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Variational solutions for fractal transmission problems

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Abstract¹ The aim of this talk is to describe second order transmission problems involving a layer of fractal type, which is imbedded in an Euclidean domain. Fractals are geometric objects with highly non-Euclidean characteristics: despite their tricky geometry, there are however large families of fractals which possess a very rich analytic structure. So we are able to study fractals both as intrinsic bodies, in which is possible to give a suitable notion of Laplacian and as boundaries of Euclidean domains supporting traces of functions belonging to classic spaces like mains supporting traces of functions belonging to classic spaces like Sobolev spaces. Or possibly as bodies and boundaries at the same time, when they occur as highly conductive layers inside a Euclidean domain, which is the situation we focus on in this presentation.

1. INTRODUCTION

The aim of this talk is to review second order transmission problems involving a layer of fractal type imbedded in an Euclidean domain.

There is a huge literature dealing with transmission problems in connection with various applications in different fields: electrostatics, magnetostatics, hydraulic fracturing, studies of absorption or irrigation techniques. In electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see the paper by H. Pham Huy and E. Sanchez-Palencia [40] and the references listed in). Another engineering application is the model problem of the flow of oil in a fractured medium in order to increase the flow of oil from a “reservoir” into a producing oil well (see the paper of J. R. Cannon and G. H. Meyer [6] for more details). Further examples can be found in the book of R. Dautray and J. L. Lions [7]. In all these applications the mathematical model is an elliptic or parabolic boundary value problem involving a transmission condition on the interface (layer) either of order zero, one or two. We also refer to the recent survey of D. E. Apushkinskaya and A. I. Nazarov (see [2]) where the boundary value problems mentioned before are seen in the more general perspective of the so-called *Venttsel problems*, which go back to the late 50’s (A. D. Venttsel [42]).

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In many applications one is indeed interested in enhancing the layer absorption and diffusion, for a given conductivity of the layer material, this could be also achieved by raising as much as possible the surface of the layer with respect to the surrounding volume. In this respect, suitable layers of fractal type may provide a new interesting setting adequate to the preceding goal.

We are particularly interested in transmission problems with *highly conductive* fractal layers: these layers enjoy higher conductivity with respect to the surrounding space and, because of that, they absorb energy and convey intrinsic diffusion more efficiently. From the analytic point of view one deals with *transmission conditions* involving, at the same time, *traces* of functions from classic Sobolev spaces and intrinsic *Laplaceans* within the layer. The matching in the same problem of Euclidean and fractal analytic notions provides a significant crossing of Euclidean and fractal theory and tools (see the conference of U. Mosco [35] for an exhaustive discussion of this interesting topic).

Transmission problems with *fractal*, highly conductive layers are new: to our knowledge the first examples have been given by M. R. Lancia [24].

We shall consider here only the stationary version of the layer problem, and we focus our attention on a simple geometry as illustrated in Figure 1. Let Q denote

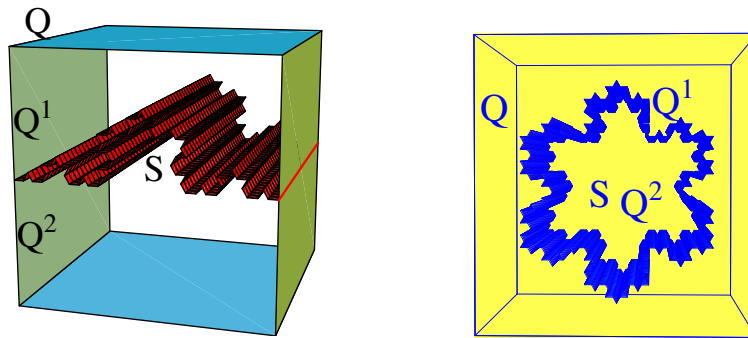


FIGURE 1. Koch roof and Koch pipe

the unit cube $Q =]0, 1[^3$ in \mathbb{R}^3 and S denote the layer in Q of the type $S = K \times I$ where $I = [0, 1]$ and K is either the von Koch curve or the Koch snowflake. We assume that S is located in a median position inside Q . The (closed) layer S divides the (open) domain Q in two (open) domains Q^1 and Q^2 , $Q = Q^1 \cup Q^2 \cup (S \cap Q)$ (see Figure 1).

As already mentioned highly conductive layers are characterized with respect to the surrounding space for having much greater conductivity, or permeability: *heat* or *flow* in the space is absorbed by the layer and starts diffusing within it much more efficiently than in the surrounding volume. The normal derivative from each side of the layer has a *jump* across the layer which acts as a *source term* for the Laplace operator generating the layer diffusion. The resulting boundary transmission condition is thus of *second order*, what is in some sense unusual for second order elliptic boundary value problems. Moreover, the condition has an *implicit* character, since the source term of the layer equation - the jump of the

normal derivatives - is not among the data of the problem, but depends on the solution itself.

Formally the equations in Q are:

$$(1.1) \quad -\Delta u^i = f \quad \text{on } Q^i$$

$$(1.2) \quad u = 0 \quad \text{on } \partial Q$$

$$(1.3) \quad u^1 = u^2 \quad \text{on } S,$$

where $u^i = u|_{Q^i}$, $i = 1, 2$, and these equations must be coupled with the equations on the layer S

$$(1.4) \quad u|_S = 0 \quad \text{on } \partial S$$

$$(1.5) \quad \frac{\partial u^1}{\partial \nu_1} + \frac{\partial u^2}{\partial \nu_2} = -\Delta_S u \quad \text{on } S$$

where $\partial/\partial \nu_i$ denotes the (formal) exterior normal derivative to the boundary of Q^i and $\Delta_S u$ the (formal) Laplacean operator on S . The rigorous definition of the operators and the spaces is one of the main technical difficulties in this kind of problems.

The previous conditions can be seen as the Euler conditions satisfied by the minimizer of a suitable energy functional; the existence of the variational solution is based on *variational principles*.

A convenient procedure, also from the numerical point of view, is to consider suitable approximating problems, to prove existence and uniqueness of the weak solution for both the pre-fractal and the fractal problem then to establish "sharp" estimates for the approximating solution u_h and finally to prove the asymptotic convergence of u_h towards the solution of the fractal problem u .

In section 2 we introduce the pre-fractal problem and we present the main regularity results, in section 3 we are concerned with the fractal problem and in section 4 we briefly deal with the asymptotic convergence.

2. PRE-FRACTAL PROBLEMS

We consider a 3-dimensional Euclidean domain Q containing a pre-fractal subset S_h , the *layer*. In our basic model the layer is of the type

$$(2.1) \quad S_h = K_h \times I,$$

where K_h is the pre-fractal Lipschitz curve occurring in the construction of the *Koch curve* in the plane, whose endpoints are A and B and $I = [0, L]$ is a real interval (for simplicity we take $L = 1$). We refer to [25] for the case of the Koch snowflake.

The layer is embedded in a 3-dimensional box:

$$Q = (0, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0, 1),$$

with coordinates (x_1, x_2, y) and the boundary of S_h belongs to the boundary of Q . S_h divides Q in two adjacent subdomains Q_h^1 (above) and Q_h^2 (below).

In order to state the variational principle, we need to define an energy functional E_h of the type

$$(2.2) \quad E_h = E_Q + E_{S_h}$$

where E_Q is the volume energy and E_{S_h} the layer energy.

We assume that the energy E_Q is simply the usual Dirichlet integral

$$(2.3) \quad E_Q[u] = \int_Q |\nabla u|^2 dQ$$

where $dQ = dx_1 dx_2 dy$ is the Lebesgue volume measure on \mathbb{R}^3

The space of functions of finite energy on Q , vanishing on ∂Q , is the usual Sobolev space $H_0^1(Q)$. It is well known that these functions have a well defined trace on the Lipschitz surface S_h (see *e.g.* Necăs [37]).

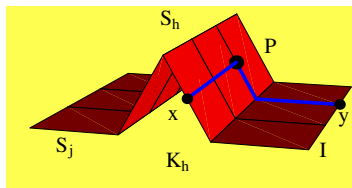


FIGURE 2. Layer coordinates

We give cartesian coordinates to points belonging to the *pre-fractal* piece-wise affine layer $S = K_h \times I$, $h = 1, 2, \dots$; K_h is the *piece-wise linear* pre-fractal approximation of K at the step h , here occurring as the section of the pre-fractal layers S_h , obtained by h iterations of the similarity mappings of K , where the initial set K_0 is the segment with endpoints $A = (0, 0, 0)$ and $B = (1, 0, 0)$. By Γ we denote the set $\{A, B\}$ and by $\overset{\circ}{K}$ the set $K \setminus \Gamma$.

By ℓ we denote the natural arc-length coordinate on each edge of K_h and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, $y = y$ on every affine “face” S_j of S_h . By $d\ell$ we denote the one-dimensional measure given by the arc-length ℓ and by dS the surface measure on each face S_j of S_h , that is, $dS = d\ell dy$. We define E_{S_h} on $S_h = K_h \times I$ by setting

$$(2.4) \quad E_{S_h}[u] = \sum_j \left(\int_{S_j} (\sigma_h^1 |D_\ell u|^2 + \sigma_h^2 |D_y u|^2) dS \right)$$

where σ_h^1, σ_h^2 are positive constants that must be chosen conveniently. By Fubini theorem, we can write this functional in the form

$$(2.5) \quad E_{S_h}[u] = \sigma_h^1 \int_I \left(\int_{K_h} |D_\ell u|^2 d\ell \right) dy + \sigma_h^2 \int_{K_h} \left(\int_I |D_y u|^2 dy \right) d\ell.$$

We now go back to the total energy E_h in (2.1), where now we take E_Q given by (2.2) and E_{S_h} given by (2.4). The functional E_h is well defined on the domain

$$(2.6) \quad D_0(E_h) = \{u \in H_0^1(Q) : u|_{S_h} \in H_0^1(S_h)\}$$

where $H_0^1(S_h)$ denotes the Sobolev space of functions on the piece-wise affine set S_h vanishing on ∂S_h . This space is defined, for instance, according to Necăs [37].

From now on, we denote by the same letter both the quadratic energy forms and the associated bilinear forms defined on the space of functions of finite energy.

Theorem 2.1. *In the previous notations the space $D_0(E_h)$ defined in (2.6) is a Hilbert space under the norm $\|u\| = (E_h[u])^{1/2}$ and the form E_h with domain $D_0(E_h)$ is a regular Dirichlet form in $L^2(Q)$.*

This result is a direct consequence of trace and density results for Sobolev spaces in polyhedra-like domains, as in Grisvard [9], [10], Nečas [37], Buffa-Ciarlet [4].

At this stage, since h is fixed, the value of the constants in (2.4) does not play any significant role. A good choice of the constants σ_h^1 and σ_h^2 , on the other hand, is essential in the asymptotic theory that sees the pre-fractals converging to the limit fractal.

As a corollary of Theorem 2.1, for any choice of f in $L^2(Q)$ there exists a unique function $u_h \in D_0(E_h)$ that minimizes the total energy:

$$(2.7) \quad \frac{1}{2} E_h[u] - \int_Q f u \, dQ .$$

This gives the *variational solution* of the transmission problem in Q with pre-fractal layer $S_h = K_h \times I$ and Dirichlet boundary conditions $u_h = 0$ on ∂Q and $u_h|_{S_h} = 0$ on ∂S_h .

The variational solution satisfies a *second order transmission condition*, which comes to light once we write the (weak) *Euler equation*. This leads to the *strong formulation* of the transmission problem, obtained by integration by parts through Green formulas in each domain Q_h^i .

We now summarize the main results for the pre-fractal transmission.

Theorem 2.2. *In the assumptions and notations of Theorem 2.1 we have that the variational solution u_h satisfies*

$$(2.8) \quad u_h \in C(\overline{Q}) ,$$

$$(2.9) \quad u_h^1 \in H^{\frac{8}{5}-\varepsilon}(Q_h^1) , \quad u_h^2 \in H^{\frac{7}{4}-\varepsilon}(Q_h^2) ,$$

$$(2.10) \quad \text{and} \quad \|u_h^i\|_{H^{s_i}(Q_h^i)} \leq c(h) \|f\|_{L^2(Q)} , \quad s_1 < \frac{8}{5} , \quad s_2 < \frac{7}{4} ,$$

$$(2.11) \quad \frac{\partial u_h^i}{\partial \nu^i} \in L^2(S_h) \quad i = 1, 2 .$$

Where $u_h^i = u_h|_{Q_h^i}$ and $\partial u_h^i / \partial \nu^i$ denote the exterior normal derivatives to the boundary of Q_h^i , $i = 1, 2$.

Let us note that the norms of the functions u_h^i in the fractional Sobolev spaces (in (2.10)) depend on h .

As a consequence the variational solution u_h is indeed the strong solution, going back to the strong problem we find that: equality (1.1) holds a. e. in Q_h^i , equalities (1.2), (1.3) and (1.4) are satisfied pointwise and the transmission condition (1.5) holds in $L^2(S_h)$. Here the Laplacean $'\Delta_S''$ in (1.5) is the linear combination of the “piece-wise” second order tangential derivative along the sides of K_h and the “usual” second order partial derivative in y :

$$(2.12) \quad ''\Delta_S'' = \sigma_h^1 D_\ell^2 + \sigma_h^2 D_y^2 .$$

Remark 2.1. *As to the different Sobolev regularity exponents for u_h^1 and u_h^2 , this discrepancy is due to the geometry of the polyhedra Q_h^1 and Q_h^2 , which have different (largest) dihedral angles, $(5/3)\pi$ and $(4/3)\pi$ respectively. As it is known from the*

regularity theory, the regularity of the solutions improves if the opening of the inner dihedral angles becomes smaller. This effect holds on despite the implicit character of the equations and the dependence of the regularity exponent on the angle remains unperturbed.

We point out that the regularity result of the preceding theorem is new even in the Euclidean case of flat or smooth layers considered by Pham Huy-Sanchez Palencia. Both in the Euclidean and in the fractal case, the difficulty in proving regularity properties of solutions stems from the implicit character of the volume equations in Q_h^1 , Q_h^2 and of the Poisson equation within the layer.

Actually we proved in the “flat” case that both the restrictions of the (variational) solution to Q_h^i belong to the Sobolev space $H^2(Q_h^i)$ see Theorem 3.2 in [27].

In order to present the main tools we split the proof of Theorem 2.2 in two steps.

• **Step 1** Consider the weak solutions w_h^i, \hat{w}_h^i in $H^1(Q_h^i)$ of the following auxiliary problems

$$(2.13) \quad \begin{cases} \Delta \hat{w}_h^i = 0 & \text{in } Q_h^i \\ \hat{w}_h^i = u_h & \text{on } \partial Q_h^i \end{cases}$$

$$(2.14) \quad \begin{cases} -\Delta w_h^i = f & \text{in } Q_h^i \\ w_h^i = 0 & \text{on } \partial Q_h^i . \end{cases}$$

As the link between u_h^i and the solutions of problems (2.13) and (2.14) is

$$(2.15) \quad u_h^i = w_h^i + \hat{w}_h^i ,$$

then the regularity of u_h^i follows from the regularity of w_h^i and \hat{w}_h^i .

From the results of Jerison and Kenig (see [12] Theorem 3 and Theorem 2), we deduce

$$(2.16) \quad \begin{aligned} & \frac{\partial \hat{w}_h^i}{\partial \nu_i} \in L^2(S_h) , \\ & \left\| \frac{\partial \hat{w}_h^i}{\partial \nu_i} \right\|_{L^2(S_h)} \leq c(h) \|\nabla_t u_h\|_{L^2(S_h)} , \quad i = 1, 2 . \end{aligned}$$

Note that the right-hand side of (2.16) can be evaluated in terms of the L^2 -norm of f in Q .

On the other we can use Kondrat’ev type results to obtain weighted estimates for the second order derivatives of w_h^i .

Nazarov and Plamenevsky (see [36]) study the asymptotics of solutions of the Dirichlet problem for the Laplace operator in a three-dimensional bounded domain having edges on the boundary, and determine the structure of asymptotic formulas for solutions near (smooth) edges or near their intersection points (vertices of polyhedra) using the (simplifying) fact that the coefficients are constant and the opening of the dihedral angles remains invariant along the edges.

Consider problem (2.14) in the bounded domain Q_h^i having several intersecting edges on the boundary. Let us denote by $\{P_\tau^i, \tau = 1, \dots, N\}$ the set of the vertices.

Near a intersection point P_τ^i the domain coincides with a cone K_τ^i cutting out a domain Ω_τ^i on the sphere \mathcal{S}^2 .

We apply Theorem 10.2.3 and Proposition 10.3.1 of [36] and by using the properties

of the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on Ω_τ^i , ($\tau = 1, \dots, N$, see e.g. [1] and [3]) we deduce the following estimates for the second order derivatives of w_h^i

$$(2.17) \quad \sum_{|\alpha|=2} \int_{Q_h^1} \rho^{2(\sigma_1-\mu_1)} r^{2\mu_1} |D^\alpha w_h^1|^2 dx dy \leq c(\mu_1, h) \|f\|_{L^2(Q)}^2$$

$$(2.18) \quad \sum_{|\alpha|=2} \int_{Q_h^2} \rho^{2(\sigma_2-\mu_2)} r^{2\mu_2} |D^\alpha w_h^2|^2 dx dy \leq c(\mu_2, h) \|f\|_{L^2(Q)}^2$$

$$(2.19) \quad \text{for any } 1 > \mu_1 > \frac{2}{5}, \quad 1 > \mu_2 > \frac{1}{4}, \quad 1 > \sigma_i \geq 0.$$

where $r = r(P)$ denotes the distance of the point P from the nearest edge and $\rho = \rho(P)$ the distance of the point P from the nearest vertex of the polyhedron. Now we choose $\sigma_1 = \mu_1$ and $\sigma_2 = \mu_2$ and, denoting by δ the distance from the boundary, we derive:

$$\|\delta^{\mu_i} D^\alpha w_h^i\|_{L^2(Q_h^i)} \leq c(\mu_i, h) \|f\|_{L^2(Q)}, \quad |\alpha| = 2, \quad i = 1, 2.$$

Interpolation techniques, according to [14], allow us to show that w_h^i belongs to usual (fractional) Sobolev spaces: more precisely, we have

$$(2.20) \quad \|w_h^i\|_{H^{2-\mu_i}(Q^i)} \leq c(\mu_i, h) \|f\|_{L^2(Q)}, \quad i = 1, 2.$$

$$\mu_1 = \frac{2}{5} + \epsilon \quad \text{and} \quad \mu_2 = \frac{1}{4} + \epsilon.$$

Remark 2.2. *In the following step we will make use of trace spaces on boundaries of polyhedra. For the present application, in our opinion, the more convenient definition of Sobolev spaces (on boundaries of polyhedra) is that given by Buffa-Ciarlet in [4] (section 2.1). If $0 < s \leq 1$ the Sobolev space $H^s(S_h)$ defined in [4] coincides, with equivalent norms, with the Sobolev space defined in Nečas [37] by local Lipschitz charts.*

Instead for large "s" the definitions of the trace spaces on "irregular" boundaries are more complicated (see Grisvard [9] and [10]) so we confine ourselves to the case which best fits to our situation and we recall that according to [4] the Sobolev space $H^{3/2}(S_h)$ coincides with the space

$$\left\{ v \in H^1(S_h) : v|_{S_j} \in H^{3/2}(S_j) \quad \forall S_j \text{ of } S_h \right\}$$

and (in particular) is embedded in $C(S_h)$.

Roughly speaking in the definition of the Sobolev space, natural in this context, compatibility conditions on the edges and on the vertices are required for the function and not for the derivatives.

An analogous characterization holds for the spaces $H^{3/2}(\partial Q_h^i)$ and $H^{3/2}(\partial Q)$.

Remark 2.3. *The extension of the classical trace theorems (see [31]) to general Lipschitz domains is tricky. Jonsson and Wallin in [18] characterize the traces of functions in $H^s(\mathbb{R}^n)$ on "bad" closed sets, the so-called d -sets, for any $s > (n-d)/2$, the limit case $s = (n-d)/2$ being considered by Triebel in [41]. If $s \geq 1 + (n-d)/2$ the trace spaces are some suitable Besov spaces which look very different from the usual Sobolev spaces when $d \leq n-1$. These spaces clearly have an important role in domains G with a fractal boundary, for example the Koch roof S , which is a special*

case of d -set, as we will see in the following section 3.

For the special case of polyhedra, as the domains Q_h^i occurring in the present section, the trace space of $H^2(Q_h^i)$ is the Sobolev space $H^{3/2}(\partial Q_h^i)$ (see [9] and [4] theorem 2.4).

Now we can back to the proof of Theorem 3.2 and we sketch the second step.

• **Step 2** We obtain (2.11) starting from (2.15), (2.16), (2.20) and using trace results (see [9] and [4]). The Green formula and the use of suitable test functions allow us to prove that $\Delta_S u_h$ belongs to $L^2(S_h)$. As $u_h \in H_0^1(S_h)$ from theorem 8 of [5] we deduce that, in particular, u_h belongs to $H^{3/2}(S_h)$.

We note that the restriction $u_h^i = u_h|_{Q_h^i}$ is the weak solution, in $H^1(Q_h^i)$, of the problem

$$(2.21) \quad n \begin{cases} -\Delta u_h^i = f & \text{in } Q_h^i \\ u_h^i = 0 & \text{on } \partial Q_h^i \setminus S_h \\ u_h^i = u_h & \text{on } S_h . \end{cases}$$

Denote by \tilde{u}_h^i the trivial extension of u_h in ∂Q_h^i - $\tilde{u}_h^i = u_h$ in S_h and $\tilde{u}_h^i = 0$ in $\partial Q_h^i \setminus S_h$ - then $\tilde{u}_h^i|_{\partial Q_h^i}$ belongs, in particular, to $H^{3/2}(\partial Q_h^i)$. Let \hat{u}_h^i be a function in $H^2(Q_h^i)$ such that $\hat{u}_h^i|_{\partial Q_h^i} = \tilde{u}_h^i$ (see [4] theorem 2.4), then $\Delta \hat{u}_h^i \in L^2(Q_h^i)$ and $\|\Delta \hat{u}_h^i\|_{L^2(Q_h^i)} \leq c(h)\|f\|_{L^2(Q)}$.

The function $v_h^i := u_h^i - \hat{u}_h^i$ is the weak solution in $H_0^1(Q_h^i)$ of the Dirichlet problem

$$\begin{cases} -\Delta v_h^i = f + \Delta \hat{u}_h^i & \text{in } Q_h^i \\ v_h^i = 0 & \text{on } \partial Q_h^i , \end{cases}$$

by proceeding as in step 1 we can obtain that $v_h^i \in H^{s_i}(Q_h^i)$ where $s_1 < 8/5$ and $s_2 < 7/4$. Then u_h^i inherits the regularity of v_h^i as \hat{u}_h^i is more regular, this yields (2.9) and (2.10). Finally $u_h|_{S_h}$ is continuous on S_h (see remark 2.2), u_h^i belongs to $H^{8/5-\varepsilon}(Q_h^i)$, $i = 1, 2$, hence from Morrey-Sobolev embedding u_h^i is continuous in $\overline{Q_h^i}$. Taking into account that $u_h^i|_{S_h} = u_h|_S$, $i = 1, 2$, the proof of theorem 2.2 is achieved.

3. FRACTAL PROBLEM

We define the energy form E_S on the fractal layer $S = K \times I$. We give a point $P \in S$ the cartesian coordinates $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plane containing K and y is the coordinate of the orthogonal projection of P on the y -line containing the interval I : $P = (x, y) \in S$, $x = (x_1, x_2) \in K$, $y \in I$. We set

$$(3.1) \quad E_S(u, v) = \sigma^1 \int_I \int_K \mathcal{L}_x(u, v)(dx) dy + \sigma^2 \int_K \int_I D_y u D_y v dy \mathcal{H}^{d_f}(dx) .$$

where σ^1 and σ^2 are positive constants. Here, $\mathcal{L}_x(\cdot, \cdot)(dx)$ denotes the (measure-valued) Lagrangean of the energy form E_K associated with the Brownian motion on K with domain $D_0(K)$, now acting on $u(x, y)$ and $v(x, y)$ as functions of $x \in K$ for a. e. $y \in I$; $\mathcal{H}^{d_f}(dx)$ is the Hausdorff measure acting on each section K of S

for a. e. $y \in I$ with $d_f = \log 4 / \log 3$. We note that the “volume measure” dS in S is the product measure $dS = d\mathcal{H}^{d_f} \times dy$.

The form E_K on the fractal K has the integral representation

$$(3.2) \quad E_K[u] = \int_K d\mathcal{L}[u]$$

where the local energy $\mathcal{L}[\cdot]$ is a (measure-valued) Lagrangean on K , see [33], [32] and [34]. E_K is a Dirichlet form in the Hilbert space $L^2(K, \mathcal{H}^{d_f}(dx))$. The Laplace operator Δ_K on K , with Dirichlet boundary condition $u|_\Gamma = 0$ on Γ , is obtained from the bilinear form

$$(3.3) \quad E_K(u, v) = \int_K d\mathcal{L}(u, v)$$

taken with domain

$$(3.4) \quad D_0(K) := \{u \in L^2(K, \mathcal{H}^{d_f}(dx)) : \int_K d\mathcal{L}[u] < \infty, u|_\Gamma = 0\}$$

by the identity

$$E_K(u, v) = \int_K -(\Delta_K u)v \mathcal{H}^{d_f}(dx).$$

The Laplacean Δ_K is a non-positive self-adjoint operator, with a domain D_{Δ_K} dense in $L^2(K, \mathcal{H}^{d_f})$, and the previous identity holds for every $u \in D_{\Delta_K}$ and every $v \in D_0(K)$. We shall denote by the same symbol “ Δ_K ” the Laplace operator as variational operator from $D_0(K) \rightarrow (D_0(K))'$ defined by the identity

$$(3.5) \quad E_K(u, v) = -\langle \Delta_K u, v \rangle, \quad u, v \in D_0(K).$$

In the following we will also make use of the non homogeneous domain

$$(3.6) \quad D(K) = \{u \in L^2(K, \mathcal{H}^{d_f}(dx)) : \int_K d\mathcal{L}[u] < \infty\}.$$

We then write the total energy E for the problem with the fractal layer S , as in (2.1), where the volume energy E_Q is still given by (2.2), while the layer energy E_S is now given by (3.1). The functional E is well defined on the domain

$$(3.7) \quad D_0(E) = \{u \in H_0^1(Q) : u|_S \in D_0(S)\}.$$

where $D_0(S)$ is the closure in the intrinsic norm $\|u\| = (E_S[u])^{1/2}$ of the set

$$(3.8) \quad C_0(S) \cap L^2(I; D_0(K)) \cap H_0^1(I; L^2(K))$$

$$\text{where } L^2(K) := L^2(K, \mathcal{H}^{d_f}).$$

Theorem 3.1. *In the previous notations and assumptions the space $D_0(E)$ in (3.7) is a Hilbert space under the intrinsic norm $\|u\| = (E[u])^{1/2}$ and the form E with domain $D_0(E)$ is a regular Dirichlet form in $L^2(Q)$.*

The proof makes use of the theory of Besov spaces on d-sets developed by Jonsson-Wallin [18], Triebel [41] and of the theory of the Besov spaces on closed sets (that are no d-sets) developed by Jonsson [15].

It is to be pointed out that interesting relations between Dirichlet forms and Brownian motion penetrating fractals have been addressed by Lindström [30], Jonsson [17] and Kumagai [22].

As in the pre-fractal case, for every $f \in L^2(Q)$ there exists a unique function $u \in D_0(E)$ that minimizes the total energy functional:

$$(3.9) \quad \frac{1}{2}E[u] - \int_Q fu \, dQ .$$

This function provides the *variational*, or *weak*, solution to the *transmission problem* in Q , with fractal layer S and Dirichlet conditions $u = 0$ on ∂Q and $u|_S = 0$ on ∂S . We now summarize the main results for the fractal transmission.

Theorem 3.2. *In the assumptions and notations of Theorem 3.1 we have that the variational solution u satisfies:*

$$(3.10) \quad \frac{\partial u^1}{\partial \nu^1} , \frac{\partial u^2}{\partial \nu^2} \in (B_{\beta,0}^{2,2}(S))'$$

where $(B_{\beta,0}^{2,2}(S))'$ is the dual space of $B_{\beta,0}^{2,2}(S)$, $\beta = d_f/2$

$$(3.11) \quad \frac{\partial u^1}{\partial \nu^1} + \frac{\partial u^2}{\partial \nu^2} = -\Delta_S u \quad \text{in the dual space } (D_0(S))' \text{ of } D_0(S) .$$

Here $u^i = u|_{Q_i}$, $i = 1, 2$ and Δ_S denotes the *variational* Laplace operator from $D_0(S)$ to the dual $(D_0(S))'$ associated with the Dirichlet forms E_S as in (3.5). Hence formula (1.5) has to be intended in the sense of the dual space of $D_0(S)$.

The proof of Theorem 3.2 is delicate and makes use of complicated results from sophisticated subspaces of the Besov spaces on d-sets S and K . The space $B_{\beta}^{2,2}(S)$, is the space of functions on S for which is finite the norm

$$(3.12) \quad \|u\|_{L^2(S,dS)} + \left(\int \int_{|x-y|<1} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2\beta}} \, dS(x) \, dS(y) \right)^{\frac{1}{2}} .$$

here $d = 1 + d_f$, and $\beta = d_f/2$. Let us recall that the definition of Besov spaces on (d-sets) with smoothness index greater or equal to 1 is more complicated than the previous one, see [18] and [41].

The space $B_{\beta,0}^{2,2}(S)$ is a subspace of $B_{\beta}^{2,2}(S)$ and is the fractal analogue of the Lions-Magenes space $H_{0,0}^{\frac{1}{2}}(S)$, namely

$$B_{\beta,0}^{2,2}(S) = \{u \in L^2(S, dS) : \exists w \in H_0^1(Q), w|_S = u \text{ on } S\}$$

taken with the norm

$$\|u\|_{B_{\beta,0}^{2,2}(S)} = \inf\{\|w\|_{H_0^1(Q)} : w \in H_0^1(Q), w|_S = u \text{ on } S\} .$$

The space $D_0(S)$ (see (3.8)) turns out to be a subspace of the Besov space $B_{\beta,0}^{2,2}(S)$, the proof can be done as in Proposition 3.1 of [25]. An important role, in proving Theorem 3.2 is played by the characterization of the domain $D(K)$ (see formula (3.6)) in terms of the “tricky” Lipschitz spaces $Lip(\cdot, 2, \infty, K)$, obtained in [26], Theorem 3.1. Jonsson introduced these Lipschitz spaces $Lip(\cdot, 2, \infty, \cdot)$ to characterize the domain of the Dirichlet form of the Brownian motion on the Sierpinski gasket and stated their “collocation” in the “Besov scale” and several interesting properties, (see [16] and, for further applications, [39] and [22]).

Remark 3.1. We address the question whether the (variational) solution u of the fractal transmission problem is continuous on \bar{Q} . Up to now we are able to prove the continuity only in the 2-dimensional case. We sketch the proof. Let Q be the unit square in $\mathbb{R}^2 : Q =]0, 1[^2$, $S = K$, $Q = Q^1 \cup Q^2 \cup \overset{\circ}{K}$ (see the following Figure 3). Let us consider the “analogous” problems as in (2.13) and (2.14) and denote by \hat{w}^i and w^i the corresponding solutions. The function $u|_K$ is Hölder continuous on K with Hölder exponent $d_f/2$ (see Corollary 3.3 of [26]) and $u(A) = u(B) = 0$ then u is (Hölder) continuous on ∂Q^i . As the domains Q^i are “non-tangentially accessible” domains according to the definition given by Jerison and Kenig (see [13] section 3) they are “regular” for the Dirichlet problem (2.13) and hence the solutions \hat{w}^i are continuous on \bar{Q}^i .

On the other hand one can prove that the solutions w^i are Hölder continuous on \bar{Q}^i with Hölder exponent $1/3$ (see Nystrom Theorem 3.1 in [38] part B). Hence $u^i = w^i + \hat{w}^i$ is continuous on \bar{Q}^i and

$$u = \begin{cases} u^1 & \text{on } Q^1 \\ u & \text{on } K \\ u^2 & \text{on } Q^2 \end{cases}$$

is continuous on \bar{Q} .

This procedure does not work for the 3-dimensional case.

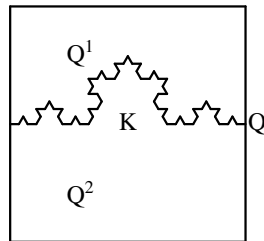


FIGURE 3. Basic model

4. ASYMPTOTICS

An asymptotic “constructive” theory for the pre-fractal approximation is an important step toward the numerical analysis of the problem.

We address the question whether the pre-fractal solutions u_h do converge to the fractal solution u , as the pre-fractal layer S_h converges to the fractal layer S for $h \rightarrow \infty$.

As h increases the pre-fractal layers develop increasing surface up to reach the infinite 2-dimensional area of the limit fractal surface. In formula (2.10) the norm in the fractional Sobolev spaces (with smoothness index greater than 1) blows up as $h \rightarrow +\infty$. So we are forced to choose a different notion of convergence, actually an energy convergence.

The Euclidean pre-fractal energies must be re-normalized in order to keep the energy finite in the limit. This amounts to choose the constants σ_h^1 and σ_h^2 in (2.5) conveniently.

The good choice of the re-normalization factors is obtained by taking into account the effect of the d_f -dimensional length intrinsic to the fractal curve K .

Theorem 4.1. *Let u be the variational solution of the fractal transmission problem in the domain Q of \mathbb{R}^3 , with layer $S = K \times I$ with K the fractal Koch curve. For every integer $h \geq 1$, let u_h be the variational solution of the transmission problems in Q with pre-fractal layer $S_h = K_h \times I$. If we scale the energy functionals (2.5), by taking $\sigma_h^1 = \sigma^1(3^{d_f-1})^h$ and $\sigma_h^2 = \sigma^2(3^{1-d_f})^h$ then as $h \rightarrow \infty$ we find:*

$$(4.1) \quad u_h \rightarrow u \text{ strongly in } H_0^1(Q),$$

$$(4.2) \quad \int_{S_h} \frac{\partial u_h^i}{\partial \nu^i} \phi dS \rightarrow \left\langle \frac{\partial u^i}{\partial \nu^i}, \phi \right\rangle_{(B_{\beta,0}^{2,2}(S))', B_{\beta,0}^{2,2}(S)}, \quad \forall \phi \in H_0^1(Q),$$

$$\beta = d_f/2, \quad i = 1, 2$$

$$(4.3) \quad \int_{S_h} \Delta_{S_h} u_h \phi dS \rightarrow \langle \Delta_S u, \phi \rangle_{(D_0(S))', D_0(S)} \quad \forall \phi \in D_0(E),$$

where Δ_{S_h} is the piecewise second operator in (2.12) and Δ_S the variational Laplacean from $D_0(S)$ to the dual $(D_0(S))'$, associated to the Dirichlet form E_S as in (3.5).

Theorem 4.1 is a consequence of a stronger result the so called M convergence of the pre-fractal energies E_h to the fractal energy E . The M convergence can be deduced from analogous 2-dimensional results (see [28] theorem 4.1) by using “technical” tools as previous mentioned trace and extension results for Besov spaces on d sets (see [18] or [41]) and “classical” arguments as Fatou’s lemma and convergence theorems.

Remark 4.1. *A quantitative estimate of the rate of convergence in (4.1) would be very useful for numerical solutions of the fractal problem.*

Remark 4.2. *From the probabilistic point of view, Brownian motions penetrating fractal sets - a probabilistic counterpart of the analytic variational approach adopted here - have been constructed by Lindström [30] and Kumagai [22], however without reference to transmission problems and related transmission conditions.*

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