result:

Proposition 3.5. *Let $u \in D_{loc}(\Omega)$ be a nonnegative subharmonic function in Ω. Let $B(x,4r) \subset \Omega)$ and $q > p-1$. Then there exists positive constants C_q, L such that for $\frac{1}{2} \le s < t \le 1$ we have*

$$sup_{B(x,\frac{sr}{2})} u \le \left(\frac{C_q}{(t-s)^L\, m(B(x,tr))} \int_{B(x,tr)} u^q m(dy)\right)^{\frac{1}{q}}$$

By Proposition 3.5 as in [5] Lemma 5.2 we obtain:

Theorem 3.6. *Let $u \in D_{loc}(\Omega)$ be a nonnegative subharmonic function in Ω. Let $B(x,4r) \subset \Omega)$ and $q > 0$. Then there exists a positive constants C_q such that we have*

$$sup_{B(x,\frac{r}{2})} u \le \left(\frac{C_q}{m(B(x,r))} \int_{B(x,r)} u^q m(dy)\right)^{\frac{1}{q}}$$

The chain rule in section 2 allow us to prove the following result:

Proposition 3.7. *Let $u \in D_{loc}(\Omega)$ be a nonnegative superharmonic function in Ω. Let $B(x,4\kappa r) \subset \Omega)$ then for every $\epsilon > 0$ we have*

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |\log(u+\epsilon) - (\log(u+\epsilon))_{x,r}|^p m(dy) \le C$$

where C depends only on c_0, c_1, κ and p.

Proof. For every $\epsilon > 0$, $v + \epsilon \in D_{loc}(\Omega)$, and

$$\int_{B(x,2r)} \mu((u+\epsilon), w)(dy) \ge 0 \tag{3.1}$$

for every $w \in D_0(B(x,2r))$, $w \ge 0$ m a.e.. Let η be the cut off function between $B(x,r)$ and $B(x,2r)$. Then by the chain rule

$$\begin{aligned}
&\int_{B(x,2r)} \eta^p \mu(\log(u+\epsilon), \log(u+\epsilon))(dy) = \\
&= \int_{B(x,2r)} \eta^p (u+\epsilon)^{-p} \mu((u+\epsilon),(u+\epsilon))(dy) = \\
&= \frac{1}{1-p} \int_{B(x,2r)} \eta^p \mu((u+\epsilon),(u+\epsilon)^{1-p})(dy) = \\
&= \frac{1}{1-p} [\int_{B(x,2r)} \mu((u+\epsilon), \eta^p (u+\epsilon)^{1-p})(dy) - \\
&- \int_{B(x,2r)} (u+\epsilon)^{1-p} \mu((u+\epsilon),\eta^p)(dy)] \le \\
&\le \frac{1}{p-1} \int_{B(x,2r)} p\eta^{p-1} (u+\epsilon)^{1-p} \mu((u+\epsilon),\eta)(dy)
\end{aligned}$$

We get:

$$\int_{B(x,2r)} \eta^p \mu(\log(u+\epsilon), \log(u+\epsilon))(dx) \leq$$

$$\leq \frac{1}{2} \int_{B(x,2r)} \eta^p \mu(\log(u+\epsilon), \log(u+\epsilon))(dy) +$$

$$+ \frac{2^{p^2-1}}{(p-1)^p}\ p \int_{B(x,2r)} \alpha(\eta)(dy)$$

Hence by the duplication property

$$\int_{B(x,2r)} \eta^p \mu(\log(u+\epsilon), \log(u+\epsilon))(dx) \leq \frac{2^{p^2} pc}{(p-1)^p r^p}\ m(B(x,2r)) \leq \frac{C}{r^p}\, m(B(x,r))$$

and since $\epsilon = 1$ on $B(x,r)$.

By Poincaré inequality we have

$$\int_{B(x,r)} |\log(u+\epsilon) - (\log(u+\epsilon))_{x,r}|^p m(dy) \leq$$

$$\leq c_2 r^p \int_{B(x,\kappa r)} \tilde{\mu}(\log(u+\epsilon), \log(u+\epsilon))(dx) \leq$$

$$\leq cm(B(x,\kappa r))$$

and the thesis follows using the duplication property.

□

Proposition 3.7 proves in particular that $\log(u+\epsilon)$ is in BMO locally in Ω with a norm in BMO, which is bounded independently from ϵ. Then recalling Proposition 5.5 and Corollary 5.6 in [5] and by the same methods used in Proposition 5.7 of [5] we obtain:

Proposition 3.8. *Let $u \in D_{loc}(\Omega)$ be a nonnegative superharmonic function in Ω (different from zero). Let $B(x_0, 12\kappa R) \subset \Omega)$ then there exists a positive constant γ for every $x \in B(x_0, R)$, $0 < r \leq R$ we have*

$$(\frac{1}{m(B((x,r))} \int_{B(x,r)} u^\gamma m(dy))(\frac{1}{m(B(x,r))} \int_{B(x,r)} u^{-\gamma} m(dy)) \leq K$$

where K, γ are positive constants depending only on c_0, c_1, κ and p.

We give now a lemma about the behavior of the power of positive harmonics, whose proof is essentially founded on the chain rule:

Proposition 3.9. *Let u be a positive harmonic in $B(x,r)$. Then the function u^q belongs to $D_{loc}(B(x,r)$ and is a positive subharmonic in $B(x,4r)$ if $q < 0$ or $q > 1$, while is a superharmonic if $q \in (0,1)$.*

We are now in position to prove the main result of the paper:

Theorem 3.10. *Let u be a nonnegative harmonic in $A\Omega$. Then for every ball $B(x,r) \subseteq B(x, 16\kappa r) \subseteq \Omega$ we have*

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u$$

where C is a constant depending on c_0, c_1, κ and p.

Proof. It is enough to prove the result in the case u positive. Let u be a local positive harmonic, by the previous lemma we know that $w = u^{-1}$ is a positive subsolution. We can thus apply Theorem 3.6 and we find that

$$\sup_{B(x,r)} w \leq C_q(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} w^q m(dy))^{1/q}$$

and

$$\inf_{B(x,r)} u \geq C_q^{-1}(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^{-q} m(dy))^{-1/q}.$$

Now we choose $q = \gamma$ where γ is the constant of Proposition 3.8 and suppose that $B(x, 16\kappa r) \subset \Omega$; then

$$inf_{B(x,r)} u \geq C_\gamma^{-1}(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^{-\gamma} m(dy))^{-\frac{1}{\gamma}} \geq$$

$$\geq C_\gamma^{-1} K^{-\frac{1}{\gamma}}(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^{\gamma} m(dy))^{\frac{1}{\gamma}} \geq C_\gamma^{-2} K^{-\frac{1}{\gamma}} \sup_{B(x,r)} u$$

where we use Theorem 3.6.

REFERENCES

[1] P. Bénilan & C. Picard, *Quelques aspects non linéaires du principe du maximum*, Séminaire de Théorie du Potentiel, IV, Lectures Notes in Math. 713, 1979, Springer Verlag, Berlin- Heidelberg-New York, 1-37.

[2] A. Beuerling & J. Deny, *Dirichlet spaces*, Proc. Nat. Ac. Sc., **45**(1959), 208-215.

[3] M. Biroli, *Strongly local nonlinear Dirichlet functionals*, in Nonlinear Partial Differential Equations, Alushta, Kryme, May 2003.

[4] M. Biroli & U. Mosco, *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, Comptes Rendus Acad. Sc. Paris, 313(1991), 593-598.

[5] M. Biroli & U. Mosco, *A Saint Venant type Principle for Dirichlet forms on discontinuous media*, Annali Mat. Pura Appl., 169(IV)(1995), 125-181.

[6] M. Biroli & P. G. Vernole, *Strongly local nonlinear Dirichlet functionals and forms*, Preprint, 2003.

[7] R. Capitanelli, *Lagrangians on homogeneous spaces*, Tesi di Dottorato di Ricerca, Università di Roma "La Sapienza", 2001.

[8] R. Capitanelli, *Nonlinear energy forms on certain fractal curves*, Journal Convex Anal., 3(2002), 67-80.

[9] R. Capitanelli, *Functional inequalities for measure valued lagrangians on homogeneous spaces*, Adv. Math. Sci. Appl., 13(2003), 301-313.

[10] R. Capitanelli & M. R. Lancia, *Nonlinear energy forms and Lipschitz spaces on the Koch curve*, Journal Convex Anal., 9(2002), 245-257.

[11] L. Capogna, D. Danielli & N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Commun. in Partial Diff. Eq., 18(1993), 1765-1794.

[12] F. Cipriani & G. Grillo, *The Markov property for classes of nonlinear evolutions*, Nonlinear Anal., 47(2001), 3549-3554.

[13] F. Cipriani & G. Grillo, *Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces*, J. Reine Angew. Math., 562(2003), 201-235.

[14] N. Dunford & J. Schwartz, *Linear Operators I*, Interscience Publ., New York, 1957.

[15] M. Fukushima, Y. Oshima & M. Takeda, *Dirichlet forms and Markov processes*, W. de Gruyter & Co., Berlin-Heidelberg-New York, 1994.

[16] P.E. Herman, R. Peirone & R. S. Strichartz, *p-Energy and p-harmonic functions on Sierpinski gasket type fractals*, Preprint, 1994.

[17] F. Hirsch, *Familles résolvantes, générateurs, cogénérateurs, potentiels*, Ann. Inst. Fourier, 22(1972), 89-210.

[18] E. H. Hille & R. S. Phillips, *Functional analysis and semigroups*, Colloqium Publications XXXI, American Mathematical Society , Providence, 1957.

[19] J. Malỳ & U. Mosco, *Remarks on measure valued Lagrangians on homogeneous spaces*, Ricerche di Matem., 48(1999), 217-231.

[20] J. Malý & W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Math. Surveys Mon., 51, American Mathematical Society, Providence, 1997.

[21] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal., 123(1994), 368-421.

[22] D. Pascali & S. Sburlan, *Nonlinear mappings of monotone type*, Ed. Academiei, Bucarest, 1978.

[23] R. S. Strichartz & C. Wong, *The p-Laplacian on the Sierpinski gasket*, Preprint, 2003.