Lecture Notes of
*Seminario Interdisciplinare di Matematica*
Vol. 3(2004), pp. 129 - 143.

# Recent results on Kolmogorov equations and applications

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**Abstract**[1] This paper contains a survey on a class of linear and non linear Kolmogorov-type operators, some applications to finance are discussed in details.

## 1. Introduction

We consider the second order partial differential equation

$$Lu(x,t) \equiv \sum_{i,j=1}^{m_0} \partial_{x_i} \left( a_{ij} \partial_{x_j} u(x,t) \right) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(x,t) - \partial_t u(x,t) = 0 \tag{1.1}$$

where $(x,t) = (x_1, \ldots, x_N, t) = z$ denotes the point in $\mathbb{R}^{N+1}$, and $1 \le m_0 \le N$, $A = (a_{ij})$ and $B = (b_{ij})$ are $N \times N$ real constant matrices, $A$ is symmetric and non-negative defined.

Equation (1.1) may be strongly degenerate, since only $m_0$ second order derivatives occur with $m_0 < N$, however the solutions to (1.1) can be regular. Indeed, Kolmogorov considered in 1934 [26] the following equation

$$\partial_{x_1}^2 u + x_1 \partial_{x_2} u - \partial_t u = 0, \qquad (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}\ , \tag{1.2}$$

and constructed an explicit fundamental solution of (1.2) which is a $C^\infty$ function outside the diagonal. This implies that (1.2) is hypoelliptic, that is every distributional solution to (1.2) is a $C^\infty$ function. Note that equation (1.2) can be written in the form (1.1) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Operators of the form (1.1) were considered by Hörmander, in his celebrated paper on hypoelliptic second order differential equations [24]. He pointed out that the Kolmogorov method, based on the Fourier transform, can also be applied to every operator in the form (1.1), under the following condition:

$$\text{rank Lie}\,(X_1, \ldots, X_N, Y) = N + 1\ , \tag{1.3}$$

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Workshop on Second Order Subelliptic Equations and Applications Cortona, June 15-21 2003.

at any point of $\mathbb{R}^{N+1}$. In (1.3), Lie $(X_1,\dots,X_N,Y)$ denotes the Lie algebra generated by $X_1,\dots,X_N,Y$ and

$$X_j=\sum_{k=1}^N a_{jk}\partial_{x_k},\ j=1,\dots,N, \qquad Y=\langle x,BD\rangle-\partial_t,$$

Under condition (1.3), Hörmander constructed in [24], page 148, an explicit fundamental solution for (1.1). Denote, for every $t\in\mathbb{R}$,

$$E(t)=\exp(-tB^T), \qquad C(t)=\int_0^t E(s)AE^T(s)ds\ . \tag{1.4}$$

As a consequence of (1.3), we have

$$C(t)>0 \quad \text{for every} \quad t>0\ , \tag{1.5}$$

then the function

$$\Gamma(x,t,\xi,\tau)=\Gamma(x-E(t-\tau)\xi,t-\tau)\ , \tag{1.6}$$

where $\Gamma(x,t)=0$ if $t\le 0$ and

$$\Gamma(x,t)=\frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}}\exp\left(-\frac{1}{4}\langle C^{-1}(t)x,x\rangle-t\ \mathrm{tr}(B)\right), \qquad \text{if } t>0 \tag{1.7}$$

is a fundamental solution for (1.1). Hereafter we use the notations

$$z=(x,t),\ \zeta=(\xi,\tau), \qquad x,\xi\in\mathbb{R}^N,\ t,\tau\in\mathbb{R}\ .$$

It is quite trivial to recognize that $\Gamma(z,\zeta)$ is a $C^\infty$ function outside $\{(z,\zeta)\in\mathbb{R}^{N+1}\times\mathbb{R}^{N+1} : z=\zeta\}$. Then, under assumption (1.3), the operator $L$ in (1.1) is hypoelliptic. It is noteworthy to remark that condition (1.3) is also necessary for (1.5) (see Theorem 3 in [27] and Proposition A.1 in [33] for an easy proof of the equivalence of (1.5) and (1.3)).

In PDE's theory, rank conditions like (1.3) are today called of Hörmander's type because they are sufficient hypoellipticity conditions for the class of second order differential operators

$$\sum_{j=1}^p X_j^2u+X_0u \qquad \text{where} \qquad X_j=\sum_{k=1}^N a_{jk}\partial_{x_k},\ j=0,\dots,p\ ,$$

and the $a_{jk}$ are real valued $C^\infty$ functions (see [24], Theorem 1.1, see also [45] for a general theory of the local regularity for Hörmander operators). Let us remark that, although operators in the form of "sum of squares of vector fields" $\sum_{j=1}^p X_j^2u$ or in the form of the corresponding "heat operators" $\sum_{j=1}^p X_j^2u-\partial_tu$ are widely studied (see for instance [48], [8] and their bibliography) only few results of the general theory of Hörmander operators apply to Kolmogorov type equations.

We next give some motivations for the study of operators of the form (1.1) from the theory of probability and mathematical physics. The operator (1.2) is the lowest dimension version of the following degenerate parabolic operator in $\mathbb{R}^{N+1}$, $N=2n$,

$$L=\sum_{j=1}^n\partial_{x_j}^2+\sum_{j=1}^n x_j\partial_{x_{n+j}}-\partial_t\ . \tag{1.8}$$

Kolmogorov introduced (1.8) in 1934 in order to describe the probability density of a system with $2n$ degree of freedom. The $2n$-dimensional space is the phase space, $(x_1, \dots, x_n)$ is the velocity and $(x_{n+1}, \dots, x_{2n})$ the position of the system. We also recall that (1.8) is a prototype for a family of evolution equations arising in the kinetic theory of gases that take the following general form

$$Yu = \mathcal{J}(u)\ . \tag{1.9}$$

Here $\mathbb{R}^{2n} \ni x \longmapsto u(x,t) \in \mathbb{R}$ is the density of particles which have velocity $(x_1, ..., x_n)$ and position $(x_{n+1}, ..., x_{2n})$ at time $t$,

$$Yu := -\sum_{j=1}^{n} x_j \partial_{x_{n+j}} u + \partial_t u$$

is the so called *total derivative of* $u$ and $\mathcal{J}(u)$ describes some kind of collisions. This last term can take different form. For instance, in the usual Fokker-Planck equation, we have

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}\partial_{x_j} u + b_i u\right) + \sum_{i=1}^{n} a_i \partial_{x_i} u + au \tag{1.10}$$

where $a_{ij}, a_i, b_i$ and $a$ are functions of $(x,t)$; $\mathcal{J}(u)$ may also occur in non-divergence form, moreover they may depend on $z \in \mathbb{R}^{2n+1}$ and on the unknown function $u$, even through some integral expressions. This kind of operator is studied as a simplified version of the Boltzmann collision operator. For the description of wide classes of stochastic processes and kinetic models leading to equations of the previous type, we refer to the classical monographies [10], [17] and [11].

Ultraparabolic differential equations with *non linear total derivative terms* appear when studying convection-diffusion models. We would like to mention the paper by Escobedo, Vázquez and Zuazua [20] in which the following equation is studied

$$\partial_y g(u) - \partial_t u = -\Delta_x u, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ y, t \in \mathbb{R}\ . \tag{1.11}$$

The linearized equation of (1.11) $g'(u)\partial_y v - \partial_t v = -\Delta_x v$, if $g'(u)$ is different from zero and smooth enough, can be reduced to the form (1.1) with $N = n + 2$,

$$A = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}\ . \tag{1.12}$$

In next section we describe the Lie group structure framework suitable for the study of the Kolmogorov operators, then we give a survey of known results. Finally, we discuss some other motivations arising in finance.

## 2. Lie groups related to Kolmogorov operators

Let $L$ be the operator in (1.1) with constant matrices $A$ and $B$. In [33] it was shown that $L$ is invariant with respect to the left translations of the Lie group $\mathcal{G} = \left(\mathbb{R}^{N+1}, \circ\right)$ with composition law defined by

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \qquad (x,t), (\xi,\tau) \in \mathbb{R}^N \times \mathbb{R}\ , \tag{2.1}$$

with $E(\cdot)$ as in (1.4). It is easily checked that $(\xi,\tau)^{-1} = (-E(-\tau)\xi, -\tau)$ in $\mathcal{G}$ then, by (1.6), we can write $\Gamma(z,\zeta) = \Gamma(\zeta^{-1} \circ z)$. The Hörmander condition (1.3) implies that, for some basis on $\mathbb{R}^N$, the matrices $A$ and $B$ take the following block form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \\ * & * & * & \cdots & * \end{pmatrix} \tag{2.2}$$

where $A_0$ is a symmetric non-singular $p_0 \times p_0$ matrix and the $B_j$'s blocks are $p_{j-1} \times p_j$ matrices of rank $p_j$, $j = 1, 2, \dots, r$. The $p_j$'s are positive integers such that

$$p_0 \geq p_1 \geq \dots p_r \geq 1, \qquad \text{and} \qquad p_0 + p_1 + \cdots + p_r = N,$$

and the blocks denoted by $*$ are arbitrary (see [33], Proposition 2.1).

In the sequel we shall call *Kolmogorov operator with constant coefficients* any operator of the type (1.1) with the matrices $A$ and $B$ satisfying the above structural conditions. The class of Kolmogorov operators with constant coefficients contains a remarkable subclass of operators which are also invariant with respect to a suitable dilation group. Indeed, there exists a group of dilations $(\delta_\lambda)_{\lambda>0}$ such that

$$L \circ \delta_\lambda = \lambda^2 \left(\delta_\lambda \circ L\right), \qquad \forall \lambda > 0, \tag{2.3}$$

*if and only if* all the $*$-blocks in (2.2) are zero matrices. In this case

$$\delta_\lambda = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}, \lambda^2), \tag{2.4}$$

where $I_{p_j}$ denotes the $p_j \times p_j$ identity matrix. The proofs of these statements are contained in [28] and [33](cf. Proposition 2.2). When the $*$-blocks in $B$ are zero, the dilations $(\delta_\lambda)_{\lambda>0}$ in (2.4) are a group of automorphisms of $\mathcal{G}$. Equipped with them, $\mathcal{G}$ becomes a homogeneous group with homogeneous dimension $Q+2$, where

$$Q := p_0 + 3p_1 + \cdots + (2r+1)p_r,$$

(see [27], page 288, and [33], Remark 2.1).

We shall call *homogeneous Kolmogorov operator* every Kolmogorov operator whose matrix $B$ has null $*$-blocks. It is easy to check that the fundamental solution $\Gamma$ of a homogeneous Kolmogorov equation is $\delta_\lambda$-homogeneous of degree $-Q$, i.e.

$$\Gamma(\delta_\lambda(z)) = \lambda^{-Q}\Gamma(z), \quad \forall z \in \mathbb{R}^{N+1} \setminus \{0\},\ \forall \lambda > 0.$$

In this case $\Gamma$ takes the following simple form:

$$\Gamma(x,t) = \frac{c_N}{t^{Q/2}} \exp\left(-\frac{1}{4}\langle C^{-1}(1) D_0(t^{-1/2})x, D_0(t^{-1/2})x\rangle\right),$$

where $C(1)$ is given by (1.4), with $t = 1$ and $D_0(\lambda)$ is the $N \times N$ matrix

$$D_0(\lambda) = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}).$$

Moreover $c_N = (4\pi)^{-N/2} \left(\det C(1)\right)^{-1/2}$ (see [33], Proposition 2.3).

In the family of Kolmogorov operators, the homogeneous ones play a central role. Indeed, any Kolmogorov operator can be approximated, in a suitable sense, by a homogeneous operator. More precisely, let $B$ in (2.2) be the matrix related to some Kolmogorov operator with constant coefficients

$$L = \text{div}\,(AD) + \langle x, BD\rangle - \partial_t. \tag{2.5}$$

If we denote by $B_0$ the matrix obtained by annihilating every $*$-block in (2.2), the operator

$$L_0 = \operatorname{div}(AD) + \langle x, B_0 D\rangle - \partial_t$$

is homogeneous. Denoting by $\Gamma$ and $\Gamma_0$ the fundamental solutions with pole at $\zeta = 0$ of $L$ and $L_0$ respectively, then, for every $b > 0$ there exists a positive constant $a$ such that

$$\frac{1}{a}\Gamma_0(z) \le \Gamma(z) \le a\Gamma_0(z) \tag{2.6}$$

on the level set $\{z : \Gamma_0(z) > b\}$ (see [33], Theorem 3.1). The operator $L_0$ is homogeneous and, due to inequalities (2.6), it could be called the *principal part of* $L$.

As we already noticed, the fundamental solution of a constant coefficients Kolmogorov operator $L$ is invariant with respect to the left translations on $\mathcal{G} = (\mathbb{R}^{N+1}, \circ)$, and homogeneous of degree $-Q$ with respect to the dilations $\delta_\lambda$ ($\mathcal{G}$ and $\delta_\lambda$ defined in (2.1) and (2.4)). Then, it is quite obvious to expect that the intrinsic geometry underlying $L$ is that one determined by $\mathcal{G}$ and $\delta_\lambda$.

Let $\alpha_1, \dots, \alpha_N$ be strictly positive integers such that

$$\delta_\lambda = \operatorname{diag}\left(\lambda^{\alpha_1}, \dots, \lambda^{\alpha_N}, \lambda^2\right)$$

and define, for every $z \in \mathbb{R}^{N+1} \setminus \{0\}$, $\|z\|_{\mathcal{G}} = \rho$ where $\rho$ is the unique positive solution to the equation

$$\frac{t^2}{\varrho^4} + \sum_{j=1}^{N} \frac{x_j^2}{\varrho^{2\alpha_j}} = 1, \qquad z = (x_1, \dots, x_N, t)\ .$$

We agree to let $\|z\|_{\mathcal{G}} = 0$ if $z = 0$. Then $z \longmapsto \|z\|_{\mathcal{G}}$ is a $\delta_\lambda$-homogeneous function of degree one, continuous on $\mathbb{R}^{N+1}$, strictly positive and of class $C^\infty$ in $\mathbb{R}^{N+1} \setminus \{0\}$. If we define

$$d_{\mathcal{G}}(z, \zeta) = \|\zeta^{-1} \circ z\|_{\mathcal{G}}, \qquad z, \zeta \in \mathbb{R}^{N+1}\ , \tag{2.7}$$

then $(\mathbb{R}^{N+1}, d_{\mathcal{G}})$ is a (pseudo-)metric space, the natural one for the operator $L$, since it is not difficult to recognize that $d_{\mathcal{G}}$ is equivalent to the *control distance* related to $L$.

## 3. A survey of results

Consider in $\mathbb{R}^{N+1}$ the second order differential operator

$$L = \sum_{i,j=1}^{p_0} a_{ij}(z)\partial_{x_i x_j} u + \langle x, BDu\rangle - \partial_t u\ , \tag{3.1}$$

where $1 \le p_0 \le N$, $z = (x, t) \in \mathbb{R}^N \times \mathbb{R}$. Assume the matrix $B$ as in (2.2) with the $*$-blocks equal to zero; the block $B_j$ has rank $p_j$ and dimension $p_{j-1} \times p_j$, $j = 1, \dots, r$, with $p_0 \ge p_1 \ge \dots \ge p_r \ge 1$. Suppose the matrix $(a_{ij})_{i,j=1,\dots,p_0}$ in (3.1) uniformly positive definite, i.e. there exists $\lambda > 0$ such that

$$\frac{1}{\lambda}\sum_{j=1}^{p_0} \xi_j^2 \le \sum_{i,j=1}^{p_0} a_{ij}(z)\xi_i\xi_j \le \lambda \sum_{j=1}^{p_0} \xi_j^2 \tag{3.2}$$

for every $(\xi_1, \dots, \xi_{p_0}) \in \mathbb{R}^{p_0}$ and for every $z \in \mathbb{R}^{N+1}$. We also assume the $a_{ij}$ Hölder continuous with exponent $\alpha \in ]0,1[$ with respect to the distance $d_{\mathcal{G}}$ in (2.7), i.e.

$$|a_{ij}(z) - a_{ij}(\zeta)| \le M d_{\mathcal{G}}(z,\zeta)^\alpha, \qquad \forall z, \zeta \in \mathbb{R}^{N+1} , \tag{3.3}$$

for some constant $M$. Under these hypotheses, a fundamental solution for the operator $L$ can be constructed by adapting the Levi's parametrix method to the Lie group and metric structures related to the matrix $B$ (see [41], Theorem 1.1, which improves and generalizes the previous results by Weber [49], Il'in [25] and Sonin [47]).

The Levi's parametrix method also provides a global upper bound for the fundamental solution $\Gamma$. It was shown in [41], Corollary 2.5, that there exists a positive constant $\mu$ such that, if $\Gamma^+$ denotes the fundamental solution of the constant coefficients Kolmogorov operator

$$L^+ = \mu \Delta_{p_0} + \langle x, BD \rangle - \partial_t ,$$

then,

$$\Gamma(z,\zeta) \le c^+ \Gamma^+(z,\zeta) \tag{3.4}$$

for every $z = (x,t),\ \zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$, $0 < t - \tau < T$ for some positive constant $c^+ = c^+(T)$.

If the operator $L$ can be written in divergence form

$$L = \sum_{i,j=1}^{p_0} \partial_{x_i} \left( a_{ij}(z) \partial_{x_j} \right) + Yu \tag{3.5}$$

then a lower bound for $\Gamma$ analogous to (3.4) also holds. This result relies on a Harnack inequality for non negative solutions to $Lu = 0$, which is invariant with respect to the translation and dilation groups, related to the matrix $B$, described in Section 2. To be more specific, let us introduce some notation. Consider the Euclidean cylinder

$$H_1 = \{ (x,t) \in \mathbb{R}^N \times \mathbb{R} \ \mid \ |x| < 1,\ |t| < 1 \} .$$

For every $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$H_r(z_0) \equiv z_0 \circ (\delta_r(H_1)) = \{ z \in \mathbb{R}^{N+1} \ \mid \ z = z_0 \circ \delta_r(\zeta),\ \zeta \in H_1 \} , \tag{3.6}$$

and

$$H_r^-(z_0) = \{ z \in H_r(z_0) \ \mid \ t = t_0 - r^2 \} .$$

Let $u$ be a non negative solution to $Lu = 0$ in $\Omega$, then

$$\sup_{H_{r\theta}^-(z_0)} u \le c_0 u(z_0) , \tag{3.7}$$

for every $H_r(z_0) \subseteq \Omega$, $0 < r \le r_0$ (see [41] Theorem 1.3; $c_0, r_0 > 0$ and $\theta \in ]0,1[$ are constant only depending on $B$, on $\lambda$ and $M$ in (3.2)-(3.3)). The above result extends some Harnack inequalities for constant coefficients Kolmogorov operators first appeared in [29], [21] and [33]. Starting from inequality (3.7), the following global lower bound estimate is proved (see [42], Main Theorem).

**Theorem 3.1.** *Let* $\Gamma$ *be the fundamental solution of the divergence form operator* (3.5). *There exists a positive constant* $\mu$ *such that, if* $\Gamma^-$ *denotes the fundamental solution of*

$$L^- = \mu^{-1}\Delta_{p_0} + \langle x, BD\rangle - \partial_t \ ,$$

*then, for every* $T > 0$, *there exists a positive constant* $c^-$ *such that*

$$c^-\Gamma^-(z,\zeta) \le \Gamma(z,\zeta) \tag{3.8}$$

*for every* $z = (x,t),\ \zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$, $0 < t - \tau < T$.

We would like again to stress that the functions $\Gamma^-$ and $\Gamma^+$ appearing in (3.4) and (3.8) have the explicit form (1.6)-(1.7), with the matrix $A$ in (1.4) replaced by $\mu^{-1}\mathrm{diag}(I_{p_0}, 0, ..., 0)$ and $\mu\ \mathrm{diag}(I_{p_0}, 0, ..., 0)$ respectively. Theorem 3.1 was proved in [42] by using a technique which was inspired by a method of Aronson and Serrin for classical parabolic operators. The core of the method used in [42] is a kind of discretization of the connectivity Theorem of Caratheodory-Razewski-Chow. The rank condition (1.3) played a crucial role.

We want to close this section by briefly recalling some interior regularity results. We first consider the Schauder type estimates proved in [35] (see also [34]). The results of these papers improve and generalizes the previous ones contained in [22], [46] and [18]. Let $L$ be as in (3.1) and $u$ be a smooth real function defined on a subset $\Omega$ of $\mathbb{R}^{N+1}$. Then, for every bounded open set $\Omega_1$ such that $\overline{\Omega}_1 \subseteq \Omega$, there exists a constant $c > 0$ such that

$$|u|_{2+\alpha,\Omega_1} \le c\left(\sup_{\Omega}|u| + |Lu|_{\alpha,\Omega}\right) . \tag{3.9}$$

Here we use the notations

$$|f|_{\alpha,\Omega} = \sup_{z,\zeta\in\Omega, z\neq\zeta} \frac{|f(z)-f(\zeta)|}{(d_{\mathcal{G}}(z,\zeta))^{\alpha}} \ ,$$

and

$$|f|_{2+\alpha,\Omega_1} = \sum_{i,j=1}^{p_0} |\partial_{x_i x_j} f|_{\alpha,\Omega_1} + |Yu|_{\alpha,\Omega_1} + \sup_{\Omega_1}|u| \ .$$

In [35], the interior Schauder estimates are also used to study a first boundary value problem for $L$.

The $L^p$ regularity theory for weak solutions to equations in non-divergence form or in divergence form has been studied in [9], [36], [43] and [44]. The main assumptions is that the coefficients $a_{ij}$ belong to the space of *vanishing mean oscillation* $\mathrm{VMO}_{\mathcal{G}}$ defined as follows. Denote by $B_r(z_0)$ the $d_{\mathcal{G}}$-ball of center $z_0$ and radius $r$:

$$B_r(z_0) = \{z \in \mathbb{R}^{N+1} \ \mid \ d_{\mathcal{G}}(z_0, z) < r\} \tag{3.10}$$

and denote the Lebesgue measure of $B_\rho$ by $\mathrm{meas}(B_\rho)$. We say that a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$ belongs to the space $\mathrm{VMO}_{\mathcal{G}}$ if

$$\lim_{r\to 0^+}\left(\sup_{\rho\le r} \frac{1}{\mathrm{meas}(B_\rho)} \int_{B_\rho} |u - u_{B_\rho}|\right) = 0 \ , \tag{3.11}$$

where

$$u_{B_\rho} = \frac{1}{\mathrm{meas}(B_\rho)} \int_{B_\rho} u(\zeta) d\zeta \ .$$

In [9] and [43], interior regularity properties of strong solutions to the non-divergence form equation $Lu = f$, with $L$ as in (3.1), were studied. The main results are some $L^p_{\text{loc}}$ estimates of the derivatives of the solution $u$ and its Hölder continuity in terms of some $L^q_{\text{loc}}$ norm of $f$. The key tools are some deep continuity results for singular integrals, and their commutators with $\text{VMO}_{\mathcal{G}}$ functions. The same methods and techniques, suitably adapted, were used in [36] and in [44] in order to prove interior regularity results for weak solutions to the equation $Lu = \sum_{i=1}^{p_0} \partial_{x_i} F_i$, where $L$ is in divergence form (3.5).

We would also like to quote the paper [30] in which a boundary value problem for a class of quasilinear operators of Fokker-Planck type was studied. In [30] the a priori estimates of [35] are used as crucial tools. The Hölder estimates for weak solutions to (3.5) have been used in [31] for studying a boundary value problem for the non linear equation

$$\sum_{i,j=1}^{p_0} \partial_{x_i} \left( a_{ij}(z,u) \partial_{x_j} \right) + Yu = 0 \ . \tag{3.12}$$

For a more detailed summary of results we refer to [32].

## 4. The Moser iterative method

As said in the previous Section, the Hölder estimates for weak solutions to (3.5) proved in [36] and in [44], have been used in [31] for studying a boundary value problem for the non linear equation (3.12). However, the dependence of the Hölder constant on the regularity of the coefficients $a_{ij}$ forces quite restrictive hypotheses on the nonlinearity. In order to remove such restrictions, regularity results for solutions to linear equations with merely measurable $a_{ij}$'s are needed. A first result in such a direction has been recently proved by the author in collaboration with A. Pascucci. In [38], the local boundedness of the weak solutions to (3.12) is proved only assuming the uniform positivity condition (3.2) for the matrix $(a_{ij})$. The main result in [38] is the following theorem.

**Theorem 4.1.** *Let $u$ be a weak solution to*

$$\sum_{i,j=1}^{p_0} \partial_{x_i} \left( a_{ij}(z) \partial_{x_j} \right) + Yu = 0 \tag{4.1}$$

*in an open set $\Omega \subseteq \mathbb{R}^{N+1}$ an assume that condition (3.2) holds. Then we have*

$$\sup_{H_\rho(z_0)} |u| \le c \left( \frac{1}{(r-\rho)^{Q+2}} \int_{H_r(z_0)} |u|^p \right)^{\frac{1}{p}} ,$$

*for every $p \ge 1$ and $r, \rho > 0$ such that $\frac{r}{2} \le \rho < r$ and $\overline{H_r(z_0)} \subseteq \Omega$. Here $H_r(z_0)$ is the $\mathcal{G}$-cylinder defined in (3.6). The constant $c$ only depends on $p, \lambda$ and the matrix $B$.*

This theorem is proved in [38] by using an iterative procedure analogous to the one introduced by Moser in the classical elliptic and parabolic cases. As it is well known, the Moser technique is based on a combination of Caccioppoli type estimates with the classical Sobolev inequality. Now, the weak solutions to (4.1) satisfy a Caccioppoli type estimate. However, it only gives $L^2_{\text{loc}}$ bound of the first order derivatives $\partial_{x_j} u$, $j = 1, \ldots, p_0$ and does not give any information on the others $(N - p_0)$ spatial derivatives. Thus, if $p_0 < N$, this lack of information cannot be restored by the usual Sobolev embedding theorem.

The key idea in [38] *is to prove a Sobolev type inequality for non negative sub- and super-solutions to* (4.1), good enough to be successfully combined with the previous "weak" Caccioppoli inequality. To be more specific, let us first recall the definition of weak sub- and super-solution to (4.1). We say that a function $u \in L^2_{\text{loc}}(\Omega)$, $\Omega$ open subset of $\mathbb{R}^{N+1}$, is a *weak sub-solution* to (4.1) if the weak derivatives $\partial_{x_1} u, \ldots, \partial_{x_{p_0}} u, Yu$ exist, belong to $L^2_{\text{loc}}(\Omega)$ and

$$\int_\Omega -\langle ADu, D\varphi\rangle + \phi Yu \geq 0, \qquad \forall \phi \in C_0^\infty(\Omega),\ \phi \geq 0\ .$$

If $-u$ is a weak sub-solution, we say that $u$ is a weak super-solution. Then, the following Caccioppoli type estimate holds (cf. [38], Proposition 3.2)

**Proposition 4.2.** *Let $u$ be a non-negative weak sub-solution of* (4.1) *in $\Omega$. Let $\rho, r > 0$, $\frac{r}{2} \leq \rho < r$, and $\overline{H_r} \subseteq \Omega$. Then, there exists a constant $c$, only dependent on $\lambda$ in* (3.2) *and on the homogeneous dimension $Q$, such that*

$$\|\partial_{x_j} u^p\|_{L^2(H_\rho)} \leq \frac{c\sqrt{1+\varepsilon}}{\varepsilon} \|u^p\|_{L^2(H_r)}, \quad \textit{where} \quad \varepsilon = \frac{|2p-1|}{4p}\ , \tag{4.2}$$

*for every $j = 1, \ldots, p_0$ and $p < 0$ or $p \geq 1$. The same inequality holds for non-negative weak super-solutions and $p \in ]0, 1/2[$.*

The key Sobolev type inequality for weak sub- and super-solutions proved in [38] is the following.

**Proposition 4.3.** *Let $u$ be a non-negative weak sub-solution to* (4.1) *and let $r, \rho$ be as in the previous Proposition 4.2. Then $u \in L^{2\kappa}_{\text{loc}}(H_\rho)$, $\kappa = 1 + \frac{2}{Q}$, and there exists a constant $c$, only dependent on $Q$ and $\lambda$, such that*

$$\|u\|_{L^{2\kappa}(H_\rho)} \leq \frac{c}{r-\rho} \left( \|u\|_{L^2(H_r)} + \sum_{j=1}^{p_0} \|\partial_{x_j} u\|_{L^2(B_r)} \right)\ . \tag{4.3}$$

*The same inequality holds for non-negative super-solutions.*

Inequalities (4.2)-(4.3) allow to start up an iterative procedure like to the classical Moser's one and to prove Theorem 4.1. We also recall that Theorem 4.1 has been used in [39] to obtain a pointwise global upper bound for the fundamental solution of (3.5)

**Theorem 4.4.** *There exists a positive constants $C$, only dependent on $\lambda$ in* (3.2) *and on $B$, such that*

$$\Gamma(x,t,\xi,\tau) \leq C\ \Gamma^+(x,t,\xi,\tau), \qquad \forall x, \xi \in \mathbb{R}^N,\ t > \tau\ .$$

*Here $\Gamma^+$ is the fundamental solution of a suitable constant coefficient Kolmogorov operator.*

Let us explicitly note that Theorem 4.4 improves inequality (3.4) since $C$ and $\Gamma^+$ do not depend on the continuity modulus of the coefficients $a_{i,j}$'s.

## 5. Applications to finance

In this section we give some motivations for the study of linear and non linear Kolmogorov equations arising in finance.

Linear equations arise in the celebrated Black & Scholes analysis [7]. Consider a "stock" whose price $S_t$ is given by the stochastic differential equation

$$d\,S_t = \mu S_t\,dt + \sigma S_t\,dW_t \tag{5.1}$$

where $\mu$ and $\sigma$ are positive constants and $W_t$ is a Wiener process. Also consider a "bond" whose price $B_t$ only depends on a constant rate of interest $r$: $B_t = B_0 e^{t\,r}$. Finally, consider an "European option", that is a contract which gives the *right* (but not the *obligation*) to buy the stock at a given "exercise price" $E$ at a given "expiry time" $T$. The problem studied in [7] is to find a fair price of the option contract. Under some assumptions on the financial market, Black & Scholes show that the price of the option, as a function of the time and of the stock price $V(t,S_t)$, is the solution of the following partial differential equation

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} = 0 \tag{5.2}$$

in the domain $(S,t) \in \mathbb{R}^+ \times ]0,T[$, with the *final condition*

$$V(T,S_T) = \max(S_T - E, 0)\ . \tag{5.3}$$

In the last decades the Black & Scholes theory has been developed by many authors and mathematical models involving Kolmogorov type equations have appeared [1], [4], [5] and [50]. Indeed, *Asian options* are options whose exercise price is not fixed as a given constant $E$, but depends on some average of the history of the stock price. In that case, the value of the option at the expiry time $T$ is:

$$V(S_T, M_T) = \max\left(S_T - e^{\frac{M_T}{T}}, 0\right), \quad M_t = \int_0^t \log(S_\tau)d\tau\ ;$$

if we use a geometric average,

$$V(S_T, M_T) = \max\left(S_T - \frac{M_T}{T}, 0\right), \quad M_t = \int_0^t S_\tau d\tau\ ;$$

if we use an arithmetic average. When considering the geometric average (suppose by simplicity that the interest rate is $r = 0$) the Black & Scholes method leads to the following degenerate equation

$$S^2\partial_S^2 V + (\log S)\partial_M V + \partial_t V = 0, \qquad S,t > 0,\ M \in \mathbb{R} \tag{5.4}$$

which can be reduced to the Kolmogorov equation (1.2) by means of an elementary change of variables (see [6], page 479). In the case of arithmetic average we find

$$S^2\partial_S^2 V + S\partial_M V + \partial_t V = 0, \qquad S,M,t \in \mathbb{R}^+\ . \tag{5.5}$$

The Cauchy problem for both equations (5.4) and (5.5) is studied in [6]. Note that the coefficient $S^2$ is unbounded and is not strictly positive, then (5.5) is somehow more degenerate than the Kolmogorov equation (1.2). However the theory of Kolmogorov equations has been adapted in [6] so that existence, uniqueness and regularity of the solution of the Cauchy problem related to (5.5) is proved (see Theorem 4.6, Remark 4.7 and Theorem 4.14 ). Moreover, some numerical methods to approximate the solution of the Cauchy problem are proposed in [6]. In order to describe the numerical method, we introduce the "discrete" Kolmogorov operator

$$L_G u(x,y,t) = -\frac{u(x,y,t)-u(x,y+\delta x,t-\delta)}{\delta} + \tag{5.6}$$

$$+\frac{u(x-h,y+\delta x,t-\delta)-2u(x,y+\delta x,t-\delta)+u(x+h,y+\delta x,t-\delta)}{h^2},$$

that is an approximation of the operator (1.2) in the sense that

$$L_G u = Lu + (h^2+\delta)O(h,\delta)$$

(here $O(h,\delta)$ denotes a bounded function, $(x,y,t) \in \mathbb{R}^2\times ]0,T[$ and $L$ is as in (1.2)). The operator $L_G$ is well defined in the grid

$$G = \left\{ (j\Delta_x, k\Delta_y, n\Delta_t) \in \mathbb{R}^3 : j,k,n \in \mathbb{Z} \right\},$$

with $\Delta_x = h, \Delta_t = \delta$ and $\Delta_y = t\delta$. In next theorem we compare the solution of the discrete Cauchy problem

$$L_G u_G = 0 \text{ in } G \cap \mathbb{R}^2\times ]0,T[, \quad u_G = \varphi \text{ in } G\cap\{t=0\}, \tag{5.7}$$

with the solution to the Cauchy problem

$$Lu = 0 \text{ in } \mathbb{R}^2\times ]0,T[, \quad u = \varphi \text{ in}\{t=0\}, \tag{5.8}$$

where $\varphi$ is a bounded continuous function.

**Theorem 5.1.** *Let $u$ be a solution of the Cauchy problem* (5.8) *and $u_G$ be a solution of* (5.7)*. Then, for every $\varepsilon_1, \varepsilon_2 > 0$ and for every $H$ compact subset of $R^2\times ]0,T[$ there exists a grid $G$, verifying the following stability condition $\frac{\Delta_t}{(\Delta_x)^2} \le \frac{1}{2}$, such that*

$$\max_{(x,y,t)\in G\cap H} |u(x,y,t) - u_G(x,y,t)| \le \varepsilon_1,$$

$$\max_{(x,y,t)\in G\cap H} \left| \frac{\partial u}{\partial x}(x,y,t) - \frac{u_G(x+\Delta_x,y,t)-u_G(x,y,t)}{\Delta_x} \right| \le \varepsilon_2 .$$

*Moreover $\varepsilon_1 = O\big((\Delta_x)^2\big)$ and $\varepsilon_2 = O\big(\Delta_x\big)$.*

The convergence of the numerical solutions has been proved also for the arithmetic average equation (5.5) (see Theorem 4.16 in [6]).

A more recent motivation arising in finance is related to the model by Hobson & Rogers [23]. In the Black & Scholes theory the hypothesis that the volatility $\sigma$ in the stochastic differential equation (5.1) is constant contrasts with the empirical observations. Many authors, aiming to overcome this problem, proposed some models based on a stochastic volatility (see [19] for a survey), however the presence of a second Wiener process leads some difficulties in the arbitrage argument underlying the Black & Scholes theory. The model due to Hobson and Rogers for European options assumes that the volatility only depends on the difference between the present stock price and the past price. This simple model seems to capture the features observed in the market and avoid the problems related to the use of many sources of randomness.

As in the study of Asian options, in the Hobson & Rogers model for European options the value of the option $V(t,S_t,M_t)$ is supposed to depend on the time $t$, on the price of the stock $S_t$, on some average $M_t$ and must satisfy the following differential equation

$$\frac{1}{2}\sigma^2(S-M)\left(\partial_S^2 V - \partial_S V\right) + (S-M)\partial_M V + \partial_t V = 0 \,, \tag{5.9}$$

that is a non-homogeneous Kolmogorov equation, whose coefficient of the second order derivative is Hölder continuous. In a recent paper by Di Francesco and Pascucci [16] the Cauchy problem related to (5.9) has been studied and the convergence of the numerical solutions has been proved. Then, Di Francesco, Foschi and Pascucci [15] made some test aiming to compare the stability and the rate of convergence of different numerical methods for solving (5.9). They show that the method proposed in [16], based on the discrete operator $L_G$ (5.6), produces precise results when compared with the Euclidean schemes. Note that (5.6) relies on the approximation of the directional derivative $Y$ with the finite difference $-(u(x,y,t) - u(x,y+\delta x, t-\delta))/\delta$; hence this method, which is respectful of the non-Euclidean geometry of the Lie group, provides a good approximation of the solution.

We conclude this survey by discussing some regularity results concerning a Kolmogorov type equation with a *non linear total derivative*

$$\partial_{x_1}^2 u + u\partial_{x_2} u - \partial_t u = f$$

that has been proposed in [2] as a mathematical model for utility functional and decision making. Note that, when $f = 0$, the above equation becomes a particular case of (1.11). In the sequel we consider the equation

$$\Delta_x u + h(u)\partial_y u - \partial_t u = f(\cdot, u), \qquad (x,y,t) \in \mathbb{R}^{p_0} \times \mathbb{R} \times \mathbb{R} \,. \tag{5.10}$$

Due to the lack of diffusion in the $y$-direction, (5.10) has mixed parabolic and hyperbolic features. Indeed, when $h(u) = u$, $f \equiv 0$ and the solution only depends on $y$, it becomes the Burger's equation. For this reason, Escobedo, Vazquez and Zuazua [20] consider the Cauchy problem related to (5.10) in the framework of the conservation laws, and prove that, under a suitable entropy condition, there exists a unique distributional solution which is not necessarily a continuous function. On the other hand, Antonelli, Barucci and Mancino in [2] find a (local in time) Hölder continuous viscosity solutions to that Cauchy problem.

The existence and regularity problem for the above weak solutions has been studied in [14], [37], [40]. These results improve and generalize the preceding ones in [20] and [2]. In [14] the following natural definition of *classical solution* to (5.10) is introduced: we say that $u$ is a classical solution to the equation in (5.10) if:

(i) $\partial_{x_i x_i} u$, $i = 1, \dots, p_0$ exists and it is a continuous function;

(ii) the directional derivative

$$\frac{\partial u}{\partial \nu_z}(z), \qquad \nu_z = (0, h(u(z)), -1) \,,$$

exists at every point and it is a continuous function of $z = (x,y,t)$;

(iii) equation (5.10) is satisfied at every point.

The main idea in the study of the regularity of the solutions to (5.10) is a modification of the classical freezing method. To be more specific, if $h$ is a Lipschitz continuous function, we consider

$$L_{\bar z} = \Delta_x u + (h(u(\bar z)) + x_1 - \bar x_1)\,\partial_y u - \partial_t u$$

which is a "good" approximation to the left hand side of (5.10). Note that, up to a straightforward change of coordinates, $L_{\bar z}$ is the Kolmogorov operator (1.1) with $N = p_0 + 1$ and the matrices $A, B$ as in (1.12).

It has to be noticed that the rank condition (1.3) is satisfied. Then $L_{\bar z}$ has a fundamental solution which takes the explicit form (1.6)-(1.7). Starting with this remark and by using analysis on Lie groups combined with standard techniques in degenerate parabolic problems, in [40] the following existence and uniqueness theorem is proved:

**Theorem 5.2.** *Let $f, g$ and $h$ be Lipschitz continuous in their domains. Then, there exists $T > 0$ and a unique function $u : \mathbb{R}^N \times ]0, T[ \longrightarrow \mathbb{R}$, classical solution to*

$$\begin{cases} \Delta_x u + h(u)\partial_y u - \partial_t u = f(\cdot, u) & \text{in } \mathbb{R}^N \times ]0, T[, \\ u(\cdot, 0) = g\ , \end{cases} \tag{5.11}$$

*and such that*

$$|u(x, y, t) - u(x', y', t)| \le c_0(|x - x'| + |y - y'|),$$
$$|u(x, y, t) - u(x, y, t')| \le c_0|t - t'|^{\frac{1}{2}}(1 + |x| + |y|),$$

*for every $(x, y, t), (x', y', t), (x, y, t') \in \mathbb{R}^{p_0} \times \mathbb{R} \times \mathbb{R}$.*

A weaker version of Theorem 5.2 was previously proved in [3] by some probabilistic technique.

Further regularity properties of the solution found in [40] can be obtained under some additional condition. We would like only mention the following optimal regularity result which follows from Theorem 3.1 in [37].

**Theorem 5.3.** *Let $u$ be a classical solution to the equation (5.10) in an open set $\Omega \subseteq \mathbb{R}^{p_0} \times \mathbb{R} \times \mathbb{R}$. If $f$ and $h$ are $C^\infty$ functions on their domains, and $b'(u)D_x u \neq 0$ at any point of $\Omega$, where $D_x = (\partial_{x_1}, \dots, \partial_{x_{p_0}})$, then $u \in C^\infty(\Omega)$.*

This theorem is an extension of a previous result in [14] and it is proved by using a suitable freezing method introduced by Citti [12] in a different context. Such a method is based on the following remark. Let us define

$$X_j = \partial_{x_j},\ j = 1, \dots, p_0, \ \text{and}\ \ Z = h(u)\partial_y - \partial_t\ . \tag{5.12}$$

Then, if

$$b'(u)D_x u \neq 0, \qquad \text{at any point of } \Omega\ ,$$

the condition

$$\dim\left(\text{span}\{X_1, \dots, X_{p_0}, Z, [X_1, Z], \dots, [X_{p_0}, Z]\}\right) = p_0 + 2 \tag{5.13}$$

holds everywhere in $\Omega$. In (5.13), $[X_j, Z]$ denotes the Lie bracket of $X_j$ and $Z$. Condition (5.13), which is a kind of Hörmander rank condition of step two, in [37] and [14] is the starting point of a bootstrap argument in suitable spaces of Hölder continuous functions. These spaces are modeled on the vector fields $X_1, \dots, X_{p_0}$ and $Z$ in (5.12), and depend on the function $u$. We directly refer to [37] and [14] for more details on such a bootstrap argument which has been used in [12] and [13] to prove the $C^\infty$ smoothness of solutions to a Levi curvature equation.

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