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Levi flatness and Stein basis

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Abstract[1]. Let Ω a bounded domain in $\mathbb{C}^n$ with smooth boundary. Let M a C^∞ Levi flat hypersurface of a neighbourhood U of $\overline{\Omega}$, such that $M \cap \emptyset$ is connected and $M \cap b\Omega$ is a connected submanifold of $b\Omega$. Let $K = \overline{\Omega} \cap M$. We consider the following problem: under what conditions K is a Stein compact set (i.e. has a Stein neighbourhoods basis)? A conjecture is: if bK is a topological $2n-1$-sphere then K is a Stein compact set. In the paper we discuss the problem when Ω is a strictly pseudoconvex domain in $\mathbb{C}^2$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary and M a topological (2n-1)-dimensional hypersurface of Ω, with boundary $bM \subset b\Omega$. M is said to be *Levi flat* if its interior is foliated by complex hypersurfaces (of complex dimension n-1). The question whether or not M has a *Stein basis* (i.e. a fundamental system of Stein neighbourhoods) is natural. We formulate the following conjecture:

> ($\star$) if Ω is strongly pseudoconvex and bM is a topological $(2n-1)$-sphere, then M has a Stein basis.

In this paper we discuss this problem assuming that Ω is a strongly pseudoconvex domain in $\mathbb{C}^2$.

More generally, we consider a topological surface $S \subset b\Omega$ defined as zero set $\{g = 0\}$ of a continuous function $g : b\Omega \to \mathbb{R}$ and we assume, for simplicity, that $b\Omega \setminus S$ has two connected components S_+, S_- and G has two peak points $p \in S_+$, $q \in S_-$, with $g(p) > 0$, $g(q) < 0$. Let $L(u)$ be the Levi operator introduced in [7] and $\mathcal{F} = \mathcal{F}_g$ be the family of all (weak) solutions of the Dirichlet problem $L(u) = 0$ in Ω, $u = g$ on $b\Omega$. Then, using the results of [7] and a generalized version of "Kontinuitätsatz", we prove that every Levi flat hypersurface M with boundary

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$bM = S$ is contained in the zero set $\{u = 0\}$ of u (cf. Theorem 3.1). It follows that

$$M \subset \bigcap_{u\in\mathcal{F}} \{u = 0\} \subset \{u^- \geq 0\} \cap \{u^+ \leq 0\} = \widehat{S}_{P(\Omega)}$$

where $u^\pm$ are the extremal solutions for the Dirichlet problem $L(u) = 0$ in Ω, $u = g$ on $b\Omega$ and $\widehat{S}_{P(\Omega)}$ is the hull of S with respect to the space of plurisubharmonic functions in Ω which are continuous on $\overline{\Omega}$ (cf. Theorem 2.1, 2.3).

In particular, if $M = \widehat{S}_{P(\Omega)}$, there exists a basis of neighbourhoods $\{U_m\}_{m\in\mathbb{N}}$ of M, for the relative topology of $\overline{\Omega}$, such that $\widetilde{U}_m = U_m \cap \Omega$, $m \in \mathbb{N}$, and $\Omega \setminus bU_m$ are Stein domains (cf. Corollary 3.3).

Finally, applying the above method to a C^1 sphere S we prove that the boundary of $\{u^- \geq 0\} \cap \{u^+ \leq 0\}$ is the union of two Levi flat hypersurfaces (cf. Theorem 4.1).

2. PRELIMINARIES

1. Let us recall the main results proved in [7] for the Dirichlet problem for the Levi operator

$$L(u) = (\delta_{\alpha\beta}|\partial u|^2 - u_{\bar\alpha}u_\beta)u_{\alpha\bar\beta}$$

$u_\alpha = u_{z_\alpha}$, $u_{\bar\alpha} = u_{\bar z_\alpha}$, $|\partial u|^2 = |u_1|^2 + |u_2|^2$.

Theorem 2.1. *Let $g \in C^0(b\Omega)$. The Dirichlet problem $L(u) = 0$ in Ω, $u = g$ on $b\Omega$ has two extremal weak* (*in the sense of viscosity*) *solutions $u^\pm \in C^0(b\Omega)$. All weak solutions u of the same problem satisfy $u^+ \leq u \leq u_-$.*

The function $u^-(u^+)$ is called the *lower* (*upper*) *weak solution* corresponding to g.

Remark 2.1. Comparison principle for $L(u) = 0$ fails to hold in general as it is shown by the solutions $u_1 = |z_1|^2$, $u_2 = 1 - |z_2|^2$ in the unit ball $\mathbb{B}$. Neverthless it is true for lower and upper solutions ([7], Proposition 4.3).

The following statement (cf. [7], Theorem 3.3) is crucial for our purposes:

Theorem 2.2. *Let $u \in C^0(\Omega)$ be a weak solution of $L(u) = 0$. Then*

- (i) *for every $c \in \mathbb{R}$, both sets $\{u \leq c\}$ and $\{u \geq c\}$ are pseudoconvex* (*whenever nonempty*)*;*
- (ii) *for every $c \in \mathbb{R}$, both sets $\{u \leq c\}$ and $\{u \geq c\}$ have local maximum property* (*whenever nonempty*)*;*
- (iii) *for every $c \in \mathbb{R}$, the sets $\{u = c\}$ have local maximum property* (*if it is nonempty*)*.*

We recall that a locally closed subset X of $\mathbb{C}^n$ has the *local maximum property* if for every point $x \in X$ there is a neighbourhood U of x with the following property: for every compact set $K \subset U$ and every function u, which is plurisubharmonic in a neighbourhood of K, one has

$$\max_{X\cap K} u = \max_{X\cap bK} u\ ,$$

where $\max\limits_{X\cap bK} u$ is meant to be $-\infty$ whenever $X \cap bK = \emptyset$. Accordingly, if X has the local maximum property then it has no isolated point.

2. Geometric properties of $u^{\pm}$ are described in terms of hulls (cf. [7], Corollary 5.3).

Let $P(\Omega)$ denote the space of all continuous functions in a bounded domain $\overline{\Omega}$ of $\mathbb{C}^n$ which are plurisubharmonic in Ω. For a compact $K \subset \Omega$ define the $P(\Omega)$-*hull* of K,

$$\widehat{K}_{P(\Omega)} = \{z \in \overline{\Omega} : u(z) \leq \max_{K} u, \forall u \in P(\Omega)\} \ .$$

Theorem 2.3. *Let $u^{\pm}$ be as in 2.2. Then, for all $c \in \mathbb{R}$, $\widehat{\{g = c\}}_{P(\Omega)} = \{u^{-} \leq c\} \cap \{u^{+} \geq c\}$.*

3. THE MAIN THEOREM

Let $M \subset \overline{\Omega}$ be as in Introduction. Let Ω_1, Ω_2 be the connected components of $\Omega \setminus K$ and $\mathcal{O}(\overline{\Omega}_1)$, $\mathcal{O}(\overline{\Omega}_2)$ the algebras of germs of holomorphic functions in a neighbourhood of $\overline{\Omega}_1$, $\overline{\Omega}_1$, respectively.

Denote $\widehat{K}_j$, $\widehat{bM}_j$ respectively, the hulls $\widehat{M}_{\mathcal{O}(\overline{\Omega}_j)}$, $\widehat{bM}_{\mathcal{O}(\overline{\Omega}_j)}$, $j = 1, 2$, of M, bM. Since every point of $b\Omega$ is a peak point for $\mathcal{O}(\overline{\Omega})$ we have the following

$$\widehat{bM}_j = \widehat{M}_j, \ \widehat{bM}_j \cap b\Omega = bM, \ j = 1, 2 \ .$$

Assume, for simplicity, that bM is connected in such a way that $b\Omega \setminus bM$ has two connected components $b\Omega_{+}$, $b\Omega_{-}$.

Let g a smooth function defining bM and having two peak points $p \in b\Omega_{-}$, $q \in b\Omega_{+}$ i.e. $bM = \{g = 0\}$, $g(q) < g(z) < g(p)$ for every $z \neq p, q$, and $g(p) > 0$, $g(q) < 0$.

We have the following

Theorem 3.1. *Let $u \in C^0(\overline{\Omega})$ be a weak solution of the problem $L(u) = 0$ in Ω, $u = g$ on $b\Omega$. Then for the zero set $\{u = 0\}$ of u we have*

$$M \subset \{u = 0\} \subset \widehat{bM}_1 \cup \widehat{bM}_2 = \widehat{M}_1 \cup \widehat{M}_2 \ .$$

If M is C^1

$$M \subset \{u = 0\} \subset \widehat{bM}_1 \cup \widehat{bM}_2 = \widehat{M}_1 \cup \widehat{M}_2 \ .$$

In particular, Theorem 2.3 implies

$$M \subset \{u^{+} \leq 0\} \cap \{u^{-} \geq 0\} \ .$$

First we prove the following

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, with C^1 boundary. Let K be a compact subset of $\overline{\Omega}$ such that $\Sigma = K \cap b\Omega$ is a C^1, connected, k-dimensional submanifold M of $b\Omega$. Assume that Σ is the boundary of a $(k+1)$-dimensional submanifold Σ of $b\Omega$. Then there exists a subset W of $\overline{\Omega}$ with the properties:*

- (i) *$W \cap b\Omega = W \cap K = \Sigma$, $\Sigma \subset \overline{W \setminus \Sigma}$;*
- (ii) *$W \setminus \Sigma$ is a C^1 submanifold.*

Proof. Let we choose ϵ and δ in such a way that the subsets $\{x \in \mathbb{R}^n : d(x, \Sigma) < \epsilon\}$ and $\{x \in \mathbb{R}^n : d(x, b\Omega) < \delta\}$ define a tubular neighbourhoods of class C^1 for Σ and $b\Omega$ respectively. Let $\{a_n\}_{n \in \mathbb{N}}, a_n \in \mathbb{R}^{+}$ be a strictly decreasing sequence such that $a_n \to 0$ and $a_1 < a < \epsilon$. Let $U_n = \{x \in \mathbb{R}^n : d(x, M) < a_n\}$. We choose then a sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of positive reals numbers such that $\epsilon_n < \min\{\epsilon, d(\overline{U}_n - U_{n+1}, K)\}$. It is immediately seen that there exists a $\psi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ of class C^1 such that

- $\psi < \delta$, $\psi(x)$ is constant for $x > a$ and $\psi(x) \to 0$ as $x \to 0$;
- $\psi < \sum_{n\in\mathbb{N}} \epsilon_n \chi_{(a_{n+1},a_n]}$.

Let $x \in b\Omega$ and $\hat{n}(x)$ be the inner unit normal vector to $b\Omega$ at x. We define $\rho : b\Omega \to \overline{\Omega}$ by

$$\rho(x) = x + \psi(d(x,\Sigma))\hat{n}(x) \ ;$$

it is clear that ρ is of class C^1 and that $W = \rho(N) \cup \Sigma$ satisfies the requested conditions ($W \setminus \Sigma$ is locally graph of a C^∞ function over a manifold).

□

Proof of Theorem 3.1. (1) We have $\{u = g(p)\} = p$, $\{u = g(q)\} = q$. Indeed $\{u = g(p)\} \cap b\Omega = \{p\}$ and assume, by a contradiction, that $\{u = g(p)\} \neq p$. Then there exists a real hyperplane H in $\mathbb{C}^n \setminus \{p\} \cup \{q\}$ such that $\{u = g(p)\} \cap H$ is a compact subset of $H \cap b\Omega$ and, denoting H^+ the open componente of $\mathbb{C}^2 \setminus H$ which does not contain p, one has $H^+ \cap \{u = g(p)\} \neq \emptyset$. In view of the extension theorem in [4], every holomorphic function in $\Omega \cap H^+ \setminus \{u = g(p)\}$ extends through $\{u = g(p)\}$ in contradiction with the fact that $\Omega \cap H^+ \setminus \{u = g(p)\}$ is a domain of holomorpy (cf. 2.2). In the same way is proved that $\{u = g(q)\} = \{q\}$.

(2) $M \subset \{u = 0\}$. We first observe that, according to 2.2, $\Omega \setminus \{u = 0\}$ is a domain of holomorphy and thus $\{u = 0\}$ has no compact connected component contained in Ω. Let us suppose that there exists a point z_0 in $(M \setminus \{u = 0\}) \cap \Omega$ and that $u(z_0) > 0$. Since $M \cap \{u = c\} = \emptyset$ for c near $g(q)$ there exist $c_0 > 0$ such that $(M \setminus bM) \cap \{u = c_0\} \neq \emptyset$ and $(M \setminus bM) \cap \{u = c_0\} = \emptyset$ for a sequence $c_\nu \searrow c_0$. This is in contradiction with "Kontinuitätsatz" (cf. [6], Theorem 5.2), since all level sets $\Omega \cap \{u = c\}$ have the local maximum property (cf. 2.2) and $\Omega \setminus M$ is pseudoconvex. Hence $M \subset \{u = 0\}$. The proof when $u(z_0) < 0$ is similar.

(3) Let M be C^1. Assume, for a contradiction, that there is a point $z_0 \in \{u = 0\} \setminus \widehat{bM}_1$ and let $W \subset \overline{\Omega}_1$ be as in Lemma 3.2. Let Ω_W be the domain bounded by W and M. Since $\mathcal{O}(\overline{\Omega}_W) \subset \mathcal{O}(\overline{\Omega}_1)$ one has

$$\widehat{bM}_{\mathcal{O}(\overline{\Omega}_W)} \subset \widehat{bM}_1 \ .$$

Let $f \in \mathcal{O}(\overline{\Omega}_1 \setminus \Omega_W)$; f is a CR function in $W \setminus bM$ and consequently, in view of [3], Theorem 1, it extends holomorphically on a open subset containing z_0. This gives a contradiction, since $\Omega_1 \setminus \{u \geq 0\}$ is pseudoconvex. This proves that $\{u = 0\} \cap \subset \widehat{bM}_1$.

The proof of the inclusion $\{u = 0\} \subset \widehat{bM}_2$ is similar.

□

Remark 3.1. In the statement of Theorem 3.1 the hypothesis that g has two peak points is crucial (cf. Example 3.1).

Corollary 3.3. *Assume that* $\widehat{\{g = 0\}}_{P(\Omega)} = M$. *Then there is a basis of neighbourhoods* $\{U_m\}_{m\in\mathbb{N}}$ *of* M *for the relative topology of* $\overline{\Omega}$ *such that* $\widetilde{U}_m = U_m \cap \Omega$, $m \in \mathbb{N}$, *and* $\Omega \subset bU_m$ *are Stein domain. In particular* $\mathring{M}$ *is union of an increasing sequence of Stein compacts* (*i.e. having a Stein basis*).

Example 3.1. Let $\mathbb{T}$ the real 2-torus $\{z \in \mathbb{C}^2 : \mid z_1 \mid = \mid z_2 \mid = 1\}$. $\{|z_1| = 1, |z_2| \leq 1\}$ and $\{|z_2| = 1, |z_1| \leq 1\}$ are two hypersurfaces of $\mathbb{C}^2$ whose boundary is $\mathbb{T}$; it is straightforward to see that they are Stein compacts. However, the (singular) hypersurface of $\mathbb{C}^2$

$$X = \{|z_1| = |z_2|, |z_1| \le 1\}\ ;$$

doesn't admit a fundamental system of holomorphically convex neighborhoods.

Proof. It suffices to show that, for every ϵ defined $V_\epsilon = \{||z_1| - |z_2|| < \epsilon\}$, then V_ϵ admits $\{|z_i| \le 1, i = 1, 2\}$ as holomorphic completion. So, let $f : V_\epsilon \to \mathbb{C}$ be an holomorphic function, and let we suppose that f extends to the set $A_M = V_\epsilon \cup \{|z_i| < M, i = 1, 2\}$[2] for some $M < 1$. Consider the set

$$H = \{z_1 = \alpha, |z_2| \le b\} \cup \{|z_1 - \alpha| \le c, |z_2| = b\}\ ,$$

with $|\alpha| = M - \epsilon/4$, $b = M + \epsilon/12$, $c = \epsilon/3$; each holomorphic function defined on a neighborhood of H extends to the interior of

$$P = \{|z_1 - \alpha| \le c, |z_2| \le b\}\ .$$

It is then clear that $\{z_1 = \alpha, |z_2| \le b\} \subset A_M$; in fact $|z_2| - |z_1| \le \epsilon/6$ and one is true among $|z_2| < M$ and $|z_1| - |z_2| \le \epsilon/4$. Moreover $\{|z_1 - \alpha| \le c, |z_2| = b\} \subset A_M$ because in that case $|z_2| - |z_1| \le 2\epsilon/3$ (certainly $|z_1| > |\alpha| - \epsilon/3$) and $|z_1| - |z_2| \le |z_1 - \alpha| + |\alpha| - |z_2| \le 0$. Then A_M is a neighborhood of H; it follows that f extends to $A_M \cup P$. For the arbitrariness of α (with $|\alpha| = M - \epsilon/4$) f extends over the set

$$A_M \cup \{|z_1| < M + \epsilon/12, |z_2| \le |z_1|\}\ .$$

Exchanging the role of z_1 and z_2 one shows that f is extendable to $A_{M+\epsilon/12}$.

□

4. AN APPLICATION

Let now assume that $S \subset b\Omega$ is a 2-dimensional C^1 sphere dividing $b\Omega$ into two connected components S_+, S_-. Let $g : b\Omega \to \mathbb{R}$ a C^1 defining function for S, as in Section 3, and let $u^\pm$ be as in Theorem 3.1.

As an application of our method we get the following

Theorem 4.1. *S is the boundary of a Levi flat hypersurface $M \subset \overline{\Omega}$ (not necessarily unique). Moreover*

- (i) *all Levi flat hypersurface with boundary S are contained in $\{u^- \ge 0\} \cap \{u^- \le 0\}$;*
- (ii) *the boundary of $\{u^- \ge 0\} \cap \{u^+ \le 0\}$ is union of two Levi flat hypersurfaces.*

Proof. (Sketch). Let $c_\nu \to 0$ be a sequence of real numbers such that the level sets $S_\nu = g = c_\nu$ are smooth 2-spheres satisfying the Bedford-Klingenberg conditions (cf. [1]). Let M_ν be the smooth Levi flat hypersurface, with boundary S_ν, as in the Main Theorem of [1]. By construction, M_ν is a union of analytic discs Δ_ν with boundaries in S_ν. Let $A(S_\nu)$ be the area of S_ν. Then, since $A(S_\nu) \to A(S)$ and analytic discs are area minimizing, it follows that $A(\Delta_\nu)$ is uniformly bounded. Now we invoke the "compactness theorem" of Gromov [2] (cf. also [5]) to deduce the existence of M, as in the statement.

(i) is a consequence of our main Theorem.

In order to prove (ii) let $\Omega \setminus M = \Omega_1 \cup \Omega_2$, Ω_1, Ω_2 be connected. Let $c_\nu \to 0$ be as above and such that $c_\nu > 0$. "Kontinuitätsatz" implies that $M_\nu \subset (\Omega_1 \setminus \{u^- \ge 0\} \cap \{u^- \le 0\})$, hence the limit hypersurface M is contained in $\overline{\Omega}_1 \cap \{u^- \ge 0\} \cap \{u^- \le 0\}$.

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[2] Of course, for small M (e.g. $M < \epsilon/2$) $A_M = V_\epsilon$.

Similarly, starting by a sequence $c_\nu \to 0$ with $c_\nu < 0$, we obtain a limit (Levi flat) hypersurface $M' \subset \overline{\Omega}_2 \cap \{u^- \geq 0\} \cap \{u^- \leq 0\}$.

$\square$

Remark 4.1. It is worth while to observe that we cannot exclude the existence of Levi flat hypersurfaces M such that $bM \subset S_+$ but $M \not\subset \overline{\Omega}_1$.

Remark 4.2. In the above theorem the existence of M, when S is only a topological 2-sphere, is an open, difficult problem, even for $\Omega = \mathbb{B}$, the standard ball in $\mathbb{C}^2$. In the same vein, another crucial problem is uniqueness of M. This is unknown even if $\Omega = \mathbb{B}$ and S is a smooth 2-sphere contained in $\mathbb{S}^3 = b\mathbb{B}$. Finally we observe that uniqueness and statement (ii) imply that M has a Stein basis. Example 3.1 shows that, in the general case, uniqueness is a stronger property than existence of a Stein basis.

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