

A class of non-divergence form operators on Carnot groups

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Abstract¹ We are concerned with the construction of the fundamental solutions and with invariant Harnack inequalities, for the sub-elliptic operators in non-divergence form $\sum_{i,j} a_{i,j}(x,t) X_i X_j - \partial_t$ and $\sum_{i,j} a_{i,j}(x) X_i X_j$ where the X_i 's form a stratified system of Hörmander vector fields and $(a_{i,j})$ is positive definite and Hölder continuous.

1. INTRODUCTION AND MAIN RESULTS

In this note we report on a series of results by A. Bonfiglioli, E. Lanconelli and the present author, concerning with some non-divergence form equations on Carnot groups. More precisely, we are mainly concerned with the existence of the fundamental solutions (along with qualitative properties, upper and lower bounds, etc.) and with invariant Harnack inequalities, for the sub-elliptic operators in non-divergence form

$$(1.1) \quad \sum_{i,j} a_{i,j}(x,t) X_i X_j - \partial_t, \quad \sum_{i,j} a_{i,j}(x) X_i X_j ,$$

where the X_i 's form a stratified system of Hörmander vector fields and $(a_{i,j})_{i,j \leq m}$ is a positive-definite matrix with Hölder continuous entries.

Before stating our results and describing our approach, we would like to give some motivation for this research. Second order linear and nonlinear PDE's of sub-elliptic type arise in many different settings: geometric theory of several complex variables, curvature problems for CR-manifolds, sub-Riemannian geometry, diffusion processes, control theory, human vision, see e.g., [1, 16, 17, 22, 23, 27, 28, 33, 34, 35, 37, 38, 39]. The underlying algebraic structures of all these equations are homogeneous stratified groups \mathbb{G} (see the definition below). Several results concerning with the divergence-form counterpart of (1.1) are present in the literature, both for linear and quasi-linear equations, see e.g., [10, 11, 12, 13, 15, 20, 29, 30, 31, 32, 40]. On the contrary, at the authors' knowledge, very few papers are devoted to non-divergence form operators of the above type, see [3, 8, 9, 14, 40]. We would like to stress that operators exactly in this form arise in studying motion by Levi curvature of hypersurfaces of \mathbb{C}^{n+1} [22, 33]. Our aim is to provide a new contribution to the study of non-divergence form operators (1.1). Indeed, the results presented here are a step in a wider project aimed to apply analysis on stratified groups to the study of *fully non-linear* PDE's of sub-elliptic type, such as the Levi-curvature equation and its parabolic counterpart, for which operators like (1.1) arise as linearizations.

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We now introduce some notation and then state our main results. We start by giving the definition of a Carnot group. Let \circ be an assigned Lie group law on \mathbb{R}^N . Suppose \mathbb{R}^N is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \dots, \lambda^r x^{(r)}) .$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \dots + N_r = N$. We denote by \mathfrak{g} the Lie algebra of (\mathbb{R}^N, \circ) i.e., the Lie algebra of left-invariant vector fields on \mathbb{R}^N . For $i = 1, \dots, N_1$, let X_i be the (unique) vector field in \mathfrak{g} that agrees at the origin with $\partial/\partial x_i^{(1)}$. We make the following assumption: the Lie algebra generated by X_1, \dots, X_{N_1} is the whole \mathfrak{g} . With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a *Carnot group*. We also say that \mathbb{G} is of *step* r and has $m := N_1$ *generators*. In literature (see e.g., [19, 21, 36, 39]) a *Carnot group* (or *stratified group*) \mathbb{H} is defined as a connected and simply connected Lie group whose Lie algebra \mathfrak{h} admits a *stratification*

$$\mathfrak{h} = \mathfrak{G}_1 \oplus \dots \oplus \mathfrak{G}_r \quad \text{with} \quad [\mathfrak{G}_1, \mathfrak{G}_i] = \mathfrak{G}_{i+1}, \quad [\mathfrak{G}_1, \mathfrak{G}_r] = \{0\} .$$

The two definitions are equivalent up to isomorphism. We denote by $Q = \sum_{j=1}^r jN_j$ the *homogeneous dimension* of \mathbb{G} . If $Q \leq 3$, then \mathbb{G} is the ordinary Euclidean group $(\mathbb{R}^Q, +)$. Hence, throughout the paper, we shall always assume $Q \geq 4$. In the sequel, d will denote the Carnot-Carathéodory control distance associated to the family of vector fields X_1, \dots, X_m . We recall that this distance is equivalent to any quasi-distance induced by a homogeneous norm on \mathbb{G} . Hereafter, we also denote $d(x, 0)$ simply by $d(x)$ and use the notation $B_d(x, r)$ for the d -ball of center $x \in \mathbb{G}$ and radius $r > 0$.

Let us fix the parameters $\Lambda > 1$, $k > 0$ and $\alpha \in (0, 1]$. Throughout the paper, all constants will be meant to depend on Λ , k , α . Let $A = (a_{i,j})_{i,j \leq m}$ be a function defined on \mathbb{R}^{N+1} and taking values in the symmetric $m \times m$ matrices. Throughout the paper we shall suppose that A satisfies the following conditions:

$$(1.2) \quad \begin{aligned} \Lambda^{-1}|\eta|^2 &\leq \langle A(x, t)\eta, \eta \rangle \leq \Lambda|\eta|^2 \quad \forall \eta \in \mathbb{R}^m, \\ |a_{i,j}(x, t) - a_{i,j}(x', t')| &\leq k(d(x, x')^\alpha + |t - t'|^{\alpha/2}), \end{aligned}$$

for every $(x, t), (x', t') \in \mathbb{R}^{N+1}$. Hereafter, we use the notation $z = (x, t)$ and $\zeta = (\xi, \tau)$ for the point of \mathbb{R}^{N+1} ($x, \xi \in \mathbb{G}$, $t, \tau \in \mathbb{R}$). We denote by \mathcal{H} the parabolic-type operator on \mathbb{R}^{N+1}

$$\mathcal{H} = \sum_{i,j=1}^m a_{i,j}(x, t) X_i X_j - \partial_t .$$

We also consider the stationary operator

$$\mathcal{L} = \sum_{i,j=1}^m a_{i,j}(x) X_i X_j$$

on \mathbb{R}^N (here the coefficients $a_{i,j}$ are supposed as in (1.2) but not depending on t). We want to discuss the existence and the well-behaved properties of the fundamental solutions for \mathcal{H} and \mathcal{L} . We construct a fundamental solution Γ for \mathcal{H} by means of the well-known Levi's parametrix method, whereas a fundamental solution γ for \mathcal{L} is derived by a t -saturation argument. We explicitly remark that several new difficulties arise in the adaptation to our setting of the cited parametrix method.

The major of these difficulties consists in obtaining suitable *uniform* estimates for the relevant frozen operators (see (1.5)). The derivation of these estimates is outlined in Section 2. Here we only point out that we make use of a lifting procedure to free stratified groups, and of an accurate analysis of the equivalence (via well-estimated automorphisms) of all sub-Laplacians on such free stratified groups. Moreover, the lack of knowledge of an explicit expression for the parametrix clearly makes Levi’s method more involved than in the classical context.

By integrating Γ over the time variable t , we are then able to construct also a local fundamental solution γ for \mathcal{L} . This can be done provided suitable *long-time* estimates of Γ are established. We stress that, whereas optimal small-time estimates of Γ and of its derivatives can be directly obtained from the construction of Γ , a more delicate matter is to show long-time estimates. We are able to obtain the latter up to a suitable modification of the coefficients $a_{i,j}$ outside a compact set (see Lemma 3.3 below).

We hereafter state our main results. For the complete proofs we refer to [3].

Theorem 1.1. *Suppose the coefficients $a_{i,j}$ of \mathcal{H} satisfy (1.2). Then there exists a fundamental solution Γ for \mathcal{H} , with the properties listed below.*

- (i) Γ is a continuous function away from the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. Moreover, for every fixed $\zeta \in \mathbb{R}^{N+1}$, $\Gamma(\cdot; \zeta) \in \Gamma_{\text{loc}}^{2+\alpha}(\mathbb{R}^{N+1} \setminus \{\zeta\})$ and we have $\mathcal{H}(\Gamma(\cdot; \zeta)) = 0$ in $\mathbb{R}^{N+1} \setminus \{\zeta\}$.
- (ii) $\Gamma(x, t; \xi, \tau) = 0$ for $t \leq \tau$. Moreover, there exists a positive constant M and, for every $T > 0$, there exists a positive constant $c(T)$ such that, for $0 < t - \tau \leq T$, the following estimates hold:

$$0 \leq \Gamma(x, t; \xi, \tau) \leq c(T) (t - \tau)^{-Q/2} \exp\left(-\frac{d^2(x, \xi)}{M(t - \tau)}\right),$$

$$|X_i(\Gamma(\cdot; \xi, \tau))(x, t)| \leq c(T) (t - \tau)^{-(Q+1)/2} \exp\left(-\frac{d^2(x, \xi)}{M(t - \tau)}\right),$$

$$(1.3) \quad |X_i X_j(\Gamma(\cdot; \xi, \tau))(x, t)|, |\partial_t(\Gamma(\cdot; \xi, \tau))(x, t)| \leq c(T) (t - \tau)^{-(Q+2)/2} \exp\left(-\frac{d^2(x, \xi)}{M(t - \tau)}\right).$$

- (iii) For every $\psi \in C_0^\infty(\mathbb{R}^{N+1})$, the convolution $w(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z; \zeta) \psi(\zeta) d\zeta$ belongs to the class $\Gamma_{\text{loc}}^{2+\alpha}(\mathbb{R}^{N+1})$ and we have $\mathcal{H}w = -\psi$ in \mathbb{R}^{N+1} .
- (iv) Let $\mu \geq 0$ and $T_2 > T_1$ be such that $(T_2 - T_1)\mu$ is small enough. Then, for every $f \in \Gamma^\beta(\mathbb{R}^N \times [T_1, T_2])$ (where $0 < \beta \leq \alpha$) and $g \in C(\mathbb{R}^N)$ satisfying the growth condition $|f(x, t)|, |g(x)| \leq c \exp(\mu d^2(x))$ for some constant $c > 0$, the function (defined for $x \in \mathbb{R}^N, t \in (T_1, T_2]$)

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; \xi, T_1) g(\xi) d\xi + \int_{\mathbb{R}^N \times [T_1, t]} \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau,$$

belongs to the class $\Gamma_{\text{loc}}^{2+\beta}(\mathbb{R}^N \times (T_1, T_2)) \cap C(\mathbb{R}^N \times [T_1, T_2])$. Moreover, u is a solution to the Cauchy problem $\mathcal{H}u = -f$ in $\mathbb{R}^N \times (T_1, T_2)$, $u(\cdot, T_1) = g$.

Theorem 1.2. *Suppose that the dimension m of \mathfrak{G}_1 is greater than two. Suppose the coefficients $a_{i,j}$ of \mathcal{L} satisfy (1.2). For every fixed bounded open set $\Omega \subset \mathbb{R}^N$, there exists a fundamental solution γ for \mathcal{L} in Ω , with the properties listed below.*

- (i) γ is a continuous function away from the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$. Moreover, for every fixed $\xi \in \mathbb{R}^N$, $\gamma(\cdot, \xi) \in \Gamma_{\text{loc}}^{2+\alpha}(\mathbb{R}^N \setminus \{\xi\})$ and we have $\mathcal{L}(\gamma(\cdot, \xi)) = 0$ in $\Omega \setminus \{\xi\}$.
- (ii) For every compact set $K \Subset \mathbb{R}^N$, there exists a positive constant c such that

$$0 \leq \gamma(x, \xi) \leq c(1 + d(x, \xi)^{2-Q}), \quad \xi \in K, \quad x \in \mathbb{R}^N.$$

- (iii) For every $\psi \in C_0^\infty(\mathbb{R}^N)$, the convolution $w(x) = \int_{\mathbb{R}^N} \gamma(x; \xi) \psi(\xi) \, d\xi$ belongs to the class $\Gamma_{\text{loc}}^{2+\alpha}(\mathbb{R}^N)$ and we have $\mathcal{L}w = -\psi$ in Ω .

In the above theorems, we have denoted by $\Gamma^\beta, \Gamma^{2+\beta}$ the appropriate sub-elliptic Hölder spaces.

As is well known, Levi’s parametrix method requires the knowledge of several good properties of the frozen constant coefficient parabolic-type operators

$$(1.4) \quad \mathcal{H}_A = \mathcal{L}_A - \partial_t = \sum_{i,j=1}^m a_{i,j} X_i X_j - \partial_t,$$

where $A = (a_{i,j})_{i,j}$ is a fixed matrix in the ellipticity class

$$\mathcal{M}_\Lambda = \{m \times m \text{ symmetric matrices } A \mid \Lambda^{-1}|\xi|^2 \leq \langle A\xi, \xi \rangle \leq \Lambda|\xi|^2 \forall \xi \in \mathbb{R}^m\}.$$

For instance, a crucial role is played by the following *uniform Gaussian estimates* for the fundamental solutions Γ_A of \mathcal{H}_A :

$$(1.5) \quad \begin{aligned} &|X_{i_1} \cdots X_{i_p}(\partial_t)^q \Gamma_A(x, t) - X_{i_1} \cdots X_{i_p}(\partial_t)^q \Gamma_B(x, t)| \leq \\ &\leq c_{\Lambda,p,q} \|A - B\|^{1/r} t^{-(Q+p+2q)/2} \exp\left(-\frac{d^2(x)}{c_\Lambda t}\right), \end{aligned}$$

for every $A, B \in \mathcal{M}_\Lambda$ (here, $\|A\|$ denotes the matrix norm $\max_{|\xi|=1} |A\xi|$). Gaussian estimates, but not uniform, for heat kernels on Lie groups were proved by Jerison and Sánchez-Calle [24], by Kusuoka and Stroock [26] and by Varopoulos, Saloff-Coste and Coulhon [39]. Uniform estimates, but not Gaussian, for families of Hörmander operators generalizing \mathcal{H} , were proved by Rothschild and Stein [36] and by Bramanti and Brandolini [8].

Since the derivation of uniform and Gaussian estimates is a non-trivial task and is a key point in proving the main results stated above, in Section 2 we briefly recall how (1.5) can be established by a direct approach. In Section 3, we briefly outline the proof of Theorem 1.1, describing our adaptation of the parametrix method and how uniform estimates naturally intervene. We then sketch how long-time estimates are used in order to prove Theorem 1.2 employing both a t -saturation and an approximation argument.

Starting from Theorem 1.1 above, it is then possible to prove the following invariant Harnack inequality for \mathcal{H} . We refer to [7] for the complete proof.

Theorem 1.3. *Let $R_0 > 0, 0 < h_1 < h_2 < 1$ and $\gamma \in (0, 1)$. There exists a positive constant $M = c(h_1, h_2, \gamma, R_0)$ such that*

$$\max_{\overline{B_d(\xi_0, \gamma R)} \times [\tau_0 - h_2 R^2, \tau_0 - h_1 R^2]} u \leq M u(\xi_0, \tau_0)$$

for every $(\xi_0, \tau_0) \in \mathbb{R}^{N+1}$, $R \in (0, R_0]$ and every $u \in \Gamma^2(B_d(\xi_0, R) \times (\tau_0 - R^2, \tau_0)) \cap C(\overline{B_d(\xi_0, R)} \times [\tau_0 - R^2, \tau_0])$ satisfying $\mathcal{H}u = 0, u \geq 0$ in $B_d(\xi_0, R) \times (\tau_0 - R^2, \tau_0)$.

We have used the notation $\Gamma^2(\Omega)$ to denote the space of continuous functions, defined on an open set $\Omega \subseteq \mathbb{R}^{N+1}$, which have continuous Lie-derivatives along the vector fields X_1, \dots, X_m up to second order and continuous derivative along ∂_t (up to first order). A main step in the proof of Theorem 1.3 is to establish a pointwise Gaussian estimate from below of Γ .

Theorem 1.4. *For every $T > 0$, there exists a positive constant $c(T)$ such that*

$$\Gamma(x, t; \xi, \tau) \geq c(T)^{-1} (t - \tau)^{-Q/2} \exp(-c d^2(x, \xi)/(t - \tau)),$$

for $0 < t - \tau \leq T$ and $x, \xi \in \mathbb{R}^N$.

Clearly from Theorem 1.3, an invariant Harnack inequality for \mathcal{L} straightforwardly follows. Moreover, by integrating the fundamental solution Γ of $\mathcal{L} - \partial_t$ over the variable t , we easily find an estimate from below of γ , by means of Theorem 1.4.

Theorem 1.5. *With the hypotheses and notation of Theorem 1.2 above, there exists a positive constant c such that*

$$\gamma(x, \xi) \geq c^{-1} d(x, \xi)^{2-Q}$$

for every $x, \xi \in \mathbb{R}^N$ such that $0 < d(x, \xi) < 1$.

We now briefly describe our approach in proving Theorem 1.3. The main tool in proving the Harnack inequality is a suitable adaptation of the method by Krylov&Safanov, also used by Fabes&Stroock [18] for classical parabolic equations and by Kusuoka&Stroock [25, 26] for parabolic equations modelled on sum of squares of smooth vector fields satisfying Hörmander condition. In the latter papers, this method has been applied in a sub-elliptic context similar to ours, jointly with probabilistic results. We do not use probabilistic techniques, but we make a crucial use of the existence of the relevant Green function on a suitable family of sets. More precisely, we construct the Green function for the operator \mathcal{H} on cylindrical domains which are regular for the Dirichlet problem related to \mathcal{H} , and we show that an arbitrary cylindrical domain can be approximated from the inside by a sequence of \mathcal{H} -regular domains. Some needed uniform properties of such approximation are non-trivial and requires a careful potential theory investigation, which is object of the paper [4]. We remark that the existence of the Green function G on such approximating domains is indeed sufficient in order to adapt the cited Krylov&Safanov method, once some well-behaved invariant estimates of G are available. We prove such invariant estimates of G as a consequence of the good behavior of our approximation with respect to the underlying Carnot-Carathéodory metric structure. We finally would like to stress that the proof of the Gaussian bound from below of the fundamental solution for \mathcal{H} (from which the estimates of G are derived) is based on the cited non-trivial uniform estimates for the relevant frozen operators and relies on the work made through [2, 5, 6].

2. BASIC RESULTS

The classical Levi's parametrix method (i.e., if the underlying stratified group is the usual Euclidean space $(\mathbb{R}^N, +)$) exploits at various levels the explicit knowledge of the fundamental solution for the strictly parabolic constant coefficient operator $\sum_{i,j=1}^N a_{i,j} \partial_i \partial_j - \partial_t$ ($A = (a_{i,j})_{i,j}$ being a positive-definite matrix). Obviously, this fundamental solution is given by the composition of the fundamental solution of the

classical heat operator $\sum_{i=1}^N (\partial_i)^2 - \partial_t$ on \mathbb{R}^{N+1} with a linear change of coordinates related to the matrix $A^{-1/2}$.

A naïve idea in studying the fundamental solutions for (1.4) and in approaching (1.5) is to ask if something similar may occur in the case of general stratified groups \mathbb{G} . Namely, we ask if all sub-Laplacians on \mathbb{G} can be put (via a diffeomorphism) into a fixed canonical form. To this end, if $A = (a_{i,j})_{i,j \leq m}$ is a positive-definite symmetric matrix, we first remark that

$$\mathcal{L}_A = \sum_{i,j=1}^m a_{i,j} X_i X_j$$

may be rewritten as $\sum_{i=1}^m Y_i^2$, where

$$Y_i = \sum_{j=1}^m (A^{1/2})_{i,j} X_j .$$

As a consequence, it is natural to ask if there exists a diffeomorphism $T_A : \mathbb{G} \rightarrow \mathbb{G}$ such that (in the new coordinate system defined by T_A) the vector field Y_i is turned into the (fixed) left-invariant vector field Z_i agreeing at the origin with ∂_i . In this way, if we set

$$\Delta_{\mathbb{G}} = \sum_{i=1}^m Z_i^2$$

(we call $\Delta_{\mathbb{G}}$ the canonical sub-Laplacian on \mathbb{G}), the sub-Laplacian $\sum_{i=1}^m Y_i^2$ is turned into $\Delta_{\mathbb{G}}$, i.e.,

$$(2.1) \quad \mathcal{L}_A(u \circ T_A) = (\Delta_{\mathbb{G}}u) \circ T_A, \quad \text{for every } u \in C^\infty(\mathbb{G}) .$$

In the classical case when $X_i = \partial/\partial x_i$, this problem always has a solution. On the contrary, counterexamples can be given showing that T_A may not exist for general stratified groups \mathbb{G} and, when it exists, it may be non-linear (see [5]). Broadly speaking, the problem relies on the commutativity properties of the X_i 's: if the linear dependence relations among commutators up to order r of X_1, \dots, X_m are the least possible, then the above problem does have a solution. More explicitly, if \mathbb{G} is a *free stratified group* (i.e., its Lie algebra is isomorphic to a free nilpotent Lie algebra) then there exists an automorphism T_A of the group \mathbb{G} satisfying (2.1).

As a straightforward consequence, when \mathbb{G} is free, we are able to obtain the fundamental solution Γ_A for $\mathcal{H}_A = \mathcal{L}_A - \partial_t$ simply as the composition of T_A with the fundamental solution $\Gamma_{\mathbb{G}}$ for the *fixed* canonical heat operator $\Delta_{\mathbb{G}} - \partial_t$. Indeed, if \mathbb{G} is free, it turns out that

$$(2.2) \quad \Gamma_A(x, t; \xi, \tau) = |\det \mathcal{J}_{T_A}(x)| \Gamma_{\mathbb{G}}(T_A(x), t; T_A(\xi), \tau), \quad x, \xi \in \mathbb{R}^N, t, \tau \in \mathbb{R}$$

(\mathcal{J}_{T_A} denotes the Jacobian matrix of T_A). Thanks to this somewhat explicit representation of the fundamental solution for \mathcal{H}_A , the next step in order to obtain the uniform estimates in (1.5) is to establish *ad hoc* uniform estimates for T_A . To this end it can be proved that $|\det \mathcal{J}_{T_A}(x)|$ turns out to be a uniformly bounded constant and that

$$d(T_A(x), T_B(x)) \leq c_\Lambda \|A - B\|^{1/r} d(x)$$

for every $A, B \in \mathcal{M}_\Lambda$ and $x \in \mathbb{G}$. Consequently (see [2]), when \mathbb{G} is free, the uniform Gaussian estimates (1.5) follow from (2.2) and from the following Gaussian

estimates of the (fixed) fundamental solution $\Gamma_{\mathbb{G}}$:

$$|X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_{\mathbb{G}}(x, t)| \leq c_{\mathbb{G}} t^{-(Q+p+2q)/2} \exp\left(-\frac{d^2(x)}{c_{\mathbb{G}} t}\right).$$

In order to handle the case of an arbitrary stratified group \mathbb{G} , our main tool is to *lift* \mathbb{G} to a free stratified group $\tilde{\mathbb{G}}$ in such a way that $\Delta_{\mathbb{G}}$ is lifted to $\Delta_{\tilde{\mathbb{G}}}$. The lifting technique introduced by Rothschild and Stein in [36] together with some further remarks on the homogeneity properties of stratified groups, allows us to prove the following lifting result (see [6]): If \mathbb{G} is an N -dimensional stratified group then there exists an H -dimensional free stratified group $\tilde{\mathbb{G}}$ (with $H \geq N$) such that, denoting by $\pi : \mathbb{R}^H \rightarrow \mathbb{R}^N$ the projection on the first N coordinates, for every $u \in C^\infty(\mathbb{G})$, we have

$$\tilde{Z}_i(u \circ \pi) = (Z_i u) \circ \pi,$$

where $\sum_{i=1}^m Z_i^2$ and $\sum_{i=1}^m \tilde{Z}_i^2$ are the canonical sub-Laplacians $\Delta_{\mathbb{G}}$ and $\Delta_{\tilde{\mathbb{G}}}$, respectively. Since $\mathfrak{G}_1 = \text{span}\{Z_1, \dots, Z_m\}$, the lifting result leads to a correspondence between the operators $\mathcal{H}_A = \sum_{i,j=1}^m a_{i,j} X_i X_j - \partial_t$ on $\mathbb{G} \times \mathbb{R}$ and

$$\tilde{\mathcal{H}}_A = \sum_{i,j=1}^m a_{i,j} \tilde{X}_i \tilde{X}_j - \partial_t$$

on $\tilde{\mathbb{G}} \times \mathbb{R}$.

The lifting result also allows us to establish a natural relation between the fundamental solution Γ_A for \mathcal{H}_A and $\tilde{\Gamma}_A$ for $\tilde{\mathcal{H}}_A$. Indeed, we have

$$(2.3) \quad \Gamma_A(x, t) = \int_{\mathbb{R}^{H-N}} \tilde{\Gamma}_A((x, \hat{x}), t) d\hat{x}, \quad \text{for every } x \in \mathbb{G}, t \in \mathbb{R},$$

where (x, \hat{x}) denotes the point of $\mathbb{R}^N \times \mathbb{R}^{H-N}$. This fact, together with the established uniform estimates for free groups, allows us to prove (1.5) in the general case. Indeed, by means of the integral representation in (2.3), it is possible to transfer the uniform estimates for $\{\tilde{\Gamma}_A\}$ to the uniform estimates for $\{\Gamma_A\}$.

3. CONSTRUCTION OF THE FUNDAMENTAL SOLUTIONS

Following (1.4), we set for brevity $\mathcal{H}_{\zeta_0} := \mathcal{H}_{A(\zeta_0)}$ (and analogously, $\Gamma_{\zeta_0} := \Gamma_{A(\zeta_0)}$). The uniform Gaussian estimates discussed in Section 2 (see, e.g., (1.5)) allow us to prove the following estimates of the fundamental solutions Γ_{ζ_0} for the frozen operators \mathcal{H}_{ζ_0} (uniform in ζ_0):

$$(3.1) \quad \begin{aligned} c^{-1} \mathbf{E}(x, c^{-1} t) &\leq \Gamma_{\zeta_0}(x, t) \leq c \mathbf{E}(x, ct); \\ |X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_{\zeta_0}(x, t)| &\leq c_{p,q} t^{-(p+2q)/2} \mathbf{E}(x, ct); \\ |X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_{\zeta_0}(x, t) - X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_{\zeta_1}(x, t)| &\leq \\ &\leq c_{p,q} (d(\xi_0, \xi_1)^{\frac{\alpha}{r}} + |\tau_0 - \tau_1|^{\frac{\alpha}{2r}}) t^{-(p+2q)/2} \mathbf{E}(x, ct), \end{aligned}$$

where

$$\mathbf{E}(x, t) = t^{-Q/2} \exp(-d(x)^2/t).$$

We remark that the parametrix method outlined below is classical, but several technical complications arise in our setting. We set (for $z \neq \zeta$)

$$Z_1(z; \zeta) = \mathcal{H}(z \mapsto \Gamma_{\zeta}(\xi^{-1} \circ x, t - \tau))(z).$$

Since $\mathcal{H} = \mathcal{L} - \mathcal{L}_\zeta + \mathcal{H}_\zeta$, we have

$$Z_1(z; \zeta) = \sum_{i,j=1}^m (a_{i,j}(z) - a_{i,j}(\zeta)) X_i X_j \Gamma_\zeta(\xi^{-1} \circ x, t - \tau),$$

whence (3.1) gives

$$|Z_1(z; \zeta)| \leq c(t - \tau)^{\frac{\alpha}{2} - 1} \Gamma_{\zeta_0}(\xi^{-1} \circ x, c(t - \tau)).$$

If we inductively define

$$Z_{j+1}(z; \zeta) = \int_{\mathbb{R}^N \times [\tau, t]} Z_1(z; \eta) Z_j(\eta; \zeta) d\eta \quad (t > \tau),$$

the following estimate holds (for suitable constants c_1, c_2, b_j)

$$|Z_j(z; \zeta)| \leq c_1^j b_j(\alpha) (t - \tau)^{-1+j\alpha/2} \Gamma_{\zeta_0}(\xi^{-1} \circ x, c_2(t - \tau)).$$

By means of (3.1), it can then be proved that the series

$$\Phi(z; \zeta) = \sum_{j=1}^{\infty} Z_j(z; \zeta)$$

totally converges on a suitable domain and satisfies the estimate (here $T > 0$ and $c(T) > 0$ is a constant)

$$(3.2) \quad |\Phi(z; \zeta)| \leq c(T) (t - \tau)^{\frac{\alpha}{2} - 1} \mathbf{E}(\xi^{-1} \circ x, c(t - \tau)), \quad 0 < t - \tau \leq T.$$

It is easy to see that (for $t > \tau$)

$$\Phi(z; \zeta) = Z_1(z; \zeta) + \int_{\mathbb{R}^N \times [\tau, t]} Z_1(z; \eta) \Phi(\eta; \zeta) d\eta.$$

A crucial tool in the adaptation of the parametrix method is played by the following non-trivial regularity properties of Φ : $\Phi(\cdot; \zeta)$ and $\Phi(z; \cdot)$ are continuous functions and (for $0 < t - \tau \leq T$)

$$(3.3) \quad \begin{aligned} & |\Phi(x, t; \xi, \tau) - \Phi(x', t; \xi, \tau)| \leq \\ & \leq c(T) d(x, x')^{\frac{\alpha}{2}} (t - \tau)^{\frac{\alpha}{4} - 1} (\mathbf{E}(\xi^{-1} \circ x, c(t - \tau)) + \mathbf{E}(\xi^{-1} \circ x', c(t - \tau))). \end{aligned}$$

We now set, for every $z, \zeta \in \mathbb{R}^{N+1}$ with $t > \tau$,

$$(3.4) \quad J(z; \zeta) = \int_{\mathbb{R}^N \times [\tau, t]} \Gamma_\eta(z; z') \Phi(z'; \zeta) dz', \quad \Gamma(z; \zeta) = J(z; \zeta) + \Gamma_\zeta(z; \zeta),$$

and extend $\Gamma(z; \zeta)$ to be zero for $t \leq \tau$. The good property (3.2) of Φ and the Gaussian estimates (3.1) ensure that Γ is well posed. Exploiting again the estimates (3.1) and (3.2), it is not difficult to show that Γ is continuous away from the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ and satisfies

$$|\Gamma(z; \zeta)| \leq c(T) \mathbf{E}(\xi^{-1} \circ x, c(t - \tau)) \quad (0 < t - \tau \leq T).$$

We now show that Γ is the fundamental solution for \mathcal{H} . The first step is to prove that

$$\mathcal{H}(\Gamma(\cdot; \zeta)) = 0 \text{ in } \mathbb{R}^{N+1} \setminus \{\zeta\}.$$

We start by giving the definition of a regularity class of functions that will be useful in the sequel: given an open set $\Omega \subseteq \mathbb{R}^{N+1}$, we shall denote by $\mathfrak{C}^2(\Omega)$ the class of functions $u(x, t)$ defined on Ω which are continuous in Ω with respect to the pair (x, t) and such that $u(\cdot, t)$ has continuous Lie-derivatives up to second order

along the vector fields X_1, \dots, X_m (with respect to x , for every fixed t) and $u(x, \cdot)$ has continuous derivative (with respect to t , for every fixed x), in their respective domains of definition. The class $\mathfrak{C}^2(\Omega)$ is larger than the natural space $\Gamma^2(\Omega)$ (of continuous functions u with continuous derivatives $X_i u, X_i X_j u, \partial_t u$); nevertheless, we are able to prove a weak maximum principle for solutions to $\mathcal{H}u \geq 0$ in the class $\mathfrak{C}^2(\Omega)$, as stated below.

Proposition 3.1. *Let Ω be a bounded open subset of \mathbb{R}^{N+1} and $t_0 \in \mathbb{R}$; if $u \in \mathfrak{C}^2(\Omega)$, $\mathcal{H}u \geq 0$ in $\Omega \cap \{t < t_0\}$ and $\limsup u \leq 0$ in $\partial\Omega \cap \{t \leq t_0\}$, then $u \leq 0$ in $\Omega \cap \{t < t_0\}$.*

With the above notation we are able to prove the following result.

Proposition 3.2. *For every fixed $\zeta \in \mathbb{R}^{N+1}$, we have $\Gamma(\cdot; \zeta) \in \mathfrak{C}^2(\mathbb{R}^{N+1} \setminus \{\zeta\})$ and $\mathcal{H}(\Gamma(\cdot; \zeta)) = 0$ in $\mathbb{R}^{N+1} \setminus \{\zeta\}$. Moreover, the estimates in (1.3) hold.*

The main tool in the proof of the above proposition is the representation of the Lie-derivatives of the convolution (3.4) defining J as principal-value (singular) integrals. Indeed, we have

$$X_i X_j (J(\cdot; \zeta))(z) = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \times [\tau, t-\varepsilon]} X_i X_j \Gamma_{z'}(x'^{-1} \circ x, t-t') \Phi(z'; \zeta) dz' ,$$

$$\partial_t (J(\cdot; \zeta))(z) = \Phi(z; \zeta) + \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \times [\tau, t-\varepsilon]} \partial_t \Gamma_{z'}(x'^{-1} \circ x, t-t') \Phi(z'; \zeta) dz' .$$

The proof of this fact is not straightforward; one of the main ingredients is the Hölder estimate of Φ in (3.3). It is now easy to see that

$$\begin{aligned} \mathcal{H}(\Gamma(\cdot; \zeta))(z) &= \mathcal{H}(\Gamma_\zeta(\cdot; \zeta))(z) + \\ &+ \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \times [\tau, t-\varepsilon]} \mathcal{H}(\Gamma_\eta(\cdot; z'))(z) \Phi(z'; \zeta) dz' - \Phi(z; \zeta) = \\ &= Z_1(z; \zeta) + \int_{\mathbb{R}^N \times [\tau, t]} Z_1(z; \eta) \Phi(\eta; \zeta) d\eta - \Phi(z; \zeta) = 0 . \end{aligned}$$

We avoid giving details here on the results (iii) and (iv) of Theorem 1.1 concerning the Cauchy problem related to \mathcal{H} .

We now turn to the study of the operator $\mathcal{L} = \sum_{i,j} a_{i,j}(x) X_i X_j$. Starting from Theorem 1.1, one can construct local fundamental solutions for \mathcal{L} , by integrating Γ over the time variable t , provided suitable *long-time* estimates of Γ can be established. We are able to find the needed long-time estimates if the dimension m of the first layer \mathfrak{G}_1 of the stratification of \mathfrak{g} is strictly greater than two. Indeed, we can prove the following technical lemma.

Lemma 3.3. *Let Ω be a bounded domain of \mathbb{R}^{N+1} . Up to a modification of the coefficients $a_{i,j}(x, t)$ of \mathcal{H} outside Ω , Γ satisfies the following estimate: for every compact set $K \Subset \mathbb{R}^m$, there exist positive constants c, M such that*

$$\Gamma(x, t; \xi, \tau) \leq c(M + t - \tau)^{-m/2} \exp\left(-\frac{\Lambda |x^{(1)} - \xi^{(1)}|^2}{M + t - \tau}\right)$$

(for $t > \tau + 1$ and $\xi^{(1)} \in K$). We have denoted by $x^{(1)}$ the vector of the first m coordinates of x .

The proof of Lemma 3.3 relies on the weak maximum principle for \mathcal{H} in the class \mathfrak{E}^2 (see Proposition 3.1) and on a direct comparison argument. The above long-time estimate of Γ is certainly not optimal, but it is indeed sufficient to ensure the convergence of the integral $\int_{\mathbb{R}} \Gamma(x, t; \xi, 0) dt$ (for $x \neq \xi$) if $m > 2$. This allows us to obtain the result in Theorem 1.2 by arguing as sketched below.

We fix a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in Ω . We next define a suitable “ \mathbb{G} -regularization” $A^\varepsilon = (a_{i,j}^\varepsilon)_{i,j}$ of the coefficients $a_{i,j}$ of \mathcal{L} , and we set

$$\tilde{A}^\varepsilon(x) = \varphi(x) A^\varepsilon(x) + (1 - \varphi(x)) (4\Lambda)^{-1} \mathbb{I}_m$$

(where \mathbb{I}_m denotes the $m \times m$ identity matrix) and

$$\tilde{\mathcal{H}}^\varepsilon = \tilde{\mathcal{L}}^\varepsilon - \partial_t = \sum_{i,j=1}^m \tilde{a}_{i,j}^\varepsilon(x) X_i X_j - \partial_t.$$

The coefficients have been suitably modified outside Ω so that we can apply Lemma 3.3 to ensure that the integral

$$\tilde{\gamma}^\varepsilon(x, \xi) = \int_0^\infty \tilde{\Gamma}^\varepsilon(x, t; \xi, 0) dt, \quad x \neq \xi \in \mathbb{R}^N,$$

is convergent. Moreover, since the approximating operator $\tilde{\mathcal{L}}^\varepsilon$ has smooth coefficients, one can prove that $\tilde{\gamma}^\varepsilon$ is a fundamental solution for $\tilde{\mathcal{L}}^\varepsilon$ by making use of the well-posedness of the adjoint operators $(\tilde{\mathcal{H}}^\varepsilon)^*$ and $(\tilde{\mathcal{L}}^\varepsilon)^*$. Indeed, for every test function $\psi \in C_0^\infty(\mathbb{R}^N)$, setting

$$\psi_n(x, t) = \psi(x) \theta(|t|/n)$$

(where θ is a smooth function defined on \mathbb{R} , such that $\theta(s) = 1$ for $s \leq 1$, and $\theta(s) = 0$ for $s \geq 1$) we obtain

$$\begin{aligned} -\psi(\xi) &= -\psi_n(\xi, 0) = \int_{\mathbb{R}^{N+1}} \tilde{\Gamma}^\varepsilon(x, t; \xi, 0) (\tilde{\mathcal{H}}^\varepsilon)^* \psi_n(x, t) d\xi dt \\ &\xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^{N+1}} \tilde{\Gamma}^\varepsilon(x, t; \xi, 0) (\tilde{\mathcal{L}}^\varepsilon)^* \psi(x) dx dt = \int_{\mathbb{R}^N} \tilde{\gamma}^\varepsilon(x, \xi) (\tilde{\mathcal{L}}^\varepsilon)^* \psi(x) dx. \end{aligned}$$

We finally find a fundamental solution $\tilde{\gamma}$ of

$$\tilde{\mathcal{L}} = \sum_{i,j=1}^m \tilde{a}_{i,j}(x) X_i X_j$$

(where $\tilde{A} = \varphi A + (1 - \varphi)(4\Lambda)^{-1} \mathbb{I}_m$), by an approximation argument, by letting ε tend to zero in $\tilde{\gamma}^\varepsilon$. This approximation argument is not trivial and it requires suitable Schauder-type a priori estimates and a careful study of the ε -dependence in the construction of $\tilde{\Gamma}^\varepsilon$. We omit details here. Finally, observing that $\mathcal{L} = \tilde{\mathcal{L}}$ in Ω , we are then able to prove Theorem 1.2.

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