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Differential forms on singular complex spaces and mixed Hodge structures

by Vincenzo ANCONA and Bernard GAVEAU

Abstract¹. On every reduced complex space X we construct a family of complexes of soft sheaves Λ_X^\bullet ; each of them is a resolution of the constant sheaf \mathbb{C}_X and induces the ordinary de Rham complex of differential forms on a dense open analytic subset of X . The construction is functorial (in a suitable sense). Moreover each of the above complexes can fully describe the mixed Hodge structure of Deligne on a compact algebraic variety.

1. INTRODUCTION

The de Rham complex \mathcal{E}_X^\bullet of differential forms is one of the fundamental tools for the study of the properties of a smooth manifold X . Among the most important applications are the de Rham and the Hodge theories. The basic elementary properties of \mathcal{E}_X^\bullet which are the foundation of these theories can be summarized as follows:

- \mathcal{E}_X^\bullet is a resolution of the constant sheaf \mathbb{C}_X ;
- \mathcal{E}_X^\bullet is a complex of fine sheaves;
- for $p > 2 \dim X$, $\mathcal{E}_X^p = 0$;
- if $f : X \rightarrow Y$ is a morphism of complex manifolds, the pullback $f^* : \mathcal{E}_Y^\bullet \rightarrow \mathcal{E}_X^\bullet$ is defined; it commutes to differentials, and is functorial.

If X is a singular complex space, the complex D_X^\bullet of differential forms in the sense of Grauert and Grothendieck (see [9] for precise definitions) shares with the classical de Rham complex on a manifold all the above listed properties, except the first: easy examples ([11], [16]) show that D_X^\bullet need not be a resolution of \mathbb{C}_X . On the other hand, the restriction $D_X^\bullet|_U$ to the open subset U of smooth points of X is the ordinary de Rham complex \mathcal{E}_U^\bullet . Though there are some relationships between the cohomology of the complex of the global sections of D_X^\bullet and the singular cohomology of X (see [9]), it is clear that the Grauert-Grothendieck forms are not useful in attempting to generalize the de Rham theory and to describe the Hodge-Deligne theory in the singular case.

¹Authors' address: V. Ancona, Università degli Studi di Firenze, Dipartimento di Matematica "Ulisse Dini", Viale Morgagni 67/A, I-50134 Firenze, Italy, e-mail: ancona@math.unifi.it. ;

B. Gaveau, UFR de Mathématiques, Université Pierre et Marie Curie, 4, Place Jussieu, F-75252 Paris Cedex 05, France, e-mail: gaveau@ccr.jussieu.fr .

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In a series of papers ([1]-[6]) we proposed a different definition of differential forms on a complex space, in order to obtain a resolution of \mathbb{C}_X by means of forms, and de Rham and Hodge type results. In order to define D_X^\bullet one makes use of local embeddings of X into open sets of \mathbb{C}^n , while we used a desingularization of X . The definition relies on a construction called "tower of desingularizations", and produces forms possessing all the above properties, with the exception of the existence of the pullback. We came to the conclusion that in order to obtain all the properties of the classical forms, including the existence of the pullback and the functoriality, it is necessary to define for any space X a family of complexes, instead of a single one. This is the subject of the present paper.

The forms we introduce here can fully describe also the mixed Hodge structure of Deligne [10] on a compact algebraic variety X ([8]). Our approach to this theory is particularly intended for the readers who have no knowledge of derived categories or cohomological descent theory. In the present paper only some proofs are sketched; the complete proofs will appear elsewhere ([7]).

Going into details, for every X we define a family of complexes $\mathcal{R}(X) = \{\Lambda_X^\bullet\}$ and for every morphism $f : X \rightarrow Y$ a family $\mathcal{R}(Y, X)$ of morphisms of complexes between the $\Lambda_Y^\bullet \in \mathcal{R}(Y)$ and some of the $\Lambda_X^\bullet \in \mathcal{R}(X)$, more precisely morphisms $\Lambda_Y^\bullet \rightarrow f_*\Lambda_X^\bullet$ which we simply denote $\Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$ and call (*admissible*) *pullback* with the following properties.

- (I) Λ_X^\bullet is a fine resolution of \mathbb{C}_X .
- (II) For $p > 2 \dim X$, $\Lambda_X^p = 0$.
- (III) If X is smooth, the ordinary de Rham complex \mathcal{E}_X^\bullet belongs to $\mathcal{R}(X)$, and for every morphism $F : X \rightarrow Y$ between smooth complex manifolds the ordinary de Rham pullback $f^* : \mathcal{E}_Y^\bullet \rightarrow f_*\mathcal{E}_X^\bullet$ is an admissible pullback.
- (IV) There exists a smooth, open, dense analytic subset $U \subset X$ such that the restriction $\Lambda_X^\bullet|_U$ is the ordinary de Rham complex \mathcal{E}_U^\bullet . Here analytic means that the complement of U in X is an analytic subspace of X .

The family of pullback will satisfy the following properties.

- (C) (Composition). Let $g : Z \rightarrow X$, $f : X \rightarrow Y$ be two morphisms, $\alpha : \Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$, $\beta : \Lambda_X^\bullet \rightarrow \Lambda_Z^\bullet$ two pullback: then the composition $\beta \circ \alpha : \Lambda_Y^\bullet \rightarrow \Lambda_Z^\bullet$ is again a pullback.

- (EP) (Existence of pullback). Let $f : X \rightarrow Y$ be a morphism, and fix $\Lambda_Y V \in \mathcal{R}(Y)$; then there exists a $\Lambda_X^\bullet \in \mathcal{R}(X)$ and a pullback $\Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$.

We denote by $(EP)_{m,n}$ the property (EP) when $\dim X \leq m$, $\dim Y \leq n$.

- (U) (Uniqueness of pullback). Let $f : X \rightarrow Y$ be a morphism, and $\alpha : \Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$, $\beta : \Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$ two pullback corresponding to f ; then $\alpha = \beta$.

As above we denote by $(U)_{m,n}$ the property (U) when $\dim X \leq m$, $\dim Y \leq n$.

- (F) (Filtering). If $\Lambda_X^{\bullet,1}, \Lambda_X^{\bullet,2} \in \mathcal{R}(X)$, there exists a third $\Lambda_X^\bullet \in \mathcal{R}(X)$ and to pullback $\Lambda_X^{\bullet,1} \rightarrow \Lambda_X^\bullet$, $\Lambda_X^{\bullet,2} \rightarrow \Lambda_X^\bullet$ corresponding to the identity.

We denote by $(F)_m$ the property (F) when $\dim X \leq m$.

The complexes Λ_X^\bullet and the pullback are defined by means of an induction procedure on the dimension of the complex spaces involved.

In the section 1 we explain the logic of the proof, in particular the recursion scheme.

In the section 2 we introduce the family $\mathcal{R}(X)$ and the family $\mathcal{R}(Y, X)$.

In the section 3 we show how to describe the Hodge-Deligne theory on compact algebraic varieties by means of our constructions.

The section 4 is devoted to the extension of the previous results to the case of logarithmic complexes, in order to treat the Hodge-Deligne theory in the case of open algebraic varieties.

2. THE INDUCTION PROCEDURE

Let $s, t \in \mathbb{N} \times \mathbb{N}$, $s = (m, n)$, $t = (p, q)$. We define the following order on $\mathbb{N} \times \mathbb{N}$: $s > t$ if $\sup(m, n) > \sup(p, q)$ or $\sup(m, n) = \sup(p, q)$ and $(m, n) > (p, q)$ in the lexicographic order.

We write $(EP)_s$, $(U)_s$ instead of $(EP)_{m,n}$, $(U)_{m,n}$. The existence theorem 3.1 in the next paragraph explains the meaning of $(E)_n$.

The proofs of our results are based on the following implications:

$$(E)_p + (F)_q + (EP)_t + (U)_t \text{ for } p < n, q < n, t < (0, n) \implies (E)_n$$

$$(E)_p + (F)_q + (EP)_t + (U)_t \text{ for } p \leq n, q < n, t < (n, 0) \implies (F)_n$$

$$(E)_p + (F)_q + (EP)_t + (U)_t \text{ for } (0, p) \leq s, (q, 0) \leq s, t < s \implies (EP)_s$$

$$(E)_p + (F)_q + (EP)_t + (U)_t \text{ for } (0, p) \leq s, (q, 0) \leq s, t < s \implies (U)_s$$

Also the definitions of the family $\mathcal{R}(X)$ and of admissible pullback are given by induction.

3. DEFINITIONS

Let X be a complex space, $E \subset X$ a nowhere dense closed subspace, and

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{i} & \tilde{X} \\ q \downarrow & & \pi \downarrow \\ E & \xrightarrow{j} & X \end{array}$$

a proper modification where $j : E \rightarrow X$ is the natural inclusion, \tilde{X} is a smooth manifold and π induces an isomorphism $\tilde{X} \setminus \tilde{E} \simeq X \setminus E$. By a theorem of Lojasiewicz, there exist a fundamental system of neighborhoods W of \tilde{E} in \tilde{X} and retractions $r : W \rightarrow \tilde{E}$; it follows that the restriction morphisms $H^k(W, \mathbb{C}) \rightarrow H^k(\tilde{E}, \mathbb{C})$ are isomorphisms. As a consequence we obtain:

Proposition 3.1. *Let $U \subset X$ be an open neighborhood of a point $x \in E$.*

- i) *Let $\eta_1, \eta_2 \in H^k(\pi^{-1}(U), \mathbb{C})$ two classes whose restriction to $H^k(\pi^{-1}(U) \cap \tilde{E}, \mathbb{C})$ agree. There exists an open neighborhood $V \subset U$ of x such that the restrictions agree in $H^k(\pi^{-1}(V), \mathbb{C})$.*
- ii) *Let $\theta \in H^k(\pi^{-1}(U) \cap \tilde{E}, \mathbb{C})$. There exists an open neighborhood $V \subset U$ of x and $\eta \in H^k(\pi^{-1}(V), \mathbb{C})$ inducing θ .*

Definition of the family $\mathcal{R}(X)$. We denote by $\Lambda_X^{\cdot}(-1)$ the complex obtained by shifting the degree in Λ_X^{\cdot} : more precisely $\Lambda_X^p(-1) = \Lambda_X^{p-1}$.

We define the family $\mathcal{R}(X)$ by induction on $n = \dim(X)$. If $\dim(X) = 0$, $\mathcal{R}(X)$ contains only the complex \mathbb{C}_X with $\mathbb{C}_X^0 = \mathbb{C}_X$ and $\mathbb{C}_X^p = 0$ for $p > 0$. We suppose $\mathcal{R}(Y)$ to be known for complex spaces Y of dimension $< n$; then:

Definition 3.1. Let X be a complex space of dimension n . An element $\Lambda_X^{\cdot} \in \mathcal{R}(X)$ is the assignment of the following data:

- i) a nowhere dense closed subspace $E \subset X$, $Sing(X) \subset E$ and a proper modification

$$(1) \quad \begin{array}{ccc} \tilde{E} & \xrightarrow{i} & \tilde{X} \\ q \downarrow & & \pi \downarrow \\ E & \xrightarrow{j} & X \end{array}$$

where $j : E \rightarrow X$ is the natural inclusion, \tilde{X} is a smooth manifold, $\tilde{E} = \pi^{-1}(E)$ and π induces an isomorphism $\tilde{X} \setminus \tilde{E} \simeq X \setminus E$;

- ii) there exist $\Lambda_E \in \mathcal{R}(E)$, $\Lambda_{\tilde{E}} \in \mathcal{R}(\tilde{E})$ and two pullback

$$\phi : \Lambda_E \rightarrow \Lambda_{\tilde{E}}$$

$$\psi : \mathcal{E}_{\tilde{X}} \rightarrow \Lambda_{\tilde{E}}$$

(corresponding respectively to q and i);

- iii) the complex Λ_X is defined by

$$(2) \quad \Lambda_X = \pi_* \mathcal{E}_{\tilde{X}} \oplus j_* \Lambda_E \oplus (j \circ q)_* \Lambda_{\tilde{E}}(-1)$$

with differential given by

$$d : \Lambda_X^p = \pi_* \mathcal{E}_{\tilde{X}}^p \oplus j_* \Lambda_E^p \oplus (j \circ q)_* \Lambda_{\tilde{E}}^{p-1} \rightarrow \Lambda_X^{p+1} = \pi_* \mathcal{E}_{\tilde{X}}^{p+1} \oplus j_* \Lambda_E^{p+1} \oplus (j \circ q)_* \Lambda_{\tilde{E}}^p$$

$$(3) \quad d(\omega, \sigma, \theta) = (d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega) - \phi(\sigma)))$$

- iv) the augmentation

$$\begin{aligned} \mathbb{C}_X &\rightarrow \Lambda_X^0 \\ c &\rightarrow (c, c, 0) \end{aligned}$$

makes Λ_X a resolution of \mathbb{C}_X ;

- v) there is a uniquely determined family $(X_l, h_l)_{l \in L}$ of smooth manifolds X_l and proper maps $h_l : X_l \rightarrow X$ such that

$$\Lambda_X^p = \bigoplus_l (h_l)_* \mathcal{E}_{X_l}^{p-q(l)}$$

where $q(l)$ is a nonnegative integer; moreover, there exist mappings $h_{lm} : X_l \rightarrow X_m$, commuting with h_l and h_m , such that the differential $\Lambda_X^p \rightarrow \Lambda_X^{p+1}$ is given by

$$d(\omega_l) = (d\omega_l + \sum_m \epsilon_{lm}^{(p)} h_{lm}^* \omega_m)$$

where $\epsilon_{lm}^{(p)}$ can take the values $0, \pm 1$.

The pullback $\phi : \Lambda_E \rightarrow \Lambda_{\tilde{E}}$ and $\psi : \mathcal{E}_{\tilde{X}} \rightarrow \Lambda_{\tilde{E}}$ in ii) are called *inner pullback* of the complex Λ_X .

The family $(X_l, h_l)_{l \in L}$ will be called the *hypercovering* of X associated to Λ_X , and $q(l)$ will be the *rank* of X_l .

To simplify the notations, in the sequel we will write $\Lambda_X = \bigoplus_l \mathcal{E}_{X_l}^{p-q(l)}$ instead of $\bigoplus_l (h_l)_* \mathcal{E}_{X_l}^{p-q(l)}$.

In order to explain the above definition, we consider the easy case where E and \tilde{E} are smooth, $\Lambda_E = \mathcal{E}_E$, $\Lambda_{\tilde{E}} = \mathcal{E}_{\tilde{E}}$ are the de Rham complexes, and ϕ and ψ are the usual pullback. Then the hypercovering reduces to $\{\tilde{X}, E, \tilde{E}\}$, of rank $(0, 0, 1)$

respectively. Let us prove that the complex $\mathcal{E}_{\tilde{X}} \oplus \mathcal{E}_E \oplus \mathcal{E}_{\tilde{E}}(-1)$ is a resolution of \mathbb{C}_X . Let $x \in X$, U a neighborhood of x in X , and $(\omega, \sigma, \theta) \in \Lambda_X^p(U)$, i.e. $\omega \in \mathcal{E}_{\tilde{X}}^p(\pi^{-1}(U))$, $\sigma \in \mathcal{E}_E^p(U \cap E)$, $\theta \in \mathcal{E}_{\tilde{E}}^{p-1}(q^{-1}(U \cap E))$, with $(d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega) - \phi(\sigma))) = (0, 0, 0)$. Then $d\sigma = 0$ implies by Poincaré lemma (on E) that $\sigma = d\sigma'$ (after possibly shrinking U). Then on $q^{-1}(U \cap E)$ we have

$$d(\theta - (-1)^p \phi(\sigma')) = -(-1)^p \psi(\omega)$$

which implies that ω gives a cohomology class in $H^p(p^{-1}(U), \mathbb{C})$ whose restriction to $H^p(p^{-1}(U \cap E), \mathbb{C})$ is zero; by proposition 3.1 we can write $\omega = d\omega'$ where $\omega' \in \mathcal{E}_{\tilde{X}}^{p-1}(\pi^{-1}(U))$ (again after possibly shrinking U); it follows

$$d[\theta + (-1)^p \psi(\omega') - (-1)^p \phi(\sigma')] = 0$$

thus $\theta + (-1)^p \psi(\omega') - (-1)^p \phi(\sigma')$ gives a cohomology class in $H^{p-1}(q^{-1}(U \cap E), \mathbb{C})$. Again by the proposition 3.1 we can write

$$\theta + (-1)^p \psi(\omega') - (-1)^p \phi(\sigma') = (-1)^{p-1} \psi(\omega'') + d\theta'$$

where $\omega'' \in \mathcal{E}_{\tilde{X}}^{p-1}(\pi^{-1}(U))$, $d\omega'' = 0$, and $\theta' \in \mathcal{E}_{\tilde{E}}^{p-2}(q^{-1}(U \cap E))$ (here we suppose $p \geq 2$; the case $p \leq 1$ needs minor modifications). As a consequence

$$(\omega, \sigma, \theta) = d(\omega' + \omega'', \sigma', \theta').$$

An easy adaptation of the above proof gives the following existence theorem for the complexes Λ_X , which essentially states that the property iv) in the definition 3.1 is in fact a consequence of i), ii), iii) and iv).

Theorem 3.1. (E)_n. *Let X be a complex space of dimension $\leq n$. Let $E \subset X$ be a nowhere dense closed subspace with $\text{Sing}(X) \subset E$, $j : E \rightarrow X$ the natural inclusion, and consider the proper modification (1). Let $\Lambda_E \in \mathcal{R}(E)$. There exists $\Lambda_{\tilde{E}} \in \mathcal{R}(\tilde{E})$, a pullback $\phi : \Lambda_E \rightarrow \Lambda_{\tilde{E}}$ (corresponding to q), a pullback $\psi : \mathcal{E}_{\tilde{X}} \rightarrow \Lambda_{\tilde{E}}$ (corresponding to i) with the following property: the complex (2) with differential given by (3) is a fine resolution of \mathbb{C}_X .*

It is also important to remark that in the definition 3.1 we allow the subspace E of X to be bigger than $\text{Sing}(X)$.

The definition of pullback is more delicate. In fact given a morphism $f : X \rightarrow Y$ there are no relationships between $f(\text{Sing}(X))$ and $\text{Sing}(Y)$, nor $\text{Sing}(X)$ and $f^{-1}(\text{Sing}(Y))$. If $\Lambda_Y = \mathcal{E}_Y \oplus \Lambda_F \oplus \Lambda_{\tilde{F}}(-1)$ is given, then by definition $\text{Sing}(Y) \subset F$, hence it seems natural (at least when X and Y are irreducible) to take $E = f^{-1}(F) \cup \text{Sing}(X)$ and try to construct Λ_X of the form $\Lambda_X = \mathcal{E}_{\tilde{X}} \oplus \Lambda_E \oplus \Lambda_{\tilde{E}}(-1)$ and a morphism $\Lambda_Y \rightarrow f_* \Lambda_X$. This is the most difficult point in our theory, so we give below a complete proof in the case where X and Y are irreducible. The following definition (primary pullback between irreducible spaces) is more restrictive than the general one, but it is sufficient for our current purposes.

Let $f : X \rightarrow Y$ be a morphism of irreducible complex spaces, $\dim X \leq m$, $\dim Y \leq n$, $\Lambda_X = \mathcal{E}_{\tilde{X}} \oplus \Lambda_E \oplus \Lambda_{\tilde{E}}^{p-1}$, $\Lambda_Y = \mathcal{E}_Y \oplus \Lambda_F \oplus \Lambda_{\tilde{F}}^{p-1}$, with $\Lambda_X \in \mathcal{R}(X)$ and $\Lambda_Y \in \mathcal{R}(Y)$. Let us consider the corresponding diagrams

$$\begin{array}{ccc} \tilde{E} & \rightarrow & \tilde{X} & & \tilde{F} & \rightarrow & \tilde{Y} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{j} & X & & F & \xrightarrow{k} & Y \end{array}$$

In order to define a primary pullback $\phi : \Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$ we proceed by double induction on (m, n) , i.e. we suppose that the definition of pullback is already known when the dimensions of the spaces are $\leq m$, $\leq n - 1$, or $\leq m - 1$, $\leq n$, respectively.

Definition 3.2. We say that $\phi : \Lambda_Y^\bullet \rightarrow \Lambda_X^\bullet$ is a *primary pullback* (corresponding to f) if it satisfies the following properties:

- P_0) ϕ is a morphism of complexes, i.e. it commutes with differentials.
 P_1) Let $(X_l, h_l)_{l \in L}$, $(Y_s, g_s)_{s \in S}$ the hypercoverings associated to Λ_X^\bullet , Λ_Y^\bullet , i.e.

$$\Lambda_X^p = \bigoplus_l \mathcal{E}_{X_l}^{p-q(l)} \quad , \quad \Lambda_Y^p = \bigoplus_s \mathcal{E}_{Y_s}^{p-q(s)}.$$

For every X_l there exist at most one Y_s , having the same rank q as X_l , and a commutative diagram

$$\begin{array}{ccc} X_l & \xrightarrow{f_{l_s}} & Y_s \\ h_l \downarrow & & g_s \downarrow \\ X & \rightarrow & Y \end{array}$$

such that the composition

$$\mathcal{E}_{X_l}^{p-q} \rightarrow \Lambda_X^p \xrightarrow{\phi} \Lambda_Y^p \rightarrow \mathcal{E}_{Y_s}^{p-q}$$

is either identically zero for every p , or coincides with the de Rham pullback $f_{l_s}^*$.

- P_2) Let $\alpha : \mathcal{E}_{\tilde{Y}}^\bullet \rightarrow \mathcal{E}_{\tilde{X}}^\bullet$ be induced by ϕ . Then $\alpha \equiv 0$ if and only if $f(X) \subset F$; moreover in this case ϕ is the composition $\Lambda_Y^\bullet \rightarrow \Lambda_F^\bullet \rightarrow \Lambda_X^\bullet$ where $\Lambda_Y^\bullet \rightarrow \Lambda_F^\bullet$ is the projection onto the summand Λ_F^\bullet and $\Lambda_F^\bullet \rightarrow \Lambda_X^\bullet$ is a pullback (inductively defined) corresponding to the induced morphism $X \rightarrow F$.
 P_3) If $\alpha : \mathcal{E}_{\tilde{Y}}^\bullet \rightarrow \mathcal{E}_{\tilde{X}}^\bullet$ is not identically zero, then according to P_2 $f(X) \not\subset F$; in that case we assume the following properties:
i) $f^{-1}(F) \subset E$;
ii) the morphism f extends to a morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ and $\alpha = \tilde{f}^*$ is the ordinary de Rham pullback;
iii) the morphism ϕ is given by

$$\begin{array}{ccccc} \mathcal{E}_{\tilde{Y}}^\bullet & \oplus & \Lambda_F^\bullet & \oplus & \Lambda_{\tilde{F}}^\bullet(-1) \\ \alpha \downarrow & \searrow \beta & \gamma \downarrow & \delta \swarrow & \varepsilon \downarrow \\ \mathcal{E}_{\tilde{X}}^\bullet & \oplus & \Lambda_E^\bullet & \oplus & \Lambda_{\tilde{E}}^\bullet(-1) \end{array}$$

where $(\beta, \gamma, \delta) : \Lambda_Y^\bullet \rightarrow \Lambda_E^\bullet$ is a pullback corresponding to the composition $f \circ j : E \rightarrow Y$ (inductively defined).

Proof of the existence of the primary pullback (the irreducible case). We suppose that $(EP)_{m-1, n}$, $(EP)_{m, n-1}$, $(EP)_{m-1, m}$, $(U)_{m-1, n}$, $(F)_m$, $(E)_q$, $q = \sup(m, n)$ are already proved.

Let $\Lambda_Y^p = \mathcal{E}_Y^p \oplus \Lambda_F^p \oplus \Lambda_{\tilde{F}}^{p-1}$ be defined by a proper modification

$$\begin{array}{ccc} \tilde{F} & \rightarrow & \tilde{Y} \\ \downarrow & & p \downarrow \\ F & \rightarrow & Y \end{array}$$

with inner pullback given by $u_1 : \mathcal{E}_{\tilde{F}} \rightarrow \Lambda_{\tilde{F}}^\bullet$ and $p_1 : \Lambda_F^\bullet \rightarrow \Lambda_{\tilde{F}}^\bullet$. If $f(X) \subset F$, by (EP) $_{m,n-1}$ there exists a pullback $\Lambda_F^\bullet \rightarrow \Lambda_X^\bullet$; hence we obtain the required pullback as composition $\Lambda_Y^\bullet \rightarrow \Lambda_F^\bullet \rightarrow \Lambda_X^\bullet$, where $\Lambda_Y^\bullet \rightarrow \Lambda_F^\bullet$ is the projection onto the summand.

From now on we suppose $f(X) \not\subset F$. Since X is irreducible, the subspace $E := f^{-1}(F) \cup \text{Sing}(X)$ is nowhere dense in X .

Lemma 3.1. *We can construct two commutative diagrams*

$$\begin{array}{ccccc} \tilde{E} & \rightarrow & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & h \downarrow & & p \downarrow \\ E & \rightarrow & X & \xrightarrow{f} & Y \end{array}$$

where h is a proper desingularization of X .

Proof of the lemma. The modification $f : \tilde{Y} \rightarrow Y$ is dominated by a blowing-up $Y_1 \rightarrow Y$ centered at an ideal \mathcal{I} of \mathcal{O}_Y supported on F ; we obtain a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & \tilde{Y} \\ u \downarrow & & p \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where u is the blowing-up of X centered at the ideal $\mathcal{I}\mathcal{O}_X \subset \mathcal{O}_X$. We consider the proper transform Q of $\text{Sing}(X)$ in X_1 and we take as \tilde{X} a desingularization of the blow-up of X_1 centered at Q . This proves the lemma.

Now, $\dim \tilde{F} < \dim \tilde{f}^{-1}(\tilde{F}) < m$, so that by (EP) $_{m-1,m}$ and (EP) $_{m-1,n-1}$ there exist two pullback $u_2 : \mathcal{E}_{\tilde{X}} \rightarrow \Lambda_{\tilde{f}^{-1}(\tilde{F})}^\bullet$ and $p_2 : \Lambda_{\tilde{F}}^\bullet \rightarrow \Lambda_{\tilde{f}^{-1}(\tilde{F})}^\bullet$ (here we need also (F) $_{m-1}$ in order to insure the same target $\Lambda_{\tilde{f}^{-1}(\tilde{F})}^\bullet$).

By (U) $_{m-1,n}$

$$p_2 \circ u_1 = u_2 \circ \tilde{f}^*.$$

From the trivial proper modification $id : (\tilde{X}, \tilde{f}^{-1}(\tilde{F})) \rightarrow (\tilde{X}, \tilde{f}^{-1}(\tilde{F}))$ we build the complex $\Lambda_{\tilde{F}}^\bullet \in \mathcal{R}(\tilde{X})$:

$$\Lambda_{\tilde{X}}^p = \mathcal{E}_{\tilde{X}}^p \oplus \Lambda_{\tilde{f}^{-1}(\tilde{F})}^p \oplus \Lambda_{\tilde{f}^{-1}(\tilde{F})}^{p-1}$$

with inner pullback given by $u_2 : \mathcal{E}_{\tilde{X}}^p \rightarrow \Lambda_{\tilde{f}^{-1}(\tilde{F})}^p$ and $id : \Lambda_{\tilde{F}}^p \rightarrow \Lambda_{\tilde{f}^{-1}(\tilde{F})}^p$. Let us define the morphism

$$\psi : \Lambda_Y^p = \mathcal{E}_Y^p \oplus \Lambda_F^p \oplus \Lambda_F^{p-1} \rightarrow \Lambda_{\tilde{X}}^p = \mathcal{E}_{\tilde{X}}^p \oplus \Lambda_{\tilde{f}^{-1}(\tilde{F})}^p \oplus \Lambda_{\tilde{f}^{-1}(\tilde{F})}^{p-1}$$

$$\psi(\omega, \sigma, \theta) = (\tilde{f}^*\omega, (p_2 \circ p_1)(\sigma), p_2(\theta)).$$

We check that ψ is a primary pullback corresponding to the morphism $f \circ h : \tilde{X} \rightarrow Y$; in fact it commutes with the differential because of $p_2 \circ u_1 = u_2 \circ \tilde{f}^*$; the condition P_1 on the hypercoverings of \tilde{X} and Y is clearly satisfied; the condition $(f \circ h)^{-1}(F) \subset \tilde{f}^{-1}(\tilde{F})$ is true because of $f \circ h = p \circ \tilde{f}$; $f \circ h$ extends trivially to $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$; finally the composition $\Lambda_Y^p \rightarrow \Lambda_{\tilde{X}}^p \rightarrow \Lambda_{\tilde{f}^{-1}(\tilde{F})}^p$ coincides with $p_2 \circ p_1$, therefore is a pullback.

Next we define

$$\begin{aligned} t : \mathcal{E}_{\tilde{X}}^p &\rightarrow \Lambda_{\tilde{X}}^p = \mathcal{E}_{\tilde{X}}^p \oplus \Lambda_{\tilde{f}^{-1}(\tilde{F})}^p \oplus \Lambda_{\tilde{f}^{-1}(\tilde{F})}^{p-1} \\ t(\rho) &= (\rho, u_2(\rho), 0) \end{aligned}$$

which is also a pullback corresponding to the identity $id : \tilde{X} \rightarrow \tilde{X}$ (left to the reader). Now we use induction: by $(EP)_{m-1,n}$, there is a pullback $w : \Lambda_Y^\cdot \rightarrow \Lambda_E^\cdot$ corresponding to the composite morphism $E \rightarrow X \rightarrow Y$: again by $(EP)_{m-1,m}$ there is a pullback $z : \Lambda_{\tilde{X}}^\cdot \rightarrow \Lambda_{\tilde{E}}^\cdot$ corresponding to the embedding $\tilde{E} \rightarrow \tilde{X}$; by $(EP)_{m-1,m-1}$ there is a pullback $q_1 : \Lambda_E^\cdot \rightarrow \Lambda_{\tilde{E}}^{\cdot,1}$ corresponding to the morphism $\tilde{E} \rightarrow E$. Using $(F)_{m-1}$ we can suppose $\Lambda_{\tilde{E}}^{\cdot,1} = \Lambda_{\tilde{E}}^\cdot$. Because of $(U)_{m-1,n}$ the two pullback $z \circ \psi, q_1 \circ w : \Lambda_Y^\cdot \rightarrow \Lambda_{\tilde{E}}^\cdot$ coincide:

$$z \circ \psi = q_1 \circ w.$$

We define

$$\Lambda_X^p = \mathcal{E}_{\tilde{X}}^p \oplus \Lambda_E^p \oplus \Lambda_{\tilde{E}}^{p-1}$$

where the inner pullback are defined by $z \circ t : \mathcal{E}_{\tilde{X}}^\cdot \rightarrow \Lambda_{\tilde{E}}^\cdot$ and $q_1 : \Lambda_E^\cdot \rightarrow \Lambda_{\tilde{E}}^\cdot$. Finally we define

$$\phi : \Lambda_Y^p = \mathcal{E}_Y^p \oplus \Lambda_F^p \oplus \Lambda_{\tilde{F}}^{p-1} \rightarrow \Lambda_X^p = \mathcal{E}_{\tilde{X}}^p \oplus \Lambda_E^p \oplus \Lambda_{\tilde{E}}^{p-1}$$

$$\phi(\omega, \sigma, \theta) = (\tilde{f}^* \omega, w(\omega, \sigma, \theta), z(0, p_2(\theta), 0)).$$

Here we notice that $p_2(\theta) \in \Lambda_{\tilde{F}^{-1}(\tilde{F})}^{p-1}$, $(0, p_2(\theta), 0) \in \Lambda_X^{p-1}$ so that $z(0, p_2(\theta), 0) \in \Lambda_{\tilde{E}}^{p-1}$. In order to prove that ϕ is a primary pullback, the only non trivial property is that it commutes with differentials. Let us check it.

$$d(\omega, \sigma, \theta) = (d\omega, d\sigma, d\theta + (-1)^p(u_1(\omega) - p_1(\sigma)))$$

$$\phi(d(\omega, \sigma, \theta)) = (\tilde{f}^* d\omega, w(d(\omega, \sigma, \theta)), z(0, p_2(d\theta + (-1)^p(u_1(\omega) - p_1(\sigma))), 0)).$$

On the other hand

$$d(\phi(\omega, \sigma, \theta)) = d(\tilde{f}^* \omega, w(\omega, \sigma, \theta), z(0, p_2(\theta), 0)) =$$

$$= (d\tilde{f}^* \omega, d(w(\omega, \sigma, \theta)), d(z(0, p_2(d\theta), 0)) + (-1)^p[(z \circ t)(\tilde{f}^* \omega) - q_1(w(\omega, \sigma, \theta))]).$$

Since $\tilde{f}^* d\omega = d\tilde{f}^* \omega$, $w(d(\omega, \sigma, \theta)) = d(w(\omega, \sigma, \theta))$, it remains to check the equality of the third components. We must be careful about signs: $d(z(0, p_2(d\theta), 0))$ in the above formula(s) is a differential on a $(p-1)$ -form.

Recalling that

$$\psi(\omega, \sigma, \theta) = (\tilde{f}^* \omega, (p_2 \circ p_1)(\sigma), p_2(\theta))$$

and

$$t(\tilde{f}^* \omega) = (\tilde{f}^* \omega, u_2(\tilde{f}^* \omega), 0) = (\tilde{f}^* \omega, (p_2 \circ u_1)(\omega), 0)$$

we obtain

$$\begin{aligned} d(z(0, p_2(d\theta), 0)) + (-1)^p[(z \circ t)(\tilde{f}^* \omega) - q_1(w(\omega, \sigma, \theta))] &= z(0, dp_2(\theta), (-1)^p p_2(\theta)) + \\ &+ (-1)^p[(z \circ t)(\tilde{f}^* \omega) - (z \circ \psi)(\omega, \sigma, \theta)] = \\ &= z\{(0, dp_2(\theta), (-1)^p p_2(\theta)) + (-1)^p[\tilde{f}^* \omega - \psi(\omega, \sigma, \theta)]\} = \\ &= z(0, dp_2(\theta) + (-1)^p[(p_2 \circ u_1)(\omega) - (p_2 \circ p_1)(\sigma)], 0) \end{aligned}$$

which gives the result.

4. THE HODGE-DELIGNE MIXED STRUCTURE

The complexes of differential forms defined above give a natural approach to the Deligne theory of mixed Hodge structure.

For a fixed $\Lambda_X \in \mathcal{R}(X)$ we define:

i) The (increasing) weight filtration

$$W_m(\Lambda_X^p) := W_m(\mathcal{E}_{\tilde{X}}^p) \oplus W_m(\Lambda_E^p) \oplus W_m(\Lambda_{\tilde{E}}^{p-1})$$

by induction on $\dim(X)$: $W_m(\Lambda_E^p)$ and $W_m(\Lambda_{\tilde{E}}^{p-1})$ are already defined because $\dim E < \dim X$, $\dim \tilde{E} < \dim X$, while $W_m(\mathcal{E}_{\tilde{X}}^p)$ is equal to $\mathcal{E}_{\tilde{X}}^p$ for $m \geq p$ and zero otherwise.

More constructively, if $(X_l, h_L)_{l \in L}$ is the hypercovering associated to Λ_X^p ,

$$W_m(\Lambda_X^p) = \bigoplus_{p-q(l) \leq m} \mathcal{E}_{X_l}^{p-q(l)}$$

ii) The (decreasing) Hodge filtration

$$F^r(\Lambda_X^p) := F^r(\mathcal{E}_{\tilde{X}}^p) \oplus F^r(\Lambda_E^p) \oplus F^r(\Lambda_{\tilde{E}}^{p-1})$$

again by induction (here $F^r(\mathcal{E}_{\tilde{X}}^p)$ is the usual Hodge filtration in the smooth case). Alternatively

$$F^r(\Lambda_X^p) = \bigoplus_l F^r(\mathcal{E}_{X_l}^{p-q(l)}).$$

It is not hard to see that $d(W_m(\Lambda_X^p)) \subset W_{m+1}(\Lambda_X^{p+1})$, while $d(F^r(\Lambda_X^p)) \subset F^r(\Lambda_X^p)$; moreover an admissible pullback preserves both the filtrations.

Theorem 4.1. *If X is a compact projective variety, for any $\Lambda_X \in \mathcal{R}(X)$ the weight and the Hodge filtrations induce the Hodge-Deligne mixed structure on the cohomology of X .*

Sketch of the proof. By induction on the dimension of X we already know that the conclusion is true for the cohomology of E and \tilde{E} ; moreover it is true for the cohomology of \tilde{X} , which is smooth. Then Λ_X coincides (up to shift) with the cone of the morphism of bifiltered complexes

$$(j \circ q)_* \Lambda_E^p \xrightarrow{(-1)^p(\psi - \phi)} \pi_* \mathcal{E}_{\tilde{X}}^p \oplus j_* \Lambda_E^p.$$

It follows by [10] that the weight and the Hodge filtrations induce a mixed Hodge structure on the cohomology groups $H^j(X, \mathbb{C})$; by the property (F) (filtering) it does not depend on the choice of the particular complex Λ_X and is functorial on X . This is enough to get the conclusion.

Remark. The above definition of the weight filtration must be suitably shifted in order to recover the more classical: $W'_q(\Lambda_X^p) = W_{q+p}(\Lambda_X^p)$.

5. LOGARITHMIC COMPLEXES

In this section we show how to modify the previous constructions in order to define complexes of forms with logarithmic poles at infinity, which describe the Deligne theory of mixed Hodge structure on an open algebraic variety [10].

By a pair (of complex spaces) (X, Q) we mean the data of a complex space X and of a closed, nowhere dense complex subspace Q . We denote the complement $X \setminus Q$ by X° . More generally, if U is an open subset of X we define $U^\circ = U \setminus Q$.

A morphism of pairs $f : (X, Q) \rightarrow (Y, R)$ is a morphism $f : X \rightarrow Y$ such that $f(Q) \subset R$ and $f(X^\circ) \subset Y^\circ$.

Let (X, Q) be a pair such that X is smooth and Q is a divisor with normal crossing. Let $\Omega_X^\bullet \langle \log Q \rangle$ be the complex of holomorphic forms with logarithmic poles along Q . We define the (smooth) logarithmic de Rham complex of the pair (X, Q) by

$$\mathcal{E}_X^p \langle \log Q \rangle = \sum_{k+q=p} \Omega_X^k \langle \log Q \rangle \otimes \mathcal{E}_X^{0,q}$$

equipped with the usual differential (the above tensor product is taken over \mathcal{O}_X). The sheaves $\mathcal{E}_X^p \langle \log Q \rangle$ are obviously fine.

A classical result (see [15] for a smooth divisor, and [14] for the general divisor with normal crossing) states the following

Proposition 5.1. *Let (X, Q) be a pair such that X is smooth and Q is a divisor with normal crossing; let $j : Q \rightarrow X$ be the embedding. The natural morphism of complexes on X*

$$\mathcal{E}_X^\bullet \langle \log Q \rangle \rightarrow j_* \mathcal{E}_{X^\circ}^\bullet$$

induces isomorphisms of the cohomology sheaves:

$$(4) \quad \mathcal{H}^p(\mathcal{E}_X^\bullet \langle \log Q \rangle) \rightarrow \mathcal{R}^p j_* \mathbb{C}_{X^\circ}$$

in particular for every open set U of X the natural morphisms

$$H^k(U, \mathcal{E}_X^\bullet \langle \log Q \rangle) \rightarrow H^k(U^\circ, \mathbb{C})$$

are isomorphisms.

For a pair (X, Q) we define a family of complexes of fine sheaves $\mathcal{R}(X \langle \log Q \rangle) = \{\Lambda_X^\bullet \langle \log Q \rangle\}$ and for every morphism $f : (X, Q) \rightarrow (Y, R)$ a family of morphisms of complexes between the $\Lambda_Y^\bullet \langle \log R \rangle \in \mathcal{R}(Y \langle \log R \rangle)$ and some of the $\Lambda_X^\bullet \langle \log Q \rangle \in \mathcal{R}(X \langle \log Q \rangle)$, more precisely $\Lambda_Y^\bullet \langle \log R \rangle \rightarrow f_* \Lambda_X^\bullet \langle \log Q \rangle$ which we simply denote $\Lambda_Y^\bullet \langle \log R \rangle \rightarrow \Lambda_X^\bullet \langle \log Q \rangle$ and call (*admissible pullback*) with the following properties.

- (I) The restriction $\Lambda_{X^\circ}^\bullet$ of $\Lambda_X^\bullet \langle \log Q \rangle$ to X° belongs to $\mathcal{R}(X^\circ)$ (as defined in the previous paragraphs), and the natural morphism of complexes on X

$$\Lambda_X^\bullet \langle \log Q \rangle \rightarrow j_* \Lambda_{X^\circ}^\bullet$$

induces isomorphisms in cohomology:

$$(5) \quad \mathcal{H}^p(\Lambda_X^\bullet \langle \log Q \rangle) \rightarrow \mathcal{R}^p j_* \mathbb{C}_{X^\circ}$$

- (II) For $p > 2 \dim X$, $\Lambda_X^p \langle \log Q \rangle = 0$.
 (III) If X is smooth and Q is a divisor with normal crossing the logarithmic de Rham complex $\mathcal{E}_X^\bullet \langle \log Q \rangle$ belongs to $\mathcal{R}(X \langle \log Q \rangle)$; for every morphism $f : (X, Q) \rightarrow (Y, R)$ with Y smooth and R a divisor with normal crossing the ordinary de Rham pullback $f^* : \mathcal{E}_Y^\bullet \langle \log R \rangle \rightarrow f_* \mathcal{E}_X^\bullet \langle \log Q \rangle$ is an admissible pullback.
 (IV) The families $\mathcal{R}(X \langle \log Q \rangle)$ and the families of pullback satisfy the properties of composition, existence, uniqueness and filtering. We leave to the reader the precise statements.

It is useful to put by definition $\Lambda_X^p \langle \log Q \rangle = 0$ at points $x \in X$ where $Q = X$ (in a neighborhood of x).

We sketch now the construction of the family $\mathcal{R}(X \langle \log Q \rangle)$ for any pair (X, Q) .

Step 1) Let X be smooth, and Q any subspace. Let

$$(6) \quad \begin{array}{ccc} \tilde{Q} & \xrightarrow{i} & \tilde{X} \\ q \downarrow & & \pi \downarrow \\ Q & \xrightarrow{j} & X \end{array}$$

be a proper modification, where \tilde{X} is smooth, \tilde{Q} is a divisor with normal crossing and π is an isomorphism outside \tilde{Q} ; we define

$$\mathcal{E}_X \langle \log Q \rangle = \pi_*(\mathcal{E}_{\tilde{X}} \langle \log \tilde{Q} \rangle)$$

which satisfies the isomorphism (4); the above complex, which depends on the choice of the diagram (6), will be called a logarithmic de Rham complex of the pair (X, Q) ; the reader should keep in mind that this gives new complexes even when Q is a divisor with normal crossing.

Step 2) Let (X, Q) be any pair, $E \subset X$ a nowhere dense closed subspace, with $\text{Sing}(X) \subset E$, and consider the diagram (1). Let us define $\tilde{Q} = \pi^{-1}(Q)$, $M = E \cap Q$, $\tilde{M} = \tilde{E} \cap \tilde{Q}$. Let $\Lambda_E \langle \log M \rangle \in \mathcal{R}(E \langle \log M \rangle)$ (which exists by induction on $\dim(X)$). Then we can find $\Lambda_{\tilde{E}} \langle \log \tilde{M} \rangle \in \mathcal{R}(\tilde{E} \langle \log \tilde{M} \rangle)$, a complex $\mathcal{E}_{\tilde{X}} \langle \log \tilde{Q} \rangle$ as in step 1), a pullback $\phi : \Lambda_E \langle \log M \rangle \rightarrow \Lambda_{\tilde{E}} \langle \log \tilde{M} \rangle$ (corresponding to q), a pullback $\psi : \mathcal{E}_{\tilde{X}} \langle \log \tilde{Q} \rangle \rightarrow \Lambda_{\tilde{E}} \langle \log \tilde{M} \rangle$ (corresponding to i) with the following property: the complex

$$\Lambda_X \langle \log Q \rangle = \pi^* \mathcal{E}_{\tilde{X}} \langle \log \tilde{Q} \rangle \oplus j_* \Lambda_E \langle \log M \rangle \oplus (j \circ q)_* \Lambda_{\tilde{E}} \langle \log \tilde{M} \rangle (-1)$$

whose differential is by definition

$$d(\omega, \sigma, \theta) = (d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega) - \phi(\sigma)))$$

induces the isomorphism of cohomology sheaves (5).

From the construction it follows that for a given complex $\Lambda_X \langle \log Q \rangle$ there is a uniquely determined family $((X_l, Q_l), h_l)_{l \in L}$ of pairs (X_l, Q_l) (X_l smooth and Q_l a divisor with normal crossing in X_l) and proper maps of pairs $h_l : (X_l, Q_l) \rightarrow (X, Q)$ such that

$$\Lambda_X^p \langle \log Q \rangle = \oplus_l \mathcal{E}_{X_l}^{p-q(l)} \langle \log Q_l \rangle$$

where $q(l)$ is a nonnegative integer; moreover, there exist mappings $h_{lm} : (X_l, Q_l) \rightarrow (X_m, Q_m)$, commuting with h_l and h_m , such that the differential $\Lambda_X^p \langle \log Q \rangle \rightarrow \Lambda_X^{p+1} \langle \log Q \rangle$ is given by

$$d(\omega_l) = (d\omega_l + \sum_m \epsilon_{lm}^{(p)} h_{lm}^* \omega_m)$$

where $\epsilon_{lm}^{(p)}$ can take the values $0, \pm 1$.

The family $((X_l, Q_l), h_l)_{l \in L}$ will be called the *hypercovering* of (X, Q) associated to $\Lambda_X \langle \log Q \rangle$, and $q(l)$ will be the *rank* of (X_l, Q_l) .

Arguing as in the previous paragraph, the reader can understand how to describe the Hodge-Deligne theory, by means of logarithmic forms, also in the case of open algebraic varieties.

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