Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 3(2004), pp. 77 - 96.

Hypersurfaces of minimal type in sub-Riemannian geometry

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Abstract[1] We introduce a notion of horizontal mean curvature on a smooth hypersurface in a Carnot group, and develop a sub-Riemannian calculus in such ambients. In this calculus a smooth surface is minimal if and only if it is a stationary point of the horizontal perimeter functional.

The development of the twentieth century's mathematics has been enormously influenced by the desire to understand minimal surfaces. The present status of the subject has reached a level of great sophistication made possible by the major developments in Calculus of Variations, Partial Differential Equations and Geometric Measure Theory which have occurred over the past seventy years. Quite recently, such developments have spurred an increasing interest in a subject which is presently still in its infancy and which is in many respects *terra incognita*: the study of minimal surfaces in sub-Riemannian geometry. This new area of interest offers beautiful and remarkable new challenges. The purpose of the present paper is to survey some of the results and ideas which have been recently set forth.

Roughly speaking, a minimal surface is a measurable set which minimizes the perimeter according to De Giorgi inside a given open set $\Omega \subset \mathbb{R}^m$ among all measurable subsets which have the same "boundary values" outside Ω. The measure theoretic existence of such "surfaces" is readily established using the basic properties of the perimeter along with a fundamental compactness theorem for BV functions. Such existence result is at the basis of the developments connected with the *problem of Plateau* which, in its simplest formulation, consists in finding a surface of least area which spans an assigned curve.

Another fundamental question is connected with the so-called Bernstein problem. The latter states that a C^2 minimal graph in $\mathbb{R}^{m+1}$ must necessarily be an affine hyperplane. In this context the denomination "minimal surface" has a quite different meaning than in the above recalled measure theoretic solution of the

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Keywords. Minimal surfaces; H-mean curvature; integration by parts; first and second variation; monotonicity of the H-perimeter.

Plateau problem. The latter involves measurable sets which minimize the perimeter, whereas in the Bernstein problem the word minimal indicates an hypersurface $S \subset \mathbb{R}^{m+1}$ which is a priori assumed at least C^2, and whose mean curvature

$$H \overset{def}{=} \frac{\kappa_1 + \cdots + \kappa_m}{m} ,$$

vanishes identically. We recall that the principal curvatures $\kappa_1, \cdots, \kappa_m$ of S are the eigenvalues of the Weingarten map $\mathcal{W} = -D\nu \circ (Dx)^{-1}$, where $x : U \subset \mathbb{R}^m \to \mathbb{R}^{m+1}$ is a parametrization of S, and $\nu : U \to \mathbb{S}^m$ denotes the outward pointing Gauss map of S. It is a classical fact that $H = (1/m)\, trace\, \mathcal{W}$.

It is natural to wonder whether the classical notion of minimal surface in the Bernstein problem is in any way connected to the more modern one, based on geometric measure theory, in the above recalled formulation of the Plateau problem. While an answer to this question would take us to the core of the subject, for which we refer the reader to the classical monographs [24], [31], [35], here we confine ourselves to recalling the following basic result. Denote by $G_\lambda(S)$ the hypersurface obtained by deforming S with the one-parameter family of local diffeomorphisms $G_\lambda(x) = x + \lambda f(x)\nu(x)$, where $f \in C_o^\infty(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^n$ an open set containing S, and $\lambda \in \mathbb{R}$ is small. In what follows H_s denotes the s-dimensional Hausdorff measure in $\mathbb{R}^{m+1}$.

Theorem A *The first variation of the area of S is given by the formula*

$$\frac{d}{d\lambda} H_m(G_\lambda(S))\Big|_{\lambda=0} = \int_S m\, H\, f\, d\sigma , \tag{0.1}$$

where H indicates the mean curvature of S. The second variation is given by

$$\frac{d^2}{d\lambda^2} H_m(G_\lambda(S))\Big|_{\lambda=0} = \int_S \left\{ |\delta f|^2 + f^2 \left[\sum_{i=1}^{m+1} |\delta \nu_i|^2 - m^2 H^2 \right] \right\} d\sigma , \tag{0.2}$$

where δ denotes the Riemannian gradient on S.

It is clear from (0.1) that S is minimal in the classical sense $H \equiv 0$ if and only if it is a stationary point of the area functional. The second variation formula (0.2) allows to study the problem of stability of minimal surfaces. Here, of course, the central notion is that of area or perimeter. In sub-Riemannian geometry there exists a corresponding notion of perimeter which plays an equally central role, and which we now recall, see [3] or also the forthcoming book [20].

Consider a Carnot group $\boldsymbol{G}$ of step r. This is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is graded and r-nilpotent. This means that there exists vector subspaces $V_1, \cdots, V_r \subset \mathfrak{g}$ such that $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$, with $[V_1, V_i] = V_{i+1}$, $i = 1, \cdots, r-1$, $[V_1, V_r] = \{0\}$. A natural family of non-isotropic dilations on $\mathfrak{g}$ associated with this grading is given by $\Delta_\lambda(\xi) = \lambda \xi_1 + \cdots + \lambda^r \xi_r$, if $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$. Using the global diffeomorphism $\exp : \mathfrak{g} \to \boldsymbol{G}$, one then lifts these dilations to the one-parameter family of group automorphisms $\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g)$, $g \in \boldsymbol{G}$. The homogeneous dimension associated with the dilations $\{\delta_\lambda\}_{\lambda>0}$ is given by $Q = \sum_{i=1}^r i\, dim\, V_i$. Such number replaces the topological dimension $N = \sum_{i=1}^r dim\, V_i$ in the analysis of $\boldsymbol{G}$. Its relevance is expressed by the fact that, if dg denotes the bi-invariant Haar measure on $\boldsymbol{G}$ obtained by pushing forward through the exponential mapping the Lebesgue measure on $\mathfrak{g}$, then $d(\delta_\lambda(g)) = \lambda^Q dg$. Here, bi-invariant means with respect to the operators of left- and right-translation $L_g(g') = gg'$, $R_g(g') = g'g$, on $\boldsymbol{G}$.

Let $\{e_{i,j_i}\}$, $i = 1, \cdots, r$, $j_i = 1, \cdots, dim\, V_i$, be an orthonormal basis of $\mathfrak{g}$, such that for each $i = 1, \cdots, r$ the vectors $\{e_{i,1}, \cdots, e_{i,dim\, V_i}\}$ constitute an orthonormal basis of V_i. Because of the special role played by the first layer V_1 in the grading of $\mathfrak{g}$, it will be convenient to have a simpler notation for the elements of its basis. We thus set henceforth $m = dim\, V_1$, and will indicate with $\{e_1, \cdots, e_m\}$ the corresponding basis of V_1. Denoting with $(L_g)_*$ the differential of left-translations, we define a family of left-invariant vector fields on $\boldsymbol{G}$ by letting $X_i(g) = (L_g)_*(e_i)$, $i = 1, \cdots, m$, $X_{i,j_i}(g) = (L_g)_*(e_{i,j_i})$, $i = 2, \cdots, r$, $j_i = 1, \cdots, dim\, V_i$. We assume that $\boldsymbol{G}$ is endowed with a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ with respect to which the vector fields $X_1, \cdots, X_m$, X_{i,j_i}, $i = 2, \cdots, r$, $j_i = 1, \cdots, dim\, V_i$ are orthonormal. In view of the grading assumption on $\mathfrak{g}$, it is clear that the vector fields $X_1, \cdots, X_m$ generate the Lie algebra of all left-invariant vector fields on $\boldsymbol{G}$. Thanks to a fundamental result of Hörmander [26], this implies that the so-called sub-Laplacian on $\boldsymbol{G}$,

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2 \ ,$$

is hypo-elliptic. Of course, in the truly non-abelian case $r > 1$ such operator fails to be elliptic at every point of $\boldsymbol{G}$. The vector fields $X_1, \cdots, X_m$ generate a sub-bundle $H\boldsymbol{G}$ of the tangent bundle $T\boldsymbol{G}$ which is usually called the horizontal bundle. The sub-gradient of u is defined by

$$\nabla_H\, u = \sum_{j=1}^{m} \langle \nabla u, X_j \rangle\, X_j \ ,$$

where we have denoted by ∇u the Riemannian gradient of u. Notice that $\nabla_H u$ is the projection of ∇u onto the subbundle $H\boldsymbol{G}$. Given a function u on $\boldsymbol{G}$, we will routinely identify $X_j u$ with $\langle \nabla u, X_j \rangle$.

Minimizing length on those absolutely continuous paths joining two points $g, g' \in \boldsymbol{G}$ and whose tangent vector lies in $H\boldsymbol{G}$ at every point, one obtains the so-called Carnot-Carathéodory distance $d(g, g')$, see [4], [7], [34], [25]. Such distance plays a predominant role in the analysis and geometry of $\boldsymbol{G}$, whereas the underlying Riemannian distance $d_R(g, g')$ is, loosely speaking, confined to the background.

Given an open set $\Omega \subset \boldsymbol{G}$, we let

$$\mathcal{F}(\Omega) = \{\zeta \in C^1_o(\Omega, H\boldsymbol{G}) \mid |\zeta|_\infty = \sup_{\Omega} |\zeta| \le 1\} \ ,$$

where for $\zeta = \sum_{j=1}^m \zeta^j X_j$, we have let $|\zeta| = (\sum_{j=1}^m (\zeta^j)^2)^{1/2}$. For a function $u \in L^1_{loc}(\Omega)$, the *H-variation* of u with respect to Ω is defined by

$$Var_H(u;\Omega) = \sup_{\zeta \in \mathcal{F}(\Omega)} \int_{\boldsymbol{G}} u \sum_{j=1}^{m} X_j \zeta^j \, dg \ .$$

A function $u \in L^1(\Omega)$ is called of bounded H-variation in Ω if $Var_H(u;\Omega) < \infty$. The space $BV_H(\Omega)$ of functions with bounded H-variation in Ω, endowed with the norm

$$\|u\|_{BV_H(\Omega)} = \|u\|_{L^1(\Omega)} + Var_H(u;\Omega) \ ,$$

is a Banach space. Let $E \subset \boldsymbol{G}$ be a measurable set, Ω be an open set. The *H-perimeter* of E with respect to Ω was introduced in [3], see also [20], following De

Giorgi's classical definition [12]

$$P_H(E;\Omega) = Var_H(\chi_E;\Omega)\ , \tag{0.3}$$

where χ_E denotes the indicator function of E. We recall that two related definitions were independently set forth in [2], and in [17]. An H-minimal surface in an open set $\Omega \subset \boldsymbol{G}$ was defined in [21] as the boundary of a set of least H-perimeter, among all those with the same boundaries outside Ω. The existence of such "surfaces" (a priori, these are just sets of locally finite H-perimeter), and a measure theoretic solution of the Plateau problem, were also established in [21] following the classical approach due to De Giorgi. Using classical arguments as for instance in [16], one obtains from the Riesz representation theorem.

Theorem B. *Given an open set $\Omega \subset \boldsymbol{G}$, let $E \subset \boldsymbol{G}$ be a set with locally finite H-perimeter in Ω. There exist a Radon measure $\|\partial_H E\|$ in Ω, and a $\|\partial_H E\|$-measurable function $\boldsymbol{\nu}_H^E : \Omega \to H\boldsymbol{G}$, such that*

$$|\boldsymbol{\nu}_H^E(g)| = 1 \qquad \textit{for} \quad \|\partial_H E\| - a.e. \quad g \in \Omega\ ,$$

and for which one has for every $\zeta \in C_o^1(\Omega; H\boldsymbol{G})$

$$\int_E \sum_{j=1}^m X_j \zeta^j\, dg = \int_\Omega \langle \zeta, \boldsymbol{\nu}_H^E \rangle\, d\|\partial_H E\| = \int_\Omega \langle \zeta, d[\partial_H E] \rangle\ .$$

Let $E \subset \boldsymbol{G}$ be a C^1 domain, with Riemannian outer unit normal $\boldsymbol{\nu}$. If $\zeta \in C_o^1(\Omega; H\boldsymbol{G})$, we have

$$\int_E \sum_{j=1}^m X_j \zeta^j\, dg = \int_{\partial E \cap \Omega} \sum_{j=1}^m \zeta^j\, \langle \boldsymbol{\nu}, X_j \rangle\, dH_{N-1}\ . \tag{0.4}$$

It is worth noting at this point that, when $\boldsymbol{G}$ is Abelian, then $r = 1$ and thus $\mathfrak{g} \cong \mathbb{R}^m \cong \boldsymbol{G}$. In this situation, we have $X_j = \partial/\partial x_j$, the standard basis of $\mathbb{R}^m$, and thus (0.4) allows to recapture the standard $(m-1)$-dimensional area of that portion of ∂E which lies inside Ω. In the truly non-Abelian situation however, it is clear from (0.4) that that portion of ∂E where $\langle \boldsymbol{\nu}, X_j \rangle = 0$ becomes relevant. This leads us to recall the following basic notion.

Henceforth, we indicate with $\mathcal{S} \subset \boldsymbol{G}$ a C^2 oriented hypersurface. We will denote by $\boldsymbol{N}$ the non-unit outer Riemannian normal to $\mathcal{S}$, and will indicate with $\boldsymbol{\nu} = \boldsymbol{N}/|\boldsymbol{N}|$ the Riemannian Gauss map of $\mathcal{S}$.

Definition 0.1. *A point $g_o \in \mathcal{S}$ is called* characteristic *if one has $H_{g_o}\boldsymbol{G} \subset T_{g_o}\mathcal{S}$. Notice that this is equivalent to saying that*

$$X_j(g_o) \in T_{g_o}\mathcal{S}\ , \qquad j = 1, \cdots, m\ . \tag{0.5}$$

The characteristic set *of $\mathcal{S}$, $\Sigma = \Sigma_{\mathcal{S}}$, is the collection of all characteristic points of $\mathcal{S}$.*

An old, yet basic, result of Derridj [14], [15] guarantees that if $S \in C^\infty$, then $H_{N-1}(\Sigma) = 0$. A considerable improvement of such result has been recently established by Balogh [1] for the Heisenberg group (in co-dimension one), and by Magnani [29], [30] for general Carnot groups (and any co-dimension). In particular, this latter author has proved that if $S \subset \boldsymbol{G}$ is a C^1 hypersurface, then $\mathcal{H}_{Q-1}(\Sigma) = 0$, where for $s > 0$ we have denoted by $\mathcal{H}_s$ the s-dimensional Hausdorff measure constructed with the Carnot-Carathéodory distance $d(g, g')$. Since it was proved in [9], see also

[10], that for a C^2 domain the H-perimeter measure is mutually absolutely continuous with respect to the Hausdorff measure $\mathcal{H}_{Q-1}$, we conclude from the results in [30] that for such domains the H-perimeter measure of the characteristic set is zero. It is often convenient to analyze the characteristic set in terms of the so-called angle function W. Let

$$p_j = \langle \boldsymbol{N}, X_j \rangle \ , \qquad j = 1, \cdots, m \ , \qquad W = \sqrt{p_1^2 + \cdots + p_m^2} \ . \tag{0.6}$$

If $g_o \in \mathcal{S}$ is characteristic, then we have $p_j(g_o) = 0$, $j = 1, \cdots, m$, and therefore one has the following alternative characterization of the characteristic set,

$$\Sigma = \{ g \in \mathcal{S} \mid W(g) = 0 \} \ .$$

We also let

$$\overline{p}_j = \frac{p_j}{W} \ , \qquad \text{so that} \qquad \overline{p}_1^2 + \cdots + \overline{p}_m^2 \equiv 1 \qquad \text{on} \qquad \mathcal{S} \setminus \Sigma \ . \tag{0.7}$$

The next definition plays a basic role in the sequel.

Definition 0.2. *We define the* horizontal normal $\boldsymbol{\mathcal{Y}}_H : \mathcal{S} \to H\boldsymbol{G}$ *by the formula*

$$\boldsymbol{\mathcal{Y}}_H = \sum_{j=1}^m \langle \boldsymbol{\nu}, X_j \rangle \, X_j \ . \tag{0.8}$$

The horizontal Gauss map $\boldsymbol{\nu}_H : \mathcal{S} \setminus \Sigma \to H\boldsymbol{G}$ *is defined by*

$$\boldsymbol{\nu}_H = \frac{\boldsymbol{\mathcal{Y}}_H}{|\boldsymbol{\mathcal{Y}}_H|} = \sum_{j=1}^m \overline{p}_j \, X_j \ . \tag{0.9}$$

We note explicitly that $\boldsymbol{\mathcal{Y}}_H$ is the projection of the normal $\boldsymbol{\nu}$ on the horizontal sub-bundle $H\boldsymbol{G} \subset T\boldsymbol{G}$. Such projection vanishes only at characteristic points, and this is why the horizontal Gauss map is not defined on Σ. An obvious consequence of the definition which is, nonetheless, important is

$$|\boldsymbol{\nu}_H|^2 \equiv 1 \ , \qquad \text{in} \quad \mathcal{S} \setminus \Sigma \ , \tag{0.10}$$

which is of course a reformulation of the second equation in (0.7). One also has

$$\langle \boldsymbol{\nu}_H, \boldsymbol{\mathcal{Y}}_H \rangle = |\boldsymbol{\mathcal{Y}}_H| \ , \qquad \boldsymbol{\mathcal{Y}}_H - \langle \boldsymbol{\mathcal{Y}}_H, \boldsymbol{\nu}_H \rangle \, \boldsymbol{\nu}_H = 0 \ . \tag{0.11}$$

It is clear from Definition 0.2 that if $\mathcal{S} \in C^k$, then $\boldsymbol{\nu}_H \in C^{k-1}(\mathcal{S} \setminus \Sigma)$. The following notion plays an important role.

Definition 0.3. *The* horizontal tangent space *at a point* $g \in \mathcal{S} \setminus \Sigma$ *is defined as follows*

$$T_{H,g} \overset{def}{=} \{ \zeta \in H_g\boldsymbol{G} \mid \langle \zeta, \boldsymbol{\mathcal{Y}}_H \rangle_g = 0 \} \ .$$

The horizontal tangent bundle *of* $\mathcal{S}$ *is defined by*

$$T_H\mathcal{S} = \bigcup_{g \in \mathcal{S} \setminus \Sigma} T_{H,g}\mathcal{S} \ .$$

It is clear that, since $dim\, H_g\boldsymbol{G} = m$, then $dim\, T_{H,g}\mathcal{S} = m - 1$, and one has in fact

$$H_g\boldsymbol{G} = T_{H,g}\mathcal{S} \oplus span\, \{ \boldsymbol{\mathcal{Y}}_H(g) \} \ .$$

Returning to Theorem B, from the latter and from (0.4) we now conclude the following basic representation result for the H-perimeter.

Proposition 0.4. *Let $E \subset \boldsymbol{G}$ be a C^1 domain. For every open set $\Omega \subset \boldsymbol{G}$, and any $\zeta \in C^1_o(\Omega; H\boldsymbol{G})$, one has*

$$\int_\Omega \langle \zeta, \boldsymbol{\nu}^E_H \rangle \, d\|\partial_H E\| = \int_{\partial E \cap \Omega} \langle \zeta, \boldsymbol{\mathcal{Y}}_H \rangle \, dH_{N-1} \,,$$

where $\boldsymbol{\mathcal{Y}}_H$ is defined in (0.8)*. Moreover,*

$$d\|\partial_H E\| = |\boldsymbol{\mathcal{Y}}_H| \, d\left(H_{N-1} \lfloor \partial E\right) \,, \tag{0.12}$$

and one has

$$\|\partial_H E\|(\Omega) = P_H(E;\Omega) = \int_{\partial E \cap \Omega} |\boldsymbol{\mathcal{Y}}_H| \, dH_{N-1} \,. \tag{0.13}$$

Many times, it is useful to define a given hypersurface as the zero set of a non-degenerate function (a defining function). We consider a C^2 bounded open set $\mathcal{U} \subset \boldsymbol{G}$ and we assume for convenience that there exists a globally defined $\phi \in C^2(\boldsymbol{G})$ such that

$$\mathcal{S} = \partial \mathcal{U} \,, \qquad \text{with} \qquad \mathcal{U} = \{ g \in \boldsymbol{G} \mid \phi(g) < 0 \} \,, \tag{0.14}$$

and for which $|\nabla \phi| \geq \alpha > 0$ in an open neighborhood $\mathcal{O}$ of $\mathcal{S}$, where $\nabla \phi$ denotes the Riemannian gradient of ϕ. In this situation the Riemannian Gauss map is given by $\boldsymbol{\nu}(g) = \nabla\phi(g)/|\nabla\phi(g)|$, $g \in \mathcal{O}$. As a consequence, we have

$$|\boldsymbol{\mathcal{Y}}_H(g)| = \frac{|X\phi(g)|}{|\nabla\phi(g)|} \,, \tag{0.15}$$

and

$$\boldsymbol{\nu}_H = \frac{X\phi}{|X\phi|} \,. \tag{0.16}$$

Henceforth, given a hypersurface $\mathcal{S} \subset \boldsymbol{G}$, we will denote by

$$d\sigma_H = |\boldsymbol{\mathcal{Y}}_H| \, dH_{N-1} \lfloor \mathcal{S} \,, \tag{0.17}$$

the *H-perimeter measure* supported on $\mathcal{S}$. For a detailed study of such measure see [9], [10]. We remark that if that $\mathcal{S} = \partial \mathcal{U}$, as in (0.14), then according to (0.12) we have

$$d\sigma_H = d\|\partial_H \mathcal{U}\| \,,$$

and this justifies the name of H-perimeter measure.

Given an open set $\Omega \subset \boldsymbol{G}$ the notation $\Gamma^k(\Omega)$ indicates the space of Folland and Stein [36] of functions which admit continuous derivatives up to order k with respect to the vector fields $X_1, \cdots, X_m$. We denote by $\Gamma^k_o(\Omega)$ the subspace of functions with compact support in Ω. We will henceforth assume that the Riemannian Gauss map $\boldsymbol{\nu}$ of $\mathcal{S}$ is defined not only on $\mathcal{S}$ but in a full open neighborhood $\mathcal{O} \subset \boldsymbol{G}$ of $\mathcal{S}$, as it is done for instance in the classical references [31], [24], [23]. With this assumption, then also the vector field $\boldsymbol{\mathcal{Y}}_H$ is defined in $\mathcal{O}$, and consequently the horizontal Gauss map $\boldsymbol{\nu}_H$ is defined in $\mathcal{O} \setminus \Sigma$.

Definition 0.5. *Consider a function $u \in \Gamma^1(\mathcal{O})$. The* tangential sub-gradient *of u on $\mathcal{S}$ is defined as follows*

$$\delta_H u \overset{def}{=} \nabla_H u - \langle \nabla_H u, \boldsymbol{\nu}_H \rangle \boldsymbol{\nu}_H$$

at every point $g \in S \setminus \Sigma$.

It is clear from Definition 0.5 that $\delta_H u$ represents the projection orthogonal to $\boldsymbol{\nu}_H$ of the sub-gradient $\nabla_H u$ onto $H\boldsymbol{G}$. One has in fact from (0.10) and Definition 0.5

$$\langle \delta_H u, \boldsymbol{\nu}_H \rangle \equiv 0 \qquad \text{in} \quad \mathcal{S} \setminus \Sigma \,, \tag{0.18}$$

which shows that $\delta_H u(g) \in T_{H,g}\mathcal{S}$ for every $g \in \mathcal{S} \setminus \Sigma$, and thus

$$|\delta_H u|^2 = |\nabla_H u|^2 - \langle \nabla_H u, \boldsymbol{\nu}_H \rangle^2 \,. \tag{0.19}$$

The tangential sub-gradient depends only on the values of the function on $\mathcal{S}$, i.e., if $v \in \Gamma^1(\mathcal{O})$ is such that $u \equiv v$ on $\mathcal{S}$, then on $\mathcal{S} \setminus \Sigma$ one has

$$\delta_H u \equiv \delta_H v \,.$$

Remark 0.6. *Before proceeding it is worth pausing here to recall the pioneering address by E. Cartan on the geometric representation of non-holonomic mechanical systems at the Bologna 1928 International Congress of Mathematicians. Cartan's ideas have been more recently taken up by several groups of mathematicians with various intents. In particular, we refer the reader to the articles* [27], [28] *and references therein, for a geometric development which is closely connected to the spirit of the results in* [11]. *In particular, the notion of tangential sub-gradient introduced in Definition 0.5 on functions, can be connected to the intrinsic framework of Cartan's non-holonomic connections. The latter, in fact, can be shown to be the projection onto the horizontal bundle $H\boldsymbol{G}$ of the Riemannian Levi-Civita connection in $\boldsymbol{G}$ and, in fact, can be uniquely determined as follows. Let $\mathcal{O} \subset \boldsymbol{G}$ be an open set, and let $X, Y \in C^1(\mathcal{O}, H\boldsymbol{G})$. If $Y = \sum_{j=1}^m Y^j X_j$, where $Y^j = \langle Y, X_j \rangle$, then*

$$D_X Y = \sum_{j=1}^m X(Y^j)\, X_j$$

uniquely identifies a non-holonomic connection on $\boldsymbol{G}$.

We now introduce another basic notion from [11], that of sub-Riemannian, or H-mean curvature.

Definition 0.7. *We define the* H-mean curvature *of $\mathcal{S}$ at points of $\mathcal{S} \setminus \Sigma$ as*

$$\mathcal{H} = \sum_{i=1}^m \delta_{H,i} \nu_{H,i} \,.$$

If $g_o \in \Sigma$ we let

$$\mathcal{H}(g_o) = \lim_{g \to g_o, g \in \mathcal{S} \setminus \Sigma} \mathcal{H}(g) \,,$$

*provided that such limit exists, finite or infinite. We do not define the H-mean curvature at those points $g_o \in \Sigma$ at which the limit does not exist. Finally, we call $\boldsymbol{\mathcal{H}} = \mathcal{H} \boldsymbol{\nu}_H$ the H-*mean curvature vector .

Definition 0.8. *A C^2 hypersurface $\mathcal{S}$ is called H-*minimal *if its H-mean curvature vanishes everywhere.*

Remark 0.9. *An important geometric interpretation of H-mean curvature is that it represents the Riemannian mean curvature of the "restriction" of $\mathcal{S}$ to the horizontal bundle $H\boldsymbol{G}$.*

The following result shows that when $\mathcal{S}$ is a cylindrical hypersurface over the first layer of the Lie algebra, then its H-mean curvature coincides with the classical Riemannian mean curvature of its projection.

Theorem 0.10. *Suppose that the hypersurface $\mathcal{S}$ is a* vertical cylinder*, i.e., it can be represented in the form*

$$\mathcal{S} = \{g \in \boldsymbol{G} \mid \mathfrak{h}(x_1(g), \cdots, x_m(g)) = 0\} , \tag{0.20}$$

where $\mathfrak{h} \in C^2(\mathbb{R}^m)$, and there exist an open set $\omega \subset \mathbb{R}^m$ and $\alpha > 0$ such that $|\nabla \mathfrak{h}| \geq \alpha$ in ω. Under these assumptions, the characteristic set of $\mathcal{S}$ is empty, and the H-mean curvature of $\mathcal{S}$ is given by

$$\mathcal{H}(g) = (m-1)\, H(x(g)) , \tag{0.21}$$

where H represents the Riemannian mean curvature of the projection $\pi_{V_1}(\mathcal{S})$ of $\mathcal{S}$ onto the horizontal layer V_1, and $x(g)$ represent the projection onto V_1 of the exponential coordinates of $g \in \boldsymbol{G}$. In particular, $\mathcal{S}$ is H-minimal if and only if $\pi_{V_1}(\mathcal{S})$ is a classical minimal surface in $V_1 \simeq \mathbb{R}^m$.

For a function $u : \boldsymbol{G} \to \mathbb{R}$ we define the *symmetrized horizontal Hessian* of u at $g \in \boldsymbol{G}$, $\nabla_H^2 u = (u_{,ij})$, is defined by

$$u_{,ij} \overset{def}{=} \frac{1}{2} \{X_i X_j u + X_j X_i u\} , \qquad i, j = 1, \cdots, m . \tag{0.22}$$

We notice that the sub-Laplacian introduced above is given by

$$\mathcal{L} u = tr\, \nabla_H^2 u .$$

We also consider the following nonlinear operator

$$\mathcal{L}_\infty u \overset{def}{=} \sum_{i,j=1}^m u_{,ij}\, X_i u\, X_j u , \tag{0.23}$$

which by analogy with its by now classical Euclidean ancestor we call the ∞-*sub-Laplacian.* The reason for introducing the operator $\mathcal{L}_\infty$ is in the following result which is often useful in computing the H-mean curvature.

Proposition 0.11. *At every point of $S \setminus \Sigma$ one has in terms of a defining function ϕ of S*

$$\mathcal{H} = \frac{1}{|X\phi|^3} \left\{ |X\phi|^2\, \mathcal{L}\phi - \mathcal{L}_\infty \phi \right\} .$$

Remark 0.12. *We stress that if we consider the $m \times m$ matrix $\mathcal{X}_H$ defined by the equation*

$$\mathcal{X}_H \overset{def}{=} \frac{1}{|X\phi|^3} \left\{ |X\phi|^2 Id_{m\times m} - X\phi \otimes X\phi \right\} \nabla_H^2 \phi ,$$

then using Proposition 0.11 it is easy to recognize that

$$\mathcal{H} = trace\, \mathcal{X}_H .$$

The matrix $\mathcal{X}_H$ incorporates geometric information on the horizontal Gauss map $\boldsymbol{\nu}_H$ of $\mathcal{S}$. In fact, one can prove that such matrix is the one associated with the horizontal second fundamental form *of $\mathcal{S}$, and the ensuing linear operator on $H\boldsymbol{G}$ should thereby be called the* horizontal shape operator.

The papers [33], [11], [22] contain many examples of H-minimal surfaces and we refer the reader to those sources for a detailed discussion. We next introduce a divergence theorem from [11] which is central to the development of sub-Riemannian calculus. Such result, and its corollaries, are formally reminiscent of the classical ones. However, an important difference is that the ordinary surface measure is replaced by the H-perimeter measure. Furthermore, they contain an additional term which is due to the non-trivial commutation relations. Such term prevents the corresponding horizontal Laplace-Beltrami operator from being formally self-adjoint in $L^2(\mathcal{S}, d\sigma_H)$ in general.

Theorem 0.13 (**Sub-Riemannian divergence theorem**). *Consider a C^2 hypersurface $\mathcal{S}$ in a Carnot group $\boldsymbol{G}$ which we assume given as in* (0.14). *If $u \in \Gamma^1_o(\mathcal{O}\setminus\Sigma)$ then we have*

$$\int_{\mathcal{S}} \left\{\delta_{H,i} u + \boldsymbol{c}^i_{\mathcal{S}} u\right\} \, d\sigma_H = \int_{\mathcal{S}} u \, \mathcal{H} \, \nu_{H,i} \, d\sigma_H \ , \qquad i = 1, \cdots, m \ . \tag{0.24}$$

where the continuous functions $\boldsymbol{c}^i_{\mathcal{S}}$ on $\mathcal{S}\setminus\Sigma$ are defined by

$$\boldsymbol{c}^i_{\mathcal{S}} = \frac{\sum_{j=1}^m [X_i, X_j]\phi \, \nu_{H,j}}{|X\phi|} \ . \tag{0.25}$$

Moreover, the horizontal vector field $\boldsymbol{c}_{\mathcal{S}} = \sum_{i=1}^m \boldsymbol{c}^i_{\mathcal{S}} X_i$ is perpendicular to the horizontal Gauss map $\boldsymbol{\nu}_H$, i.e., one has

$$\langle \boldsymbol{c}_{\mathcal{S}}, \boldsymbol{\nu}_H \rangle = 0 \ . \tag{0.26}$$

and thereby $\boldsymbol{c}_{\mathcal{S}} \in T_H\mathcal{S}$.

Remark 0.14. *We emphasize that in the abelian case $\boldsymbol{G} = \mathbb{R}^m$, we have $X_i = \partial/\partial x_i$, $i = 1, \cdots, m$, and thereby $\boldsymbol{c}_{\mathcal{S},i} \equiv 0$. In this case formula* (0.24) *recaptures the classical integration by parts formula on a hypersurface, see for instance* [31], [23].

The basic prototype of a Carnot group is the Heisenberg group $\mathbb{H}^n$, see [36]. For $n \in \mathbb{N}$, the underlying manifold of such group is $\mathbb{C}^n \times \mathbb{R}$, with group law

$$g \circ g' = (z,t) \circ (z',t') = \left(z + z', t + t' - \frac{1}{2} Im \, \langle z, \overline{z'} \rangle \right) \ . \tag{0.27}$$

Here, we have let $z = x + iy, z' = x' + iy' \in \mathbb{C}^n$, $t, t' \in \mathbb{R}$, $\langle z, \overline{z'} \rangle = \sum_{j=1}^n z_j \overline{z'_j}$. Denoting by $L_g(g') = g \circ g'$ the left-translation associated with (0.27), and by $(L_g)_*$ its differential, one readily recognizes that the generators of the (real) Heisenberg algebra $\mathfrak{h}_n$ are the left-invariant vector fields

$$\begin{aligned} X_j(g) &= (L_g)_*(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t} \ , \\ X_{n+j}(g) &= (L_g)_*(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t} \ , \end{aligned} \tag{0.28}$$

$j = 1, \cdots, n$. The sub-Laplacian on $\mathbb{H}^n$ is the second-order partial differential operator given by $\mathcal{L} = \sum_{j=1}^{2n} X_j^2$. Such operator is the real part of the Kohn sub-Laplacian in $\mathbb{C}^{n+1}$, see [36]. One has

$$[X_i, X_{n+j}] = T \, \delta_{ij} \ , \qquad i, j = 1, \cdots, n \ , \tag{0.29}$$

where $T = \partial/\partial t$ represents the characteristic direction, all other commutators being trivial. One can thus decompose the Heisenberg algebra as follows $\mathfrak{h}_n = V_1 \oplus V_2$, with $V_1 = \mathbb{C}^n \times \{0\}$, $V_2 = \{0\} \times \mathbb{R}$, and since (0.29) implies $[V_1, V_1] = V_2$, we see that Hörmander's finite rank condition [26]

$$rank\, Lie\{X_1, \cdots, X_{2n}\} \equiv dim\, \mathbb{H}^n = 2n + 1 \ , \tag{0.30}$$

is fulfilled at step $r = 2$.

Remark 0.15. *We note explicitly that when* $\boldsymbol{G}$ *is the Heisenberg group* $\mathbb{H}^n$ *the matrix* $([X_i, X_j]\phi)_{i,j=1}^m$ *appearing in the definition* (0.25) *is given by* $T\phi\, \mathcal{A}$*, where*

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the symplectic matrix in $\mathbb{R}^{2n}$ *(we have denoted with* I *the identity matrix in* $\mathbb{R}^n$*). If* $\zeta = \sum_{i=1}^m \zeta^i X_i$ *denotes a horizontal vector, then we have*

$$\langle \boldsymbol{c}_{\mathcal{S}}, \zeta \rangle \ = \ \frac{\sum_{i,j=1}^m [X_i, X_j]\phi\, X_j\phi\, \zeta^i}{|X\phi|^2} \ = \ T\phi \frac{\langle \mathcal{A}(X\phi), \zeta \rangle}{|X\phi|^2} \ . \tag{0.31}$$

Remark 0.16. *We emphasize that the statement of Theorem 0.13 can be casted in completely intrinsic terms, thus avoiding explicit mention of the defining function* ϕ*. However, we have preferred to use the above formulation since it is better suited for computations.*

We next derive from Theorem 0.13 an integration by parts formula for the Heisenberg group $\mathbb{H}^1$ which is, in fact, equivalent to it. In the sequel, given a surface $\mathcal{S} \subset \mathbb{H}^1$, we consider the quantities $p_1, p_2, W, \overline{p}_1$ and $\overline{p}_2$ introduced in (0.6), (0.7). For the sake of simplicity we will drop the subscripts and set instead $p = p_1, q = p_2, \overline{p} = \overline{p}_1, \overline{q} = \overline{p}_2$. We will indicate with Z, and Y, the vector fields defined by $Y = \overline{p}X_1 + \overline{q}X_2$, $Z = \overline{q}X_1 - \overline{p}X_2$. We emphasize that Y is just a different notation for the horizontal Gauss map (0.9), whereas Z is the orthonormal complement of Y in the sub-bundle $H\mathbb{H}^1$, and thereby $Z \in T_H\mathcal{S}$. It is worth observing that the (Riemannian) divergence in $\mathbb{H}^1$ of these vector fields is given by

$$div\, Y = X_1\overline{p} + X_2\overline{q} = \mathcal{H} \ \ , \ \ div\, Z = X_1\overline{q} - X_2\overline{p} \ . \tag{0.32}$$

One obtains for any function F on $\mathcal{S} \setminus \Sigma$

$$X_1 F = \overline{q}\, ZF + \overline{p}\, YF \ \ , \ \ X_2 F = \overline{q}\, YF - \overline{p}\, ZF \ . \tag{0.33}$$

One also has

$$\delta_{H,1} F = \overline{q}\, ZF \ \ , \ \ \delta_{H,2} F = -\overline{p}\, ZF \ . \tag{0.34}$$

From (0.34) we obtain

$$|\delta_H F|^2 = (ZF)^2 = (\overline{q}\, X_1 F - \overline{p}\, X_2 F)^2 \ . \tag{0.35}$$

Theorem 0.17. *Let* $F \in \Gamma^1_o(\mathcal{O} \setminus \Sigma)$*, then*

$$\int_{\mathcal{S}} ZF\, d\sigma_H = - \int_{\mathcal{S}} F \frac{T\phi}{W}\, d\sigma_H \ .$$

We next recall another interesting integration by parts formula from [11] which involves the vector fields Y and T.

Theorem 0.18. *For any function* $F \in \Gamma^1_o(\mathcal{S} \setminus \Sigma)$ *one has*

$$\int_{\mathcal{S}} \left\{ TF - Y\left(F \frac{T\phi}{W} \right) \right\} d\sigma_H =$$

$$= \int_{\mathcal{S}} \mathcal{H}\, F\, \frac{T\phi}{W}\, d\sigma_H + \int_{\mathcal{S}} F \left\{ \frac{YW}{W}\, \frac{T\phi}{W} - \frac{TW}{W} \right\} d\sigma_H \ .$$

As a consequence of the latter identity we obtain the equivalent one

$$\int_{\mathcal{S}} TF\, d\sigma_H = \int_{\mathcal{S}} YF\, \frac{T\phi}{W}\, d\sigma_H + \int_{\mathcal{S}} F\, \frac{T\phi}{W}\, \mathcal{H}\, d\sigma_H \ .$$

Having recalled the notion of H-perimeter, and that of H-mean curvature it is at this point worth analyzing the connection between the notion of H-minimal surface introduced in Definition 0.8, and the measure theoretic one introduced in [21]. The next result shows that, for foliated deformations of the hypersurface $\mathcal{S}$, H-minimal hypersurfaces are critical points of the H-perimeter.

Theorem 0.19. *Let $\mathcal{O} \subset \boldsymbol{G}$ be an open set and consider $\phi \in C^1(\mathcal{O})$ with $|\nabla\phi| \geq \alpha > 0$, and set $\mathcal{U}_\rho = \{g \in \mathcal{O} \mid \phi(g) < \rho\}$. We then have*

$$\frac{d}{d\rho}\, P_H(\mathcal{U}_\rho\ ;\ \boldsymbol{G}) = \int_{\partial\mathcal{U}_\rho} \frac{\mathcal{H}}{|\nabla\phi|}\, dH_{N-1} \ .$$

In particular, if $\mathcal{S} = \partial\mathcal{U}_0$ is H-minimal, then $\mathcal{U} = \mathcal{U}_0$ is a critical point of the H-perimeter since

$$\frac{d}{d\rho}\ P_H(\mathcal{U}_\rho; \boldsymbol{G})\big|_{\rho=0} = 0 \ .$$

We next consider a hypersurface in the Heisenberg group $\mathbb{H}^n$, and compute the first and second variation of the H-perimeter for general deformations of $\mathcal{S}$. To keep the presentation transparent, we will focus on the first Heisenberg group $\mathbb{H}^1$, although our formulas generalize to the higher dimensional case. We consider general deformations of $\mathcal{S}$, and notice that any such deformation can be achieved by suitably combining one along the horizontal normal $\boldsymbol{\nu}_H$ with one along the characteristic direction T. Our main results are contained in the following theorem, whose proof uses, among other ingredients, Theorems 0.17 and 0.18.

Theorem 0.20. *Let $\mathcal{S} \subset \mathbb{H}^1$ be a C^2 surface, and suppose that $h, k \in C^\infty_o(\mathcal{S} \setminus \Sigma)$. For small $\lambda \in \mathbb{R}$ consider the family of diffeomorphisms*

$$J_\lambda(g) = g + \lambda \Big(h(g)\boldsymbol{\nu}_H + k(g)T \Big) \ , \tag{0.36}$$

and the corresponding surfaces $\tilde{\mathcal{S}}^\lambda = J_\lambda(\mathcal{S})$. The first variation of the H-perimeter with respect to the deformation (0.36) *is given by*

$$\frac{d}{d\lambda} P_H(\tilde{\mathcal{S}}^\lambda))\Big|_{\lambda=0} = \int_{\mathcal{S}} \mathcal{H} \left(h + \frac{T\phi}{W}\, k \right) d\sigma_H \ . \tag{0.37}$$

In particular, $\mathcal{S}$ is stationary with respect to (0.36) *if and only if it is H-minimal.*

The second variation of the H-perimeter is given by the formula

$$\frac{d^2}{d\lambda^2} P_H(\tilde{\mathcal{S}}^\lambda)\Bigg|_{\lambda=0} = \tag{0.38}$$

$$= \int_{\mathcal{S}} \Bigg\{ \left| \delta_H \left(h + \frac{T\phi}{W}\, k \right) \right|^2 + \left(h + \frac{T\phi}{W}\, k \right)^2 \Bigg\{ \left(\frac{T\phi}{W} \right)^2 +$$

$$+ 2\, \frac{T\phi}{W}\, (\overline{q} Y \overline{p} - \overline{p} Y \overline{q}) - 2\, \frac{Z(T\phi)}{W} \Bigg\} \Bigg\}\, d\sigma_H + \int_{\mathcal{S}} k^2 \left(\frac{T\phi}{W} \right)^2 \frac{Z(T\phi)}{W}\, d\sigma_H +$$

$$+\int_{\mathcal{S}}\mathcal{H}\left\{2\,k\,Yh\,\frac{T\phi}{W}-2\,k\,Th\right\}d\sigma_H+$$

$$+\int_{\mathcal{S}}k^2\left\{T\mathcal{H}\,\frac{T\phi}{W}-Y\mathcal{H}\left(\frac{T\phi}{W}\right)^2-\left(\frac{T\phi}{W}\right)^2\mathcal{H}^2\right\}d\sigma_H\ .$$

In particular, if $\mathcal{S}$ is $H-$minimal, then we obtain from (0.38)

$$\frac{d^2}{d\lambda^2}P_H(\tilde{\mathcal{S}}^\lambda)\bigg|_{\lambda=0}= \tag{0.39}$$

$$=\int_{\mathcal{S}}\left\{\left|\delta_H\left(h+\frac{T\phi}{W}\,k\right)\right|^2+\left(h+\frac{T\phi}{W}\,k\right)^2\left\{\left(\frac{T\phi}{W}\right)^2+\right.\right.$$

$$\left.\left.+\,2\,\frac{T\phi}{W}\,(\overline{q}Y\overline{p}-\overline{p}Y\overline{q})-2\,\frac{Z(T\phi)}{W}\right\}\right\}d\sigma_H+\int_{\mathcal{S}}k^2\left(\frac{T\phi}{W}\right)^2\frac{Z(T\phi)}{W}\,d\sigma_H\ .$$

Returning to the above raised question of the connection between H-minimality and H-perimeter, we see as a notable consequence of (0.37) that $\mathcal{S}$ is a stationary point of the H-perimeter if and only if it is H-minimal. Clearly, formula (0.38) allows to attack the question of the stability of H-minimal surfaces, and for this aspect we refer the reader to [11].

Remark 0.21. *In* (0.37), (0.38) *and* (0.39) *we have denoted by ϕ a defining function for $\mathcal{S}$ as in* (0.14). *The explicit reference to such a function has been suggested by considerations of brevity, and computational ease. The reader should bear in mind that the quantity $T\phi$ can be replaced throughout by a more intrinsic (but also more complicated) expression which makes no reference to a defining function. Thus, in particular, Theorem 0.20 holds for parametric surfaces as well.*

The proof of Theorem 0.20 is rather involved. Our approach is based on first establishing the case of deformations along $\boldsymbol{\nu}_H$ only (this corresponds to setting $k\equiv 0$ in (0.38)), and then use such result as a building block for the more general case. In view of this, and because of its interest, we state separately below the relevant result.

Theorem 0.22. *Let $\mathcal{S}\subset\mathbb{H}^1$ be a C^2 surface, let $h\in C_o^\infty(\mathcal{S}\setminus\Sigma)$, and consider the family of surfaces $G_\lambda(\mathcal{S})$, where $G_\lambda(g)=g+\lambda h(g)\boldsymbol{\nu}_H(g)$. The first variation of the H-perimeter with respect to the horizontal deformation G_λ is given by*

$$\frac{d}{d\lambda}P_H(G_\lambda(\mathcal{S}))\Big|_{\lambda=0}=\int_{\mathcal{S}}\mathcal{H}\,h\,d\sigma_H\ . \tag{0.40}$$

The second variation of the H-perimeter is given by the formula

$$\frac{d^2}{d\lambda^2}P_H(G_\lambda(\mathcal{S}))\Big|_{\lambda=0}= \tag{0.41}$$

$$=\int_{\mathcal{S}}\left\{|\delta_H h|^2+h^2\left[\frac{\left(T\phi\right)^2}{W^2}+2\,\frac{T\phi}{W}\left(\overline{q}Y\overline{p}-\overline{p}Y\overline{q}\right)-2\,\frac{Z(T\phi)}{W}\right]\right\}d\sigma_H\ .$$

The reader should note the remarkable discrepancy between (0.41) in Theorem 0.22 and its classical ancestor (0.2) in Theorem A. There are two striking differences.

First, a term such as $\sum_{i=1}^{m+1} |\delta \nu_i|^2 - m^2 H^2$ is missing in (0.41). Actually, the proof of Theorem 0.22 produces the following formula

$$\frac{d^2}{d\lambda^2} P_H(G_\lambda(\mathcal{S}))\Big|_{\lambda=0} = \int_{\mathcal{S}} \Bigg\{ |\delta_H h|^2 + h^2 \Bigg[|\delta_H \boldsymbol{\nu}_{H,1}|^2 + |\delta_H \boldsymbol{\nu}_{H,2}|^2 - \mathcal{H}^2 + \\ + \frac{\big(T\phi\big)^2}{W^2} + 2\,\frac{T\phi}{W}\,\big(\overline{q} Y \overline{p} - \overline{p} Y \overline{q}\big) - 2\,\frac{Z(T\phi)}{W} \Bigg] \Bigg\}\, d\sigma_H \ ,$$

which, at the formal level, is more similar to (0.2). But then, in view of the following

Proposition 0.23. *One has on* $\mathcal{S} \setminus \Sigma$

$$|\delta_H \boldsymbol{\nu}_{H,1}|^2 + |\delta_H \boldsymbol{\nu}_{H,2}|^2 = (Z\overline{p})^2 + (Z\overline{q})^2 \equiv \mathcal{H}^2 \ ,$$

which says that one of the principal sub-Riemannian curvatures is always zero, the above formula reduces to (0.38). The second difference is the additional term

$$\frac{\big(T\phi\big)^2}{W^2} + 2\frac{T\phi}{W}\big(X_2\overline{p} - X_1\overline{q}\big) - 2\frac{Z(T\phi)}{W}$$

which is not present in the Riemannian case. Such term, which arises from the sub-Riemannian integration by parts formulas (see Theorems 0.17, 0.18, and also Theorem 0.13), is produced by the non-trivial commutation relations of the ambient space.

It is also useful to note the following corollary of Theorem 0.20 which expresses the stability of the H-perimeter with respect to deformations along the characteristic direction T.

Corollary 0.24. *Let* $k \in C_o^\infty(\mathcal{S} \setminus \Sigma)$*, and consider the family of surfaces* $F_\lambda(\mathcal{S})$*, where* $F_\lambda(g) = g + \lambda k(g) T$*. The first variation of the* H*-perimeter with respect to the deformation* F_λ *is given by*

$$\frac{d}{d\lambda} P_H(F_\lambda(\mathcal{S}))\Big|_{\lambda=0} = \int_{\mathcal{S}} \mathcal{H}\, k\, \frac{T\phi}{W}\, d\sigma_H \ . \tag{0.42}$$

The second variation of the H*-perimeter is given by the formula*

$$\frac{d^2}{d\lambda^2} P_H(F_\lambda(\mathcal{S}))\Big|_{\lambda=0} = \int_{\mathcal{S}} |\delta_H k|^2 \left(\frac{T\phi}{W}\right)^2 d\sigma_H \ . \tag{0.43}$$

As we have seen from (0.37), H-minimal surfaces arise as stationary points of the H-perimeter. Another fundamental question in Calculus of Variations is that of minimizing the H-perimeter under a volume constraint. This is of course the sub-Riemannian version of the celebrated isoperimetric problem, and even for the basic model of the Heisenberg group the latter still constitutes a challenging open question. In [32] Pansu conjectured that in $\mathbb{H}^1$ the isoperimetric profiles should be given by (left-translations and dilations of) the sets obtained by rotating around the t-axis a geodesic joining the two points at height $\pm\pi R^2/8$. In [8] the following (partial) solution of the isoperimetric problem in $\mathbb{H}^n$ was established. Consider the collection of sets $\mathcal{E} = \{E \subset \mathbb{H}^n \mid E \quad \text{satisfies } (i) - (ii)\}$, where

(i) $|E \cap \mathbb{H}^n_+| = |E \cap \mathbb{H}^n_-|$;

(ii) There exist $R > 0$, and C^1 functions $u, v : [0, R] \to \mathbb{R}$ satisfying $u(R) = v(R) = 0$, such that $\partial E \cap \mathbb{H}^n_+ = \{(z,t) \,|\, t = u(|z|^2/4)\}$ and $\partial E \cap \mathbb{H}^n_- = \{(z,t) \,|\, t = v(|z|^2/4)\}$.

Theorem 0.25. *Given $V > 0$, the variational problem*

$$min\{P_H(E;\mathbb{H}^n) \mid E \in \mathcal{E},\ |E| = V\}$$

has a unique solution $E_o \in \mathcal{E}$. Furthermore, ∂E_o is given explicitly as the graph

$$t = \pm \left\{ \frac{1}{4}|z|\sqrt{R^2 - |z|^2} - \frac{R^2}{4} \tan^{-1}\left(\frac{|z|}{\sqrt{R^2-|z|^2}} \right) + \frac{\pi R^2}{8} \right\} \quad , \quad |z| \le R \ . \tag{0.44}$$

The sign $\pm$ depends on whether one considers $\partial E_o \cap \mathbb{H}^n_+$, or $\partial E_o \cap \mathbb{H}^n_-$. Finally, the set E_o is only of class C^2 (but not C^3!) near its two characteristic points $\big(0, \pm\pi R^2/8\big)$, it is C^∞ away from them, and the C^2 compact hypersurface without boundary $\mathcal{S} = \partial E_o$ has positive constant H-mean curvature given by

$$\mathcal{H} = \frac{Q-2}{R} \ .$$

In striking contrast with the classical case, Theorem 0.25 shows in particular that, because of characteristic points, it is possible for a surface of constant H-mean curvature to have only the minimum amount of required regularity.

Returning to Theorem 0.13, we are lead to introduce a partial differential operator on $\mathcal{S}$ which plays an important role in the development of the theory. For the abelian group $\boldsymbol{G} = \mathbb{R}^m$ such operator is the standard Laplace-Beltrami operator on a hypersuface.

Definition 0.26. *The* tangential sub-Laplacian *on $\mathcal{S}$ is defined as follows*

$$\Delta_H u \overset{def}{=} \sum_{i=1}^m \delta_{H,i}\delta_{H,i} u \ . \tag{0.45}$$

The modified tangential sub-Laplacian *is defined by*

$$\mathcal{L}_H u \overset{def}{=} \sum_{i=1}^m \delta_{H,i}\delta_{H,i} u + \sum_{i=1}^m \boldsymbol{c}^i_{\mathcal{S}}\, \delta_{H,i} u = \Delta_H u + \langle \boldsymbol{c}_{\mathcal{S}},\ \delta_H u\rangle \ , \tag{0.46}$$

where the functions $\boldsymbol{c}^i_{\mathcal{S}}$ are given by (0.25).

Remark 0.27. *It is important to keep in mind that for a vertical hypersurface given by*

$$\mathcal{S} = \{g \in \boldsymbol{G} \mid \mathfrak{h}(x_1(g), \cdots, x_m(g)) = 0\} \ ,$$

with $\mathfrak{h} \in C^2(\mathbb{R}^m)$, the operators Δ_H and $\mathcal{L}_H$ coincide

$$\mathcal{L}_H = \sum_{i=1}^m \delta_{H,i}\delta_{H,i} = \Delta_H \ .$$

In such case it is easy to show that $\mathcal{L}_H$ is formally self-adjoint in $L^2(\mathcal{S}, d\sigma_H)$.

One basic raison d'être for the operator $\mathcal{L}_H$ is in the following sub-Riemannian Stokes' theorem which follows from Theorem 0.13.

Corollary 0.28. *Let $u \in \Gamma^2_o(\mathcal{O} \setminus \Sigma)$, then we have*

$$\int_{\mathcal{S}} \mathcal{L}_H u \, d\sigma_H = 0 \ . \tag{0.47}$$

When $\mathcal{S}$ is a compact hypersurface without boundary, then (0.47) *holds for any $u \in \Gamma^2(\mathcal{O} \setminus \Sigma)$.*

Corollary 0.29. *Let $u \in \Gamma^1(\mathcal{O} \setminus \Sigma)$, then for every $\zeta \in \Gamma^2_o(\mathcal{O})$ we have*

$$\int_{\mathcal{S}} \langle \delta_H u, \delta_H \zeta \rangle \, d\sigma_H = - \int_{\mathcal{S}} u \, \mathcal{L}_H \zeta \, d\sigma_H \ . \tag{0.48}$$

A useful mean for computing $\mathcal{L}_H u$ on $\mathcal{S}$ using the vector fields $X_1, \cdots, X_m$ in the ambient group $\boldsymbol{G}$ is provided by the following proposition.

Proposition 0.30. *Let $u \in \Gamma^2(\mathcal{O} \setminus \Sigma)$, then we have on $\mathcal{S} \setminus \Sigma$*

$$\Delta_H u = \mathcal{L} u - \langle \nabla^2_H u \, \boldsymbol{\nu}_H, \boldsymbol{\nu}_H \rangle - \langle Xu, \boldsymbol{\nu}_H \rangle \, \mathcal{H} \ , \tag{0.49}$$

$$\mathcal{L}_H u = \mathcal{L} u + \langle \boldsymbol{c}_{\mathcal{S}}, Xu \rangle - \langle \nabla^2_H u \, \boldsymbol{\nu}_H, \boldsymbol{\nu}_H \rangle - \langle Xu, \boldsymbol{\nu}_H \rangle \, \mathcal{H} \ . \tag{0.50}$$

The first elementary example of solutions of the tangential sub-Laplacian Δ_H on $\mathcal{S}$ is provided by the following proposition.

Proposition 0.31. *Let $\mathcal{S} \subset \boldsymbol{G}$ be a H-minimal hypersurface, then if $x(g) = (x_1(g), \cdots, x_m(g))$ denote the projection onto the horizontal layer of the exponential coordinates of $g \in \boldsymbol{G}$, one has*

$$\Delta_H(x_i) = 0 \ , \qquad\qquad i = 1, \cdots, m \ .$$

Another less trivial situation of special interest is when $\boldsymbol{G}$ is a Carnot group of step $r = 2$, and one has a hypersurface $\mathcal{S}$ given as a graph over the first layer of the Lie algebra. In such case, identifying via the exponential map $g = \exp \xi(g)$ with $\xi(g) \cong (x(g), y(g))$, we can find an open set $\Omega \subset V_1$, and a C^2 function $h : \Omega \to \mathbb{R}$, such that for some $s \in \{1, \cdots, k\}$, $\mathcal{S}$ can be written as

$$\mathcal{S} = \{(x(g), y(g)) \in \boldsymbol{G} \mid x(g) \in \Omega, \ y_s(g) = h(x(g))\} \ . \tag{0.51}$$

For instance, in the special case of the Heisenberg group $\mathbb{H}^n$ we would be considering a graph over $\mathbb{C}^n$, i.e., $\mathcal{S} = \{(z,t) \in \mathbb{H}^n \mid z \in \Omega \subset \mathbb{C}^n, \ t = h(z)\}$. Henceforth, we will drop the dependence of x and y from $g \in \boldsymbol{G}$.

Theorem 0.32. *Let $\boldsymbol{G}$ be a Carnot group of step $r = 2$, and $\mathcal{S} \subset \boldsymbol{G}$ be a H-minimal hypersurface of the type* (0.51)*, then outside the characteristic set Σ the coordinate functions $x_1, \cdots, x_m, y_1, \cdots, y_k$ are solutions of the tangential sub-Laplacian on $\mathcal{S}$.*

Corollary 0.33. *In the Heisenberg group let*

$$\mathcal{S} = \{z, t) \in \mathbb{H}^n \mid z \in \Omega \, , \, t = h(z)\} \ ,$$

where $\Omega \subset \mathbb{R}^{2n}$ is an open set, and $h \in C^2(\Omega)$. Is $\mathcal{S}$ is H-minimal, then the coordinate functions $x_1, \cdots, x_n, \ y_1, \cdots, y_n, \ t$ are solutions of Δ_H on $\mathcal{S}$.

The following notions of p-Dirichlet integral and of p-harmonic function on an hypersurface are central to the development of geometric subelliptic pde's on $\mathcal{S}$.

Definition 0.34. *Given $1 < p < \infty$ we define the (H,p)-*Dirichlet integral *of a function $u \in \Gamma^1(\mathcal{O} \setminus \Sigma)$ on the hypersurface $\mathcal{S}$ as*

$$\mathcal{E}_{H,\mathcal{S}}(u) = \frac{1}{p} \int_{\mathcal{S}} |\delta_H u|^p \, d\sigma_H \ .$$

*We say that u is (H,p)-*subharmonic *(-*superharmonic*) in $\mathcal{S}$ if for every $\zeta \in \Gamma^1_o(\mathcal{O} \setminus \Sigma)$, $\zeta \geq 0$, one has*

$$\int_{\mathcal{S}} |\delta_H u|^{p-2} \langle \delta_H u, \delta_H \zeta \rangle \, d\sigma_H \leq 0 \ (\geq 0) \ .$$

We say that u is (H,p)-harmonic *in $\mathcal{S}$ if u is simultaneously (H,p)-subharmonic and superharmonic. When $p=2$ we simply say that u is H*-harmonic *in $\mathcal{S}$.*

According to Corollary 0.29 we can adopt the following alternative notion of H-subharmonicity.

Definition 0.35. *A function $u \in L^1(\mathcal{S}, d\sigma_H)$ is called H-subharmonic in $\mathcal{S}$ if*

$$0 \le \int_{\mathcal{S}} u\, \mathcal{L}_H \zeta \, d\sigma_H \,, \qquad \textit{for every} \quad \zeta \in \Gamma^2_o(\mathcal{O} \setminus \Sigma) \,,\ \zeta \ge 0 \,. \tag{0.52}$$

We close this survey article with briefly discussing the recent solution of the sub-Riemannian Bernstein problem in $\mathbb{H}^1$ by Scott Pauls and the second named author [22]. To motivate the relevant results we mention that in [11] the following sub-Riemannian Bernstein problem was formulated: consider a complete H-minimal surface $\mathcal{S} \subset \mathbb{H}^n$. Under which assumptions on $\mathcal{S}$, and on the dimension of $\mathbb{H}^n$, is it true that $\mathcal{S}$ must be a vertical hyperplane, i.e., there exist $(a,b) = (a_1, \cdots, a_n,\ b_1, \cdots, b_n) \in \mathbb{R}^{2n} \setminus \{0\}$, and $\lambda \in \mathbb{R}$, such that

$$\mathcal{S} = \{(x,y,t) \in \mathbb{H}^n \mid \langle a,x\rangle + \langle b,y\rangle = \lambda\} \ ? \tag{0.53}$$

One easily recognizes that the vertical hyperplanes in (0.53) are H-minimal, and they have empty characteristic set (however, *any* hyperplane in $\mathbb{H}^n$ is H-minimal. In particular, such is the characteristic hyperplane $\{(x,y,t) \in \mathbb{H}^n \mid t = 0\}$). The following conjecture was proposed.

Conjecture: *In the Heisenberg group $\mathbb{H}^n$, at least in low dimension, the Bernstein property should hold provided that the surface $\mathcal{S}$ is an entire graph, and that its characteristic set Σ be empty.*

We stress that substantial evidence seems in favor of the above conjecture. On one hand, there is the close relation between the Bernstein property and the classical Liouville theorem for harmonic functions. Such connection continues to hold in the sub-Riemannian setting. Now, the Liouville property in $\mathbb{H}^n$ presents a striking new phenomenon with respect to the classical setting, namely that if u is a bounded entire solution of the sub-Laplacian in $\mathbb{H}^n$, then u depends *only* on the horizontal variables $(x,y) \in \mathbb{R}^{2n}$. As a consequence, such a function must, in fact, be an ordinary harmonic function in $\mathbb{R}^{2n}$, and therefore by the classical Liouville theorem it is constant. Secondly, and perhaps more importantly, a beautiful structure theorem in [18], [19] shows that when one adapts De Giorgi's method of the blow-up to the sub-Riemannian setting of $\mathbb{H}^n$, one obtains in the limit blow-up sets which are vertical (non-characteristic) planes as in (0.53). By imposing the non-characteristic assumption in the conjecture one rules thus out the undesired H-minimal characteristic hyperplanes such as $\{(x,y,t) \in \mathbb{H}^n \mid t = 0\}$.

In light of this evidence, the initial efforts of the authors of [22] went in the direction of proving the conjecture true. In the process of establishing its veracity, a basic representation result for a graph-like H-minimal surface was developed, see Theorem 0.37 below. Such result is based on the notions of seed curve (see Definition 0.36) and height function. Such notions constitute in fact a powerful tool for constructing new H-minimal surfaces. In this process, and analyzing various possibilities, we have made the striking discovery that the above conjecture is in fact not true.

Counterexample to the conjecture: *The real analytic surface*

$$\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid y = -x\tan(\tanh(t))\}\ , \tag{0.54}$$

is an entire H-minimal graph, with empty characteristic locus, over the coordinate (x, t)-plane in $\mathbb{H}^1$.

This counterexample shows the failure of the above formulated sub-Riemannian counterpart of the classical Bernstein property. We also stress that, being smooth and with empty characteristic set, the surface in (0.54) is a regular intrinsic surface in the sense of [18]. Nonetheless, in [22] we prove a result, Theorem 0.41 below, which is in fact closest in spirit to the classical theorem of Bernstein. To precisely state such result, we introduce some definitions, and present some results of independent interest which are fundamental in the proof of the main theorem.

For an H-minimal surface of the type

$$\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid (x, y) \in \Omega\, ,\ t = h(x, y)\}\ , \tag{0.55}$$

where $\Omega \subset \mathbb{R}^2$ and $h \in C^2(\Omega)$, we consider the horizontal Gauss map (0.16). Letting $p = X_1(t-h)$ and $q = X_2(t-h)$, with the above meaning of W, $\overline{p}$ and $\overline{q}$, we have

$$\boldsymbol{\nu}_H = \overline{p}\, X_1 + \overline{q}\, X_2\ .$$

As the H-minimal surface in question is a graph over the xy-plane, we may, by abuse of notation, think of $\boldsymbol{\nu}_H$ as a vector field on a domain in $\mathbb{R}^2$.

Definition 0.36. *A* seed curve *of the H-minimal surface $\mathcal{S}$ of the type* (0.55) *is defined to be an integral curve of $\boldsymbol{\nu}_H$. We denote a seed curve by $\gamma_z(s)$, i.e., $\gamma_z(0) = z \cong (x, y)$ and $\gamma_z'(s) = \boldsymbol{\nu}_H(\gamma_z(s))$. If a base point is understood or irrelevant, we simply denote a seed curve by $\gamma(s)$. We will indicate by $\mathcal{L}_z(r)$ the integral curve of $\boldsymbol{\nu}_H^{\perp}$ (again, thought of as a vector field in the plane) starting at the point z.*

We define the *signed curvature* of γ by

$$\kappa(s) \overset{def}{=} \langle \gamma''(s), \boldsymbol{\nu}_H^{\perp}(\gamma(s))\rangle = \gamma_1''(s)\gamma_2'(s) - \gamma_2''(s)\gamma_1'(s)\ . \tag{0.56}$$

We note explicitly that, thanks to $|\boldsymbol{\nu}_H| = 1$, a seed curve is always parameterized by arc-length. Using $\{\mathcal{L}_z, \gamma_z\}$ as our coordinate curves, we obtain a new local parameterization of the xy-plane $F : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$(s, r)\ \to (x(s, r), y(s, r)) \overset{def}{=} F(s, r) = \gamma(s) + r\, \boldsymbol{\nu}_H^{\perp}(\gamma(s))\ . \tag{0.57}$$

Keeping in mind that $\boldsymbol{\nu}_H^{\perp}(\gamma(s)) = (\gamma_2'(s), -\gamma_1'(s))$, we have

$$F(s, r) = (\gamma_1(s) + r\,\gamma_2'(s)\ ,\ \gamma_2(s) - r\,\gamma_1'(s))\ . \tag{0.58}$$

It is shown in [22] that $F(s, r)$ defines a local diffeomorphism over a region of the (s, r)-plane, up to the curve $\mathcal{C}_\gamma = \{(s, r) \in \mathbb{R}^2 \mid r = 1/\kappa(s)\}$, which is denominated the singular locus of the parametrization.

The first result in [22] is the following basic representation theorem for H-minimal surfaces which are graphs with empty characteristic set over a portion of the xy-plane.

Theorem 0.37. *Let $k \geq 2$. A patch of a C^k surface $\mathcal{S} \subset \mathbb{H}^1$ of the type* (0.55), *with empty characteristic locus over Ω, is an H-minimal surface if and only if, for every $z \in \mathcal{S}$, there is a neighborhood of z which can be parameterized by*

$$(\gamma_1(s) + r\gamma_2'(s),\ \gamma_2(s) - r\gamma_1'(s),\ h(s, r))\ , \tag{0.59}$$

where

$$h(s,r) = h_0(s) - \frac{r}{2}\langle \gamma(s), \gamma'(s) \rangle \ . \tag{0.60}$$

and

$$\gamma \in C^{k+1}, \qquad h_0 \in C^k \ .$$

Thus, to specify such a patch of a smooth H-minimal surface of this type, one must specify a single curve in $\mathbb{H}^1$ determined by a seed curve $\gamma(s)$, parameterized by arc-length, and an initial height function $h_0(s)$.

One consequence of the representation in Theorem 0.37 is that H-minimal surfaces of the type (0.55) are, in fact, ruled surfaces. In particular, for fixed s, the straight line $(\gamma_1(s)+r\gamma_2'(s), \gamma_2(s)-r\gamma_1'(s))$ in the (s,r)-plane lifts to a geodesic line in $\mathbb{H}^1$. We stress that Theorem 0.37 is useful in both the study of known examples as well as in the construction of new H-minimal surfaces. For a detailed analysis we refer the reader to [22].

Seed curves associated to H-minimal surfaces are the fundamental objects of study in [22].

Definition 0.38. *If no portion of a C^2, complete, connected H-minimal surface can be written as a graph over the xy-plane, we say that $\mathcal{S}$ has* trivial seed curve*. Otherwise, $\mathcal{S}$ has a* non-trivial seed curve*.*

We show that if $\mathcal{S}$ has trivial seed curve, then $\mathcal{S}$ must be a vertical plane as in (0.53). The next theorem shows that, suitably generalized, seed curves completely determine H-minimal surfaces.

Theorem 0.39. *Let $\mathcal{S} \subset \mathbb{H}^1$ be a C^2, complete, connected H-minimal surface. Then, either $\mathcal{S}$ is a vertical plane, or $\mathcal{S}$ is determined by a* generalized seed curve*.*

Loosely speaking, a generalized seed curve is a collection of seed curves and associated height functions together with patching data which define a single curve in $\mathbb{H}^1$. This single curve completely determines an H-minimal surface. In the following definition the signed curvature $\kappa(s)$ of a seed curve is that given in (0.56).

Definition 0.40. *An H-minimal surface is said to have* constant curvature *if either it has trivial seed curve (in which case $\mathcal{S}$ is a vertical plane), or if the signed curvature $\kappa(s)$ of each seed curve which is part of the generalized seed curve defining the H-minimal surface is constant.*

We can now state our main result.

Theorem 0.41 (of Bernstein type). *Let $\mathcal{S}$ be a C^2 connected H-minimal surface which is a graph over some plane P, then $\mathcal{S}$ has constant curvature according to Definition 0.40.*

When the complete H-minimal surface $\mathcal{S}$ fails to be a graph, then it need not have constant curvature. An example is given by the sub-Riemannian catenoids in [33]. We stress that in Theorem 0.41 we have made no assumption concerning the characteristic set of $\mathcal{S}$. Theorem 0.41 points out a rigidity of the seed curve under the assumption that the H-minimal surface is a graph: it must be composed of circles or lines. In the case where all the seed curves in the generalized seed curve are circles, $\mathcal{S}$ may or may not have empty characteristic set. However, in the case where at least one seed curve in the generalized seed curve is a straight line, we show that

$\mathcal{S}$ always has non-empty characteristic locus. We emphasize that there are many graph-like H-minimal surfaces given simply by specifying different initial height functions. For instance, the H-minimal surface (0.54) in the counterexample above, is a graph over the xt-plane, it has empty characteristic set, and its generalized seed curve consists of a single seed curve which is a circle. The plane $\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid t = 0\}$ is a graph over the xy-plane, it has non-empty characteristic locus, and again its single seed curve is a circle. On the other hand, the surface surface $\mathcal{S} = \{(x, y, t) \in \mathbb{H}^1 \mid t = xy/2\}$ found in [33] is a graph over the xy-plane, has non-empty characteristic set and seed curve a straight line. In connection with this latter example, if we further restrict our attention to graphs over the xy-plane, then we can completely classify the possible H-minimal surfaces. We mention that the relevant result, Theorem 0.42 below, has been first established in the recent paper [6], but with an approach completely different from that in [22].

Theorem 0.42. *Suppose $\mathcal{S}$ is a C^2 connected H-minimal graph over the entire xy-plane, then:*

1. *Either $\mathcal{S}$ has seed curve a circle, and $\mathcal{S}$ is a plane of the form $ax+by+ct = d$ for some real numbers a, b, c, d, with $c \neq 0$, (with characteristic set formed by the isolated point $\Sigma = \{(-2b/c, 2a/c, d/c)\}$).*
2. *Or, $\mathcal{S}$ has seed curve a straight line, and (modulo left-translation and rotation about the t-axis) $\mathcal{S}$ can be written as*

$$\Big(s + r, s - r, h_0(s) - \frac{sr}{2}\Big)$$

We note that the both of these classes of surfaces were found in [33], and the second set of examples can be written as

$$t = \alpha x^2 - \frac{xy}{2} + f(x - \alpha y)$$

for some real number α, and some function f (which is of course equivalent to the choice of an height function $h_0(s)$).

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