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Strong comparison principle for q-pseudoconvex functions

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Abstract[1] We announce some recent results, jointly obtained with E. Lanconelli, about a new class of curvature PDO's describing relevant properties of real hypersurfaces of $\mathbb{C}^{n+1}$. In our setting the pseudoconvexity and the Levi form play the same role as the convexity and the real Hessian matrix play in the real Euclidean one. Our curvature operators are second order fully nonlinear PDO's not elliptic at any point. However, when computed on generalized q-pseudoconvex functions, we shall show that their characteristic form is nonnegative definite with kernel of dimension one. Moreover, we shall show that the missing ellipticity direction can be recovered by commutation. These properties allow us to prove a strong comparison principle, leading to symmetry theorems for domains with constant curvatures and to identification results for domains with comparable curvatures. These results will appear in a definitive form in [8].

1. INTRODUCTION

Let $D = \{z \in \mathbb{C}^{n+1} : f(z) < 0\}$ be a C^2 domain with boundary a real manifold

$$bD = \{z \in \mathbb{C}^{n+1} : f(z) = 0\} .$$

Here f is a real value function and

$$\partial_p f := (f_{z_1}(p), \dots, f_{z_{n+1}}(p)) \neq 0$$

at any point $p \in bD$, with $f_{z_j} = \dfrac{\partial f}{\partial z_j}$. We shall also write f_j instead of f_{z_j}, and use analogous notations for second order derivatives. We shall denote by $\Pi_p^{\mathbb{C}}(bD)$ the complex tangent space to bD at the point p

$$\Pi_p^{\mathbb{C}}(bD) = \{h \in \mathbb{C}^{n+1} : \langle h, \bar{\partial}_p f\rangle = 0\} ,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{C}^{n+1}$. Let us denote by

$$\mathcal{H}_p(f) := \Big(f_{j,\bar{k}}(p)\Big)_{j,k=1,\dots,n+1} .$$

the complex Hessian matrix of the function f at p. When we restrict the complex Hessian form to the complex tangent space we obtain a Hermitian form which

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is invariant under biholomorphic maps and which is called the *Levi form* of the function f at the point p

$$\zeta \longmapsto L_p(f,\zeta) := \langle \mathcal{H}_p^T(f)\zeta,\zeta\rangle = \sum_{j,k=1}^{n+1} f_{j,\bar{k}}\zeta_j\bar{\zeta}_k, \quad \forall \zeta \in \Pi_p^{\mathbb{C}}(bD) \ .$$

Remark 1. *It is a standard fact that $L_p(f,\cdot)/\Pi_p^{\mathbb{C}}(bD)$ is the biholomorphic invariant part of the real Hessian form of f, and we suggest the books* [5], [6], [9] *for details.*

We recall that the domain D is strictly *Levi-pseudoconvex* if the Levi form of f is strictly positive definite at any point of bD.

Let B be an orthonormal basis of $\Pi_p^{\mathbb{C}}(bD)$. We shall call *the B normalized Levi matrix* of bD at $p \in bD$ the Hermitian matrix

$$L_p(f,B) := \frac{1}{|\partial_p f|}\left(\langle \mathcal{H}_p^T(f)\zeta_j,\zeta_k\rangle\right)_{j,k=1,\cdots,n} \ .$$

If $p \in bD$ we denote by $\lambda_1(p),\cdots,\lambda_n(p)$ the eigenvalues of the normalized Levi form.

Proposition 1. *The eigenvalues of the normalized Levi form $L_p(f,B)$ are independent of the defining function f for the domain D and of the basis B. They only depend on the domain D.*

In view of the previous proposition it is quite natural to expect that symmetric functions in the eigenvalues of the normalized Levi form describe the geometric feature of a domain. Thus, we give the following definition.

Definition 1. *A domain D of $\mathbb{C}^{n+1}$ is q-pseudoconvex if for all $j \in \{1,\ldots,q\}$*

$$\sigma^{(j)}(\lambda_1,\ldots,\lambda_n) := \sum_{1\le i_1<\cdots<i_j\le n} \lambda_{i_1}\cdots\lambda_{i_j} > 0$$

at any point $p \in bD$.

For all $q \in \{1,\ldots,n\}$ we define the q-pseudocurvature of D

$$k_p^{(q)}(bD) = \sigma^{(q)}(\lambda)/\binom{n}{q} \ .$$

$k_p^{(n)}(bD) = \prod_{j=1}^n \lambda_j$ is the total Levi curvature introduced in [10], [7] and can be viewed as the complex analogous of the Gauss curvature. We shall call $k_p^{(1)}(bD) = (\lambda_1 + \cdots + \lambda_n)/n$ the Levi mean curvature to emphasize that it is the complex analogous of the Euclidean mean curvature. In the Euclidean case the intermeddle curvatures have been studied in [3].

Remark 2. *The notions of q-pseudoconvexity and of q-pseudocurvature are independent of the particular choice of the defining function of D.*

We remark that a domain D is strictly Levi-pseudoconvex iff it is n-pseudoconvex.

Classical elementary symmetric functions of the eigenvalues of the Levi form were considered by Bedford and Gaveau in [1]. Taking into account the definitions given in that paper, we recognize that

$$k_p^{(q)}(bD) = -\frac{1}{\binom{n}{q}} \frac{1}{|\partial_p f|^{q+2}} \sum_{1 \le i_1 < \cdots < i_{q+1} \le n+1} \Delta_{(i_1,\cdots,i_{q+1})}(f)$$

where

$$\Delta_{(i_1,\cdots,i_{q+1})}(f) = \det \begin{pmatrix} 0 & f_{\bar{i}_1} & \cdots & f_{\bar{i}_{q+1}} \\ f_{i_1} & f_{i_1,\bar{i}_1} & \cdots & f_{1,\bar{i}_{q+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{q+1}} & f_{i_{q+1},\bar{i}_1} & \cdots & f_{i_{q+1},\bar{i}_{q+1}} \end{pmatrix}.$$

In particular the total Levi curvature can be explicitly written as

$$k_p^{(n)}(bD) = -\frac{1}{|\partial_p f|^{n+2}} \det \begin{pmatrix} 0 & f_{\bar{1}} & \cdots & f_{\overline{n+1}} \\ f_1 & f_{1,\bar{1}} & \cdots & f_{1,\overline{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+1} & f_{n+1,\bar{1}} & \cdots & f_{n+1,\overline{n+1}}. \end{pmatrix}.$$

Example 1 (q-pseudocurvature of the ball). *Let B_R be the ball of radius R with center at the origin in $\mathbb{R}^{2n+1}$. Since $f(z) = |z|^2 - R^2$ is a defining function for B_R, we have*

$$L_p(f, B) = \frac{1}{R} I_n, \quad \forall p \in bB_R,$$

for any orthonormal basis B of the complex tangent space. Then, all the eigenvalues of the normalized Levi form are equal to $1/R$, so that

$$k_p^{(q)}(bB_R) = \left(\frac{1}{R}\right)^q, \quad \forall p \in bB_R. \tag{1.1}$$

Our main result is the following strong comparison principle.

Theorem 1 (Strong comparison Principle). *Let D and D' be q-pseudoconvex domains of $\mathbb{C}^{n+1}$ with connected boundaries. Suppose the following conditions are satisfied*

- **a.** $D' \subseteq D$
- **b.** $k_{p'}^{(q)}(bD') \le k_p^{(q)}(bD)$ $\forall p \in bD$ *and* $p' \in bD'$.

Then $D' = D$.

By taking into account (1.1) from Theorem 1 we immediately get the following symmetry result.

Corollary 1. *Let $D \subseteq \mathbb{C}^{n+1}$ a q-pseudoconvex domain with connected boundary, $1 \le q \le n$. Assume there exists a ball $B_R(z_0) \subseteq D$ tangent to bD at some point of bD. Then, if*

$$k_p^{(q)}(bD) \ge \left(\frac{1}{R}\right)^q, \quad \forall p \in bD,$$

we have $D = B_R(z_0)$.

We want to remark that all the previous definitions can be "localized" in a quite obvious way. Then, we can extend the notion of q-pseudoconvexity to the graphs of functions defined in an open subset of $\mathbb{R}^{2n+1}$. Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and let $u \in C^2(\Omega, \mathbb{R})$. Denote by

$$\Gamma(u) := \{(x,t) \in \Omega \times \mathbb{R} : u(x) < t\}$$

the epigraph of u and by

$$\gamma(u) := \{(x, u(x)) : x \in \Omega\}$$

the graph of u.

Definition 2. *We say that u is q-pseudoconvex if $\Gamma(u)$ is q-pseudoconvex at any point of $\gamma(u)$.*

From Theorem 1 we get an estimate of the domain over which the graph of a function u has prescribed q-pseudocurvature.

Corollary 2. *Let $u : B_R \to \mathbb{R}$ be a C^2 q-pseudoconvex function in the ball of $\mathbb{R}^{2n+1}$*

$$B_R := \{x \in \mathbb{R}^{2n+1} : |x| < R\}.$$

Denote by $k^{(q)}(x, u)$ the q-pseudocurvature of the graph of u at the point $(x, u(x))$. Then

$$R \leq \sup_{(x,u)\in B_R\times\mathbb{R}} \Big(\frac{1}{k^{(q)}(\xi, u)}\Big)^{1/q}.$$

Corollary 3. *Let $u : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a C^2 and q-pseudoconvex function. Then*

$$\inf_{(x,u)\in\mathbb{R}^{2n+2}} k^{(q)}(x, u) = 0.$$

2. Sketch of the proof of Theorem 1

Since the q-pseudocurvature is invariant for translations, it is not restrictive to assume $D' \subset D$ and $bD \cap bD' \neq \emptyset$. Let $p \in bD \cap bD'$ and U be a neighborhood of p such that $U \cap D$ and $U \cap D'$ are the epigraph of u and v respectively. We set $w = u - v$ and recognize that w satisfies

$$\mathcal{L}(w) = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j w + b_j(x) X_j w \geq 0, \tag{2.1}$$

where

$$X_j = \sum_{\ell=1}^{2n+1} b_j^\ell(x)\partial_{x_\ell},$$

are $2n$ first order partial differential operators whose coefficients depend on the Euclidean gradient Du : $b_j^\ell(x) = b_j^\ell(Du(x))$. Moreover, $b_j(x) = b_j(Du(x), Dv(x))$ and $a_{ij}(x) = a_{ij}(Du(x), D^2u(x), Dv(x), D^2v(x))$ satisfy

$$\sum_{i,j=1}^{2n} a_{ij}(x)\xi_i\xi_j > 0, \quad \forall\xi = (\xi_1, \dots, \xi_{2n}) \in \mathbb{R}^{2n}, \xi \neq 0, \tag{2.2}$$

at every point $x \in \Omega$.

Hence, the operator $\mathcal{L}$ in (2.1) is *"elliptic" only along $2n$ linearly independent directions.* The missing ellipticity direction can be recovered by commutation. Indeed, given the structure of the vector fields X_j's, the commutator $[X_j, X_k]$ takes the following form

$$[X_j, X_k] = v_{jk}(x)T,$$

where T is a first order partial differential operator such that, by identifying the first order operators with the vector fields having the same coefficients,

$$\mathrm{d}im\ (\mathrm{S}pan\{X_j, T : j = 1, \dots, 2n\}) = 2n + 1$$

at any point of Ω. Precisely $\{X_j,\, j = 1, \ldots, 2n\}$ is a real basis of $\Pi^{\mathbb{C}}_{(x,u(x))}(\gamma(u))$, viewed as a $2n$-dimensional real vector space, and $T \notin \Pi^{\mathbb{C}}_{(x,u(x))}(\gamma(u))$. Moreover, for every point $x \in \Omega$ there exists a pair (j, k) such that $v_{jk}(x) \neq 0$. Hence

$$\mathrm{d}im\ (\mathrm{S}pan\{X_j, [X_j, X_k] : j, k = 1, \ldots, 2n\}) = 2n + 1 \tag{2.3}$$

at any point of Ω.

The proof will then be a consequence of the following Bony's type maximum principle [2] (see also [4] for the case $n = 1$).

Theorem 2 (**Maximum Principle**). *Let $\Omega_0 \subseteq \Omega$ be open and connected and let $\mathcal{L}$ be the linear operator in* (2.1) *with continuous coefficients and satisfying conditions* (2.2) *and* (2.3). *Suppose $w \in C^2(\Omega_0, \mathbb{R})$ and*

$$\begin{cases} \mathcal{L}w \geq 0 & in\, \Omega_0 \\ w \leq 0 & in\, \Omega_0. \end{cases}$$

Then $w < 0$ in Ω_0 or $w \equiv 0$ in Ω_0.

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