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Flat convergence of Jacobians and Γ-convergence for the Ginzburg-Landau functionals

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Abstract[1]. I try to offer a brief account of some of the main ideas in the Γ-convergence result for the Ginzburg-Landau functionals, proved in [3] by the author, G. Alberti (Pisa) and G. Orlandi (Verona). This paper is an expanded version of the material I used for a Seminar in Potenza.

1. Introduction

In this expository paper, I wish to give a somewhat informal account of the results and methods of a joint research with Giovanni Alberti (Pisa) and Giandomenico Orlandi (Verona), on the Γ-convergence of the Ginzburg-Landau functionals.

For a more complete description of the results with full proofs, the interested readers are referred to our paper [3]: the aim of the present survey is rather to emphasize some of the main ideas, avoiding too much technical detail.

The problem we want to study is the following. Consider the Ginzburg-Landau functionals

$$F_\varepsilon(u) = \int_\Omega \big(\frac{1}{k}|Du|^k + \frac{1}{\varepsilon^2}W(u)\big)\, dx,$$

where Ω is a bounded domain in $\mathbb{R}^{n+k}$ $(n \geq 0,\ k \geq 2)$, $W : \mathbb{R}^k \to \mathbb{R}$ is a non-negative continuous function which vanish only on the unit sphere S^{k-1}, ε is a positive parameter and $u : \Omega \to \mathbb{R}^k$ is a sufficiently regular function (typically, in the Sobolev space $W^{1,k}$).

This functional is a close relative of some slightly more complex energies, which are used in theoretical physics to model vortices in superfluidity and superconductivity: for this reason, it has been widely studied in recent years, starting from the pioneering work by F. Bethuel, H. Brezis and F. Hélein [5]. For the physical background, see for instance [13], [7].

In particular, we are interested in the asymptotic behavior of the minimizers $\tilde{u}_\varepsilon$ of $F_\varepsilon(\cdot)$ (subject to a suitable boundary condition[2]), when the parameter ε goes to 0.

Clearly, when ε is very small the minimizers $\tilde{u}_\varepsilon$ will prefer to take values on the unit sphere S^{k-1} in a large portion of Ω, because this way the second term $\frac{1}{\varepsilon}W(\tilde{u}_\varepsilon)$ of the energy density vanishes.

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[2]Typically $u = g$ on $\partial\Omega$, with $g \in W^{1-1/k,k}(\partial\Omega, S^{k-1})$.

Suppose now that $\tilde{u}_\varepsilon \to \tilde{u}_0$ when $\varepsilon \to 0$ (we will show that this is indeed true in a suitably weak topology): we may well expect the limit function $\tilde{u}_0$ to take values on S^{k-1} almost everywhere in Ω. Now, the map $\tilde{u}_0 : \Omega \to S^{k-1}$ will satisfy the boundary datum g, and due to the non-trivial topology of the sphere we will often find that $\tilde{u}_0$ must have a nonempty singular set inside Ω.[3]

In particular, we easily see that if $\tilde{u}_0$ is "as smooth as possible", in a generic situation the singular set will be supported by a *manifold of codimension* k. Indeed, the singular set of $\tilde{u}_0$ is strongly related to a concentration phenomenon of the Ginzburg-Landau energies, and we will see that it is located on a *minimal surface of codimension* k which solves a Plateau-like problem (with a boundary datum depending on the "topology" of g).

As I noticed above, the mathematical research on the Ginzburg-Landau functional has a long history. The case $n = 0$, $k = 2$ was studied by F. Bethuel, H. Brezis and F. Hélein in [5], where a complete description of the asymptotic behavior of minimizers of $F_\varepsilon(\cdot)$ is provided. Some of these results on the concentration of the Ginzburg-Landau energies on singular points have been extended to $n = 0$ and general k in [10], [12]. In the case $n = 1$, a concentration result on line vortices was obtained by T. Rivière [18] for $k = 2$, and later by E. Sandier [20] for arbitrary k. Finally, concentration on minimal surface of arbitrary dimension n and codimension $k = 2$ was proved by T. Rivière and F.H.Lin in [17].

In all of the above papers, the asymptotic behavior of minimizers of $F_\varepsilon(\cdot)$ is derived through a careful exploitation of the properties of the solutions to the Euler-Lagrange equations of the functionals themselves: the aim of our research was to give an alternative proof by using De Giorgi's Γ-convergence as a basic tool. This alternative way of looking at the problem provides a good description of the asymptotic behavior of the minimizers of $F_\varepsilon(\cdot)$ in general dimension n and codimension k.

I briefly recall the definition of Γ-convergence: if X is a sufficiently nice topological space and $F_j : X \to \mathbb{R} \cup \{+\infty\}$ is a sequence of functions, we say that F_j Γ-*converges* to $F : X \to \mathbb{R} \cup \{+\infty\}$ if the following two conditions are satisfied:

(Γ.1) $\liminf_{j\to+\infty} F_j(u_j) \geq F(u)$ whenever $u \in X$ and $u_j \to u$ in X;

(Γ.2) for every $u \in X$ there exists an "optimal sequence" $v_j \to u$ in X such that $\lim_{j\to+\infty} F_j(v_j) = F(u)$.

Γ-convergence is a *variational convergence* in the following sense: suppose that for each j we have a minimizer $\tilde{u}_j$ of F_j, and that $\tilde{u}_j \to \tilde{u}$ in X for $j \to +\infty$. It is very easy to check that then $\tilde{u}$ is a minimizer of the Γ-limit F, and that we have convergence of the minimum values.

In other words, Γ-convergence ensures that a "limit of minimizers" is a always a minimizer of the Γ-limit.

We wish to apply this idea to the Ginzburg-Landau functionals $F_\varepsilon(\cdot)$ (of course, we will compute the Γ-limit for $\varepsilon \to 0$).

To achieve that, we must first rescale the functionals. Indeed, the convergence of the minimum energies to the minimum of the Γ-limit has an important implication:

[3] This may happen even if g itself is regular, because in many situations there are obstructions at extending g to a regular map $u_0 : \Omega \to S^{k-1}$. Take for instance $n = 0$, $k = 2$, Ω the unit ball in $\mathbb{R}^2$. If $g : \partial\Omega \to S^1$ is the identity map (and hence $deg(g) = 1$), it cannot be extended to a continuous map from Ω into S^1.

if we wish to recover some non-trivial information from Γ-convergence, we need to know that the minima of the functionals are equibounded. This is *not* the case for the Ginzburg-Landau functionals, because in fact an estimation of the minimum values shows that $F_\varepsilon(\tilde{u}_\varepsilon) = O(|\log\varepsilon|)$: we are thus forced to compute the Γ-limit of the *rescaled sequence of functionals* $F_\varepsilon(\cdot)/|\log\varepsilon|$.

We must also choose a suitable topology in our function space, because this is required by the definition of Γ-convergence. The above discussion shows that a natural choice is any topology (the stronger the better) in which we have *compactness of all sequences of functions* $\{v_\varepsilon\}$ *which satisfy the bound* $F_\varepsilon(v_\varepsilon) = O(|\log\varepsilon|)$: this would of course ensure compactness of minimizers.

Unfortunately, a sequence $\{v_\varepsilon\}$ satisfying the above estimate is not necessarily equibounded in any Sobolev norm (because the L^k norm of the gradient is allowed to diverge like $|\log\varepsilon|^{1/k}$), and we must look for a still weaker topology.

The key idea is that the energy density of the maps v_ε is closely related with their *Jacobians* Jv_ε, and that when $\varepsilon \to 0$ these Jacobians concentrate on a singular measure supported by a *n-dimensional oriented surface without boundary.*

If $v = (v_1, \ldots, v_k) \in W^{1,k}(\Omega, \mathbb{R}^k)$, we recall that the jacobian of v is the k-form $dv_1 \wedge \ldots \wedge dv_k$, whose components are simply the $k \times k$ minors of the matrix Dv. Denoting by $\star : \bigwedge^k(\mathbb{R}^{n+k}) \to \bigwedge_n(\mathbb{R}^{n+k})$ the usual identification of k-covectors with n-vectors in $\mathbb{R}^{n+k}$, we can dually see the k-form Jv as the n-vector field $\star Jv$: this way, the jacobian has a natural structure of n-current and can be regarded as a sort of *diffuse n-dimensional surface*[4].

The topology we will adopt in our main Γ-convergence result is precisely the *convergence of the Jacobians* $\star Jv$ *in the flat norm* $\mathbf{F}_\Omega$[5].

We are now in position to state our main result. For the sake of simplicity, in this first formulation we will not include the boundary condition g: this would cause some technical problems, and I prefer to delay the discussion to the last Section of this paper. Of course, the boundary condition is necessary to apply the Gamma-convergence result to non-trivial variational problem.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^{n+k}$ with $n \geq 0$ and $k \geq 2$. Then the following statements hold*

(i) Compactness and lower-bound inequality
Given a (countable) sequence of maps $\{u_\varepsilon\}$ such that $F_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$ for some constant C, we can extract a subsequence such that the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_\Omega$ to $\alpha_k M$, where M is an n-dimensional integral boundary in Ω and α_k is the measure of the unit ball in $\mathbb{R}^k$. For every such subsequence we have the estimate

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} F_\varepsilon(u_\varepsilon) \geq \beta_k \|M\|,$$

where $\beta_k := (k-1)^{k/2}\alpha_k$, and $\|M\|$ denotes the mass of M.

(ii) Upper bound inequality
For every n-dimensional integral boundary M in Ω there exist maps u_ε such

[4]Moreover, as the jacobian Jv is always an exact form, it turns out that $\star Jv$ is indeed the boundary of a $(n+1)$-dimensional current. In particular, $\partial(\star Jv) = 0$.

[5]See the next Section for a definition of this convergence.

that $\mathbf{F}_\Omega(\star Ju_\varepsilon - \alpha_k M) \to 0$ *and*

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} F_\varepsilon(u_\varepsilon) = \beta_k \|M\| \ .$$

Remark 1.1. In the case $k = 2$ and for general n, an independent proof of the compactness result and of the lower-bound inequality was provided by R.L. Jerrard and H.M. Soner [16]. They also obtained the upper bound inequality in the case $k = 2$ and $n = 0$ (in this case, the singular manifold M is a finite set of points).

Remark 1.2. The above theorem can be rephrased as a genuine Γ-convergence result in the following way.

As ambient space we choose $X = W^{1,k-1}(\Omega, \mathbb{R}^k)$ (this is the summability we need in order to give a distributional definition of the Jacobian, see for instance [15]), and the functionals $F_\varepsilon(\cdot)$ are extended to the whole of X by putting $F_\varepsilon(u) = +\infty$ whenever $u \in W^{1,k-1}(\Omega, \mathbb{R}^k) \setminus W^{1,k}(\Omega, \mathbb{R}^k)$.

The limit functional is defined as

$$F_0(u) = \begin{cases} \beta_k \| \star Ju\| & \text{if } u \in W^{1,k-1}(\Omega, S^{k-1}) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

We put on X the topology corresponding to the flat convergence of the *Jacobians* of the maps[6]. In this situation, our main theorem tells us that the functionals $F_\varepsilon(\cdot)/|\log\varepsilon|$ Γ-converge to F_0 as $\varepsilon \to 0$[7]. Moreover, any sequence of maps with energies bounded by $C|\log\varepsilon|$ is compact in the topology considered.

2. Sketch of the proof of the compactness and Γ-convergence theorem

As promised, I will try to give here an idea the proof of Theorem 1.1, but I wish to mention a very nice exposition [1] G. Alberti wrote some time ago on a preliminary stage of our research (the main result was subsequently improved, together with some of the proofs). For this reason, I will touch only cursorily some of the points Giovanni discussed at length in his work.

To begin with, we need to understand the "flat norm" in the statement of Theorem 1.1. Let T be a n-dimensional *boundary* in an open set $A \subset \mathbb{R}^{n+k}$: we define

$$\mathbf{F}_\Omega(T) = \inf\{\|S\|_\Omega : S \text{ is a } (n+1) - \text{dimensional current in } A \text{ and } T = \partial S\},$$

while we put $\mathbf{F}_\Omega(T) = +\infty$ if T is not a boundary. Thus, we will say that $T_i \to T$ in the flat norm if $\mathbf{F}_\Omega(T_i - T) \to 0$ (and, in particular, this will imply that T_i and T have the same boundary). Notice that our definition of the flat norm is strictly related with the flat seminorms used in [9], but is not exactly the same. The induced convergence is rather weak, because we may have flat convergence even if the masses of the currents T_i are not equibounded: this is a key point in the application to the Ginzburg-Landau functionals.

Proof of compactness in Theorem 1.1. An important point in proving compactness, is a beautiful and very sharp estimate by R.L. Jerrard [14], which basically

[6] In particular, this topology identifies any two maps having the same jacobian. In this sense, we are dealing with a quotient space of X.

[7] Indeed, by Theorem 5.6 of [ABO1], any rectifiable boundary M in Ω is the jacobian of a map in the space X (divided by α_k).

tells us that the cost of making a *point singularity* (in terms of Ginzburg-Landau energy) in the case $n = 0$, is at least $\beta_k |\log\varepsilon|$.

The estimate (in the form we need it, compare Lemma 3.9 in [3]), can be stated informally as follows:

Let Q be a $k-$dimensional cube in $\mathbb{R}^k$, with edge $\ell(\varepsilon) = 1/|\log\varepsilon|^2$ (the cube should be "big enough" with respect to ε). If $u \in W^{1,k}(Q, \mathbb{R}^k)$ has $|u| > 1/2$ on $\partial\Omega$, and the degree of the map $v(x) = u(x)/|u(x)| : \partial Q \to S^{k-1}$ is $d \in \mathbb{Z}$, then

$$\int_Q \big(\frac{1}{k}|Du|^k + \frac{1}{\varepsilon^2}W(u)\big)\, dx \geq \beta_k\ |d|\ |\log\varepsilon| + additional\ stuff. \tag{1}$$

Unfortunately, the "additional stuff" in the inequality is not too easy to handle, and (1) is only true for $\varepsilon < \varepsilon_0$, where ε_0 depends *on the degree* d: this generates some technical difficulties in the proof, without changing the essential nature of the estimate.

Remark 2.1. Estimate (1) shows that the Ginzburg-Landau energy needed to build a singularity grows like $|\log\varepsilon|$ as $\varepsilon \to 0$.

Unfortunately, with so much energy at our disposal, beside creating a singularity we can also build a sequence of wildly oscillating maps, thus destroying any hope of a *pointwise* compactness for a sequence of maps $\{u_\varepsilon\}$ with $F_\varepsilon(u_\varepsilon) = O(|\log\varepsilon|)$. This fact was first noticed in [8], and it is the reason we are looking for some weak compactness of the *Jacobians* and not of the maps[8].

The situation would be different for *minimizers* of the Ginzburg-Landau functionals: in case $k = 2$, compactness in Sobolev spaces has been proved in [5] for $n = 0$, and in [4] for general n.

Our idea to prove flat compactness of the Jacobians is fairly simple. Fix a sequence $\{u_\varepsilon\}$ of maps such that $F_\varepsilon(u_\varepsilon) = O(|\log\varepsilon|)$. For every ε, we cover Ω with a cubic grid $\mathcal{G}_\ell$ with edge $\ell(\varepsilon) = |\log\varepsilon|^{-2}$: the position of the grid is chosen in such a way that the following two conditions hold (at least for a countable subsequence of ε).

1. Let $R_k(\ell)$ be the k-skeleton of the grid $\mathcal{G}_\ell$ (i.e., the union of its k-dimensional faces). Then

$$\ell^n F_\varepsilon(u_\varepsilon; R_k(\ell)) \approx \begin{pmatrix} k \\ n+k \end{pmatrix} F_\varepsilon(u_\varepsilon), \tag{2}$$

   where $F_\varepsilon(u_\varepsilon; R_k(\ell))$ is the slice of the functional on $R_k(\ell)$ (namely, the integral of the Ginzburg-Landau energy density on the k-skeleton, with respect to the Haudorff measure $\mathcal{H}^k$).

2. If $R_{k-1}(\ell)$ denotes the $(k-1)$-skeleton of the grid, then we have

$$|u_\varepsilon| \to 1 \text{ uniformly on } R_{k-1}(\ell) \text{ as } \varepsilon \to 0. \tag{3}$$

We can easily obtain (2) by means of a Fubini argument: the binomial coefficient is needed because we have to slice the energy in the direction of every k-plane

[8]Our quest is further complicated by the fact that, in general, the Jacobians are not equibounded, not even in L^1

belonging to the grid. If we denote by $\tilde{R}_k(\ell)$ the union of only the k faces of $\mathcal{G}_\ell$ which are parallel to the vector $e_{n+1}\wedge\ldots\wedge e_{n+k}$, we would have a sharper estimate:

$$\ell^n F_\varepsilon(u_\varepsilon;\tilde{R}_k(\ell)) \approx F_\varepsilon(u_\varepsilon). \tag{4}$$

The idea to obtain (3) is explained in [3], Lemma 3.4 and also in [1].

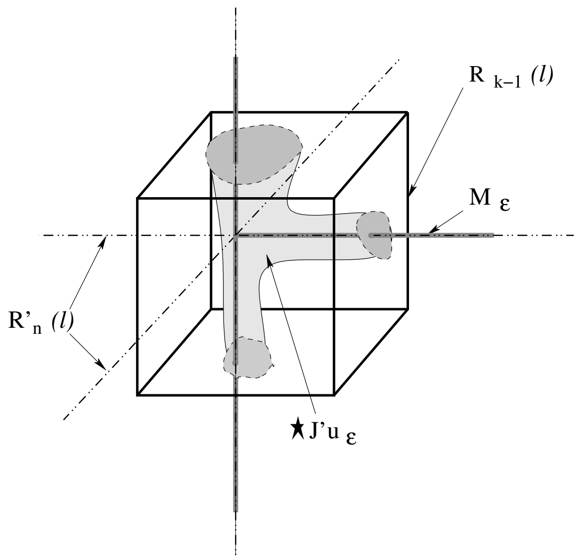
Now, we replace the sequence of the Jacobians $\star Ju_\varepsilon$ with a sequence of *slightly modified Jacobians* $\star J'u_\varepsilon$, defined in [3], Section 3.5. The modified Jacobians are asintotically flat-close to the original ones (in the sense that $\mathbf{F}_\Omega(\star Ju_\varepsilon - \star J'u_\varepsilon)\to 0$ as $\varepsilon\to 0$), and so we only have to prove compactness for the new sequence. The currents $\star J'u_\varepsilon$ have indeed some advantages: first of all, as a consequence of (3) they are supported away from $R_{k-1}(\ell)$. Moreover, if P is a k-dimensional face of $\mathcal{G}_\ell$, we have

$$\int_P J'u_\varepsilon = \alpha_k\cdot \deg(u_\varepsilon;\partial P), \tag{5}$$

where $\deg(u_\varepsilon;\partial P)$ is the degree of the map $u_\varepsilon(\cdot)/|u_\varepsilon(\cdot)| : \partial P \to S^{k-1}$.

We now use a modified version ([3], Lemma 3.7) of Federer's deformation theorem ([9], 4.2.9; see also Section 29 of [21]), to project the currents $\star J'u_\varepsilon$ on the *n-skeleton* $R'_n(\ell)$ of the dual grid to $\mathcal{G}_\ell$. We thus obtain a sequence of real polyhedral chains M_ε, which are asyntotically flat-close to $\star J'u_\varepsilon$.

The following picture shows what happens in a cube of the grid $\mathcal{G}_\ell$ (in this case we have $n=1$ and $k=2$):



Moreover, by (5) together with (1), the multiplicity of M_ε on each face Q of $R'_n(\ell)$ is in $\alpha_k\mathbb{Z}$, and it is controlled by the rescaled Ginzburg-Landau energy on the k-face P of $R_k(\ell)$ which is dual to Q. This remark, together with (2), tells us that the currents M_ε have equibounded masses (and are boundaries), and this ensures that there is a subsequence converging in the flat norm to $\alpha_k M$, where M is an integral boundary.

Proof of the lower-bound inequality in Theorem 1.1. If we look carefully at the steps in the proof of compactness, we see that we have really proved the following estimate for the masses of the projected Jacobians M_ε:

$$\frac{\beta_k}{\alpha_k}\|M_\varepsilon\| \approx \left(\begin{array}{c} k \\ n+k \end{array}\right) \frac{1}{|\log\varepsilon|} F_\varepsilon(u_\varepsilon). \tag{6}$$

Recalling the lower semicontinuity of mass with respect to flat convergence, this gives almost exactly the lower-bound inequality, were it not for the unfortunate factor $\left(\begin{array}{c} k \\ n+k \end{array}\right)$ in front of the right-hand side.

But keeping in mind what we observed in (4), estimate (6) becomes

$$\frac{\beta_k}{\alpha_k}\|M_\varepsilon \cdot (e_1 \wedge \ldots \wedge e_n)\| \approx \frac{1}{|\log\varepsilon|} F_\varepsilon(u_\varepsilon),$$

where $M_\varepsilon \cdot (e_1 \wedge \ldots \wedge e_n)$ denotes the component of the current M_ε in the direction of the plane spanned by $e_1, \ldots, e_n$.

The proof of the lower-bound inequality is then concluded by means of a localization argument: we cover most of Ω with small, disjoint open sets. Inside each of these open sets we repeat the above construction by choosing the direction of the coordinate vectors in such a way that the tangent planes to the limit current M are almost parallel to $(e_1 \wedge \ldots \wedge e_n)$.

Proof of the upper-bound inequality in Theorem 1.1. Notice first that it is enough to construct the optimal sequence $\{u_\varepsilon\}$ when the current M is an integral polyhedral boundary in Ω with multiplicity 1: indeed, every integral boundary can be approximated in the flat norm by currents of this kind, with convergence of the masses.

Another simple, but important observation is the following: in case $n = 0$, if the 0-dimensional current M is supported by a single point x_0 (with multiplicity 1), the optimal sequence is easily constructed as follows:

$$u_\varepsilon(x) = \begin{cases} \dfrac{x - x_0}{|x - x_0|} & if\ |x - x_0| \geq \varepsilon \\[2ex] \dfrac{x - x_0}{\varepsilon} & if\ |x - x_0| < \varepsilon \end{cases}$$

Let now M be an integral polyhedral boundary in Ω, with multiplicity 1 on every face (with a slight abuse of notation, we also denote by M the supporting polyhedron). By the above remark on the case $n = 0$, the upper-bound inequality is easily proved if we are able to construct a map $u : \Omega \to S^{k-1}$ such that

- $\star Ju = \alpha_k M$;
- for "most" of the points $x_0 \in M$, if we look at the restriction of u to the normal space of M at x_0 (a k-dimensional plane), this restriction behaves like $(x - x_0)/|x - x_0|$ in a neighborhood of x_0;
- u is globally of class $W^{1,k-1}$, it is smooth except on $M \cup S$ (where S is a $(n-1)$-dimensional polyhedral set), and

$$|Du(x)| \leq C/dist(x, M \cup S) \quad \forall x \in \Omega.$$

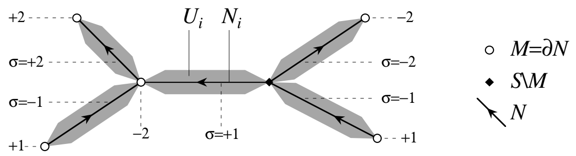
Indeed, if we have a map like that, we define

$$u_\varepsilon(x) = g_\varepsilon(x)u(x) \qquad \text{where } g_\varepsilon(x) = \min\{1, \operatorname{dist}(x, M \cup S)/\varepsilon\},$$

and some easy computations show that this sequence is indeed as required in the upper-bound inequality.

The construction of a map u with the above properties was realized in our paper [2], Theorem 5.10: the sketch of the construction can be summarized as follows.

Let N be a $(n+1)$-dimensional polyhedral current such that $\partial N = M$, and let P be a face of N (with the same multiplicity and orientation). By means of the so-called *dipole construction* (see [6] or [2], Proposition 5.2), it is not difficult to construct a map $u : \mathbb{R}^{n+k} \to S^{k-1}$ which is smooth outside ∂P and has $\star Ju = \alpha_k \partial P$. Moreover, this map is constantly equal to the north pole of S^{k-1}, except on a closed set U_P containing P and contained in a neighborhood of P. This set U_P has a fairly simple shape: near the center of P, it looks like a tubular neighborhood of the face, but its thickness goes to 0 (in a way we can adjust) in approaching $\partial\Omega$. For this reason, it is possible to repeat the dipole construction around each polyhedral face of M without an overlapping of the sets U (except, of course, at the boundary of the faces): see the following picture.



Thus, the dipole construction gives a global map $\tilde{u} : \Omega \to S^{k-1}$ with $\star J\tilde{u} = \alpha_k M$. This is not yet the map we need, because it does not behave like $\frac{x-x_0}{|x-x_0|}$ around the points of M, and moreover it has a huge singular set besides M. Indeed, the map $\tilde{u}$ will be singular at each face of the n-skeleton of N, even if this face does not belong to M. But in this case the total jacobian at the face should be zero, and the singularity is topologically trivial.

Indeed, we have a procedure that allows us to remove all unnecessary singularity from the n-skeleton of N, by pushing them on the $(n-1)$-skeleton (the polyhedral set S mentioned above). This procedure has also the additional benefit of making the map u like $\frac{x-x_0}{|x-x_0|}$ at the points of M, thus concluding the construction: the picture below is an attempt of visualizing what happens in a low-dimensional case.

3. Application to variational problems

In this section, we want to state a version of Theorem 1.1 with built-in boundary condition, in order to apply the Γ-convergence result to non trivial variational problems.

Let $g : \partial\Omega \to S^{k-1}$ be a map in the trace space $W^{1-1/k,k}$. The limiting variational problem will be defined in terms of a *cobordism class of rectifiable currents*, depending on the boundary datum g. In case Ω satisfies a homological hypothesis, namely $H_n(\Omega, \mathbb{Z}) = 0$, this variational problem in cobordism classes will be interpreted as a more familiar Plateau problem.

We say that two rectifiable currents M, M' are *cobordant* in $\overline{\Omega}$ if $M - M' = \partial N$, with N a rectifiable current supported in $\overline{\Omega}$. Now, fix any smooth map $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ with trace g, and take any regular value y (with $|y| < 1$) for the map u. The level set $u^{-1}(y)$ is a n-dimensional manifold, and can be given the structure of a rectifiable current which we denote $\tilde{M}$. It can be shown that the cobordism class of $\tilde{M}$ in $\overline{\Omega}$ *does not depend on the choice of the extension* u, *nor of the regular value* y.

The main result is then the following:

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^{n+k}$, $g : \partial\Omega \to S^{k-1}$ be a map in the trace space $W^{1-1/k,k}$, $\tilde{M}$ be a rectifiable current of dimension n, constructed from g as explained above.*

Then the following statements hold.

(i) Compactness and lower-bound inequality
Given a (countable) sequence of maps $\{u_\varepsilon\}$ such that $u_\varepsilon = g$ on $\partial\Omega$ and $F_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$ for some constant C, we can extract a subsequence such that the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$ to $\alpha_k M$, where M is an n-dimensional integral current in $\overline{\Omega}$ belonging to the cobordism class of $\tilde{M}$. For every such subsequence we have the estimate

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} F_\varepsilon(u_\varepsilon) \ge \beta_k \|M\|.$$

(ii) Upper bound inequality
For every n-dimensional integral current M, belonging to the cobordism class of $\tilde{M}$ in $\overline{\Omega}$, there exist maps u_ε such that $u_\varepsilon = g$ on $\partial\Omega$, $\mathbf{F}_{\mathbb{R}^{n+k}}(\star Ju_\varepsilon - \alpha_k M) \to 0$ and

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} F_\varepsilon(u_\varepsilon) = \beta_k \|M\| \ .$$

The ideas involved in the proof of Theorem 3.1 are basically those outlined in Section 2. Nevertheless, the proof of the upper bound inequality is much more difficult, due to the need of constructing maps u_ε satisfying the boundary condition g: for this reason, one is forced to use a subtle construction, and a larger arsenal of results from Geometric Measure Theory. The interested reader is again referred to [3].

Remark 3.1. As we anticipated above, the geometrical meaning of the cobordism class determined by g becomes a bit more familiar if we assume that $H_n(\Omega, \mathbb{Z}) = 0$. This means that any n-dimensional rectifiable cycle is a boundary: thus, any two currents in $\overline{\Omega}$ having the same boundary are cobordant. The cobordism condition for the currents M in the theorem becomes then $\partial M = \partial\tilde{M}$.

Now, for a map $g \in W^{1-1/k,k}(\partial\Omega, S^{k-1})$ it is possible to define the distributional jacobian $\star Jg$, which is a $(n-1)$-dimensional current: it is not too hard to check that $\partial\tilde{M} = \frac{1}{\alpha_k} \star Jg$ (compare [3], Section 5.2; see also [11] for the definition of the jacobian in trace spaces).

This has a fairly simple interpretation in case g is smooth outside a singular manifold of dimension $n-1$: the jacobian $\star Jg$ is then supported by the singular set (and the multiplicity of the current can be computed in terms of the degree of the map g around the singularity).

As a corollary, we obtain the following result on the asymptotic behavior of the constrained variational problems, and on the concentration of the energy densities:

Corollary 3.1. *Let* Ω*, g and* $\tilde{M}$ *be as in Theorem 3.1, and let* $\{\tilde{u}_\varepsilon\}$ *be a family of minimizers for the Ginzburg-Landau functionals* $F_\varepsilon(\cdot)$ *with boundary datum* g*. Then there is a countable subsequence of* $\tilde{u}_\varepsilon$ *(not relabelled) such that the currents* $\star J\tilde{u}_\varepsilon$ *converge in the flat norm in* $\mathbb{R}^{n+k}$ *to a current* $\alpha_k M$*. Moreover,* M *is rectifiable, and minimizes the mass among all currents in the cobordism class of* $\tilde{M}$ [9]*.*

Moreover, $F_\varepsilon(\tilde{u}_\varepsilon) = O(|\log\varepsilon|)$*, and the rescaled energy densities converge to* $\beta_k \mathcal{H}^n|_M$ *in the sense of measures.*

References

[1] G. Alberti, *Un risultato di convergenza variazionale per funzionali di tipo Ginzburg-Landau in dimensione qualunque*, Boll. Unione Mat. Ital. Sez. B, Art. Ric. Mat., (8)4(2001), 289-310.

[2] G. Alberti & S. Baldo & G. Orlandi, *Functions with prescribed singularities*, J. Eur. Math. Soc. (JEMS), 5(2003), 275-311.

[3] G. Alberti & S. Baldo & G. Orlandi, *Variational convergence for functionals of Ginzburg-Landau type*, Preprint 2003.

[4] F. Bethuel & J. Bourgain & H. Brezis & G. Orlandi, $W^{1,p}$ *estimates for solutions to the Ginzburg-Landau equation with boundary data in* $H^{1/2}$, C. R. Acad. Sci. Paris, Sér. I, Math., 333(2001), 1069-1076.

[5] F. Bethuel & H. Brezis & F. Hélein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, 13, Birkhäuser, Boston, 1994.

[6] H. Brezis & J.-M. Coron & E. Lieb, *Harmonic maps with defects*, Commun. Math. Phys., 107(1986), 649-705.

[7] F. Bethuel & T. Rivière, *Vortices for a variational problem related to superconductivity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 12(1995), 243-303.

[8] H. Brezis & P. Mironescu, *Sur une conjecture de E. De Giorgi relative a l'energie de Ginzburg-Landau*, C. R. Acad. Sci. Paris, Sér. I Math., 319(1994), 167-170.

[9] H. Federer, *Geometric measure theory*, Grundlehren der mathematischen Wissenschaften, 153, Springer, Berlin-New York, 1969. Reprinted in the series Classics in Mathematics, Springer, Berlin-Heidelberg, 1996.

[10] Z.-C. Han & Y.-Y. Li, *Degenerate elliptic systems and applications to Ginzburg-Landau type equations, I*, Calc. Var. Partial Differential Equations, 4(1996), 171-202. *Erratum*, ibid., 4(1996), 497.

[11] F.-B. Hang & F.-H. Lin, *A remark on the Jacobians*, Commun. Contemp. Math., 2(2000), 35-46.

[12] M.-C. Hong, *Asymptotic behavior for minimizers of a Ginzburg-Landau-type functional in higher dimensions associated with n-harmonic maps*, Adv. Differential Equations, 1(1996), 611-634.

[13] A. Jaffe & C. Taubes, *Vortices and monopoles. Structure of static gauge theories*, Progress in Physics, 2, Birkhäuser, Boston, 1980.

[14] R. L. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, Siam J. Math. Anal., 30(1999), 721-746.

[15] R. L. Jerrard & H. M. Soner, *Functions of bounded higher variation*, Indiana Univ. Math. J., 51(2002), 645-677.

[16] R. L. Jerrard & H. M. Soner, *The Jacobian and the Ginzburg-Landau energy*, Calc. Var. Partial Differential Equations, 14(2002), 151-191.

[17] F.-H. Lin & T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents*, J. Eur. Math. Soc., 1(1999), 237-311. *Erratum*, Ibid., 2(2000), 87-91.

[9] In particular, if $H_n(\Omega, n) = 0$, then M minimizes the mass among all currents with boundary $\star Jg$: M is a minimal surface solving a Plateau problem.

[18] T. Rivière, *Line vortices in the $U(1)$-Higgs model*, ESAIM Contrôle Optim. Calc. Var., 1(1995/96), 77-167. Electronic publication, http://www.emath.fr/cocv/ .

[19] E. Sandier, *Lower bounds for the energy of unit vector fields and applications*, J. Funct. Anal., 152(1997), 379-403. *Erratum*, ibid., 171(2000), 233.

[20] E. Sandier, *Ginzburg-Landau minimizers from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and minimal connections*, Indiana Univ. Math. J., 50(2001), 1807-1844.

[21] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis 3, Australian National University, Canberra, 1983.