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A cortical based model of perceptual completion in the roto-translation space

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Abstract[1]. We present a mathematical model of perceptual completion and formation of subjective surfaces, which is at the same time inspired by the architecture of the visual cortex, and is the lifting in the 3-dimensional rototranslation group of the phenomenological variational models based on elastica functional. The initial image is lifted by the simple cells to a surface in the rototraslation group and the completion process is modelled via a diffusion driven motion by curvature. The convergence of the motion to a minimal surface is proved. Results are presented both for modal and amodal completion in classic Kanizsa images.

1. Introduction

1.1. **The completion problem.** It is know that the human visual system has the ability of recover missing parts of an image, complete objects and create contours which are not characterized by image gradients. These boundaries may be thought of as "apparent" or "subjective" contours, and have been deeply studied after the works of Kaznitza.

Mathematical models of perceptual completion take into account main phenomenological properties as described by psychology of Gestalt. Since subjective boundaries could be linear or curvilinear, their reconstruction is classically performed minimizing the elastica functional

$$\int_\gamma (1+k^2)ds, \tag{1.1}$$

where the integral is computed on the missed boundary, and k is its curvature (see [50]). The extension of this functional to the level set of the image I, has been applied in problems of inpainting by [51], [2] :

$$\int_\Omega |\nabla I|\Big(1+\Big|div\Big(\frac{\nabla I}{|\nabla I|}\Big)\Big|^2\Big)dxdy, \tag{1.2}$$

where the integral is extended to the domain of the image. In this way each level line of the image is completed either linearly or curvilinearly as elastica curve. In order to make occluded and occluding objects present at the same time in the

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image, in [50] (and then in [6], [16] [64]) a third dimension is introduced, and the objects present in the image are represented as a stack of sets, ordered by depth. However these representations fail where a clear depth ordering is not present in the image as in figure 1.

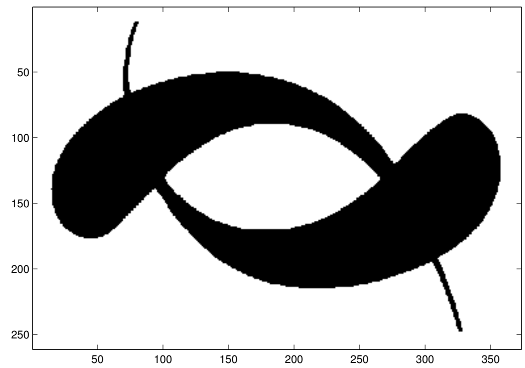

FIGURE 1. The two fishes of Kanizsa

From the neurophysiological point of view, there is considerable evidence that these kinds of perceptual completion phenomena are accomplished by the first layer of the visual cortex by actively filling in the missing information. The dominant thinking is that there are two cascade mechanism, the first one extracting the existing information ('real' boundaries, image gradients and complex features) by way of feed-forward filtering and the second one completing the missing information with recursive circuitry (even if in the past also feed forward mechanisms for completion have been proposed). The first mechanism is accomplished especially by simple cells in the primary cortex and extracts information about module and orientation of the brightness gradient of the visual stimulus. The second mechanism propagates extracted information in an orientation specific modality by means of long-range horizontal connections (Field et al. in [30], Kovacs and Julesz, in [44, 45], Kapadia et al, in [41], Gilbert et al., [27]). The specificity of this information propagation is described by the "association fields" (Field, Heyes, Hess, [30]) that indicate boundary collinear directions as privileged diffusion directions to the detriment of orthogonal ones.

Petitot and Tondut in [60] introduced a new perspective to the problem, in which the perceptual completion problem is considered as a problem of naturalizing phenomenological models on the basis of biological and neurophysiological evidence. They start from the consideration that orientation sensitive simple cells induce a fibration of orientations and that the natural space in which completion is performed is the 3-dimensional image-orientation manifold. Consequently the second order functional (1.1) is replaced by a first order length minimizing functional in the natural contact structure induced in the 3D space.

1.2. **A completion model in the rototraslation group.** In this paper we further develop this point of view, and build up a model directly inspired by the architecture of the visual cortex, which at the same time can be considered as the lifting in the rototraslation group of the models based on elastica functional. The completion algorithm can be considered as a diffusion driven motion by curvature in this the Lie algebra of rototraslations.

The considered image defines a function $I : D \to \mathbb{R}$, and the points of the domain D are denoted as (x, y). At every point we detect the the normal direction

$$\nabla I/|\nabla I| = (-sin(\theta), \cos(\theta))$$

and lift the domain D to a surface in the 3-D dimensional space

$$\Sigma_0 = \{(x, y, \theta(x, y)) : (x, y) \in D\}. \tag{1.3}$$

On this surface we lift the gradient of the function I:

$$u_0 : \Sigma_0 \to \mathbb{R} \quad u_0(x, y, \theta(x, y)) = |\nabla I(x, y)|. \tag{1.4}$$

This first process models the extraction of existing information operated by simple cells in the primary cortex, according to biological models. Level lines of u lie on the plane orthogonal to the gradient, so that they are tangent at every point to the vector fields

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad X_2 = \partial_\theta. \tag{1.5}$$

These are the generators of the Lie algebra of rototranslation, whose natural geometry models the circuitry of the first layers of visual cortex.

The completion of missing boundaries is performed in this setting, via a diffusion-concentration process, which models the diffusion in the direction of association fields, and an orientation selectivity process, the second step in the biological models above recalled.

The model takes into account the main phenomenological characteristics of perceptual completion, including the following:

1) the subjective boundaries can be curvilinear.

2) the modally completed parts of the image (inpainting) are lifting of the model of Ambrosio, Masnou [2], and Morel, Masnou.

3) occluding and occluded objects are present at the same time in the segmentation, and the completion allows the segmentation of partially overlapped objects without any depth ordering.

1.3. **Diffusion driven motion by curvature in the rototraslation group.** The choice of two vector fields X_1 X_2 defined in (1.5) at every point selects a bidimensional subset of the tangent space, called horizontal plane. Since

$$X_3 = [X_1, X_2] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$

is linearly independent of X_1, and X_2, the Lie algebra generated by X_1 and X_2 has maximum rank at every point. This allows to consider the space a Carnot Caratheodory metric space, naturally endowed with a control metric d. In this setting the notion of derivative is replaced by the Lie derivative in the direction of the vector fields, the horizontal gradient is defined as

$$\nabla_R u(x) = (X_1 u, X_2 u) \quad \Delta_R = X_1^2 u + X_2^2 u, \tag{1.6}$$

and the sub-Riemannian heat operator is defined as:

$$\partial_t u = X_1^2 u + X_2^2 u. \tag{1.7}$$

The study of these type of operators (represented as sum of squares of vector fields) has been deeply developed after the first celebrated papers of Hormander [34], Folland [23] and Rothshilds and Stein [63], who established regularity properties in Sobolev and C^α spaces. The explicit expression of the heat kernel is known only in a few cases, for example if

$$X_1 = \partial_x + \theta\partial_y, X_2 = \partial_\theta \ , ,$$

which are the generators of the Heisenberg group. For general vector fields, however, estimates have been established: non gaussian estimates have provided by [54], and [63], Gaussian estimates, not uniform were proved by Jerison and Sanchez Calle [37], Kusuoka and Strook [43], Varopoulos Saloff-Coste and Coulhon [72], uniform, gaussian estimates by Bonfiglioli, Lanconelli Uguzzoni [10], for a class of operators, represented in terms of homogeneous and nilpotent vector fields.

In the rototraslation group the generating vector fields X_i are not nilpotent, so that the previous result do not apply directly. However Rothschild and Stein introduced in [63] a general technique which allows to represent any family (X_i) of vector fields as

$$X_i = Y_i + R_i \ ,$$

where (Y_i) is a family of homogeneous nilpotent vector fields and (R_i) are higher order terms. In our situation, the approximating vectors Y_i turn out to be the generators of the Heisenberg group. The fundamental solution $\Gamma_H((x, y, \theta), t)$ is then explicitly known, and using the technique of [10] we can prove that:

Theorem 1.1. *Denote* Γ *the fundamental solution of the heat equation* (1.7). *In a neighborhood of each point* $\xi_0 = (x_0, y_0, \theta_0)$, *there exists a regular change of variables* ϕ *such that the function*

$$\Gamma_{\xi_0}((x, y, \theta), t) = \Gamma_H(\phi((x, y, \theta), t)) \ ,$$

is a parametrix of the fundamental solution. Precisely

$$|X_2\Gamma - X_2\Gamma_{\xi_0}|((x, y, \theta), t) \leq (t-\tau)^{-1/2}\Gamma_{\xi_0}((x, y, \theta), t)$$

in a neighborhood of the point ξ_0. *Besides*

$$|\nabla_X \Gamma((x, y, \theta), t)| \leq \frac{C}{t^{Q/2}} \exp\Big(-\frac{d^2(x, y, \theta)}{t}\Big)$$

where $Q = 4$.

The notion of regular surface has been extended in this setting. The first definition, given by Federer ([20]), see also [1]) was that a regular surface is the image of a open set of $\mathbb{R}^{n-1}$ through a Lipschitz continuous function. However the Heisenberg group turn out to be completely non rectifiable in this sense ([1]). A more natural definition of regular surface was given in [21]. It is a subset Σ of $\mathbb{R}^n$ which can be locally represented as

$$\Sigma = \{(x, y, \theta) : u(x, y, \theta) = 0, \nabla_R u(x) \neq 0\}. \tag{1.8}$$

The normal vector to Σ at a point (x, y, θ) is $\nu_R = \frac{\nabla_R u}{|\nabla_R u|}$. The mean curvature is

$$H_R(x, y, \theta) = \Big(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_R u|^2}\Big)\frac{X_i X_j u}{|\nabla_R u|}, \tag{1.9}$$

Non linear differential equations have been studied in this setting. In particular [9], [47] gave the definition of viscosity solution, replacing the standard gradient with the horizontal one. With this definition they extended to this degenerate setting the definition of motion of a surface by curvature. If the surface is the level set of a function u, the motion by curvature is described by the equation

$$u_t = \Big(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_R u|^2}\Big) X_i X_j u, \quad v(0) = d_{\Sigma_0} \ .$$

A first relation between heat equation and curvature equation in the euclidean setting goes back to paper of Bence, Merrimann and Osher [8] who describe the evolution of surface by mean curvature in terms of heat diffusion. Formal proof of the convergence of the motion of Σ_n to the motion by curvature has been provided independently by Evans [19] and Barles and Georgelin [3]. Here we further develop these ideas.

Indeed the model we propose can be formalized as a diffusion driven motion by curvature. We are given an initial surface Σ_0, defined in (1.3) and an initial measure on this surface: $u_0 \delta_{\Sigma_0}$ where the density u_0 is defined in (1.4) and δ is the dirac measure, concentrated on the surface Σ_0. For a fixed integer n , and real number t, the given measure is evolved for an interval of time of length $h = t/n$:

$$\partial_t u = \Delta_R u \ \ (\mathbb{R}^2 \times S^1) \times [0, t/n] \quad u(\cdot\ , 0) = u_0 \delta_{\Sigma_0}. \tag{1.10}$$

After the diffusion, we built a new surface Σ_1, with a concentration procedure:

$$\Sigma_1 = \{\partial_{\nu_{\Sigma_0}} u_1 = 0, \partial^2_{\nu_{\Sigma_0}} u_1 < 0\} \tag{1.11}$$

and a new density function in terms of the solution of the problem (1.11) at time t/n:

$$u_1\Big(\frac{t}{n}\Big) = \sqrt{\frac{t}{n}} u_{|\Sigma_1} \tag{1.12}$$

At time t, after n steps of length t/n, we obtain a surface $\Sigma_n(t)$, and a measure $u_n(t)\delta_{\Sigma_n(t)}$ concentrated on $\Sigma_n(t)$. We prove that

Theorem 1.2. *The sequences Σ_n and u_n converge respectively to minimal surface Σ and the Beltrami flow on Σ.*

The paper is organized as follows:

- In section 2 we present our cortical based model of perceptual completion
- In section 3 we provide the proof of the converge result.
- In section 4 we prove that the model expresses in the new space the variational model of Ambrosio Masnou
- In section 5 we present a numerical scheme to approximate the model equation and provide results by applying the algorithm to classic gestalt images.

2. The Model

2.1. **1. Simple cells and directional derivatives.** It is well known that in primates the contour extraction is provided by the area V1 of the visual cortex, which input arrives from the retina with the intermediation of the Lateral Geniculate Nucleus. The area V1 is the place of the visual cortex in which for the first time one finds cells with elongated receptive fields. These present oriented receptive fields and exhibit even or odd symmetric patterns. Figure 2.1 shows the odd ones. Simple cells can be considered as units sensitive to brightness gradients, independently if the gradient is originated by a boundary or a smooth region (Mingolla in Kanizsa

Lecture 2001). Simple cells are sensitive to space scale, position and orientation of the contrast gradient and moreover to the polarty of the gradient with respects to the elongation axis of the receptive field. Receptive fields of simple cells are modelled usually as convolution kernels of even and odd filters such as Gabor filters (Jones and Palmer [38], Daugman [14], Marcelja [48]), steerable filters (Perona [58]). Grossberg and Mingolla in [29] have interpreted the functionality of odd receptive fields as gradient indicators and even receptive fields as polarity indicators. A Gabor filter with orientation θ has the expression

$$G(x,y,\theta) = \frac{2}{s}\exp\Big(-\frac{(\tilde{x}^2+\tilde{y}^2)}{s} + i\tilde{y}\Big)\ ,$$

where

$$\tilde{x} = x\cos(\theta) + y\sin(\theta) \quad \tilde{y} = -x\sin(\theta) + y\cos(\theta). \tag{2.1}$$

We will consider here the imaginary part of the filter, which is its odd part. Condition (2.1) describes a rotation of the axis of an angle θ, so that the Gabor filters are obtained from a fixed function, via a rotation.

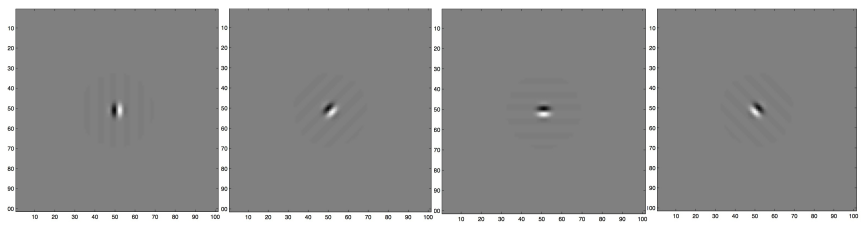

FIGURE 2. Odd part of Gabor filters with different orientations $\theta = 0$, $\theta = \pi/4$, $\theta = \pi/2$, $\theta = 3/2\pi$

The odd part of the filter can be locally approximated as

$$\frac{2\sin(\tilde{y})}{s} exp(-(\tilde{x}^2+\tilde{y}^2)/s) \simeq \frac{2\tilde{y}}{s} * exp(-(\tilde{x}^2+\tilde{y}^2)/s) = -\partial_{\tilde{y}} exp(-(\tilde{x}^2+\tilde{y}^2)/s)\ .$$

A derivative in the direction $\tilde{y}$ can be expressed in the original variables (x, y, θ) as a directional derivative, in the direction of the vector $(-sin(\theta), cos(\theta))$. We will denote it

$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y. \tag{2.2}$$

This derivative, applied to a function I, expresses the projection of the gradient in the direction $(-sin(\theta), cos(\theta))$. With this notation the filtering generates by convolution with the image I a function

$$O(x,y,\theta) = -X_3 exp(-(\tilde{x}^2+\tilde{y}^2)/s) * I = -X_3(\theta) I_s, \tag{2.3}$$

where we have denoted I_s the convolution of I with a smoothing kernel:

$$I_s = I * exp(-(\tilde{x}^2+\tilde{y}^2)/s)\ .$$

Note that $O(x, y, \theta)$ depends on the orientation θ. Due to the expression of the Gabor filter, the function O exponentially decays from its maxima. Hence for θ fixed it selects a neighborhood of the points where the component of the gradient in the direction $(-sin(\theta), cos(\theta))$, is sufficently big (see Figure 2.1).

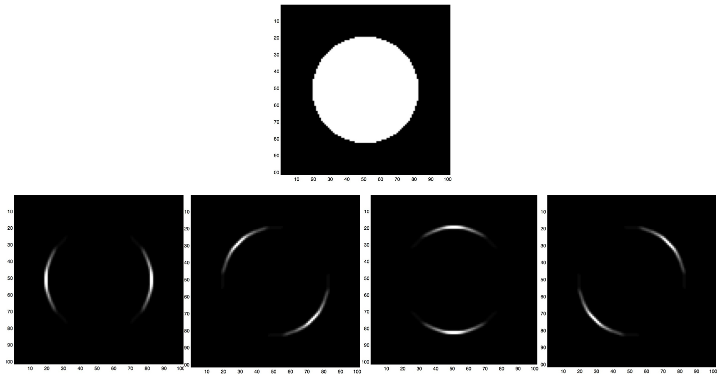

FIGURE 3. The original image showing a white disk (upper) and a sequence of convolutions with different orientations Gabor filters.

2.2. **Orientation selectivity and "non maximal" suppression.** The convolution mechanism (2.3) (that is known by neurophysiologists as orientation bias of the thalamic input) is insufficient to explain the strong orientation tuning exhibited by most simple cells. For these reasons, the classic feedforward mechanism must be integrated with additional mechanisms, in order to provide the sharp tuning experimentally observed. In the past years several models have been presented to explain the emergence of orientation selectivity in the primary visual cortex. These models use different combinations of feedforward (thalamic) and feedback (intracortical) inputs and consider different involvement of excitatory and inhibitory short range connections (Miller, Worgotter, Carandini, Bar, Priebe, Shelley, Nelson [52, 75, 12, 4, 62, 66, 55]). Even if the basic mechanism producing strong orientation selectivity is controversial ("push-pull" models [52, 62], "emergent" models [55], "recurrent" models [66] only to cite a few), nevertheless it is evident that the intracortical circuitry is able to filter out all the spurious directions and to strictly keep the direction of maximum response of the simple cells. Since $X_3 I_\sigma$ is the projection of the gradient in the direction of the vector $(-sin(\theta), cos(\theta))$, the maximum will be achieved at a value $\bar{\theta}$, which is the direction of the gradient. Then, if we call the point of maximum $\bar{\theta}$, we get

$$|X_3(\bar{\theta})| = max_\theta |X_3(\theta)|. \tag{2.4}$$

Besides only strict maxima are selected, so that $|X_3(\bar{\theta})I_s| > 0$. In this process each point (x, y) in the 2D domain of the image is lifted to the point $(x, y, \bar{\theta})$, the section $(-\sin(\theta), \cos(\theta))$ is lifted to the vector field

$$\vec{X_3}(\bar{\theta}) = (-\sin(\bar{\theta}), \cos(\bar{\theta}), 0) \ ,$$

applied at the point $(x, y, \bar{\theta})$. The whole image domain is lifted to:

$$\Sigma = \{(x, y, \bar{\theta}) : |X_3(\bar{\theta})I_s| = \max_\theta |X_3(\theta)I_s| > 0\} = \tag{2.5}$$

$$= \{(x, y, \bar{\theta}) : \partial_\theta O = 0, \partial^2_{\theta\theta} O < 0\} \ .$$

Note that this is a regular surface, in the sense of definition (1.8) zero level set of $v = \partial_\theta O$. This lifted set corresponds to the maximum of activity of the output of

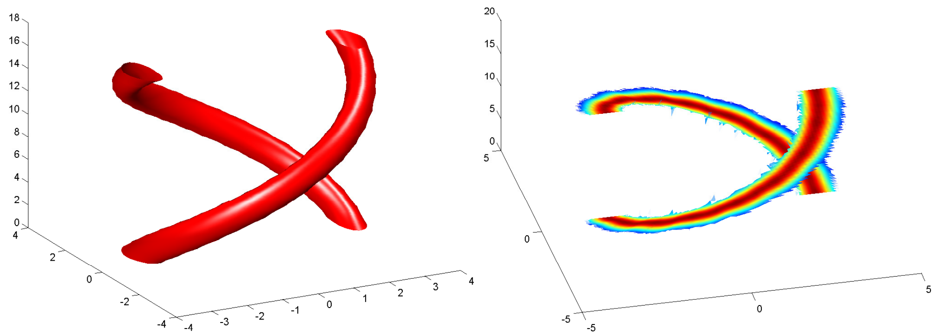

FIGURE 4. A level set of the function $O(x, y, \theta)$ (left) and the resulting surface after non maximal suppression, called lifted surface (right).

the simple cells, and can be modelled as a dirac mass concentrated on Σ itself, with a density u, given by the value of the activity:

$$u(x, y, \theta) = O(x, y, \bar{\theta})\delta_{\Sigma}. \tag{2.6}$$

2.3. **Association fields and metric of the rototraslation group.** The lifted points of the image would remain isolated without an integrative process allowing to connect local tangent vectors to form integral curves. This process is at the base of both regular contours and illusory contours formation [60].

The most plausible model of connection is based on a mechanism of "local induction". The specificity of this local induction is described by the association field (Field [30]).

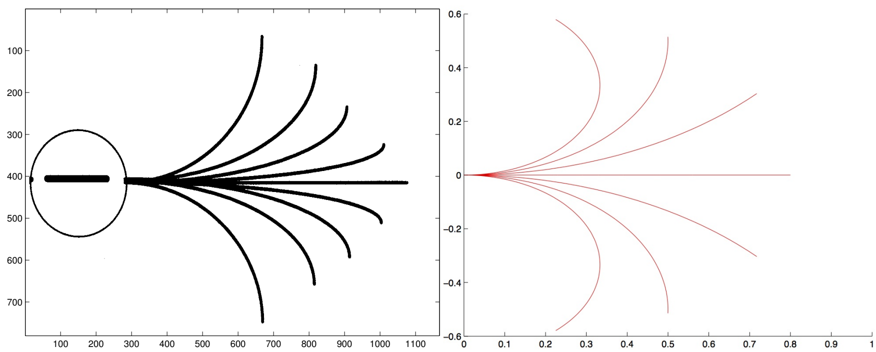

FIGURE 5. On the left association fields from the experiment of Field, Heyes and Hess. On the right integral curves of the fields by varying the parameter k.

The anatomical network of horizontal long-range connections has been proposed as the implementation of association fields, i.e. the magnitude of synaptic interactions depend upon the positions and orientations of the target cells accordingly to the association field. These allow to connect different points of the cortex.

The local association field is shown in fig. (5) and is can be interpreted as a family of integral curves of the vector fields X_1 and X_2, starting at a fixed point (x, y, θ):

$$\gamma'(t) = \vec{X_1}(\gamma) + k\vec{X_2}(\gamma), \quad \gamma(0) = (x, y, \theta), \tag{2.7}$$

obtained by varying the parameter k in $\mathbb{R}$ (fig. (5)). Recall that this is the family of curves which defines the control distance in the rototraslation group (see [54]). For every k fixed this is an horizontal curve of the rototraslation group. Long-range connections can consequently been modelled as horizontal curves, but with piecewise constant coefficients k.

2.4. **Activity propagation in the sub-Riemannian space.** Up to now we have built up a geometric space inspired by the functional geometry of the primary cortex. In the cortex neural activity develops and propagates itself in this sub-Riemannian space. For seek of semplicity, in this study we consider an extremely simple model of activity propagation, i.e. a simple linear diffusion in the geometric structure. It exactly means to diffuse with respect to the sublaplacian operator

$$\partial_t u = \Delta_R u. \tag{2.8}$$

2.5. **Completion model and minimal surfaces in the rototranslation space.** The joint work of sub-Riemannian diffusion (2.8) and non maximal suppression (2.5) allows to propagate existing information and then to complete boundaries and surfaces. Starting from the lifted surface Σ_0 defined in (2.5), and the measure $u_0\delta_{\Sigma_0}$ defined in (2.6), the two mechanisms are simultaneously applied until the completion is reached, according to the algorithm described in (1.10), (1.11), (1.12).

The sequences $\Sigma_n(t)$ and $u_n(t)$ obtained in this way converge respectively to motion by curvature surface $\Sigma(t)$ and the Beltrami flow on $\Sigma(t)$. As t tends to $+\infty$ we get from $\Sigma(t)$ a minimal surface:

$$\Delta_R u - \langle Hess_R\, u\, \nu_\Sigma, \nu_\Sigma\rangle = 0, \quad div_R(\nu_\Sigma) = 0$$

3. Proof of convergence

In this section we give a sketch of the proof of the convergence of our diffusion concentration algorithm. Using the fundamental solution we can represent the solution of the sublaplacian heat equation, with initial datum $u_0\delta_0$ as an integral on the surface Σ_0:

$$u((x, y, \theta), t) = \int_{\Sigma_0} \Gamma(\zeta) u_0(\zeta \underset{R}{-} (x, y, \theta)) d\sigma(\zeta) =$$

$$= \int_{\Sigma_0} \Gamma(\underset{R}{-}\zeta \underset{R}{+} (x, y, \theta), t) u_0(\zeta) d\sigma(\zeta)\ ,$$

where σ denotes the element of area on the surface Σ_0, and ζ is a 3D point of the space. The estimate of the fundamental solution is provided in Theorem 1.1, in terms of the fundamental solution Γ_H of the Heisenberg operator.

We explicitly remark that the function Γ_H has the usual homogeneity property:

$$\Gamma_H(\delta_{\sqrt{t}}(x, y, \theta), t) = \Gamma_H((x, y, \theta), 1) \tag{3.1}$$

with respect to the non homogeneous dilations defined as:

$$\delta_{\sqrt{t}}(x, y, \theta) = (\sqrt{t}x, ty, \sqrt{t}\theta). \tag{3.2}$$

In order to show that the surface is moving by curvature, we take $(x_0, y_0, \theta_0) \in \Sigma_0$ and denote $\nu = \nu_0$ the normal to Σ at (x_0, y_0, θ_0). Then we select $v \in \mathbb{R}$ so that $(x_0, y_0, \theta_0)\underset{R}{+}tv\nu \in \Sigma_1$, in other words:

$$\partial_\nu u((x_0, y_0, \theta_0)\underset{R}{+}tv\nu, t) = 0$$

Then vt is the normal increment from (x_0, y_0, θ_0) to the new surface Σ_1. We have to prove the following theorem

Theorem 3.1.

$$v = H(x_0, y_0, \theta_0) + o(1)$$

as $t \to 0$, *where* H *is the curvature of the initial surface at the point* (x_0, y_0, θ_0), *defined in* (1.9).

Proof. Up to a change of coordinates, we may assume that $(x_0, y_0, \theta_0) = 0$, $\nu = (0, 1, 0)$, and $\partial_\nu = X_2$, and Σ is a graph:

$$\{\theta = h(x, y) : (x, y) \in Q'\} ,$$

for some function h, and a suitable cube Q'. We then have

$$h(0) = 0, X_1 h(0) = 0, X_1^2 h(0) = H(0) .$$

Calling $s = (x_0, y_0, \theta_0)\underset{R}{+}tv\nu = (0, 0, tv)$ and differentiating the expression of u we get

$$0 = X_2 u(s) = \int_\Sigma X_2\Gamma(\underset{R}{-}\zeta\underset{R}{+}s, t)d\sigma(\zeta) =$$

where we have denoted $\zeta = (x, y, \theta)$

$$= \int_{\Sigma\cup C} X_2\Gamma(\underset{R}{-}\zeta\underset{R}{+}s, t)d\sigma(\zeta) + O(e^{-t}) =$$

using the approximation Theorem 1.1, and denoting ζ_1 the image of ζ in the change of variable,

$$= \int_{\Sigma\cup C} X_2\Gamma_H\Big(\underset{R}{-}\zeta_1 \underset{R}{+}s, t\Big)u(\zeta)d\sigma(\zeta) + O(e^{-t}) =$$

(the derivative of Γ_H is of the form θK, for a suitable kernel never vanishing K, exponentially decaying)

$$= \int_{\Sigma\cup C} \frac{1}{(2\pi t)^4}\frac{(\theta - vt)}{t}K\Big(\underset{R}{-}\zeta_1 \underset{R}{+}s, t\Big)d\sigma(\zeta) + O(e^{-t})$$

Since the integral is performed on the surface Σ, graph of h

$$= \int_C \frac{1}{(2\pi t)^4}\frac{(h(x, y) - vt)}{t}K\Big(\underset{R}{-}\zeta_1 \underset{R}{+}s, t\Big)\sqrt{1 + |\nabla h(x, y)|^2}dxdy + O(e^{-t})$$

With the change of variable $x = \sqrt{t}p$ $y = tq$,

$$= \int_C \frac{1}{(2\pi t)^4}\frac{(h(\sqrt{t}p, tq) - vt)}{t}K\Big(\underset{R}{-}\zeta_1 \underset{R}{+}s, t\Big)\sqrt{1 + |\nabla h(\sqrt{p}, tq)|^2}dxdy + O(e^{-t})$$

Since this expression is 0, the first non zero term of the Taylor development of

$$vt - h(\sqrt{t}p, tq)$$

must vanishes. Since the first derivative of h is 0, this implies that

$$v = H(x_0, y_0, \theta_0) + O(\sqrt{t}) .$$

We explicitly note that this is only a local approximation result. However the convergence proof performed by [3], [19], are based on the semigroup theory, and can be repeated in the rototraslation one. We refer to [13] for details.

4. Relation with phenomenological models

4.1. **Lifted curves and elastica.** Petitot and Tondut have already noted in [60] that for a lifted curve γ expressed in the form:

$$\gamma' = X_1 + kX_2, \tag{4.1}$$

the coefficient k is is the curvature of the projection of γ on the (x, y) plane.

Indeed if we denote $\gamma(t) = (x(t), y(t), \theta(t))$, by definition of integral curve we have

$$x' = cos(\theta), \quad y' = \sin(\theta), \quad \theta' = k$$

From the first two relations it follows that

$$\theta = \arctan\left(\frac{y'}{x'}\right) .$$

Differentiating this expression we get

$$\theta' = \sqrt{x' + y'}\frac{y''x' - x''y'}{((x')^2 - (y')^2)^{3/2}} .$$

In particular the length of γ is the elastica functional, suitably modified:

$$\int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2} = \int \sqrt{\dot{x}^2 + \dot{y}^2}\sqrt{1 + k^2} .$$

4.2. **Relation of our model with Morel-Masnou one.** Here we show the relation between our model and the one proposed by Morel-Masnou, Ambrosio-Masnou, recalled in (1.2).

As we stated in (2.5) we lift the initial image I to a surface

$$\Sigma = \{(x, t, \theta) : \partial_\theta |X_3(\theta) I_s| = 0 \ \ \partial^2_{\theta\theta} |X_3(\theta) I_s| \neq 0\} =$$

(since $\partial_\theta X_3 I_s = -X_1 I_s$, and $\partial_\theta X_1 I_s = X_3 I_s$,)

$$= \{(x, t, \theta) : X_1(\theta) I_s = 0 \ \ \partial_\theta |X_1(\theta) I_s| \neq 0\}$$

On this surface we evolve the function

$$v = X_1 I_s + |X_3 I_s|$$

Note that this means that we do not consider the intensity of I_s, but only its level sets. However this function has exactly the same level sets as the function

$$\tilde{v} = X_1 I_s + I_s ,$$

and coincides with I_s on the surface. A simple differentiation show that (for both v and $\tilde{v}$)

$$|X_1 v| = |\nabla I||curv(I_s)| \quad |X_2 v| = |\nabla I_s| ,$$

so that substituting in 1.2 we get

$$\int |\nabla I_s|(1 + |curv(I_s)|^2) = \int \sqrt{|\nabla I_s|(1 + |curv(I_s)|^2)}\sqrt{(1 + k^2)(\theta(x, y))} =$$

$$= \int_{v=0} \sqrt{(|X_1 v|^2 + |X_2 v|^2)} \ ,$$

where θ is the parametrisation previously introduced. Applying the coarea formula we can extend the integral on the whole space:

$$\int_{\mathbb{R}^2 \times S^1} |X_1 v|^2 + |X_2 v|^2 \quad if \ X_1 I_s = 0$$

This functional is simply the gradient squared, and its associated steepest descent equation is the heat equation. Hence this describes a diffusion. However, since we only want to consider the values on the surface, we also need to concentrate again on the surface. In this sense our model ca be considered a lifting of Masnou Model in $\mathbb{R}^2 \times S^1$. Note that it drastically reduces the complexity of the minimisation problem.

5. Numerical scheme and computational results

In this section, we show how to approximate the model equations with finite differences. Let us consider a rectangular uniform grid in space-time (t, x, y, θ); then the grid consists of the points $(t_n, x_l, y_m, \theta_q) = (n\Delta t, l\Delta x, m\Delta y, q\Delta\theta)$. The relation between the increments will be deduced from the structure of the local dilations we have defined. in (3.2). Since the grid is rectangular, while the natural increments are linear in direction $\vec{X_1}$ and $\vec{X_2}$, and quadratic in direction $\vec{X_3}$, we are forced to choice

$$\Delta x = \Delta y, \quad \Delta\theta = \Delta x^2 \quad \Delta t = \Delta x^2 \ .$$

Following standard notation, we denote by u^n_{lmq} the value of the function u at the grid point $(t_n, x_l, y_m, \theta_q)$. We approximate time derivative with a first order forward difference and space derivative with second order centered scheme. First and second derivatives in the direction of the sub-Riemannian fields are approximated with

$$D_1 u^n_{lmq} = \cos(\theta_q) D_x u^n_{lmq} + \sin(\theta_q) D_y u^n_{lmq}, \quad D_2 u^n_{lmq} = D_\theta u^n_{lmq}$$

where D is the usual finite difference operator on the discretized function and the subscripts x, y, θ indicate the direction of differentiation. Consequently the second derivative become:

$$D_{11} u^n_{lmq} = \cos(\theta_q)^2 D_{xx} u^n_{lmq} + 2\cos(\theta_q)\sin(\theta_q) D_{xy} u^n_{lmq} + \sin(\theta_q)^2 D_{yy} u^n_{lmq} \ ,$$
$$D_{22} u^n_{lmq} = D_{\theta\theta} u^n_{lmq} \ .$$

The lifted function (2.6) is initially diffused with the discrete heat equation, in order to provide an initial datum

$$u^{n+1}_{lmq} = u^n_{lmq} + \Delta_t (D_{11} u^n_{lmq} + D_{22} u^n_{lmq}) \quad 0 \le n \le N_1 1 \ .$$

The initial surface, is, according to (2.5), the zero level set of the derivative of this function:

$$v^{N_1}_{lmq} = D_2 u^{N_1}_{lmq} \ .$$

After that, since the algorithm approximate a Laplace Beltrami motion on the surface, at every step of the algorithm, we apply directly this equation, on the surface defined at the previous step: for every n, $N_1 \le n \le N_2$,

$$u^{n+1}_{lmq} = u^n_{lmq} + \Delta_t * \Big(\frac{(D_2 v^n_{lmq})^2 D_{11} u^n_{lmq} + (D_1 v^n_{lmq})^2 D_{22} u^n_{lmq}}{(D_{11} v^n_{lmq})^2 + (D_{22} v^n_{lmq})^2} -$$

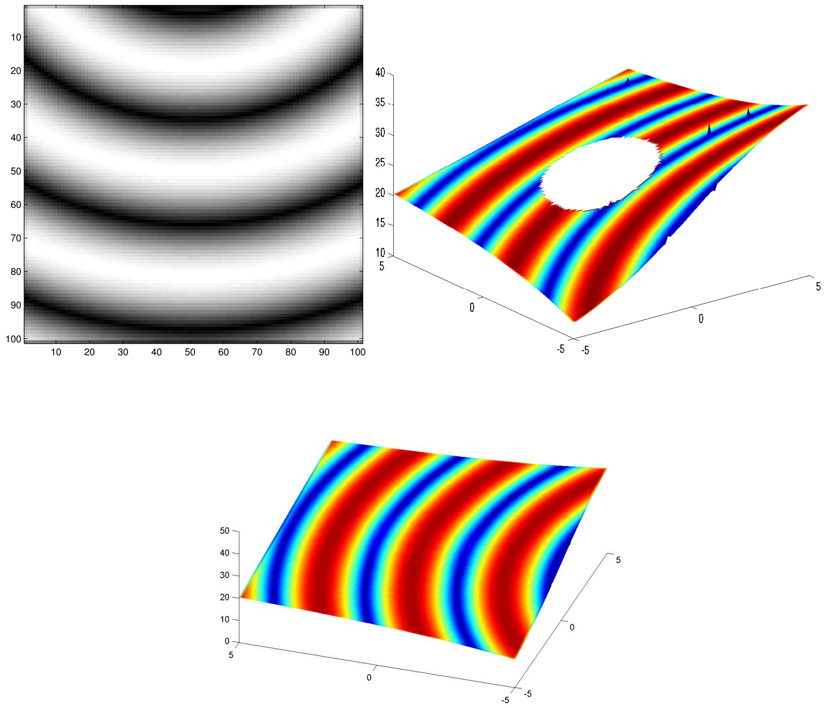

FIGURE 6. The original image (top left) is lifted in the rototranslation space with missing information in the center, like in the phenomenon of macula cieca (top right). The surface is completed by the algorithm (bottom).

$$-\frac{(D_{12}u^n_{lmq} + D_{21}u^n_{lmq})D_1 v^n_{lmq} D_2 v^n_{lmq}}{(D_{11}v^n_{lmq})^2 + (D_{22}v^n_{lmq})^2}\Big)$$

$$v^{n+1}_{lmq} = D_2 u^n_{lmq} \; .$$

We impose Neumann boundary conditions on x and y and periodic boundary conditions on the third direction θ. The time step Δt is upper bounded by the CFL (Courant-Friedrich-Levy) condition that ensures the stability of the evolution [46].

In the first numerical experiment we consider the completion of a figure that has been only partially lifted in the roto-translation space. This example mimics the missing information due to the presence of the macula cieca (blind spot) that is modally completed by the human visual system, as outlined in [39]. The original image (see Figure 6), top left) is lifted in the rotranslation space with missing information in the center (top right). The lifted surface is completed by iteratively applying eqs until a steady state is achieved. As proved in Theorem 1.1, the final surface is minimal with respects to the sub-Riemannian metric.

Finally a classical cognitive image, like the Kanizsa fishes, has been considered. This image has been deeply studied in the past because it induces several perception

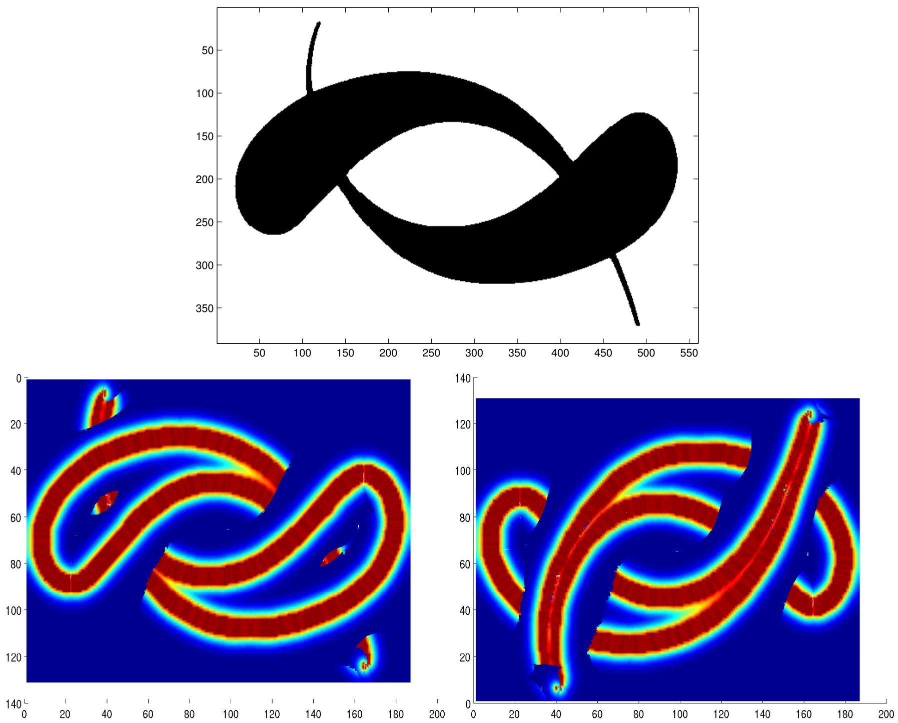

FIGURE 7. The original image (top) is lifted in the rototranslation space. Modal and amodal completion are performed at the same time by the algorithm. Two different view of the completed image are shown: in the view from the top the modally completed parts are visible (middle). In the view from the bottom the amodally completed parts are shown (bottom).

phenomena. First the fishes heads are modally completed and they appear to occlude the tails that are in turn a-modally completed. This double phenomenon does not allow to introduce a depth ordering between the two objects, but it does not prevent a clear perception of the two fishes. The original image (Figure 7 top) and is lifted in the rototranslation space. Modal and amodal completion are performed at the same time by the algorithm. Two different view of the completed image are shown in Figure 7: in the view from the top the modally completed parts are visible (middle). In the view from the bottom the amodally completed parts are shown (bottom). The two fishes are contemporary present in the 3D RT space.

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