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Commuting polarities and applications

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Abstract[1]. As far as we know the notion of commuting polarities was introduced by J. Tits in 1955 [Sur certaines classes d'espaces homogènes de groupes de Lie, Acadèmie royale de Belgique, Classes des sciences, Mèmoires, Coll in -8 - Tome XXIX]. Ten years later, B. Segre in his monumental paper [Forme e geometrie hermitiane con particolare riguardo al caso finito, Ann. Mat. Pura Appl. 70, 1-201] completely devoted to the geometry of Hermitian varieties, developed the theory of polarities permuting with a unitary polarity defined on a finite projective space.

After some basic preliminaries on polar geometry and Hermitian forms, we introduce the geometry of symplectic and orthogonal polarities commuting with a unitary polarity.

Some combinatorial and group-theoretic applications are given. In particular, some questions on complete spans of Hermitian varieties, maximal symplectic groups of finite unitary groups, and ovoids of the Hermitian surface, are addressed.

1. COMMUTING POLARITIES

In this section we give an introduction to symplectic and orthogonal polarities commuting with a unitary polarity.

1.1. **Polar Geometry.** Let V be a finite-dimensional vector space over a field K. The partially ordered set of *all* subspaces of V is the *projective geometry* $PG(V)$. If U and W are subspaces, we use $U+W$ or $\langle U, W\rangle$ to denote the subspace *spanned* by U and W. This is the smallest subspace of V that contains both U and W. Similarly, if $v_1, v_2, \ldots, v_r \in V$ we use $\langle v_1, v_2, \ldots, v_r\rangle$ to denote the subspace spanned by these vectors.

The one-dimensional subspaces of V are called *points*, the two-dimensional subspaces are called *lines*, and so on; if $\dim_K V = n$, the $(n-1)$-dimensional subspaces are called *hyperplanes*. In conformity with this notation the (projective) *dimension* of an element of $PG(V)$ is defined to be one less then the dimension of V. Thus points have dimension 0, lines have dimension 1, planes have dimension 2, and so on. If U and W are subspaces, Grassmann's relation

$$\dim(U+W) + \dim(U\cap W) = \dim U + \dim W$$

remains true when dim is interpreted as the projective dimension.

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A *collineation* of $PG(V)$ is an order-preserving bijection. A *correlation* of $PG(V)$ is a bijection from $PG(V)$ to $PG(V)$ which *reverses* inclusion. Thus a correlation sends points to hyperplanes and vice versa. Note that the composition of two correlations is a collineation. A correlation of order two is called *polarity*.

Denote by V^* the *dual* space of V that is, the vector space of linear functionals $\varphi : V \to K$.

If X is a subset, the *annihilator* of X is the subspace

$$X^\circ := \{\varphi \in V^* | \varphi(x) = 0 \text{ for all } x \in X\}.$$

If X is a subspace of V, restriction of linear functional to X defines a linear transformation $V^* \to X^* : \varphi \mapsto \varphi_{|X}$ with kernel X°. This map is surjective, hence $V^*/X^\circ \simeq X^*$ and therefore $\dim V^* - \dim X^\circ = \dim X^*$. It is known that $\dim V = \dim V^*$ and consequently we have the simple but important relation:

$$\dim X + \dim X^\circ = \dim V.$$

Further, also $X^{\circ\circ} = X$ holds.

The map from $PG(V)$ to $PG(V^*)$ which sends X to X° is a bijection which reverses inclusion. Therefore, if π is a correlation of $PG(V)$, the map

$$PG(V) \to PG(V^*) : X \mapsto \pi(X)^\circ$$

is a collineation. From the Fundamental Theorem of Projective Geometry there is a semilinear map $f : V \to V^*$ with associated automorphism $\sigma : K \to K$ such that $\pi(X)^\circ = f(X)$ for all $X \in PG(V)$. In other words,

$$\pi(X) := \{v \in V | f(x)v = 0 \text{ for all } x \in X\}.$$

If σ is an automorphism of K, a σ*-sesquilinear form* on V is a map $\beta : V \times V \to K$ such that

$$\beta(u_1 + u_2, v) = \beta(u_1, v) + \beta(u_2, v)$$
$$\beta(u, v_1 + v_2) = \beta(u, v_1) + \beta(u, v_2)$$

and

$$\beta(au, bv) = a\beta(u, v)\sigma(b),$$

for all $u, u_1, u_2, v_1, v_2 \in V$ and all $a, b \in K$. If $\sigma = 1$, the form is said to be *linear*.

A semilinear isomorphism $f : V \to V^*$ induces a σ-sesquilinear form defined by

$$\beta(u, v) := f(v)u.$$

Moreover β is *non-degenerate* in the sense that $\beta(u, v) = 0$ for all u implies $v = 0$, or equivalently, $\beta(u, v) = 0$ for all v implies $u = 0$. Conversely, a non-degenerate σ-sesquilinear form induces a semilinear map f from V to V^* defined by $f(v) := \beta(-, v)$.

A pair of vectors (u, v) such that $\beta(u, v) = 0$ is said to be *orthogonal*. For $X \in PG(V)$, the set

$$X^\perp := \{u \in V \mid \beta(u, v) = 0 \text{ for all } v \in X\}$$

is the *orthogonal complement* of X.

Again from the Fundamental Theorem of Projective Geometry, a σ-sesquilinear form β and a σ'-sesquilinear form β' induce the same correlation of $PG(V)$ if and only if, for some $b \in K$, we have $\beta'(u, v) = b\beta(u, v)$ and $\sigma'(a) = \sigma(a)$.

The correlation π of $PG(V)$ corresponding to f maps X to $X^\perp$. Further, π is a polarity if and only if $\pi = \pi^{-1}$.

Note that since $X^\perp = f(X)^\circ$ we have

$$\dim X + \dim X^\perp = \dim V,$$
$$X \subseteq Y \text{ implies } Y^\perp \subseteq X^\perp,$$
$$(X+Y)^\perp = X^\perp \cap Y^\perp \text{ and } (X \cap Y)^\perp = X^\perp + Y^\perp.$$

Two subspaces $U, W \in PG(V)$ are said to be *conjugate* if $U^\perp \subseteq W$ or, equivalently, $W^\perp \subseteq U$.

A correlation π with associated σ-sesquilinear form is a polarity if and only if $\pi = \pi^{-1}$. If the correlation π is a polarity then the pair $(PG(V), \pi)$ is called a *polar geometry.*

A sesquilinear form β such that $\beta(u,v) = 0$ implies $\beta(u,v) = 0$ for all $u, v \in V$ is said to be *reflexive.* Thus π is a polarity if and only if β is reflexive. And β is reflexive if and only if $X = X^{\perp\perp}$ for all $X \in PG(V)$.

Theorem 1.1. [18] *If* $\dim V \geq 3$ *and if* π *is a polarity of* $PG(V)$*, then* π *arises from a non-degenerate reflexive* σ*-sesquilinear form* β *of one of the following types:*

i) *Alternating. In this case* $\sigma = 1$ *and* $\beta(v,v) = 0$ *for all* $v \in V$.
ii) *Symmetric. In this case* $\sigma = 1$ *and* $\beta(u,v) = \beta(v,u)$ *for all* $u, v \in V$.
iii) *Hermitian. In this case* $\sigma^2 = 1$, $\sigma \neq 1$ *and* $\beta(u,v) = \sigma(\beta(v,u))$ *for all* $u, v \in V$.

The polar geometry $(PG(V), \pi)$ is known as a *symplectic*, *orthogonal* and *unitary* geometry according to whether *i)*, *ii)*, *iii)* holds.

When the characteristic of K is 2, the given definition of an orthogonal geometry as a polar geometry corresponding to a symmetric bilinear form is not quite general enough. It needs to rectify this as follows. A *quadratic form* on V is a function $Q : V \to K$ such that

$$Q(av) = a^2 Q(v)$$

and

$$\beta(u,v) := Q(u+v) - Q(u) - Q(v)$$

is a bilinear form. We say that β is the *polar form* of Q or that Q *polarizes* to β. Thus, one defines an *orthogonal geometry* to be a vector space V together with a quadratic form Q which is *non-degenerate* in the sense that its polar form β has the property that $\beta(u,v) = Q(u) = 0$ for all $v \in V$ implies $u = 0$. When the characteristic of K is not 2 this coincides with the first definition.

A non-zero vector u is *isotropic* if $\beta(u,u) = 0$. A subspace W is *totally isotropic* if $W \subseteq W^\perp$. A non-zero vector u is *singular* if $Q(u) = 0$ and a subspace W is *totally singular* if $Q(u) = 0$ for all $u \in W$.

A pair of vector (u,v) such that u and v are isotropic and $\beta(u,v) = 1$ is called a *hyperbolic pair*. The line $\langle u, v\rangle$ in $PG(V)$ is called a *hyperbolic line*.

A subspace W is *non-degenerate* if $W \cap W^\perp = \{0\}$. If $V = U \oplus W$ and $\beta(u,w) = 0$ for all $u \in U$ and $w \in W$, we write $V = U \perp W$ and say that V is the *orthogonal* direct sum of U and W.

The *radical* of a reflexive σ-sesquilinear form β is the subspace

$$\operatorname{rad} V := V^\perp = \{u \in V \mid \beta(u,v) = 0 \text{ for all } v \in V\}.$$

The form β is non-degenerate if and only if $\operatorname{rad} V = \{0\}$. If β is the polar form of a quadratic form Q, then Q is non-degenerate if and only if 0 is the only element of $\operatorname{rad} V$ which Q maps to 0.

Let β_1 and β_2 be reflexive sesquilinear forms on vector spaces V_1 and V_2 defined over K. A σ-semilinear map $f : V_1 \to V_2$ is called an *isometry* if it is one-to-one function and

$$\beta_2(f(u), f(v)) = \sigma(\beta_1(u, v))$$

for all $u, v \in V_1$.

If V_1 and V_2 are provided by quadratic forms Q_1 and Q_2, an isometry is defined to be a σ-semilinear map $f : V_1 \to V_2$ such that f is one-to-one and $Q_2(f(u)) = \sigma(Q_1(u))$ for all $u \in V_1$.

In all cases a *linear isometry* is an isometry whose associated field automorphism is the identity.

Theorem 1.2. (Witt's Theorem for non-degenerate sesquilinear forms) *Let β a non-degenerate form on V. For every $U \leq V$ and $f : U \to V$ an isometry there is an isometry $g : V \to V$ such that $g(u) = f(u)$ for all $u \in U$.*

Corollary 1.1. *Any two maximal totally isotropic subspaces of V have the same dimension.*

Proof. Suppose there exist two maximal totally isotropic subspaces W_1 and W_2 of V with $\dim W_1 < \dim W_2$. By Witt's Theorem for every linear map $f : W_1 \to W_2$ there is an isometry $g : V \to V$ such that $g(W_1) = f(W_1)$. Therefore W_1 is a proper subspace of $g^{-1}(W_2)$ which is also totally isotropic, a contradiction.

This common dimension is called the *Witt index* of the form β. If M is a totally isotropic subspace and β is non-degenerate, then $M \subseteq M^\perp$ and the Witt index of β is at most $\frac{1}{2}\dim V$.

Corollary 1.2. *If W_1 and W_2 are two totally isotropic subspaces of same dimension, then there is an isometry $g : V \to V$ such that $g(W_1) = W_2$.*

Suppose that β_1 and β_2 are non degnerate reflexive σ-sesquilinear forms on V. If π_1 and π_2 are the correlations of $PG(V)$ corresponding to β_1 and β_2, the product $\pi_1\pi_2$ provides a collineation of $PG(V)$. We say that π_1 and π_2 *commute* if $\pi_1\pi_2 = \pi_2\pi_1$.

1.2. **Hermitian forms.** Let $(PG(V),\mathcal{U})$ denote a unitary geometry where V is a vector space of dimension n over K and suppose that the unitary polarity $\mathcal{U}$ is induced by a σ-hermitian form β, where σ is the automorphism of K of order 2.

Let $e_0, e_1, \ldots, e_{n-1}$ be a basis for V and let $J := (\beta(e_i, e_j))$ be the matrix of β. It turns out that J is a hermitian matrix, i.e., $J^t = \overline{J}$. The coordinates of any isotropic vector satisfy the equation

$$X^t J \bar{X} = 0. \tag{1}$$

In $PG(V)$ the latter equation defines the set $\mathcal{H}$ of all totally isotropic points which is called *hermitian variety.*

For every vector $v = v_0e_0 + \ldots + v_{n-1}e_{n-1}$, the equation of $\langle v\rangle^\perp$ is $u_0x_0 + \ldots + u_{n-1}x_{n-1} = 0$ where $U = (u_0, \ldots, u_{n-1})^t$ is the vector $J\bar{v}$. Two hyperplanes with equations $U^tX = 0$ and $W^tX = 0$ are conjugate with respect to β if and only if

$$W^t(J^t)^{-1}\overline{U} = 0.$$

If v is an isotropic vector, then $v^\perp : U^tX = 0$ is a self-conjugate hyperplane, i.e.,

$$U^t(J^t)^{-1}\overline{U} = 0. \tag{2}$$

We can also consider bases of non-isotropic vectors. Given a non-isotropic vector $u_0 \neq 0$, we have $V = \langle u_1 \rangle \perp \langle u_1 \rangle^{\perp}$ and by induction we can write

$$V = \langle u_0 \rangle \perp \langle u_1 \rangle \perp \ldots \perp \langle u_{n-1} \rangle,$$

where the u_i's are non-isotropic. A set of n mutually orthogonal non-isotropic points is called an *orthogonal frame* of $PG(V)$. With respect to this basis, the matrix J is a diagonal matrix.

From now on we shall write $\bar{a}$ instead of $\sigma(a)$ and we let K_0 denote the fixed field of σ, i.e.,

$$K_0 := \{a \in K \mid a = \bar{a}\}.$$

Thus K_0 is a subfield of K and $\dim_{K_0} K = 2$.

The functions

$$Tr : \begin{array}{ccc} K & \longrightarrow & K_0 \\ a & \longmapsto & a + \bar{a} \end{array}$$

and

$$N : \begin{array}{ccc} K^* & \longrightarrow & K_0^* \\ a & \longmapsto & a\bar{a} \end{array}$$

are called the *trace* and the *norm* respectively.

Lemma 1.1. i) *Tr is a K_0-linear map onto K_0.*
ii) *$Tr(a) = 0$ if and only if $a = b - \bar{b}$ for some $b \in K$.*
iii) *N is a homomorphism.*
iv) *$N(a) = 1$ if and only if $a = b/\bar{b}$ for some $b \in K^*$.*
v) *If K is finite, then N is onto.*

Proof. i) Certainly Tr is K_0-linear and as Tr does not map every element of K to 0, it must map K onto K_0 since $\mathrm{im}(Tr)$ is a vector subspace contained in K_0. Thus $K_0 = \ker(\mathbf{1} - \sigma) = \mathrm{im}(\mathbf{1} + \sigma)$.

ii) We have $\mathrm{im}(\mathbf{1} - \sigma) \subseteq \ker(\mathbf{1} + \sigma)$, whence equality holds, both spaces being of K_0-dimension 1. In fact, $\ker(1+\sigma)$ is a hyperplane in K and $\mathrm{im}(1-\sigma) = \langle \theta - \bar{\theta} \rangle_{K_0}$ if we write $K = K_0[\theta]$.

iii) This is clear.

iv) if $a \neq -1$ and $N(a) = 1$, put $b = 1 + a$. If $a = -1$, choose b such that $Tr(b) = 0$.

v) If $|K_0| = q$, then $|K| = q^2$ and $\bar{a} = a^q$ for all $a \in K$. Then

$$\ker N = \{a \in K^* \mid a^{q+1} = 1\}.$$

The multiplicative group K_0^* of a finite field is cyclic and therefore $\ker N$ is a cyclic group of order $q + 1$. It follows that $\mathrm{im}\, N = K_0^*$.

Lemma 1.2. *If $\dim V \geq 2$ and the norm map is onto, then V contains isotropic vectors.*

Proof. Let $v \neq 0$ be a non-isotropic vector and set $b = \beta(v, v)$. Then $b = \bar{b}$ and for $a \in K$, $u \in \langle v \rangle^{\perp}$ we have

$$\beta(u + av, u + av) = \beta(u, u) + a\bar{a}b.$$

Since $-b^{-1}\beta(u, u) \in K_0$, it follows from Lemma 1.1 *v)* that there exists $a \in K$ such that $u + av$ is isotropic.

Corollary 1.3. *If $\dim V \geq 2$ and K is finite, then V contains isotropic vectors.*

Remark 1.1. In general, if V is a unitary geometry over an infinite field there is no guarantee that V will contain any isotropic vector. For example, let V be a vector space over the field $\mathbb{C}$ of complex numbers and take $\bar{a}$ to be the usual complex conjugate of a. Suppose that $e_0, e_1, \ldots, e_{n-1}$ is a basis of V and let β be the hermitian form such that $\beta(e_i, e_j) = \delta_{ij}$. It is easy to see that V does not have any isotropic vectors.

Lemma 1.3. *If $K_0 = GF(q)$ then V contains $(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})$ isotropic vectors.*

Proof. Let i_n be the number of isotropic points in $PG(V)$. Then the number of isotropic vectors in V is $(q^2 - 1)i_n$. We shall establish a recurrence relation for i_n. First observe that $i_1 = 0$ and, from the description of the isotropic points of a hyperbolic line given above, $i_2 = q + 1$.

Suppose that $n > 2$ and that P and Q are isotropic points such that $P + Q$ is non-degenerate. Since $(P + Q)^{\perp} = P^{\perp} \cap Q^{\perp}$, it follows that $(P + Q)^{\perp}$ is a hyperplane of $P^{\perp}$. Thus, every line of $P^{\perp}$ through P meets $(P + Q)^{\perp}$. For each isotropic point $R \in (P + Q)^{\perp}$ there are q^2 isotropic points on $P + R$ other then P. It follows that there are $1 + q^2 i_{n-2}$ isotropic points in $P^{\perp}$.

If $R \notin P^{\perp}$, then $P + R$ is a hyperbolic line which contains $q + 1$ isotropic points. There are $q^{2(n-2)}$ lines through P not in $P^{\perp}$ and hence q^{2n-3} isotropic points not in $P^{\perp}$. It follows that

$$i_n = q^2 i_{n-2} + q^{2n-3} + 1.$$

It is easy to check that the solution to this recurrence relation is

$$i_n = (q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})/(q^2 - 1).$$

1.3. **Commuting unitary and symplectic polarities.** Throughout this section $PG(n, q^2) := PG(V)$ will denote a projective space of dimension $n > 2$ over a finite field $GF(q^2)$.

Let $\mathcal{U}$ be a non-degenerate unitary polarity of $PG(V)$ with associated σ-sesquilinear (hermitian) form $\beta_{\mathcal{U}}$. For $X \in PG(V)$, $X^{\perp_{\mathcal{U}}}$ will denote the orthogonal complement of X with respect to $\mathcal{U}$.

Let $\mathcal{A}$ be a non-degenerate symplectic polarity commuting with $\mathcal{U}$ with associated sesquilinear (alternating) form $\beta_{\mathcal{A}}$. For $X \in PG(V)$, $X^{\perp_{\mathcal{A}}}$ will denote the orthogonal complement of X with respect to $\mathcal{A}$. Since non-degenerate simplectic polarities exist only if n is even, we have $n = 2r \geq 2$.

It is easily seen that the collineation $\mathcal{V} := \mathcal{A}\mathcal{U} = \mathcal{U}\mathcal{A}$ is a semilinear involution of $PG(V)$ with associated automorphism σ.

Proposition 1.1. *The collineation $\mathcal{V}$ fixes setwise $\mathcal{H}$ and acts fixed point free on the points of $PG(V)$ not in $\mathcal{H}$.*

Proof. Let P be an $\mathcal{U}$-isotropic point. Then $P \in P^{\perp_{\mathcal{U}}}$. If $Q = \mathcal{V}(P)$, then $Q^{\perp_{\mathcal{U}}} = \mathcal{U}(Q) = \mathcal{U}\mathcal{V}(P) = \mathcal{V}\mathcal{U}(P) = \mathcal{V}(P^{\perp_{\mathcal{U}}})$. It follows that also Q is $\mathcal{U}$-isotropic.

Let P be a fixed point under $\mathcal{V}$. Then $P^{\perp_{\mathcal{A}}} = \mathcal{A}(P) = \mathcal{U}\mathcal{V}(P) = \mathcal{U}(P) = P^{\perp_{\mathcal{U}}}$. As every point of $PG(V)$ is $\mathcal{A}$-isotropic, we have that P is a $\mathcal{U}$-isotropic point.

From now on, we will refer to fixed points under $\mathcal{V}$ as $\mathcal{V}$-fixed points.

Let e_0 be a non $\mathcal{U}$-isotropic vector. Then $a = \beta_{\mathcal{U}}(e_0, e_0) \neq 0$ and by Lemma 1.1 v) there exists $\alpha \in K$ such that $N(\alpha) = a^{-1}$. Replacing e_0 by αe_0 we may suppose $\beta_{\mathcal{U}}(e_0, e_0) = 1$.

Consider $P_0 = \langle e_0 \rangle$ and $\langle f_0 \rangle = Q_0 := \mathcal{V}(P_0)$ with $\beta_{\mathcal{U}}(f_0, f_0) = 1$. Then $Q_0^{\perp_{\mathcal{U}}} = \mathcal{U}(Q_0) = \mathcal{U}\mathcal{V}(P_0) = \mathcal{A}(P_0) = P_0^{\perp_{\mathcal{A}}}$. Since $P_0 \subset P_0^{\perp_{\mathcal{A}}}$, we get that (e_0, f_0) is an $\mathcal{U}$-orthogonal but a non $\mathcal{A}$-orthogonal pair. It is easy to see that $\langle e_0, f_0 \rangle^{\perp_{\mathcal{A}}} = \langle e_0, f_0 \rangle^{\perp_{\mathcal{U}}}$. Thus, we can simply write $\langle e_0, f_0 \rangle^{\perp}$. Further, the line $\langle e_0, f_0 \rangle$ is both non $\mathcal{A}$-degenerate and non $\mathcal{U}$-degenerate. Thus, we can write $V = \langle e_0, f_0 \rangle \perp \langle e_0, f_0 \rangle^{\perp}$.

We can continue this process until we find that V can be written as an orthogonal sum of orthogonal pairs (e_i, f_i); i.e.,

$$V = \langle e_0, f_0 \rangle \perp \langle e_1, f_1 \rangle \perp \ldots \perp \langle e_{r-1}, f_{r-1} \rangle.$$

We remark that $\{P_i = \langle e_i \rangle, Q_i = \langle f_i \rangle, i = 0, \ldots, r-1\}$ is an orthogonal frame of $PG(V)$.

With respect to the basis $e_0, f_0, \ldots, e_{r-1}, f_{r-1}$, the matrix $J_{\mathcal{U}}$ of $\beta_{\mathcal{U}}$ is the identity matrix I_{2r} and straightforward calculations show that the hermitian form is

$$\beta_{\mathcal{U}}\left(\sum_{i=0}^{r-1}(x_{2i}e_i + x_{2i+1}f_i), \sum_{i=0}^{r-1}(y_{2i}e_i + y_{2i+1}f_i))\right) =$$

$$\sum_{i=0}^{r-1}(x_{2i}\bar{y}_{2i} + x_{2i+1}\bar{y}_{2i+1}).$$

In particular, the hermitian variety $\mathcal{H}$ has equation

$$X_0\overline{X}_0 + X_1\overline{X}_1 + \ldots + X_{n-1}\overline{X}_{n-1} = 0. \tag{3}$$

The alternating form $\beta_{\mathcal{A}}$ is

$$\beta_{\mathcal{A}}\left(\sum_{i=0}^{r-1}(x_{2i}e_i + x_{2i+1}f_i), \sum_{i=0}^{r-1}(y_{2i}e_i + y_{2i+1}f_i))\right) =$$

$$\sum_{i=0}^{r-1} a_i(x_{2i}y_{2i+1} - x_{2i+1}y_{2i});$$

here $a_i = \beta_{\mathcal{A}}(e_i, f_i)$.

In $PG(V)$, the collineation $\mathcal{V}$ has equations

$$\mu\overline{X}' = AX$$

where A is the matrix

$$A = \begin{pmatrix} 0 & a_0 & & & \\ -a_0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{r-1} \\ & & & -a_{r-1} & 0 \end{pmatrix}$$

up to a non-zero scalar. Since $\mathcal{V}$ has order 2, we find that $A\bar{A} = \mu I_{2r}$, for some $\mu \in GF^*(q^2)$, i.e.,

$$a_0\bar{a}_0 = a_1\bar{a}_1 = \ldots = a_{r-1}\bar{a}_{r-1}.$$

A straightforward calculation shows that the homogeneous coordinate vector of $\mathcal{V}$-fixed points by $\mathcal{V}$ satisfy the following equations:

$$x_{2i+1} = \rho\bar{x}_{2i} \quad i = 0, 1, \ldots, r-1$$

where ρ is a $(q+1)$-th root of -1. Every such a root determines $(q^{n+1}-1)/(q^2-1)$ $\mathcal{V}$-fixed points. Hence, the number of $\mathcal{V}$-fixed points is $(q^{n+1}-1)/(q-1)$.

Proposition 1.2. *Let P, Q be two $\mathcal{V}$-fixed points. Then the line $P+Q$ contains exactly $q+1$ $\mathcal{V}$-fixed points. For each $\mathcal{V}$-fixed point P, let $\mathcal{L}_P$ the set of lines generated by P and another $\mathcal{V}$-fixed point. Then, $\mathcal{L}_P$ contains $(q^{n-1}-1)/(q-1)$ elements and $(q^{n-2}-1)/(q-1)$ of these lines are totally $\mathcal{U}$-isotropic. Each other line in $\mathcal{L}_P$ meets $\mathcal{H}$ in precisely $q+1$ $\mathcal{V}$-fixed points.*

Further, every $\mathcal{V}$-fixed line is generated by two $\mathcal{V}$-fixed points and the number of $\mathcal{V}$-fixed lines is $(q^n-1)(q^{n-1}-1)/(q^2-1)(q-1)$.

Proof. The collineation $\mathcal{V}$ induces on the line $P+Q$ a semilinear non-trivial involution with associated automorphism σ. Such an involution has $q+1$ $\mathcal{V}$-fixed points.

Suppose that P and Q are $\mathcal{V}$-fixed (isotropic) points such that $P+Q$ is $\mathcal{U}$-nondegenerate. For each $\mathcal{V}$-fixed point $R \in (P+Q)^{\perp}$, the totally $\mathcal{U}$-isotropic line $P+R$ meets $(P+Q)^{\perp}$ in exactly one point, it follows that $\mathcal{L}_P$ contains $(q^{n-2}-1)/(q-1)$ elements.

If $R \notin P^{\perp}$, then $P+R$ is a hyperbolic line which contains q isotropic $\mathcal{V}$-fixed points other then P.

A simple counting argument shows that the number of $\mathcal{V}$-fixed lines is $(q^n-1)(q^{n-1}-1)/(q^2-1)(q-1)$.

Proposition 1.3. *Let P be an isotropic $\mathcal{V}$-fixed point and set $Q := \mathcal{V}(P)$. Then, the line $P+Q$ is totally $\mathcal{U}$-isotropic and contains $q+1$ $\mathcal{V}$-fixed points.*

Proof. The line $P+Q$ is a $\mathcal{V}$-fixed line. From the proof of Proposition 1.2, $P+Q$ contains $q+1$ isotropic $\mathcal{V}$-fixed points. By way of contradiction, suppose that $P+Q$ is non totally $\mathcal{U}$-isotropic. As P and Q are isotropic, it follows that P and Q should be $\mathcal{V}$-fixed.

By Lemma 1.3 and Proposition 1.3, a simply counting argument shows that the number of totally $\mathcal{U}$-isotropic lines which are $\mathcal{V}$-fixed by the collineation $\mathcal{V}$ is $(q^n-1)(q^{n-2}-1)/(q^2-1)(q-1)$.

Now, let us look at the 3-dimensional geometry $PG(V)$ in somewhat more detailed way. The $\mathcal{V}$-fixed points are as many as the totally $\mathcal{U}$-isotropic $\mathcal{V}$-fixed lines. This number is q^3+q^2+q+1. There are $q^2(q^2+1)$ more non totally $\mathcal{U}$-isotropic $\mathcal{V}$-fixed l ines and we know that each of these lines contains exactly $q+1$ $\mathcal{V}$-fixed points.

For each $\mathcal{V}$-fixed point there are $q+1$ totally $\mathcal{U}$-isotropic lines and q^2 $\mathcal{V}$-fixed but non totally $\mathcal{U}$-isotropic lines. It follows from the Veblen-Young Axiom that the set of $\mathcal{V}$-fixed points and $\mathcal{V}$-fixed lines define a 3-dimensional projective geometry over $GF(q)$.

Finally, for every $n \geq 3$, the set of subspaces fixed by the collineation $\mathcal{V}$ provides a geometry isomorphic to the n-dimensional projective geometry over $GF(q)$.

1.4. **Commuting unitary and orthogonal polarities.** Let $(PG(n,q^2),\mathcal{U})$ be a unitary geometry of dimension $n-1$ over $GF(q^2)$, qq odd. Here the unitary polarity $\mathcal{U}$ is induced by the non-degenerate form $\beta_{\mathcal{U}}$. If $J_{\mathcal{U}}$ denotes the matrix of $\beta_{\mathcal{U}}$, we have seen that the equation of $\langle v\rangle^{\perp_{\mathcal{U}}}$ is $a_0X_0+\ldots+a_{n-1}X_{n-1}=0$ where $(a_0,\ldots,a_{n-1})^t = J_{\mathcal{U}}\bar{v}$.

Let $\mathcal{Q}$ be a correlation of $PG(V)$ with equation

$$U = KX \tag{4}$$

where K is a non-singular $n \times n$ matrix defined up to a non-zero scalar and suppose that $\mathcal{Q}$ commutes with $\mathcal{U}$, that is $\mathcal{UQ} = \mathcal{QU}$. Denote by $\langle w\rangle$ the image of $\langle v\rangle^{\perp_{\mathcal{U}}}$ under the correlation $\mathcal{Q}$. If we let v to be an $\mathcal{U}$-isotropic vector, then $\langle v\rangle^{\perp_{\mathcal{U}}}$ is a self-conjugate hyperplane and the coordinates of w satisfy the equation

$$X^t K^t (J_{\mathcal{U}}^t)^{-1} \overline{KX} = 0. \tag{5}$$

Suppose that

$$K^t (J_{\mathcal{U}}^t)^{-1} \overline{K} = \mu J_{\mathcal{U}} \tag{6}$$

for some $\mu \in GF^*(q^2)$.

From Lemma 1.1*v)* we can take $\mu = 1$; thus, equation (5) coincides with (1). In this case we say that $\mathcal{Q}$ *preserve* the hermitian variety $\mathcal{H}$.

Take $\mathcal{Q}$ to be a non-degenerate orthogonal polarity $\mathcal{B}$ of $PG(V)$ defined by a quadratic form Q which polarizes to $\beta_{\mathcal{B}}$. For $X \in PG(V)$, $X^{\perp_{\mathcal{B}}}$ will denote the orthogonal complement of X with respect to $\mathcal{B}$. By Theorem 1.2, the matrix $J_{\mathcal{B}}$ of $\beta_{\mathcal{B}}$ is a $n \times n$ symmetric matrix, i.e., $J_{\mathcal{B}}^t = J_{\mathcal{B}}$.

Proposition 1.4. *Suppose that $\mathcal{B}$ commutes with the unitary polarity $\mathcal{U}$. Then $\mathcal{B}$ preserves $\mathcal{H}$.*

Proof. For every point P of $PG(V)$, we have $\mathcal{UB}(P) = \mathcal{BU}(P)$.
Then, $(J_{\mathcal{U}}^t)^{-1}\overline{J}_{\mathcal{B}}\overline{X} = \mu J_{\mathcal{B}}^{-1} J_{\mathcal{U}}\overline{X}$, i.e., $J_{\mathcal{B}}^t (J_{\mathcal{U}}^t)^{-1}\overline{J}_{\mathcal{B}} = \mu J_{\mathcal{U}}$, for some $\mu \in GF^*(q^2)$.

If we choose a basis for V as in Section 1.3, equation (6) assume the simpler form

$$J_{\mathcal{B}}^t \overline{J}_{\mathcal{B}} = I_{n+1}. \tag{7}$$

In [35], B. Segre showed that a matrix K for which (7) holds can be written in the form

$$(b(B+D) - aI)(\bar{b}(B+D) - \bar{a}I)^{-1} \tag{8}$$

for some matrices B and D such that $\overline{B} = B = B^t$ and $\overline{D} = -D = D^t$ and some $a \in GF(q^2)$, $b \in GF^*(q^2)$, $ab^{-1} \notin GF(q^2)$ such that neither a/b nor $\bar{a}/\bar{b}$ is an eigenvalue of $B + D$.

Since $J_{\mathcal{Q}}$ is a symmetric matrix, it follows that

$$B + D = B^t + D^t = B - D.$$

As q is odd, D must be the null matrix; hence, the matrix $J_{\mathcal{B}}$ as the form

$$(bB - aI)(\bar{b}B - \bar{a}I)^{-1}. \tag{9}$$

Theorem 1.3. *Let $J_{\mathcal{B}}$ be the matrix associated with a quadratic form of a vector space over $GF(q^2)$, q odd. Then, there existsa basis for V such that, up to a non-zero scalar, the matrix $J_{\mathcal{B}}$ has the form*

$$(bB - aI)(\bar{b}B - \bar{a}I)^{-1}$$

for some $n \times n$ symmetric matrix over $GF(q)$ and some $a, b \in GF(q^2)$, $a\bar{b} \neq \bar{a}b$ such that neither a/b nor $\bar{a}/\bar{b}$ is an eigenvalue of B.

Next we study the geometry of the $\mathcal{V}$-fixed points of the collineation $\mathcal{V} = \mathcal{UB}$. Let P be a $\mathcal{V}$-fixed point and let X denote its homogeneous coordinate vector. By Theorem 1.3, the following equation

$$(bB - aI)X = \rho(\bar{b}B - \bar{a}I)\overline{X} \tag{10}$$

holds for some $\rho \in GF(q^2)$. By applying the field automorphism σ we find that the following equation

$$(\rho\bar{\rho} - 1)(bB - aI)X = 0 \tag{11}$$

also holds. As the matrix $bB - aI$ is non-singular, then $\rho\bar{\rho} = 1$. By Lemma 1.1 *iv)*, there exists $c \in GF(q^2)$ such that $\rho = c/\bar{c}$. Replacing X by cX, we can assume $\rho = 1$. Thus,

$$(bB - aI)X = (\bar{b}B - \bar{a}I)\overline{X} \tag{12}$$

is the equation of the set of $\mathcal{V}$-fixed points.

Theorem 1.4. *The set of $\mathcal{V}$-fixed points of the collineation $\mathcal{V}$ is a n-dimensional geometry over $GF(q)$.*

Proof. Set $bX - \overline{bX} = (\bar{a}b - a\bar{b})Y$ and $aX - \overline{aX} = (\bar{a}b - a\bar{b})Z$. Then $BY = Z$. By applying the field automorphism σ, we see that Y and Z are defined over $GF(q)$. Hence, we can consider Y and Z as homogeneous coordinate vectors of points of $PG(V')$ where V' is a n-dimensional vector space over $GF(q)$.

Conversely, let us consider two distinct points of $PG(V')$ such that $BY = Z$. Then we get $X = \bar{a}Y - \bar{b}Z$, i.e.,

$$X = (\bar{a}I - \bar{b}B)Y. \tag{13}$$

Further, as $\bar{a}/\bar{b}$ is not an eigenvalue of B, then $X \neq 0$.

Remark 1.2. A $\mathcal{V}$-fixed point with homogeneous coordinates given by (13) is in the hermitian variety (3) if and only if

$$Y^t[(\bar{b}B - \bar{a}I)(bB - aI)]Y = 0. \tag{14}$$

The latter equation represents a non-degenerate quadric of $PG(V')$. Indeed, $(\bar{b}B - \bar{a}I)(bB - aI)$ is a non-singular, symmetric matrix over $GF(q)$. We will denote by $\theta(n)$ the number of points of such a quadric. More precisely, if $n = 2r + 1$ then

$$\theta(n) = \frac{q^{n-1} - 1}{q - 1} \quad (\textit{parabolic quadric});$$

if $n = 2r$ the either

$$\theta(n) = \frac{(q^{r-1} - 1)(q^r - 1)}{q - 1} = \frac{q^n + q^r - q^{r-1} - 1}{q - 1} \quad (\textit{hyperbolic quadric})$$

or

$$\theta(n) = \frac{(q^r + 1)(q^{r-1} - 1)}{q - 1} = \frac{q^n - q^r + q^{r-1} - 1}{q - 1} \quad (\textit{elliptic quadric});$$

for more details on quadric of projective geometries over finite fields see [23].

By arguing as in Section 1.3 we can prove the following result.

Proposition 1.5. *Let P, Q be two $\mathcal{V}$-fixed points. Then the line $P+Q$ contains exactly $q+1$ $\mathcal{V}$-fixed points. Through any $\mathcal{V}$-fixed point there are $(q^{n-1}-1)/(q-1)$ $\mathcal{V}$-fixed lines and $\theta(n)$ of these lines are totally $\mathcal{U}$-isotropic. Further, there exists a $1-1$ correspondence from the subspaces in $PG(V')$ and the $\mathcal{V}$-fixed subspaces in $PG(V)$. In particular, such correspondence maps subspaces contained in the quadric defined by (14) into $\mathcal{V}$-fixed subspaces contained in the hermitian variety $\mathcal{H}$.*

It is possible to choose a basis for V better adapted to calculations. Choose a non $\mathcal{U}$-isotropic $\mathcal{V}$-fixed point $P=\langle e_0\rangle$. By the previous Proposition, P is also a non $\mathcal{B}$-isotropic point. As we have done in Proposition 1.1, it is easy to see that $P^{\perp_{\mathcal{B}}}=P^{\perp_{\mathcal{U}}}$. Then, we can write

$$V=\langle e_0\rangle \perp \langle e_0\rangle^{\perp}.$$

As $\beta_{\mathcal{U}}$ and $\beta_{\mathcal{B}}$ are non-degenerate form, $\langle e_0\rangle^{\perp}$ always contains non $\mathcal{U}$-isotropic and non $\mathcal{B}$-singular vectors. It follows that we can write

$$V=\langle e_0\rangle \perp \ldots \perp \langle e_{n-1}\rangle^{\perp},$$

where e_i's are non $\mathcal{U}$-isotropic and non $\mathcal{B}$-singular vectors. Hence, the matrices $J_{\mathcal{U}}$ and $J_{\mathcal{B}}$ are both diagonal matrices. Let $\beta_{\mathcal{U}}(e_i,e_i)=b_i\in GF(q)^*$ and let $\beta_{\mathcal{B}}(e_i,e_i)=c_i\in GF^*(q^2)$. By (7), we have $c_i\bar{c}_i=b_i^2$, $i=0,\ldots,n$. Then

$$c_i^{(q^2-1)/2}=c_i^{(q+1)(q-1)/2}=b_i^{q-1}=1.$$

This implies that $c_i=a_i^2$ for some $a_i\in GF^*(q^2)$. Replacing e_i with $a_i^{-1}e_i$ we can assume $c_i=1$ and $b_i=\pm1$, $i=0,\ldots,n$. It follows that, up to a permutation of coordinates in V, the equation of the quadric which represents the set of singular points of Q is

$$X_0^2+\ldots+X_{n-1}^2=0$$

and the equation of the hermitian variety associated with $\beta_{\mathcal{U}}$ is

$$X_0\overline{X}_0+\ldots X_{n-m-1}\overline{X}_{n-m-1}-(X_{n-m}\overline{X}_{n-m}\ldots X_{n-1}\overline{X}_{n-1})=0$$

where $0\le m\le (n+1)/2$.

2. Some Applications

In this Section, some combinatorial, geometrical and group-theoretic applications of the geometry of commuting polarities, are given.

2.1. **Complete spans of Hermitian varieties.** As we have seen, a *non-singular Hermitian variety* of $PG(n-1,q^2)$ is defined to be the set of all isotropic points of a non-degenerate unitary polarity, and it is denoted by $\mathcal{H}(n-1,q^2)$.

Definition 2.1. A *t-span* of $\mathcal{H}(n-1,q^2)$ is a set of t disjoint generators (that is, spaces of largest dimension lying on $\mathcal{H}(n-1,q^2)$), and it is said to be *complete* if it is not contained in a $(t+1)$-span.

If a t-span partitions the points of $\mathcal{H}(n-1,q^2)$, then it is called a **spread**. A spread of $\mathcal{H}(n-1,q^2)$ (if it exists) has size $q^{n-1}+1$ if n is even and q^n+1 if n is odd.

Assume now that $n=4$. We need the following lemma.

Lemma 2.1. *The number of lines on $\mathcal{H}(3,q^2)$ meeting two skew lines of $\mathcal{H}(3,q^2)$ is q^2+1, all of which are mutually skew.*

Proof. Let L_1, L_2 be the two lines, and let P be a point on L_1. A line on $\mathcal{H}(3,q^2)$ through P lies in the tangent plane π_P to $\mathcal{H}(3,q^2)$ at P. This plane π_P meets L_2 in a point P'; as P and P' are conjugate, so PP' is on $\mathcal{H}(3,q^2)$. Thus, through each point of L_1 there is one line meeting L_2. As these q^2+1 lines all lie in distinct planes, any two of which meet in L_1, any two lines are skew.

The following Proposition is due to J.A. Thas [38, Theorem 1] and provides an upper bound on the size of a t-span of $\mathcal{H}(3,q^2)$.

Proposition 2.1. *If S is a t-span of $\mathcal{H}(3,q^2)$, then $t \leq q^3 - q^2 + q + 1$.*

Proof. Let S be a t-span of $\mathcal{H}(3,q^2)$ and let L_1, L_2 be two distinct lines of S. The lines of $\mathcal{H}(3,q^2)$ concurrent with L_1 and L_2 are denoted by $M_1, M_2, \ldots, M_{q^2+1}$. Now count in two ways the number α of pairs $\{R_1, R_2\}$ with $R_1 \in S \setminus \{L_1, L_2\}$, $R_2 \in \{M_1, M_2, \ldots, M_{q^2+1}\}$ and $|R_1 \cap R_2| = 1$. First, fix a line $R_1 \in S \setminus \{L_1, L_2\}$. There are $q+1$ lines M_i concurrent with L_1,L_2,R_1. This depends on the fact that the lines L_1, L_2, R_1 uniquely determine a regulus $\mathcal{R}$ whose associated quadric commutes with $\mathcal{H}(3,q^2)$ and there are exactly $q+1$ lines of $\mathcal{H}(3,q^2)$ in the opposite regulus $\mathcal{R}'$. It follows that $\alpha = (t-2)(q+1)$. Next, fix a line M_i. Since there are at most q^2-1 lines of $S \setminus \{L_1, L_2\}$ concurrent with M_i, it follows that $\alpha \leq (q^2+1)(q^2-1)$. Hence $(t-2)(q+1) \leq (q^2+1)(q^2-1)$, and so $t \leq q^3 - q^2 + q + 1$.

An immediate consequence of the previous result is the following corollary.

Corollary 2.1. *The Hermitian surface $\mathcal{H}(3,q^2)$ has no spread.*

Actually, the previous Corollary was originally proved by B. Segre [35] using a completely different approach.

It is also known that for odd dimensions $n-1 > 3$, $\mathcal{H}(n-1,q^2)$ does not admit a spread (see Table AVI.2 in [27]), and that $\mathcal{H}(4,4)$ does not admit a spread. For $q \neq 2$ the existence of spreads in $\mathcal{H}(n-1,q^2)$, for even $n-1 \geq 4$, is an open problem.

When spreads do not exist, quite naturally the emphasis is on constructing complete t-spans and obtaining reasonable upper and lower bounds for the size t of such spans. Here, we investigate the cases $n-1 = 3$ and $n-1 = 5$ in detail, beginning with $n-1 = 3$.

A Hermitian surface $\mathcal{H} \cong \mathcal{H}(3,q^2)$ has the following properties, for which [24, Chapter 19] is an excellent source.

(1) The number of points on the Hermitian surface $\mathcal{H}$ is $(q^2+1)(q^3+1)$.
(2) Any line of $\Sigma = PG(3,q^2)$ meets $\mathcal{H}$ in 1 or $q+1$ or q^2+1 points. The latter lines are the *generators* of $\mathcal{H}$, and they are $(q+1)(q^3+1)$ in number. The intersections of size $q+1$ are Baer sublines, whereas lines meeting $\mathcal{H}$ in one point are called *tangent lines*.
(3) Through every point P of $\mathcal{H}$ there pass exactly $q+1$ generators, and these generators are coplanar. The plane containing these generators, say π_P, is the polar plane of P with respect to the unitary polarity defining $\mathcal{H}$. The tangent lines through P are precisely the remaining q^2-q lines of π_P incident with P, and π_P is called the *tangent plane* of $\mathcal{H}$ at P.
(4) Every plane of Σ which is not a tangent plane of $\mathcal{H}$ meets $\mathcal{H}$ in a non-degenerate Hermitian curve $\mathcal{H}(2,q^2)$.

(5) The incidence structure formed by the points and lines of $\mathcal{H}$ is the dual of the incidence structure formed by the points and lines of an elliptic quadric $Q^-(5,q)$ of $PG(5,q)$ [24], [33].

Let $\mathcal{A}$ be a symplectic polarity commuting with the Hermitian polarity $\mathcal{U}$ associated with $\mathcal{H}$. Set $\mathcal{V} = \mathcal{AU} = \mathcal{UA}$. As pointed out in Section 1 (see also [35, p. 128, 132]), $\mathcal{V}$ is a non-linear collineation, fixes q^3+q^2+q+1 points on $\mathcal{H}$, but fixes no point outside $\mathcal{H}$, and leaves invariant the same number of generators of $\mathcal{H}$. Each fixed point is incident with $q+1$ invariant generators, and each invariant generator is incident with $q+1$ fixed points. This symmetric configuration $\mathcal{W}$ on $\mathcal{H}$ extends to a 3-dimensional projective space $\Sigma_0 \cong PG(3,q)$ by adding the $q^2(q^2+1)$ $\mathcal{V}$-invariant lines which are not generators of $\mathcal{H}$. In this context, Σ_0 is naturally equipped with the symplectic polarity $\mathcal{A}$ whose isotropic lines are the lines of the above symmetric configuration $\mathcal{W}$, and $\mathcal{W} \cong \mathcal{W}_3(q)$ is a symplectic polar space. If $\mathcal{H}$ has canonical equation $X_0^{q+1}+X_1^{q+1}+X_2^{q+1}+X_3^{q+1}=0$, then $\mathcal{W}$ can be described as the subset of points of $\mathcal{H}$ whose coordinates are of the form $(a, \rho a^q, b, \rho b^q)$, where $a, b, \rho \in GF(q^2)$ with $\rho^{q+1} = -1$. The projective symplectic group $PSp_4(q)$ associated with $\mathcal{A}$ turns out to be a subgroup of the projective unitary group $PGU_4(q^2)$ associated with $\mathcal{U}$. The number of symplectic polarities commuting with the unitary polarity associated with $\mathcal{H}$ is $q^2(q^3+1)$. The importance of such commuting symplectic polarities is that they map generators to generators.

In terms of forms, we start from a symplectic geometry $(PG(V), \mathcal{A})$, where V is a four-dimensional vector space over $GF(q)$ and $\mathcal{A}$ is a polarity associated with a non-degenerate alternating bilinear form A of V. Let ω be an element of $GF(q^2)\setminus\{0\}$ such that $\omega^q = -\omega$ (of course this makes sense if q is odd!). Then $GF(q^2) = GF(q) \oplus GF(q)\omega$, and we define $W = \{(\alpha+\beta\omega)v | \alpha, \beta \in GF(q), v \in V\} \cong V \otimes GF(q^2)$. Any vector $w \in W$ can be written as $w = \sum v_i \otimes (a_i + b_i\omega) = \sum(v_i \otimes 1)a_i + \sum(v_i \otimes \omega)b_i = (\sum v_i a_i) \otimes 1 + (\sum v_i b_i) \otimes w$, which we abbreviate as $w_1 + w_2\omega$. Using this notation, we have $(\alpha+\beta\omega)(w_1+w_2\omega) = (\alpha w_1 + \beta\omega^2 w_2) + (\beta w_1 + \alpha w_2)\omega$, and we define $C : W \times W \to GF(q^2)$ as follows:

$$C(w_1 + w_2\omega, v_1 + v_2\omega) =$$
$$= A(w_1, v_1) + \omega\omega^q A(w_2, v_2) + \omega A(w_2, v_1) + \omega^q A(w_1, v_2).$$

Straightforward computations show that C is a non-degenerate (anti-)Hermitian form on W and $C_{|V} = \omega A$. Thus we obtain the embedding $Sp_4(q) \le GU_4(q^2)$, where $Sp_4(q)$ is the group of all isometries of A and $GU_4(q^2)$ is the group of all similarities of C. In particular, this embedding does not depend on the choice of ω. If q is even, then we can take $\omega = 1$ and obtain the embedding $Sp_4(q) \le U_4(q^2)$. Factoring out scalars, we get the embedding $PSp_4(q) \le PSU_4(q^2)$.

The above embeddings of groups and geometries suggest that one should study the properties of various configurations on the symplectic polar space $\mathcal{W}$, after extending scalars from $GF(q)$ to $GF(q^2)$. For instance, in [14], Cossidente and Korchmáros showed that for q even, any ovoid in $\mathcal{W}$ is a complete partial ovoid of size q^2+1 in $\mathcal{H}$ (the definition of ovoid will be given later!). We continue along this direction, thereby obtaining more information about the above embeddings.

There are several characterizations of Hermitian surfaces as point sets in $\Sigma = PG(3,q^2)$. One useful characterization is given by the following theorem.

Theorem 2.1. [24, Theorem 19.5.13] *Suppose H is a set of points in $PG(3,q^2)$, where q is any prime power, such that every line meets H in $1, n$ or q^2+1 points for* *some fixed integer n, where $1 \le n \le q^2 - 1$ and $n \ne \frac{1}{2}q^2 + 1$. Suppose further that every point in H lies on at least one n-secant. Then $n = q + 1$ and $H \cong \mathcal{H}(3, q^2)$.*

The points of $\Sigma_0 = PG(3, q)$ and the $(q + 1)(q^2 + 1)$ lines of a general linear complex of Σ_0 form a symplectic polar space $\mathcal{W} = \mathcal{W}_3(q)$, and conversely any symplectic polar space determines a general linear complex (if $\mathcal{K}$ is the Klein quadric representing the lines of Σ_0, a general linear complex of Σ_0 is represented by a non-degenerate hyperplane section of $\mathcal{K}$). We will use the above theorem to prove the following result, which implicitly was known from the point of view of generalized quadrangles or from the above group embeddings. Here we provide a purely geometrical proof in the setting of $\Sigma = PG(3, q^2)$.

Theorem 2.2. *Let $\mathcal{L}$ be a general linear complex of $PG(3, q)$, where q is any prime power. If H is the point set of $PG(3, q^2)$ covered by the lines of $\mathcal{L}$ when extended to $PG(3, q^2)$, then H is a Hermitian surface.*

In [21] Ebert and Hirschfeld found a lower bound for the size of complete t-spans of $\mathcal{H}(3, q^2)$, and they also proved the following theorem.

Theorem 2.3. [21, Theorem 3.1, Theorem 3.2] *The q^2+1 generators meeting each of two skew generators of $\mathcal{H}$ form a complete span.*

Proof. Let ℓ, ℓ' be two generators of $\mathcal{H}$, and let P be a point on ℓ. A line on $\mathcal{H}$ through P lies in the tangent plane π_P to $\mathcal{H}$ at P. This plane π_P meets ℓ' in a point P'; as P and P' are conjugate, so PP' is on $\mathcal{H}$. Thus, through each point of ℓ, there is one line meeting ℓ'. As these $q^q + 1$ lines all lie in distinct planes, any two of which meet in ℓ, any two of the lines are skew. Call S the $(q^2 + 1)$-span consisting of the lines meeting ℓ and ℓ'. Now, let Q be a point not on any line of S. The lines of $\mathcal{H}$ through Q lie in the tangent plane π_Q at Q. It must be shown that each of these $q + 1$ lines through Q meets some line of S. Let $R = \ell \cap \pi_Q$, $R' = \ell' \cap \pi_Q$. If RR' contains Q, then QRR' is a line of S, a case already accounted for. Also, no line m of $\mathcal{S}$ meets QR in a point other than R, as this would give a triangle with sides m, ℓ, QR on $\mathcal{H}$; similarly, no line of $\mathcal{S}$ meets QR'. This also shows that $RR' \notin S$. Consider how the lines of S meet the lines of $\pi_Q \cap \mathcal{H}$. If three lines of S meet a line l of $\pi_Q \cap \mathcal{H}$, then the hyperbolic quadric formed by these lines commutes with $\mathcal{H}$ and has exactly $q + 1$ lines of each of its reguli on $\mathcal{H}$; this is equivalent to saying that the corresponding plane in $PG(5, q)$ meets $Q^-(5, q)$ in a conic. Hence, any line of $\pi_Q \cap \mathcal{H}$ meets at most $q + 1$ lines of S. However, apart from QR and QR', there are $q - 1$ lines in $\pi_Q \cap \mathcal{H}$. Hence, the $q^2 + 1$ lines of S meet π_Q precisely in one point on QR, one point on QR' and $q + 1$ points on each of the other $q - 1$ lines of $\pi_Q \cap \mathcal{H}$. As Q was an arbitrary point on no line of S, it can be concluded that every line on $\mathcal{H}$ meets some line of S.

In their investigation Ebert and Hirschfeld did not find complete spans of the Hermitian surface $\mathcal{H}(3, q^2)$ of size less than $q^2 + 1$. Here, we provide a geometric construction for other complete (q^2+1)-spans of $\mathcal{H}(3, q^2)$, and also discuss examples of smaller complete spans. Our main tool will be general linear complexes. Every general linear complex of $\Sigma_0 = PG(3, q)$ admits a spread; that is, a collection of $q^2 + 1$ mutually skew lines in the complex. This is equivalent to the fact that through a point not on the Klein quadric $Q^+(5, q)$ in $PG(5, q)$ there is a line missing the quadric. Such a spread is often called a *symplectic spread* of the underlying projective space Σ_0. We will prove the following theorem.

Theorem 2.4. *Let $\mathcal{S}$ be a symplectic spread of $PG(3,q)$ for any prime power q, and let $\mathcal{L}$ denote the general linear complex containing the lines of $\mathcal{S}$. Let $\mathcal{H}$ be the Hermitian surface constructed as in Theorem 2.2. Then the extended lines of $\mathcal{S}$ form a complete (q^2+1)-span of $\mathcal{H}$.*

With a t-span of $\mathcal{H}$ there corresponds a set of points of $Q^-(5,q)$, no two of which are on a line of the quadric. Such a set of points is called a *partial ovoid* of $Q^-(5,q)$. Hence, if we find a complete t-span of $\mathcal{H}$, we also have found a complete partial ovoid of $Q^-(5,q)$. Notice that $Q^-(5,q)$ has no ovoids. From Theorem 2.4, using the regular and Lüneburg symplectic spreads, it follows that $Q^-(5,q)$ contains complete partial ovoids of size q^2+1 admitting the groups $Sz(q)$ and $P\Omega_4^-(q)$, respectively. However, there are other possibilities for a partial ovoid of size q^2+1 on $Q^-(5,q)$ (see Remark 2.1 in Section 2.1.2).

In [20] R.H. Dye proved that if p is a prime and $n+1 = kp+r$, with $k \geq 1$ and $0 \leq r \leq p-1$, then the maximal size M of a complete partial ovoid of a nonsingular quadric $Q_n(p)$ in $PG(n,p)$ satisfies the inequality

$$\binom{n}{p-1} + \binom{n}{r-1} \leq M \leq \binom{n+p-1}{p-1} - \binom{n+p-3}{p-3} + 1$$

Assuming $n = 5$ and letting $p = 2, 3$ and 5, we obtain $M \geq 5, 10$ and 21, respectively. Hence for $p = 2, 3$ our complete partial ovoids of $Q^-(5,p)$ show that Dye's lower bound is sharp in these cases.

2.1.1. *Extending a general linear complex.* We first provide a proof of Theorem 2.2. As above, we denote by Σ_0 the point set of $PG(3,q)$ and by Σ the point set of $PG(3,q^2)$. We let H denote the points of Σ lying on the lines of a general linear complex $\mathcal{L}$ of Σ_0, when extended over the field $GF(q^2)$. Clearly Σ_0 is contained in H from the properties of a general linear complex.

Note first that the extended lines of $\mathcal{L}$, which are $(q^2+1)(q+1)$ in number, cannot meet in a point of $\Sigma \setminus \Sigma_0$ since they are lines of Σ_0. This implies that

$$|H| = (q^2+1)(q+1)(q^2-q) + (q^2+1)(q+1) = (q^2+1)(q^3+1),$$

which is the number of points lying on a Hermitian surface $\mathcal{H}(3,q^2)$. Now let ℓ be an arbitrary line of Σ and consider the three possibilities for ℓ with respect to Σ_0, treated as a Baer subgeometry of Σ.

Lemma 2.2. *If ℓ is a line of Σ_0, then either $|\ell \cap H| = q+1$ or $|\ell \cap H| = q^2+1$.*

Proof. If ℓ is a line of $\mathcal{L}$, then ℓ is contained in H by definition and $|\ell \cap H| = q^2+1$. Thus assume that ℓ is a line of Σ_0 which is not contained in $\mathcal{L}$. Then $|\ell \cap H| \geq q+1$ as Σ_0 is contained in H. Suppose that $P \in \ell \cap H$ and $P \notin \Sigma_0$. Thus P lies on a unique extended line m of $\mathcal{L}$ by definition of H. Now ℓ is not a line of $\mathcal{L}$ and thus $\ell \neq m$. Then $P = \ell \cap m$, and we have $P \in \Sigma_0$ since ℓ and m are lines of Σ_0. This contradicts the assumption that $P \notin \Sigma_0$, and hence $|\ell \cap H| = q+1$.

Lemma 2.3. *If $|\ell \cap \Sigma_0| = 1$, then $|\ell \cap H| = 1$ or $|\ell \cap H| = q+1$.*

Proof. Let $P = \ell \cap \Sigma_0$. Note that $|\ell \cap H| \geq 1$ as Σ_0 is contained in H. Suppose $Q \in \ell \cap H$ and $Q \notin \Sigma_0$. Then Q lies on a unique line m of $\mathcal{L}$ by definition of H. Clearly $\ell \neq m$ since ℓ is not a line of Σ_0. Hence the plane $\pi = \langle \ell, m \rangle$ meets Σ_0 in at least the $q+1$ points of $m \cap \Sigma_0$ plus the point P. Therefore π is a plane of Σ_0.

Now suppose Q' is another point of $\ell \cap H$, other than P, and hence $Q' \notin \Sigma_0$. As above, this would generate a plane π' of Σ_0 containing ℓ. Since ℓ is not a line of Σ_0, necessarily $\pi = \pi'$. Thus the points of $\ell \cap H$, other than P, arise from extended lines of $\mathcal{L}$ meeting ℓ, all of which must lie in π. Since every plane of Σ_0 contains exactly $q+1$ lines of $\mathcal{L}$, and these lines form a planar pencil, it follows that $|\ell \cap H| = q+1$. Note that P lies on one of the lines in the above planar pencil. The result now follows.

Lemma 2.4. *If $\ell \cap \Sigma_0$ is empty, then either $|\ell \cap H| = 1$ or $|\ell \cap H| = q+1$ or $|\ell \cap H| = q^2+1$*

Proof. Since $\ell \cap \Sigma_0$ is empty, there is a unique line of Σ_0 passing through each point of ℓ. Moreover these q^2+1 lines are mutually skew and form a regular spread $\mathcal{S}_0$ of Σ_0 (see Theorem 5.3 in [7]). By considering Plücker coordinates for the lines of Σ_0 (see Table 1.5.10 in [24]), we see that every regular spread meets a general linear complex in 1, $q+1$ or all of its q^2+1 lines. Thus $|\ell \cap H| = 1$, $q+1$ or q^2+1 from the definition of $\mathcal{H}$.

Theorem 2.2 now follows from Lemmas 2.2, 2.3, 2.4 and Theorem 2.1.

2.1.2. *Construction of complete (q^2+1)-spans.* In this section we prove Theorem 2.4 and make a few comments on computational results.

Proof of Theorem 2.4: First we note that the extended lines of the symplectic spread $\mathcal{S}$ are mutually skew generators of $\mathcal{H}$ and thus form a (q^2+1)-span of $\mathcal{H}$. Suppose that ℓ is a generator of $\mathcal{H}$ disjoint from each extended line of $\mathcal{S}$. Since $\mathcal{S}$ covers all points of Σ_0, necessarily $\ell \cap \Sigma_0$ is empty. Let $\mathcal{S}_0$ be the regular spread of Σ_0 obtained as transversals to ℓ. Since the points of ℓ are all contained in $\mathcal{H}$, each point lies on a unique line of $\mathcal{L}$, and this is the only line of Σ_0 through a given point of ℓ. Therefore $\mathcal{S}_0$ is a regular spread contained in $\mathcal{L}$, and $\mathcal{S}_0$ and $\mathcal{S}$ are two symplectic spreads lying in the same linear complex. From [4] $\mathcal{S}_0$ and $\mathcal{S}$ share $1 \bmod p$ lines, where q is a power of the prime p. In particular, this implies that some extended line of $\mathcal{S}$ must meet ℓ, contradicting our assumption on ℓ. Thus the extended lines of $\mathcal{S}$ form a complete (q^2+1)-span of the Hermitian surface $\mathcal{H}$, proving Theorem 2.4.

Remark 2.1. Extensive searching using the software package MAGMA [8] revealed many complete spans of size q^2+1 for various "small" values of q. Although the projective equivalences have not been sorted out, there are many different types. Namely, inequivalent symplectic spreads will typically yield inequivalent complete spans via Theorem 2.4. The stabilizer of the spread inherits as a collineation group of the Hermitian surface leaving the associated complete span invariant. In addition, many complete spans of size q^2+1, for various values of q, were found which had a trivial stabilizer. Moreover, most of these spans turned out to be mutually inequivalent under the stabilizer of the Hermitian surface.

Remark 2.2. In [21] a lower bound of $2q+2$ is proven for the size t of any complete span of $\mathcal{H}(3,q^2)$, $q \geq 4$, whereas a general upper bound of q^3-q^2+q+1 is shown in Lemma 2.1. Our searching for $q = 4, 5, 7$ and 8, extensive but not exhaustive, indicates that perhaps both upper and lower bounds for the size of a complete span in $\mathcal{H}(3,q^2)$ should be quadratic in q. Namely, for $q=4$ the complete spans we found had sizes between $17=q^2+1$ and $25=(q+1)^2$. For $q=5$ the sizes of the complete spans found were between $26=q^2+1$ and 39. For $q=7$ the sizes found were between 46 and 60, while for $q=8$ the sizes found were between 57 and 74. Note that, in particular, complete spans of size less than q^2+1 were found for $q=7$ and 8. Our searching was random, so the probability of finding "special" or "unusual" complete spans was low. Hence it is conceivable that there are much bigger or much smaller complete spans than the ones we found for the given values of q. Nonetheless, it seems to us that the known bounds are not very good, and perhaps the model for $\mathcal{H}(3,q^2)$ given in Theorem 2.2, together with Lemmas 2.2, 2.3 and 2.4, could be useful in improving these bounds. Current efforts in this direction have so far been unsuccessful.

Remark 2.3. Let Q be any hyperbolic quadric of $PG(3,q^2)$. If Q has at least three skew lines on $\mathcal{H}$, then $\mathcal{H}\cap Q$ consists of $2q^3+q^2+1$ points lying on $2(q+1)$ generators. These generators are partitioned into two (extended) subreguli, actually a subregulus and its opposite, over the subfield $GF(q)$. Such a quadric Q *commutes* with $\mathcal{H}$. In [35, p. 167] Segre claims that one of the two extended subreguli, say $\mathcal{R}$, forms a complete $(q+1)$-span of $\mathcal{H}$. Unfortunately, this is incorrect and it is easy to find generators of $\mathcal{H}$ which are skew to each line of $\mathcal{R}$. For instance, assume $q=3$ and $GF(9)=GF(3)[\omega]$, where $\omega^2+2\omega+2=0$. If $\mathcal{H}$ is the Hermitian surface with equation $X_1^4+X_2^4+X_3^4+X_4^4=0$ and Q is the hyperbolic quadric with equation $X_1X_4-X_2X_3=0$, then Q commutes with H. The lines

$$r_1=\{(1,\omega^7,0,0),(0,0,1,\omega^7),(1,\omega^7,\omega^2,\omega),(1,\omega^7,\omega^5,2),(1,\omega^7,\omega^3,\omega^2),$$

$$(1,\omega^7,\omega^6,\omega^5),(1,\omega^7,\omega^7\omega^6),(1,\omega^7,2,\omega^3),(1,\omega^7,1,\omega^7),(1,\omega^7,\omega,1)\},$$

$$r_2=\{(1,\omega^3,\omega^6,\omega),(1,\omega^3,1,\omega^3),(0,0,1,\omega^3),(1,\omega^3,\omega^3,\omega^6),(1,\omega^3,2,\omega^7),$$

$$(1,\omega^3,0,0),(1,\omega^3,\omega^7,\omega^2),(1,\omega^3,\omega,2),(1,\omega^3,\omega^2,\omega^5),(1,\omega^3,\omega^5,1)\},$$

$$r_3=\{(1,\omega,\omega^5,\omega^6),(1,\omega,\omega^7,1),(1,\omega,\omega^3,2),(1,\omega,\omega^6\omega^7),(1,\omega,0,0),$$

$$(0,0,1,\omega),(1,\omega,1,\omega),(1,\omega,\omega,\omega^2),(1,\omega,2,\omega^5),(1,\omega,\omega^2,\omega^3)\},\text{ and}$$

$$r_4=\{(1,\omega^5,\omega^6,\omega^3),(1,\omega^5,\omega^3,1),(1,\omega^5,2,\omega),(1,\omega^5,\omega^2,\omega^7),(1,\omega^5,1,\omega^5),$$

$$(1,\omega^5,0,0),(0,0,1,\omega^5),(1,\omega^5,\omega^7,2),(1,\omega^5,\omega^5,\omega^2),(1,\omega^5,\omega,\omega^6)\}$$

form an extended subregulus $\mathcal{R}_0$ of Q on $\mathcal{H}$. The line

$$r_5=\{(1,0,\omega,0),(1,1,\omega,\omega^3),(1,\omega^5,\omega,1),(1,\omega^3,\omega,\omega^6),(1,\omega^6,\omega,\omega),$$

$$(1,\omega,\omega,2),(1,2,\omega,\omega^7),(0,1,0,\omega^3),(1\omega^2,\omega,\omega^5),(1,\omega^7,\omega,\omega^2)\}$$

is a generator of $\mathcal{H}$ skew to each line in $\mathcal{R}_0$. Hence $\mathcal{R}_0$ is not a complete 4-span. We found that $\mathcal{R}_0$ can be extended to a complete 13-span.

Exercise 2.1. Prove that, for any q, a subregulus of Q is not a complete $(q+1)$-span of the Hermitian surface.

2.1.3. *Generalizations.* In this section we discuss the construction of t-spans of the Hermitian variety $\mathcal{H}(n-1, q^2)$, where $n-1 \geq 5$ is odd. Recall that it is known that spreads do not exist in this case. One such construction is the immediate generalization of that given in Section 2.1. Indeed, we have the group embedding $PSp_{2m}(q) \leq PSU_{2m}(q^2)$, $n = 2m$, for any value of q. This means that it is always possible to construct a symplectic subgeometry $\mathcal{W}_{2m-1}(q)$ as a subset of the point set of $\mathcal{H}(2m-1, q^2)$. The symplectic geometry $\mathcal{W}_{2m-1}(q)$ has $(m-1)$-spreads, and extending scalars from $GF(q)$ to $GF(q^2)$ yields a (q^m+1)-span of $\mathcal{H}(2m-1, q^2)$.

While it is presently unknown if the above spans are complete for arbitrary m, we can prove this is true when $m = 2, 3$. As the work in the previous section handles the case when $m = 2$, we now consider $m = 3$.

Theorem 2.5. *Let $\mathcal{S}$ be a spread of $\mathcal{W} = \mathcal{W}_5(q)$ for any prime power q, and let $\mathcal{L}$ denote the collection of generators of $\mathcal{W}$. Let $\mathcal{H} = \mathcal{H}(5, q^2)$ be the Hermitian variety obtained by extending the planes of $\mathcal{L}$ over the extension field $GF(q^2)$, similarly to the construction in Theorem 2.2. Then the extended planes of $\mathcal{S}$ form a complete (q^3+1)-span of $\mathcal{H}$.*

Proof. Using notation similar to that used above, let $\Sigma = PG(5, q^2)$ be the ambient space for the Hermitian variety $\mathcal{H}$, and let $\Sigma_0 \cong PG(5, q)$ be the Baer subgeometry of Σ covered by the points of $\mathcal{W}$. The spread $\mathcal{S}$ of $\mathcal{W}$ alternatively may be described as a symplectic 2-spread of Σ_0. Since the planes of $\mathcal{S}$ are mutually skew in Σ_0, their extensions are necessarily skew in Σ and thus form a (q^3+1)-span of the Hermitian variety $\mathcal{H}$.

Any plane π of Σ meets Σ_0 in a Baer subplane, a Baer subline, a point or the empty set. Consider a generator (plane) π of $\mathcal{H}$, and suppose that $\pi \cap \Sigma_0 = \pi_0$ is a Baer subplane. Let P be any point of $\pi \setminus \pi_0$, and let ℓ be the unique line of π_0 incident with P. Let $\ell_0 = \ell \cap \pi_0$. We claim that π_0 is a generator of $\mathcal{W}$. Let σ be an extended generator of $\mathcal{W}$ incident with P. If $\sigma = \pi$, our claim is proven. If $\sigma \neq \pi$, then $\sigma \cap \pi$ is the unique line of Σ_0 incident with P, and thus $\sigma \cap \pi = \ell$. Since $\ell \subset \sigma$, ℓ_0 is necessarily a line of $\mathcal{W}$. Straightforward counting shows that the number of generators of $\mathcal{W}$ containing ℓ_0 is $q+1$, which is also the number of generators of $\mathcal{H}$ containing ℓ. Thus these two sets of generators, when the former ones are extended, must be the same, and π must be an extended generator of $\mathcal{W}$. This proves our claim, and hence any generator of $\mathcal{H}$ meeting Σ_0 in q^2+q+1 points must be an extended generator of $\mathcal{W}$.

Next suppose that π is a generator of $\mathcal{H}$ which meets Σ_0 in a Baer subline ℓ_0. Let ℓ be the extension of ℓ_0 over $GF(q^2)$, so that ℓ is a line of π. Let Q be any point of $\ell \setminus \ell_0$. More straightforward counting shows that Q lies on precisely $q+1$ extended generators of $\mathcal{W}$ (as does any point of $\mathcal{H} \setminus \Sigma_0$). Let σ_1 and σ_2 be two distinct extended generators of $\mathcal{W}$ incident with Q. Then $\sigma_1 \cap \sigma_2$ must be the unique line of Σ_0 incident with Q, and hence $\sigma_1 \cap \sigma_2 = \ell$. In particular, this implies that ℓ_0 is a line of $\mathcal{W}$. But ℓ_0 lies on precisely $q+1$ generators of $\mathcal{W}$. As there are precisely $q+1$ generators of $\mathcal{H}$ containing ℓ, we see that π must be an extended generator of $\mathcal{W}$. This contradicts the assumption that π meets Σ_0 in a Baer subline. Thus no generator of $\mathcal{H}$ meets Σ_0 in $q+1$ points.

Now we count the number of generators of $\mathcal{H}$ that meet Σ_0 in 1 or q^2+q+1 points. There are $(q^3+1)(q+1)$ generators of $\mathcal{H}$ incident with a point of $\mathcal{H}$, and there are $(q^2+1)(q+1)$ generators of $\mathcal{W}$ incident with a point of $\mathcal{W}$. Since the points of $\mathcal{W}$ are the points of Σ_0 and since these points all lie on $\mathcal{H}$, there are precisely

$q^2(q^2-1)$ generators of $\mathcal{H} \setminus \mathcal{W}$ incident with each point of Σ_0. As we vary over the $(q^6-1)/(q-1)$ points of Σ_0, no generator is double counted as no generator of $\mathcal{H} \setminus \mathcal{W}$ meets Σ_0 in a Baer subline or Baer subplane from our computations above. Adding in the $(q+1)(q^2+1)(q^3+1)$ generators of $\mathcal{W}$, each of which extends to a generator of $\mathcal{H}$, we obtain $(q+1)(q^3+1)(q^5+1)$ distinct generators of $\mathcal{H}$. But this is the total number of generators of $\mathcal{H}$, and thus no generator of $\mathcal{H}$ is disjoint from Σ_0.

Therefore, since the planes of the spread $\mathcal{S}$ cover all the points of Σ_0, no generator of $\mathcal{H}$ can be disjoint from the points covered by $\mathcal{S}$. Thus the extended planes of $\mathcal{S}$ form a complete (q^3+1)-span on $\mathcal{H}$.

The counting argument in the above proof breaks down in $\mathcal{H}(7,q^2)$, and thus it appears that there are generators (solids) of $\mathcal{H}(7,q^2)$ that are disjoint from the Baer subgeometry covered by the embedded symplectic polar space $\mathcal{W}(7,q)$. It is still possible that an extended spread of $\mathcal{W}$ is a complete span, but we have not been able to prove this.

A general upper bound of $q^2(q^2+q-1)$ for the size of a complete span in $\mathcal{H}(5,q^2)$ was proven in [38]. We did a bit of computer searching for complete spans in $\mathcal{H}(5,9)$, and were unable to find examples of cardinality larger than $q^3+1=28$. The smallest example we found had size 17. While this is very limited data, and again our searching was random rather than targeted, it seems to us that the above upper bound is far from tight. Perhaps, as in the 3-dimensional setting previously discussed, the above model for $\mathcal{H}(5,q^2)$ and the proof technique used in the above theorem will shed some light on how to improve this upper bound.

Problem 2.1. The number of totally isotropic lines of a symplectic polar space $\mathcal{W}=\mathcal{W}(n-1,q)$ is $((q^n-1)/(q-1))((q^{n-2}-1)/(q^2-1))$. Extending such lines to lines over $GF(q^2)$ we find that the set $\mathcal{H}$ of points on the extended lines in $PG(n-1,q^2)$ has exactly the same size of the set of singular points of a Hermitian variety $\mathcal{H}(n-1,q^2)$. Algebraically, it is easy to prove that $\mathcal{H}$ is a Hermitian variety. Please, find a combinatorial proof!

2.2. **Maximal symplectic subgroups of unitary groups.** The seminal contribution to the classification of the maximal subgroups of the finite classical groups was Aschbacher's theorem [3]. Aschbacher defines eight "geometric" classes $\mathcal{C}_1 \dots$, $\mathcal{C}_8$, of subgroups of the finite classical groups and proves that a maximal subgroup either belongs to one of these classes or has a non-abelian simple group as its generalized Fitting subgroup.

In their book [29], P.B. Kleidman and M.W. Liebeck have identified the members of the eight classes for modules with dimension greater than 12, and P.B. Kleidman [28] completed the work for modules with dimension up to 12. However, their analysis relies heavily upon the classification of finite simple groups. Various authors, too many to be quoted here, have used Aschbacher's theorem to elucidate much of the maximal subgroup structure on the finite classical groups.

At least seven of the eight Aschbacher's classes can be described as stabilizers of geometric configurations. Consequently, one might prefer a direct approach to the classification of maximal subgroups, which is free of the classification of finite simple groups, using the natural representations of classical groups.

Here, we are interested in Aschbacher's class $\mathcal{C}_5$. For a classical group G acting on a n-dimensional vector space V over a field F, the class $\mathcal{C}_5$ is the collection of normalizers of the classical groups acting on the n-dimensional vector spaces V_K over maximal subfields K of F such that $V = F \otimes_K V_K$.

Apart form the work of Kleidman and Liebeck, very little has been done for subgroups belonging to this class.

Here, we study maximal symplectic subgroups belonging to the class $\mathcal{C}_5$ of the unitary group $PSU_n(K)$, n even, K any field admitting a non-trivial involutory automorphism.

As we have seen, when the ground field is finite, such symplectic groups naturally arise from the geometry of a symplectic polarity commuting with a unitary polarity.

2.2.1. *The embedding $Sp_n(K_0) \leq SU_n(K)$.* Let K be a commutative field admitting a non-trivial involutory automorphism $\lambda \mapsto \bar{\lambda}$, with K_0 the fixed subfield. In this Subsection we prove the maximality of certain symplectic groups inside the unitary group $PSU_n(K)$.

Suppose that V is a n-dimensional vector space over K_0 and A is a non-degenerate alternating bilinear form on V. Let ω be an element of $K \setminus K_0$. Then $K = K_0 \oplus K_0\omega$ and there is a vector space $W = V \otimes_{K_0} K = \{(\alpha + \beta\omega)v | \alpha, \beta \in K_0, v \in V\}$. Any vector $w \in W$ can be written as $w = \sum v_i \otimes (a_i + b_i\omega) = \sum (v_i \otimes 1)a_i + \sum (v_i \otimes \omega)b_i = (\sum v_i a_i) \otimes 1 + (\sum v_i b_i) \otimes w = w_1 + w_2\omega$. [Also if $\omega^2 = \gamma\omega + \delta$, then $(\alpha + \beta\omega)(w_1 + w_2\omega) = (\alpha w_1 + \beta\delta w_2) + (\beta w_1 + \beta\gamma w_2 + \alpha w_2)\omega$.] There is a natural extension of A to an anti-hermitian form C on W given by:

$$C(w_1 + w_2\omega, v_1 + v_2\omega) = A(w_1, v_1) + \omega\bar{\omega}A(w_2, v_2) + \omega A(w_2, v_1) + \bar{\omega}A(w_1, v_2).$$

If $char K = 2$, then C is already a hermitian form. In all cases there exists a $\tau \in K$ such that $\bar{\tau} = -\tau$ (as follows from Hilbert's Theorem 90) and τC is a hermitian form with the same group as C. We write D for τC, $U_n(K)$ for the unitary group of D, $Sp_n(K_0)$ for the symplectic group of A. We obtain the embedding $Sp_n(K_0) \leq SU_n(K)$. Note that D does not depend on the choice of ω. Factoring out scalars, we get the embedding $PSp_n(K_0) \leq PSU_n(K)$.

Let $x = w_1 + w_2\omega \in W$. Then, with respect to D, x is isotropic if and only if $D(x, x) = 0$, i.e., if and only if $\omega A(w_2, w_1) + \bar{\omega}A(w_1, w_2) = 0$, i.e., if and only if $\omega A(w_2, w_1) = \bar{\omega}A(w_2, w_1)$, i.e., if and only if $A(w_1, w_2) = 0$. In particular every vector in V is isotropic with respect to D. Suppose that $0 \neq v \in V$ and that t is a unitary transvection centered on v. Then $t : x \mapsto x + \lambda D(x, v)v$ for some $\lambda \in K$ such that $\bar{\lambda} = -\lambda$. If $x \in V$, then $t(x) = x + \lambda\tau A(x, v)v$ with $\lambda\tau \in K_0$, so t fixes V globally and the restriction of t to V is a symplectic transvection, i.e., $t \in Sp_n(K_0)$.

Let $\mathcal{H}$ be the Hermitian variety of $PG(n-1, K)$ associated with D. Let Σ_0 be the set of points of the $PG(n-1, K_0)$ corresponding to V, considered as a subset of $\mathcal{H}$ inside $PG(n-1, K)$. We can regard $Sp_n(K_0)$ and $SU_n(K)$ as acting on $PG(n-1, K)$. Then $Sp_n(K_0)$ fixes $\mathcal{H}$ globally and has Σ_0 as one orbit. Suppose that x and y are isotropic vectors in W corresponding to points of $\mathcal{H} \setminus \Sigma_0$. Then $x = w_1 + w_2\omega$, $y = v_1 + v_2\omega$, for some linearly independent $w_1, w_2 \in V$ and some linearly independent $v_1, v_2 \in V$ and by Witt's Theorem there is an element of $Sp_n(K_0)$ taking w_i to v_i for each i, i.e., taking x to y. Hence $Sp_n(K_0)$ has exactly two orbits on $\mathcal{H}$.

Let G_n denote the stabilizer of Σ_0 in $SU_n(K)$ and let F be a subgroup of $SU_n(K)$ such that $G_n < F$. Then F has a single orbit of points on $\mathcal{H}$. If t is any unitary transvection in $SU_n(K)$, centered on y say, then there exists $f \in F$ such that $f(y) \in V$ and ftf^{-1} is a transvection centered on $f(y)$. Thus $ftf^{-1} \in Sp_n(K_0)$ and $t \in F$. It is well known that $SU_n(K)$, $n \geq 4$, is generated by its transvections [17], [18] and so $F = SU_n(K)$, and G_n is maximal in $SU_n(K)$. By the standard theorem for subgroups of quotients groups, the stabilizer $\overline{G_n}$ of Σ_0 in $PSU_n(K)$ is maximal in $PSU_n(K)$.

It is of some interest to know the structure of G_n. Suppose that $g \in G_n$ and that $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ (with $n = 2m$) is a symplectic basis for V with respect to A (i.e., $A(v_i, v_{m+j}) = \delta_{ij}$). Then $A(g(v_i), g(v_{m+j})) = 0$ if and only if $i = j$ and by Witt's Theorem there exists $h_1 \in Sp_n(K_0)$ such that $h_1 g(v_i) = \lambda_i v_i$ for some $\lambda_i \in K$ $(1 \leq i \leq n)$. As $h_1 g$ fixes Σ, it follows that for all $i > 1$, $\lambda_i = \beta_i \lambda_1$ for some $\beta_i \in K_0$. Hence $h_1 g = \lambda_1 I_n h_2$, where $h_2 \in GL_n(K_0)$ and fixes Σ_0, i.e., $h_2 \in GSp_n(K_0)$ (the general symplectic group, consisting of elements of $GL_n(K_0)$ that preserve A up to a scalar). It is now clear that g can be expressed as the product of a scalar matrix and an element of $GSp_n(K_0)$. Indeed all such products stabilize Σ_0. Hence G_n consists of all such products lying in $SU_n(K)$. The image of G_n in $PGL_n(K)$ is then simply $PGSp_n(K_0) \cap PSU_n(K)$.

We conclude that:

Theorem 2.6. *G_n is a maximal subgroup of $SU_n(q^2)$ containing $Sp(n, K_0)$ and $G_n = (GSp(n, K_0).G_K) \cap SU(n, K)$ where G_K is the group of scalar matrices in $GL(n, K)$. The stabilizer $\overline{G_n}$ of Σ_0 in $PSU_n(q^2)$ is a maximal subgroup of $PSU_n(q^2)$ containing $PSp(n, K_0)$ and $\overline{G_n} = PGSp(n, K_0) \cap PSU(n, K)$.*

With the commuting polarities geometric setting, the main result of Section 2.2.1 can be easily re-formulated in a more geometric way. For instance, to prove that the group $\overline{G_n}$ has exactly two orbits on totally isotropic points of $\mathcal{H}(n-1, q^2)$ one can argue as follows.
Let P_1 be a point of $\mathcal{H}(n-1, q^2) \setminus \Sigma_0$. The tangent hyperplane $P_1^{\perp}$ meets Σ in a $(n-3)$-dimensional subspace Σ_1, see [37]. Then $\Sigma_1^{\perp}$ is a line, say ℓ, of Σ_0. Since G_n is transitive on totally isotropic lines of Σ_0 (they are $((q^n-1)/(q-1))((q^{n-2}-1)/(q^2-1)$ in number), the stabilizer of ℓ in $\overline{G_n}$ is transitive on imaginary points on totally isotropic lines and the number of points on extended lines coincides with the number of points on $\mathcal{H}(n-1, q^2)$, we have proved that $\overline{G_n}$ has two orbits on $\mathcal{H}(n-1, q^2)$.

The maximality proof is essentially the same as in Section 2.2.1 rephrased in projective terms.

2.2.2. *Symplectic subgeometries and their groups.* Here, we study some geometry of the embedding $\overline{G_4} \leq PSU_4(q^2)$. This gives us information on possible intersection sizes of two symplectic groups $PSp_4(q)$ inside the unitary group $PSU_4(q^2)$.

Theorem 2.7. *Two symplectic subgeometries in $\mathcal{H}(3, q^2)$ meet in 0, $q+1$ or $2(q+1)$ points. In the case of $q+1$ points, the points lie on a totally isotropic line. In the case of $2(q+1)$ points, the points lie on a hyperbolic pair. If $q = 2$, no two disjoint symplectic subgeometries exist.*

Proof. We use the duality between $\mathcal{H}(3, q^2)$ and $Q^-(5, q)$ [17], [18], [33]. Symplectic subgeometries of $\mathcal{H}(3, q^2)$ correspond to hyperplanes of $PG(5, q)$ meeting

$Q^-(5,q)$ in a parabolic quadric. Thus the intersection of two subgeometries corresponds to the intersection of two such hyperplanes, which is a solid. There are three types of solid on $Q(4,q)$, accordingly as the quadric is met in an elliptic quadric, a cone or a hyperbolic quadric. A solid meeting the quadric in an elliptic quadric contains 0 lines of the quadric; a solid meeting the quadric in a cone contains $q+1$ lines of the quadric and, finally, a solid meeting the quadric in a hyperbolic quadric contains $2(q+1)$ lines of the quadric. In the second case, the vertex of the cone is on all $q+1$ lines, so this case corresponds to two symplectic subgeometries meeting in $q+1$ points, all on a line. In the third case, each regulus of the hyperbolic quadric corresponds to a hyperbolic line, and as the reguli are opposite, the two hyperbolic lines are polar. The number of parabolic quadrics $Q(4,q)$ on an elliptic quadric $Q^-(3,q)$ in $Q^-(5,q)$ is $q-1$ and $q-1 \geq 2$ if $q > 2$. Thus the case of disjoint subgeometries does not occur if $q = 2$.

The previous theorem yields information on possible intersection sizes of two copies of $PSp_4(q)$ inside $PSU_4(q^2)$. We have the following theorem.

Theorem 2.8. *Let $\overline{G_4}, \overline{G_4'}$ be the stabilizers in $PSU_4(q^2)$ of two symplectic geometries embedded in $\mathcal{H}(3,q^2)$. Set $K = \overline{G_4} \cap \overline{G_4'}$. Then one of the following cases occur. K is either the stabilizer of an elliptic congruence or, the stabilizer of a totally isotropic line or, the stabilizer of a hyperbolic pair. In all cases K is a maximal subgroup of $\overline{G}_4$.*

Proof. It follows from Theorem 2.7 and the classification of maximal subgroups of $PSp_4(q)$, [31], [22]. It turns out that the stabilizer of an elliptic congruence is actually the stabilizer of a complete (q^2+1)-span of $\mathcal{H}(3,q^2)$ as shown in [1], [2].

Notice that $PSU_4(q^2)$ has one or two classes of subgroups isomorphic to $PSp_4(q)$ according as q is even or odd. This depends on the fact that the group $P\Omega_6^-(q)$ has either one or two orbits on parabolic quadric sections of $Q^-(5,q)$.

Problem 2.2. It would be interesting to have information about the intersection of two distinct symplectic subgeometries embedded in $\mathcal{H}(n-1,q^2)$, for $n \geq 6$.

2.3. **Commuting polarities and ovoids of the Hermitian surface.** In the general theory of finite polar spaces important objects are the ovoids. An *ovoid* $\mathcal{O}$ of a non-degenerate Hermitian variety $\mathcal{H}(n-1,q^2)$, $n-1 \geq 3$, is a set of points of $\mathcal{H}(n-1,q^2)$ which has exactly one common point with every generator of $\mathcal{H}(n-1,q^2)$. Here *generator* means a maximal totally singular projective subspace of $\mathcal{H}(n-1,q^2)$. In even dimensions $n-1$, J. A. Thas [39] proved that $\mathcal{H}(n-1,q^2)$ has no ovoid. In odd dimensions $n-1$, the existence problem is still open for $n-1 > 3$, apart from some special cases settled with a negative answer by A. Blokhuis and G.E. Moorhouse (see [32], [41]).

In this Section we are interested in ovoids of $\mathcal{H}(3,q^2)$. As we have seen, lines lying on $\mathcal{H}(3,q^2)$ are the generators of $\mathcal{H}(3,q^2)$, and the size of an ovoid is q^3+1. The intersection of the Hermitian surface $\mathcal{H}(3,q^2)$ with any of its non-tangent planes is a Hermitian curve $\mathcal{H}(2,q^2)$, which is easily seen to be an ovoid (called the *classical ovoid*). The following construction due to Payne and Thas [33] provides non-classical ovoids of $\mathcal{H}(3,q^2)$ for every q. Given a classical ovoid $\mathcal{O}$ of $\mathcal{H}(3,q^2)$, choose two distinct points P_1 and P_2 on $\mathcal{O}$. Then the line ℓ through P_1 and P_2

meets $\mathcal{O}$ in $q+1$ points. Replace these points with those in the intersection of $\mathcal{H}(3,q^2)$ with the polar line of ℓ. The resulting set contains no conjugate pairs of points and has the same size as $\mathcal{O}$. Hence it is an ovoid. More generally, by starting from any ovoid $\mathcal{O}$ of $\mathcal{H}(3,q^2)$, one can consider the hyperbolic line L spanned by two points P_1 and P_2 of $\mathcal{O}$. Then, if all isotropic points of L lie on $\mathcal{O}$, one can show that $(\mathcal{O} \cup L^\perp) \setminus L$ is an ovoid of $\mathcal{H}(3,q^2)$. Indeed, suppose that a point P_1 of $L^\perp$ is conjugate to a point P_2 of $\mathcal{H}(3,q^2)$. Then $P_2 \subseteq P_1^\perp$, so $P_1 = P_1^{\perp\perp} \subseteq P_2^\perp$. Thus $L^\perp \subseteq P_2^\perp$, so $P_2 \subseteq L$. Hence, no point of $L^\perp$ is conjugate to a point of $(\mathcal{O} \cup L^\perp) \setminus L$, and $(\mathcal{O} \cup L^\perp) \setminus L$ is an ovoid in this case. We will call such a procedure *derivation*.

There are so many mutually inequivalent ovoids of $\mathcal{H}(3,q^2)$ known that their classification seems to be possible only under some extra condition(s) (see Section 2.3.2). For instance, in [14] all transitive ovoids of $\mathcal{H}(3,q^2)$, q even, have been classified.

Here we are concerned with the construction of a class of ovoids of $\mathcal{H}(3,q^2)$, q odd, which admit the linear group $PGL_2(q)$ in their automorphism group. Our construction mainly relies on the theory of quadrics that commute with a Hermitian surface $\mathcal{H}(3,q^2)$ of $PG(3,q^2)$. Throughout this paper $q = p^h$ will denote an odd prime power.

Let $\mathcal{B}$ be an orthogonal polarity commuting with the Hermitian polarity $\mathcal{U}$ associated with $\mathcal{H}(3,q^2)$. Set $\mathcal{V} = \mathcal{BU} = \mathcal{UB}$. As we know (see also Segre pointed out [35, p. 136]), $\mathcal{V}$ fixes either $(q+1)^2$ or q^2+1 points on $\mathcal{H}(3,q^2)$, yielding a hyperbolic quadric $Q^+(3,q)$ or an elliptic quadric $Q^-(3,q)$, respectively, embedded in a Baer subgeometry B of $PG(3,q^2)$. In both cases $B \cap \mathcal{H}(3,q^2) = Q^\pm(3,q)$. Notice that the points of $\mathcal{H}(3,q^2)$ fixed under $\mathcal{V}$ are those admitting the same tangent plane with respect to both the orthogonal polarity and the unitary polarity.

Hence, the projective orthogonal group $PGO_4^\epsilon(q)$, $\epsilon = \pm$, associated with $\mathcal{B}$ is seen to be a subgroup of the projective unitary group $PGU_4(q^2)$ associated with $\mathcal{U}$. In terms of forms, let us assume that (V,g) is a 4-dimensional unitary space over $GF(q^2)$. Let F be the subfield of $GF(q^2)$ of index two. Choose a basis $\mathbf{b} = \{v_1,\ldots,v_4\}$ of V such that $g(v_i,v_j) \in F$ for all i and j, and let W denote the F-span of $\mathbf{b}$. The restriction $\bar{g}$ of g to W is a non-degenerate symmetric bilinear form. If $\mathbf{b}$ is an orthonormal basis, then the discriminant of $\bar{g}$ is a square. Replacing v_1 by βv_1, where β is a generator of $GF(q^2)^*$, the discriminant of $\bar{g}$ becomes a non-square. Therefore, we obtain embeddings $O_4^\epsilon < GU_4(q^2)$ for both $\epsilon = +$ and $\epsilon = -$. For more details on these group embeddings, see [29].

2.3.1. *The ovoid construction.* We start this section with the following technical lemma.

Lemma 2.5. *Let $\mathcal{H}(2,q^2)$ be a Hermitian curve of $PG(2,q^2)$. Let B be a Baer subplane of $PG(2,q^2)$ intersecting $\mathcal{H}(2,q^2)$ in a conic C of B. Then the stabilizer G of C in $PGU_3(q^2)$ has three orbits on $\mathcal{H}(2,q^2)$.*

Proof. It is sufficient to show that G has two orbits on the points of $\mathcal{H}(2,q^2) \setminus C$. Given such a point P, by [37, Theorem 2] $P^\perp \cap B$ is a point Q. The two orbits correspond to the following two cases: Q is external to C and Q is internal to C. There are $q(q+1)/2$ external points to C and $q(q-1)/2$ internal points to C in B. By Witt's theorem, the isometry group of C is transitive both on external points and on internal points, and these isometries extend to $\mathcal{H}(2,q^2)$ via the embedding

$PO_3(q) \le PGU_3(q^2)$. Hence, it is sufficient to show in each case that the stabilizer of Q in $PO_3(q)$ is transitive on the points of $\mathcal{H}(2,q^2) \setminus C$ conjugate to Q. If Q is external to C, then $|Q^\perp \cap C| = 2$. Denoting by $\bar{Q}^\perp$ the extension of $Q^\perp$ to $GF(q^2)$, we have $|\bar{Q}^\perp \cap \mathcal{H}(2,q^2)| = q+1$ and the cyclic group of order $q-1$ contained in the stabilizer of Q in $PO_3(q)$ is regular on points of $(\bar{Q}^\perp \cap \mathcal{H}(2,q^2)) \setminus C$. Similarly, if Q is an internal point to C, then $|Q^\perp \cap C| = 0$ and the cyclic group of order $q+1$ contained in the stabilizer of Q in $PO_3(q)$ is regular on the $q+1$ points of $(\bar{Q}^\perp \cap \mathcal{H}(2,q^2)) \setminus C$, proving the result. Note that each of the two orbits distinct from C has size $q(q^2-1)/2$.

Let $\mathcal{H} = \mathcal{H}(3,q^2)$ be the Hermitian surface of $PG(3,q^2)$, q odd, with equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$, where $X_0, \cdots, X_3$ are homogeneous coordinates in $PG(3,q^2)$. Let $\{Q_a \mid a \in GF(q^2)^* := GF(q^2) \setminus \{0\},\ a^{q+1} = 1\}$ denote a family of $q+1$ quadrics of $PG(3,q^2)$, where Q_a has equation $aX_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$. Straightforward computations show that each of these quadrics in hyperbolic and any two of them intersect in the conic $\bar{C}$, given by equation $X_1^2+X_2^2+X_3^2 = 0$, lying in the plane $\bar{\pi}$ with equation $X_0 = 0$. Let π denote the Baer subplane of $\bar{\pi}$ whose normalized point coordinates lie in the subfield $F = GF(q)$, and let $C = \bar{C} \cap \pi$ denote the associated subconic of $\bar{C}$ in π. Furthermore, let $\mathcal{U} = \mathcal{H} \cap \bar{\pi} \cong \mathcal{H}(2,q^2)$ be the Hermitian curve, given by equation $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$, that one obtains by intersecting the Hermitian surface $\mathcal{H}$ with the plane $\bar{\pi}$.

Lemma 2.6. *Using the above notation,* $C = \mathcal{H} \cap \pi = \mathcal{U} \cap \bar{C} = \mathcal{H} \cap \bar{C}$.

Proof. Since $x^{q+1} = x^2$ for $x \in F$, it suffices to show that $\mathcal{U} \cap \bar{C} \subseteq C$. Let $P = (0, x_1, x_2, x_3) \in \mathcal{U} \cap \bar{C}$. Then $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0 = x_1^2 + x_2^2 + x_3^2$. If $x_1 = 0$, then without loss of generality we may assume $x_2 = 1$ and $x_3^{q+1} = -1 = x_3^2$, implying that $x_3 \in F$ and $P \in \pi \cap \mathcal{U} = C$. If $x_1 \neq 0$, then we may assume $x_1 = 1$ and $x_2^{q+1}+x_3^{q+1} = -1 = x_2^2+x_3^2$. In particular, $x_2^{2q}+x_3^{2q} = (-1)^q = -1$, $x_3^{2q} = -(1+x_2^{2q})$ and $x_3^2 = -(1+x_2^2)$. Thus $x_3^{2q+2} = (1+x_2^2)(1+x_2^{2q})$ on the one hand, and $x_3^{2q+2} = (1+x_2^{q+1})^2$ on the other. This implies that $(x_2 - x_2^q)^2 = 0$ and $x_2 \in F$. This further implies $x_3 \in F$ from the equation $x_2^{q+1} + x_3^{q+1} = x_2^2 + x_3^2$, and hence $P \in C$ as before.

From [35, p. 146] each quadric Q_a commutes with $\mathcal{H}(3,q^2)$. In particular, $(q+1)/2$ of them, say $Q_{a_1}, \cdots, Q_{a_{(q+1)/2}}$, are such that $\mathcal{H}(3,q^2) \cap Q_{a_i}$ is an elliptic quadric $\mathcal{O}_i$ embedded in a Baer subgeometry B_i of $PG(3,q^2)$, for $i = 1,2,3,\ldots,(q+1)/2$.

Also, from [35, Section 75] $B_i \cap \mathcal{H}(3,q^2) = \mathcal{O}_i$ for each i. Here, a_i is such that $a_i^{(q+1)/2} = -1$, and hence by appropriate renumbering we may assume that $a_i = \beta^{(q-1)(2i-1)}$ for $i = 1,2,\ldots,(q+1)/2$, where β is a primitive element of $GF(q^2)$.

In fact, one can explicitly describe the above Baer subgeometries B_i. With a_i defined as above, let $\eta_i = \beta^{-(2i-1)}$ for $i = 1,2,\ldots,(q+1)/2$, and consider the Baer subgeometry $B_i = \{(x_0, x_1\eta_i, x_2\eta_i, x_3\eta_i) : x_0, x_1, x_2, x_3 \in F\}$. To show that B_i is the Baer subgeometry described above, it suffices to show that $\mathcal{H} \cap Q_{a_i} \subseteq B_i$.

Lemma 2.7. *Using the above notation, we have* $\mathcal{H} \cap Q_{a_i} \subseteq B_i$ *for* $i = 1,2,\ldots,(q+1)/2$.

Proof. We use the ordered basis $\{(1,0,0,0),(0,\eta_i,0,0),(0,0,\eta_i,0),(0,0,0,\eta_i)\}$, where is i is temporarily fixed, to represent points of $PG(3,q^2)$.
Let $P=(y_0,y_1,y_2,y_3)\in\mathcal{H}\cap Q_{a_i}$, where the homogeneous coordinates are with respect to the above basis. If $y_0=0$, the argument is similar to the proof of Lemma 2.6 and is left as an exercise. Hence we assume $y_0\neq 0$, and thus $y_0=1$ without loss of generality. Therefore the equations $1+\eta_i^{q+1}(y_1^{q+1}+y_2^{q+1}+y_3^{q+1})=0=a_i+\eta_i^2(y_1^2+y_2^2+y_3^2)$ hold. To simplify notation, we let $\lambda=\omega^{2i-1}$, where $\omega:=\beta^{q+1}$ is a primitive element of the subfield F. Note that λ is a nonsquare in F. Direct computation shows that

$$\frac{a_i}{\eta_i^2}=\lambda=\frac{1}{\eta_i^{q+1}}.$$

Hence we have the two equations $y_1^{q+1}+y_2^{q+1}+y_3^{q+1}+\lambda=0$ and $y_1^2+y_2^2+y_3^2+\lambda=0$.

From the second equation and its q^{th} power, we obtain $\frac{y_1^2}{\lambda}+\frac{y_2^2}{\lambda}=-(1+\frac{y_3^2}{\lambda})$ and $\frac{y_1^{2q}}{\lambda}+\frac{y_2^{2q}}{\lambda}=-(1+\frac{y_3^{2q}}{\lambda})$. Multiplying these two equations and clearing denominators yields

$$y_1^{2q+2}+y_2^{2q+2}-y_3^{2q+2}=\lambda^2+\lambda y_3^2+\lambda y_3^{2q}-y_2^2y_1^{2q}-y_1^2y_2^{2q}.$$

On the other hand, from the first equation above we have $(y_3^{q+1}+\lambda)^2=(y_1^{q+1}+y_2^{q+1})^2$, which upon simplification yields

$$y_1^{2q+2}+y_2^{2q+q}-y_3^{2q+q}=\lambda^2+2\lambda y_3^{q+1}-2y_1^{q+1}y_2^{q+1}.$$

Comparing the latter two equations yields $\lambda(y_3-y_3^q)^2=(y_2y_1^q-y_1y_2^q)^2$, implying that $y_3,\ y_2y_1^q\in F$ since λ is a nonsquare in F.

From the original pair of equations we now have $y_1^{q+1}+y_2^{q+1}=y_1^2+y_2^2\in F$. An argument analogous to that given in the proof of Lemma 2.6 then shows that $y_2\in F$ and hence $y_1\in F$, implying that $P\in B_i$ and completing the proof.

It should be remarked that using the above coordinates one can directly (and easily, without the use of [35]) show that $Q_{a_i}\cap\mathcal{H}=Q_{a_i}\cap B_i=\mathcal{H}\cap B_i$ is an elliptic quadric $\mathcal{O}_i$ defined over F. In so doing, the fact that λ is a nonsquare in F is used once again. Note that the coordinate description of the Baer subgeometries implies that for $i\neq j$, $B_i\cap B_j=\pi\cup\{(1,0,0,0)\}$. It then follows from Lemma 2.6 that $\mathcal{O}_i\cap\mathcal{O}_j=C$ for $i\neq j$.

Proposition 2.2. *The union $\mathcal{E}=\bigcup\mathcal{O}_i$, $i=1,\cdots,(q+1)/2$, is a partial ovoid of $\mathcal{H}$ of size $(q^3+q+2)/2$.*

Proof. Since each quadric Q_{a_i} commutes with $\mathcal{H}$, each elliptic quadric $\mathcal{O}_i$ is a partial ovoid of $\mathcal{H}$. Let $\mathcal{O}_i,\mathcal{O}_j$ be two distinct elliptic quadrics of $\mathcal{E}$. Let P be a point of $\mathcal{O}_i$. Then by [37, Theorem 2] $P^\perp$ (with respect to the unitary polarity) meets the Baer subgeometry B_j in a line ℓ. Since Q_{a_i} commutes with $\mathcal{H}$, $P^\perp$ coincides with the tangent plane to the elliptic quadric $\mathcal{O}_i$ at P. Furthermore, since $\mathcal{O}_i$ is an ovoid of B_i, $P^\perp$ meets the plane of the conic C, namely π, in a line that must be external to C. This means that ℓ is an external line to $\mathcal{O}_j$, and $\mathcal{O}_i\cup\mathcal{O}_j$ is a partial ovoid of $\mathcal{H}$. From this it follows that $\mathcal{E}$ is a partial ovoid of $\mathcal{H}$. Since $\mathcal{O}_i\cap\mathcal{O}_j=C$ for $i\neq j$, $|\mathcal{E}|=\frac{1}{2}(q+1)(q^2-q)+(q+1)=\frac{1}{2}(q^3+q+2)$.

It should be remarked that using the above coordinates, one can give a direct proof of this proposition.

The next step in our construction is to adjoin to the partial ovoid $\mathcal{E}$ one of the two orbits of G on the Hermitian curve $\mathcal{U} = \mathcal{H} \cap \bar{\pi}$, where G is the stabilizer of C in $PGU_3(q^2)$ as defined in Lemma 2.5.

Theorem 2.9. *The union of $\mathcal{E}$ and the G-orbit $\mathbf{B}$ on $\mathcal{U} \cong \mathcal{H}(2, q^2)$ corresponding to external points of C is an ovoid $\mathcal{O}$ of $\mathcal{H} = \mathcal{H}(3, q^2)$.*

Proof. Let P be a point of $\mathcal{E}$. Then $P \in \mathcal{O}_i$, for some i with $1 \leq i \leq (q+1)/2$. The plane $P^\perp$ (with respect to the unitary polarity) meets π in a line L. Since $P^\perp$ coincides with the tangent plane to $\mathcal{O}_i$ at P, as observed above, the line L meets B_i in a line r. Since $\mathcal{O}_i$ is a partial ovoid, r cannot be secant or tangent to C, and so r is an external line to C. Thus $r^\perp$ (with respect to the orthogonal polarity) is an internal point of C, and hence P is not conjugate to any point of $\mathbf{B}$. Since $\mathcal{E}$ and $\mathcal{B}$ are both partial ovoids, so is $\mathcal{O} = \mathcal{E} \cup \mathcal{B}$. But $\mathcal{O}_i \cap \bar{\pi} = C$ for all i, and hence $\mathcal{E} \cap \mathcal{B} = \emptyset$. Therefore $|\mathcal{O}| = (q^3+q+2)/2 + (q^3-2)/2 = q^3+1$ and $\mathcal{O}$ is an ovoid of $\mathcal{H}$.

Alternative Proof. It suffices to show that no two points of $\mathcal{O}$ determine a generator of $\mathcal{H}$. To that end, suppose $P \in \mathbf{B}$ and $R \in \mathcal{E}$ such that $m = PR$ is a generator of $\mathcal{H}$. Then $P \in \mathcal{U}$ lies on some secant line ℓ of C. Let $\ell \cap C = \{A_1, A_2\}$. Also $R \in \mathcal{O}_i \setminus C$ for some i. Thus $\Gamma = \langle \ell, m \rangle$ is a plane of B_i, and we let $\Gamma_0 = \Gamma \cap B_i$ denote the corresponding Baer subplane. Now Γ_0 meets $\mathcal{O}_i$ in a conic D_0 defined over the field F. Moreover, Γ meets the hyperbolic quadric Q_{a_i} in a conic D which contains D_0 as a subconic. Hence $D \cap \mathcal{H} = \Gamma \cap Q_{a_i} \cap \mathcal{H} = \Gamma \cap \mathcal{O}_i = \Gamma \cap B_i \cap Q_{a_i} = D \cap B_i = D_0$. Recall that $\Gamma \cap \mathcal{H}$ is a tangent plane to $\mathcal{H}$ at some point $V \in m$. Since A, B and P are distinct collinear points of $\mathcal{H} \cap \Gamma$ (lying on the line ℓ) and since $V \notin \ell$, the distinct lines VA_1, VA_2 and $VP = m$ are necessarily generators of $\mathcal{H}$. But $\mathcal{O}_i$ is a partial ovoid of $\mathcal{H}$, and hence none of these generators can meet $\mathcal{O}_i$ in more than one point. As $D_0 \subset \mathcal{O}_i$ and $D_0 = D \cap \mathcal{H}$, each of these three generators must be tangent to the conic D in the plane Γ. Hence we have three concurrent tangents to a conic in a plane of odd characteristic, a contradiction. This proves the result. □

Proposition 2.3. *The automorphism group $G = Aut(\mathcal{O})$ of $\mathcal{O}$ is a subgroup of the stabilizer of the point $P = (1,0,0,0)$ in $PGU_4(q^2)$. In particular, $G/K \simeq PGL_2(q)$, where K is the cyclic homology group of order $(q+1)$ with center P and axis π.*

Proof. Clearly, G is a subgroup of the stabilizer of P in $PGU_4(q^2)$, the stabilizer being isomorphic to $GU_3(q^2)$. Let K be the homology group with center P and axis π. Then $GU_3(q^2)/K$ is isomorphic to $PGU_3(q^2)$. By construction K fixes $\mathcal{O}$ and thus $G/K < PGU_3(q^2)$. By construction again G stabilizes the conic C. From [31], the stabilizer of C in $PGU_3(q^2)$, which is isomorphic to $PGL_2(q)$, is maximal in $PGU_3(q^2)$. Now, consider the induced action of G on the plane π. The kernel of this action is exactly K. Since $G/K < PGU_3(q^2)$ and G fixes C, [31] implies that $G/K \simeq PGL_2(q)$.

Now, we need the following Lemma due to L. Giuzzi and G. Korchmàros, see [26].

Lemma 2.8. *Let $\mathcal{O}$ be an ovoid of $\mathcal{H}(3, q^2)$. A necessary and sufficient condition for $\mathcal{O}$ to be obtainable from a classical ovoid $\mathcal{H}(2, q^2)$ of $\mathcal{H}(3, q^2)$ through derivation* *is that $\mathcal{O}$ is preserved by the (α, A)-homology group of $PGU_4(q^2)$ for a non-tangent plane α and its pole A.*

Proof. Choose a pair (α, A) consisting of a non-tangent plane α of $\mathcal{H}(3, q^2)$ and the pole A under the unitary polarity H associated to $\mathcal{H}(3, q^2)$. Let $\mathcal{U}$ be the classical ovoid given by all common points of $\mathcal{H}(3, q^2)$ and α. Denote by K the homology group of $PGU_4(q^2)$ with axis α and center A. It is easily verified that if an ovoid $\mathcal{O}$ arises from $\mathcal{U}$ by (multiple) derivation , then Δ preserves $\mathcal{O}$. Conversely, we prove that if K preserves an ovoid $\mathcal{O}$ different form $\mathcal{U}$, then $\mathcal{O}$ can be obtained from $\mathcal{U}$ by (multiple) derivation. Let $P \in \mathcal{O}$ be a point not in α. Then the orbit of P under K consists of the common points of $\mathcal{H}(3, q^2)$ with the line ℓ' joining A and P. Hence, $\mathcal{H}(3, q^2) \cap \ell'$ is contained in $\mathcal{O}$. Let now $\ell'_1, \ldots, \ell'_m$ be the lines through A which meet $\mathcal{H}$ outside α, and let $\ell_1, \ldots, \ell_m$ be the corresponding polar lines. The latter lines are chords of the Hermitian curve $\mathcal{U}$, and any two of them intersect outside $\mathcal{U}$. This proves that $\mathcal{O}$ arises from $\mathcal{U}$ by multiple derivation.

Proposition 2.4. *The ovoid $\mathcal{O}$ constructed above can be obtained from multiple derivation from the classical ovoid.*

Proof. From the previous Proposition $Aut(\mathcal{O})$ contains the cyclic homology group of order $q + 1$ K with center P and axis π. The result now follows from Lemma 2.8.

We call the family of ovoids of $\mathcal{H}(3, q^2)$ constructed above *permutable ovoids*.

2.3.2. *Spreads of $Q^-(5, q)$ and 1-systems of $Q(6, q)$.* Let $\mathcal{L}$ denote the set of lines of $PG(3, q^2)$. As we have already observed, in the Grassmannian mapping $\Phi : \mathcal{L} \to Q^+(5, q^2)$, by which lines of $\mathcal{L}$ are mapped to points of $Q^+(5, q^2)$, the incidence structure consisting of all points and lines of $\mathcal{H}(3, q^2)$ is isomorphic to the dual of the incidence structure consisting of all points and lines of $Q^-(5, q)$. This also proves a classical isomorphism of the corresponding collineations groups. Under such an isomorphism, ovoids $\mathcal{O}$ of $\mathcal{H}(3, q^2)$ and 1-spreads of $Q^-(5, q)$ are equivalent objects. We recall that a 1-spread of $Q^-(5, q)$ is a partition of the point set of $Q^-(5, q)$ into lines. Let L be a fixed line of some 1-spread $\mathcal{S}$ of $Q^-(5, q)$. For each line M of $\mathcal{S}$, the subspace $\langle L, M\rangle$ has dimension three and intersects $Q^-(5, q)$ in a non-singular hyperbolic quadric $Q^+(3, q)$. Let $\mathcal{R}_{L,M}$ be the regulus of $Q^+(3, q)$ containing L and M. Then, $\mathcal{S}$ is *locally Hermitian* with respect to L if $\mathcal{R}_{L,M}$ is contained in $\mathcal{S}$ for all lines M of $\mathcal{S}$ different from L. If $\mathcal{S}$ is locally Hermitian with respect to all the lines of $\mathcal{S}$, then $\mathcal{S}$ is called *Hermitian* (see [40]). In [5] the Hermitian spread of the generalized hexagon $H(q)$ has been characterized as a spread of $Q^-(5, q)$ which is locally Hermitian with respect to all its lines. In [30], it has been proved that Hermitian spreads of $Q^-(5, q)$ are equivalent and, therefore, we can characterize a Hermitian spread of $Q^-(5, q)$ as a spread, which is locally Hermitian with respect to all its lines. In particular, $\mathcal{S}$ is Hermitian if and only if the corresponding ovoid $\mathcal{O}$ of $\mathcal{H}(3, q^2)$ is classical.

We have the following theorem.

Theorem 2.10. *The 1-spread $\mathcal{S} = \mathcal{O}^\Phi$ of $Q^-(5, q)$, where $\mathcal{O}$ is a permutable ovoid constructed as in Theorem 2.9, admits the group $PGL(2, q)$ and it is not locally Hermitian.*

Proof. The first part of the statement is clear. Let H_i be the stabilizer of $\mathcal{O}_i$ in $PGU_4(q^2)$, $i = 1, \ldots, (q+1)/2$. Then $\mathrm{Stab}_{H_i}(C)$ acts transitively on $\mathcal{O}_i \setminus C$ and since Φ permutes the elliptic quadrics $\mathcal{O}_i$, it follows that $Aut(\mathcal{O})$ has three orbits on $\mathcal{O}$, namely C, $\mathcal{E} \setminus C$ and $\mathbf{B}$. Suppose that $\mathcal{S}$ is locally Hermitian with respect to the line L. Let P denote the point of $\mathcal{O}$ corresponding by duality to L. We distinguish three cases depending upon the orbit to which P belongs. Assume first that $P \in C$. Let ℓ be any one of the many hyperbolic lines through P that meets $C \cup \mathbf{B}$ only in P. Then $|\ell \cap (\mathcal{E} \setminus C)| = q$. From [35, p. 136], if ℓ meets an elliptic quadric $\mathcal{O}_i$ in a further point, then $|\ell \cap B_i| = q + 1$; namely, $\ell \cap B_i$ is a line of B_i. Since there are $(q-1)/2$ elliptic quadrics in $\mathcal{E}$ distinct from $\mathcal{O}_i$, there exists j, with $1 \leq j \leq (q+1)/2$, such that $|\ell \cap \mathcal{O}_j| = 3$, a contradiction. Assume next that $P \in \mathcal{E} \setminus C$, and so $P \in \mathcal{O}_i$ for some i, $1 \leq i \leq (q+1)/2$. In this case, there exists an hyperbolic line ℓ through P such that $|\ell \cap (C \cup \mathcal{B})| = 1$. Put $Q = \ell \cap (C \cup \mathbf{B})$. If $Q \in C$, then since ℓ cannot meet $\mathcal{O}_i$ in any further points as above, it follows that $|((\bigcup_{j \neq i} \mathcal{O}_j) \setminus C) \cap \ell| = q$, and again an elliptic quadric contained in $\mathcal{E}$ would contain three collinear points. On the other hand, if $Q \in \mathbf{B}$, then ℓ cannot intersect π in any further points. Arguing as above, ℓ intersects an elliptic quadric $\mathcal{O}_i$ of $\mathcal{E}$ in at least two points, and hence $B_i \cap \ell$ is a line of B_i. The line $B_i \cap \ell$ intersects the Baer subplane containing C in a point and so ℓ lies in π, a contradiction. Finally, assume that $P \in \mathbf{B}$. Then there exists a hyperbolic line through P that does not meet π in any further points, and we fall into the previous case. We conclude that $\mathcal{S}$ cannot be locally Hermitian.

2.4. **Other constructions.** In this Section we show other interesting connections between the geometry of commuting polarities and some combinatorial objects.

2.4.1. *Special sets and translation ovoids.* On February 2003, at the First Irsee conference, Ernie Shult [36] gave a very nice talk and, among various things, he introduced the notion of special set of a Hermitian surface.

Definition 2.2. A *special set* S of $\mathcal{H}(3, q^2)$ is a subset of $q^2 + 1$ points of $\mathcal{H}(3, q^2)$ such that each point of $\mathcal{H}(3, q^2) \setminus S$ is conjugate to 0 or 2 points of S, or equivalently, any three points of S generate a non-tangent plane to $\mathcal{H}(3, q^2)$.

Exercise 2.2. Prove the equivalence in the previous Definition.

Proposition 2.5. *A commuting elliptic quadric $\mathcal{Q}$ embedded in $\mathcal{H}(3, q^2)$, q odd, is a special set.*

Proof. If $P \in \mathcal{H}(3, q^2) \setminus \mathcal{Q}$, then $P^\perp \cap B$, where B is the Baer subgeometry containing $\mathcal{Q}$, is a line that is either external to $\mathcal{Q}$ or secant to $\mathcal{Q}$. Equivalently, if $P, Q, R \in \mathcal{Q}$, then their span over $GF(q)$ is a conic section of $\mathcal{Q}$ and its extension to $GF(q^2)$ turns out to be a non-tangent plane of $\mathcal{H}(3, q^2)$.

In his paper, Shult, asks for the existence of "non-classical" special sets of $\mathcal{H}(3, q^2)$, namely, special sets different from an elliptic quadrics embedded in $\mathcal{H}(3, q^2)$. Of course any elliptic quadric Q embedded in $\mathcal{H}(3, q^2)$ commutes, in the sense that there exists an hyperbolic quadric $\bar{Q}$ in $PG(3, q^2)$ which commutes with $\mathcal{H}(3, q^2)$ and such that $\bar{Q} \cap \mathcal{H}(3, q^2) = Q$. We strongly believe that no non-classical example exists:

Conjecture 2.1. Any special set of $\mathcal{H}(3,q^2)$, q odd, is a commuting elliptic quadric.

Now, choose homogeneous coordinates in $PG(3,q^2)$ in such a way that the Hermitian surface has equation $X_1X_4^q + X_1^qX_4 - X_2^{q+1} - X_3^{q+1} = 0$. Let $\mathcal{Q}$ be a commuting elliptic quadric embedded in $\mathcal{H}(3,q^2)$. The quadric $\mathcal{Q}$ may be chosen as the set of points

$$\{(1,t,t^q,t^{q+1}) : t \in GF(q^2)\} \cup \{P = (0,0,0,1)\}.$$

Indeed, let $GF(q^2) = GF(q)[i]$, where $i^2 = ai + b$, $a, b \in GF(q)$. Then notice that $2i - a$ is not zero (otherwise $i \in GF(q)$). We have that $i^q = a - i$. Notice that $i+(a-i) = a$, and $i(a-i) = -b$. Let $t = ui+v$, $u, v \in GF(q)$. Then $t^q = (a-i)u+v$, and $t^{q+1} = -bu^2+auv+v^2 \in GF(q)$. The polynomial $-bu^2+auv+v^2$ is irreducible over $GF(q)$. Now, take the collineation ϕ given by

$$(X_0, X_1, X_2, X_3) \to ((X_1 - X_2)/(2i - a), ((i - a)X_1 + iX_2)/(2i - a), X_3, X_4).$$

The collineation ϕ takes C to $\{(u, v, -bu^2 + auv + v^2, 1) : u, v \in GF(q)\} \cup \{(0, 0, 1, 0)\}$, which is an elliptic quadric of $PG(3, q)$. Indeed, the tangent plane at the point $(0, 0, 0, 1)$ is $X_3 = 0$, intersecting the quadric in an imaginary line-pair.

Now, the line L_t joining the point P with a point of $\mathcal{Q} \setminus \{P\}$ is

$$\{(\lambda, \lambda t, \lambda t^q, \lambda t^{q+1} + \mu) : t, \lambda, \mu \in GF(q^2)\}.$$

Clearly, since $\mathcal{Q}$ is a partial ovoid of $PG(3,q^2)$, L_t is a secant line to $\mathcal{H}(3,q^2)$ and $L_t \cap \mathcal{H}(3,q^2)$ is the point-set $\{(\lambda, \lambda t, \lambda t^q, \lambda t^{q+1}+\mu) : t, \lambda, \mu \in GF(q^2), \lambda\mu^q+\lambda^q\mu = 0\}$. This way, varying $t \in GF(q^2)$, we get a set $\mathcal{O}$ of q^3+1 points of $\mathcal{H}(3,q^2)$. A useful representation of the set $\mathcal{O}$ is the following:

$$\{(1, t, t^q, t^{q+1} + \alpha) : t, \alpha \in GF(q^2), \alpha^q = -\alpha\}.$$

Proposition 2.6. *The set $\mathcal{O}$ is an ovoid of $\mathcal{H}(3,q^2)$. The automorphism group of $\mathcal{O}$ contains an elementary abelian group of order q^3 fixing the point P and acting transitively on the remaining points of $\mathcal{O}$.*

Proof. We only need to prove that if $P_1 \in L_{t_1}$ and $P_2 \in L_{t_2}$, then P_1 and P_2 cannot be collinear. This follows from the facts that $\mathcal{Q}$ is a special set and the plane $\langle P, P_1, P_2\rangle$ is a secant plane to $\mathcal{H}(3,q^2)$. The second assertion is clear by construction.

Remark 2.4. The ovoid $\mathcal{O}$ gives rise to a so-called locally hermitian 1-spread of $Q^-(5,q)$.

2.4.2. *Commuting conics and designs.* A 2-$(n^3+1, n+1, 1)$-design is called a unital. A classical example of a unital, U, arises from a Hermitian polarity $\mathcal{U}$ of the desarguesian projective plane of order q^2, $PG(2,q^2)$, by taking as points the isotropic points of $\mathcal{U}$ in $PG(2,q^2)$ and as blocks the non-isotropic lines of $\mathcal{U}$ in $PG(2,q^2)$. The incidence relation is that of $PG(2,q^2)$. The unital U is also called *Hermitian curve.* In [25], G. Hölz defined two new 2-designs U_1 and U_2 such that the block set of U_1 is the disjoint union of the block sets of U and U_2. To do this, he used the representation of U in a 4-dimensional projective space over $GF(q)$ due to Buekenhout [6]. Our aim is to give a very short proof of the main theorem in [25], namely:

Theorem 2.11. *Let U be a Hermitian unital in the desarguesian projective plane $PG(2,q^2)$ of odd order q^2.*
i) There exists a 2-$(q^3+1, q+1, q+2)$*-design containing U as a substructure;*
ii) There exists a 2-$(q^3+1, q+1, q+1)$*-design.*

As we have seen in Section 1 (see also [35, p. 142]), a Hermitian curve $\mathcal{H}(2,q^2)$ of $PG(2,q^2)$ admits $q^2(q^3+1)$ commuting conics $\mathcal{P}_2$, on which $PGU_3(q^2)$ acts transitively.

Each of the above conics $\mathcal{P}_2$ meets $\mathcal{H}(2,q^2)$ in a sub-conic of a Baer subplane $PG(2,q)$ of $PG(2,q^2)$. It follows that the number of such sub-conics is $q^2(q^3+1)$ as well, and, each of them contains $q(q+1)/2$ pairs of distinct points of $\mathcal{H}(2,q^2)$. Since $\mathcal{H}(2,q^2)$ has q^3+1 points, and since $PGU_3(q^2)$ acts 2-transitively on points of $\mathcal{H}(2,q^2)$, it is easy to see that through any two distinct points of $\mathcal{H}(2,q^2)$ there pass $q+1$ conics Q commuting with $\mathcal{H}(2,q^2)$.

Now, we define two incidence structures U_1 and U_2, as follows. The points of both U_1 and U_2 are those of $U=\mathcal{H}(2,q^2)$. The blocks of U_2 are the sub-conics of U obtained as above, and the blocks of U_1 are the already mentioned sub-conics and all non-isotropic lines of $\mathcal{U}$ in $PG(2,q^2)$. Of course, U_2 is a 2-$(q^3+1,q+1,q+1)$-design and U_1 is a 2-$(q^3+1,q+1,q+2)$-design, and Hölz's result is proved. For more details, see [9].

Remark 2.5. One can prove (see [13]) that the pointset of U can be partitioned into q^2-q+1 Baer subconics, and such a partition is induced by the Singer cyclic group S of order q^2-q+1 of $PGU_3(q^2)$. In particular, under the action of S on the set of $q^2(q^3+1)$ Baer subconics, q^3+q^2 subsets of size q^2-q+1 are obtained, each yielding a partition of U. We conclude that U_2 is resolvable. Clearly the designs U, U_1 and U_2 are 1-rotational.

2.4.3. *Graphs and commuting polarities.* Let $\mathcal{H}(3,4)$ be the Hermitian surface of $PG(3,4)$ with equation $X_0^3+X_1^3+X_2^3+X_3^3=0$. Then $\Sigma_0=\{(a,a^2,b,b^2): a,b\in GF(4)\}$ is a symplectic subgeometry embedded in $\mathcal{H}(3,4)$. From the general theory we know that $\mathcal{H}(3,4)$ contains 36 symplectic subgeometries. Furthermore, we recall that $\mathcal{H}(3,4)$ and $Q^-(5,2)$ are dual structures; the projective space $PG(5,2)$ has 63 points, the polar space $Q^-(5,2)$ has 27 singular points and 36 non-singular points.

We define a graph $\mathcal{G}$ as follows. Vertices of the graph are the symplectic subgeometries embedded in $\mathcal{H}(3,4)$. Two vertices are adjacent if the corresponding subgeometries meet in a totally isotropic line.

In the orthogonal representation, vertices of the graph are the non singular points of $Q^-(5,2)$, and two vertices are adjacent if and only if they are orthogonal. Indeed, as we have seen in Theorem 2.7, in this case the two symplectic subgeometries correspond to two parabolic sections of $Q^-(5,2)$ meeting in a 3-dimensional cone.

For a nonsingular point P of $Q^-(5,2)$, the 4-space $P^\perp$ has 15 points, and so $\mathcal{G}$ is a graph having order 36 and valency 15.

Proposition 2.7. *The graph $\mathcal{G}$ has diameter* 2 *and girth* 3.

Proof. Recall that the diameter of a graph is the maximum distance occurring between two vertices and the girth is the length of the shortest circuit. The assertion follows from the fact that given any two non-singular points (orthogonal or not) there always exists a third point orthogonal to both.

Proposition 2.8. *The graph $\mathcal{G}$ is distance-transitive*

Proof. Let P_1, P_2, Q_1, Q_2 vertices of $\mathcal{G}$ such that $d(P_1, P_2) = d(Q_1, Q_2) = 1$ This means that P_1 is orthogonal to P_2 and Q_1 is orthogonal to Q_2. Consider the lines $\ell_1 = P_1P_2$ and $\ell_2 = Q_1Q_2$. Then the third point of ℓ_i, $i = 1, 2$, is singular ($Q(P_1 + P_2) = Q(P_1) + Q(P_2) + B(P_1, P_2) = 0$, where B denotes the symmetric bilinear form associated with Q). There exists an element of $\alpha \in PGO_6^-(2)$ sending l_1 to ℓ_2. Twi possibilities occur: $\alpha(P_1) = Q_1$ and $\alpha(P_2) = Q_2$ or $\alpha(P_1) = Q_2$ and $\alpha(P_2) = Q_1$. In the first case we are done. In the second case, there exists an element in the stabilizer of ℓ_2 in $PGO_6^-(2)$ exchanging Q_1 and Q_2. Suppose now that $d(P_1, P_2) = d(Q_1, Q_2) = 2$. Since P_1 is not orthogonal to P_2 and Q_1 is not orthogonal to Q_2, the lines ℓ_1 and ℓ_2 are anisotropic. Again there exists $\alpha \in PGO_6^-(2)$ sending ℓ_1 to ℓ_2.

2.4.4. *Sets of type* $(0, 1, 2, q + 1)_1$. Let $PG(n - 1, q)$, $q = p^h$, p prime, $h \geq 1$, the n-dimensional projective space over $GF(q)$. If $\mathcal{K}$ is a k-set of $PG(n - 1, q)$ and $x_1, \ldots, x_t$ are natural integers such that $0 \leq x_1 < \cdots < x_t$, we say that $\mathcal{K}$ is of type $[x_1, \ldots, x_t]_d$ if each d-dimensional subspace of $PG(n - 1, q)$ meets $\mathcal{K}$ in either x_1, x_2,..., or x_t points. If W_d is a d-dimensional subspace of $PG(n-1, q)$, we say that W_d is x_i-secant to $\mathcal{K}$ if $x_i = |W_d \cap \mathcal{K}|$. In a recent paper [19] Donati and Durante defined three types of subsets of $PG(2, q^2)$ of type $[0, 1, 2, q + 1]_1$ and called them C_F-sets, K-sets and H-sets. Using the well-known Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$, such sets correspond to a 3-dimensional elliptic quadric, a 3-dimensional quadratic cone and a 3-dimensional hyperbolic quadric, respectively. They have size $q^2 + 1$, $q^2 + q + 1$ and $(q + 1)^2$, respectively.

In particular, the C_F-sets can be obtained as the intersection of a Hermitian curve $\mathcal{H}(2, q^2)$ with a suitable Baer sub-pencil of lines centered on a point external to $\mathcal{H}(2, q^2)$.

Here, assuming q odd, we provide new geometric constructions of C_F-sets.

2.4.5. *The first construction.* Let X_0, X_1, X_2 denote homogeneous coordinates in $PG(2, q^2)$.Let $\mathcal{H}(2, q^2)$ be the Hermitian curve of $PG(2, q^2)$, q odd, with equation $X_0X_2^q - 2X_1^{q+1} + X_2X_0^q = 0$.

Consider the family $\mathcal{C} = \{C_a\}$ of non-singular conics of $PG(2, q^2)$:

$$C_a : X_0X_2 - aX_1^2 = 0,$$

where a is a $(q + 1)$-st root of unity in $GF(q^2) \setminus \{0\}$. It is easily seen that C_a commutes with $\mathcal{H}(2, q^2)$ and any two of them share only the two points $P_1 = (1, 0, 0)$ and $P_2 = (0, 0, 1)$. Set $Q_a = C_a \cap \mathcal{H}(2, q^2)$. Since $|\mathcal{C}| = q + 1$, $\mathcal{C}$ is the set of all conics of $PG(2, q^2)$ commuting with $\mathcal{H}(2, q^2)$ and passing through P_1 and P_2. Set $\mathcal{K} = \cup C_a$. Then $\mathcal{K}$ has size $q^2 + 1$. It is easily seen that each point of C_a, $a^{q+1} = 1$ lies on the curve $\mathcal{X}$ with equation $X_0X_2^q - X_1^{q+1} = 0$ which has $q^2 + 1$ points over $GF(q^2)$ and so $\mathcal{K}$ coincides with $GF(q^2)$-rational points of $\mathcal{X}$. The set $\mathcal{K}$ coincides with the C_F-sets studied by Donati and Durante in [19].

2.4.6. *The second construction.* In this section, we give an alternative description of the C_F-set constructed in the previous section.

Let X_0, X_1, X_2, X_3 be homogeneous coordinates in $PG(3, q^2)$, q odd. Consider the two Hermitian surfaces $\mathcal{H}$ and $\mathcal{H}'$ of $PG(3, q^2)$, with equations:

$$(X_0\bar{X}_2 + X_1\bar{X}_3) + (X_2\bar{X}_0 + X_3\bar{X}_1) = 0. \qquad (15)$$

(16) $$(X_0\bar{X}_2 + X_1\bar{X}_3) - (X_2\bar{X}_0 + X_3\bar{X}_1) = 0.$$

From [35], it follows that (15) and (16) are the equations of a pair of commuting Hermitian surfaces. Also, from $\mathcal{H} \cap \mathcal{H}'$ is the ruled variety $\mathcal{S}$ with equation $X_0X_2^q + X_1X_3^q = 0$. In [2], the authors proved that $\mathcal{S}$ comprises a complete (q^2+1)-span of $\mathcal{H}$ [24]. In particular, $\mathcal{S}$ is the determinantal variety represented by the set of all 2×2 matrices $\begin{pmatrix} X_0 & -X_1 \\ \bar{X}_3 & \bar{X}_2 \end{pmatrix}$ such that

$$det \begin{pmatrix} X_0 & -X_1 \\ \bar{X}_3 & \bar{X}_2 \end{pmatrix} = 0.$$

Let π be the plane of $PG(3,q^2)$ with equation $X_1 = -X_3$. Then $\mathcal{H} \cap \pi$ is the Hermitian curve $\mathcal{H}(2,q^2)$ with equation $X_0X_2^q - 2X_1^{q+1} + X_2X_0^q = 0$ and, of course, $\mathcal{K} = \mathcal{S} \cap \mathcal{H}(2,q^2)$ is a subset of $\mathcal{H}(2,q^2)$ consisting of q^2+1 points with equation $X_0X_2^q - X_1^{q+1} = 0$, which is the set of type $[0,1,2,q+1]_1$ studied in the previous section.

Alternatively, we can argue as follows (for any q!!!).

Let $\mathcal{H}$ be the Hermitian surface of $PG(3,q^2)$ with equation

(17) $$\omega X_0X_2^q + X_1X_3^q + \omega^q X_2X_0^q + X_3X_1^q = 0,$$

and let π be the plane $X_3 = -\omega^q X_1$. Then $\mathcal{H} \cap \pi$ is the Hermitian curve $\overline{\mathcal{H}}$ with equation $\omega X_0X_2^q + (\omega + \omega^q)X_1^{q+1} + \omega^q X_2X_0^q = 0$. Consider the following Baer subgeometry of $\mathcal{H}$:

$$\Sigma_0 := \{(\omega^{-1}\rho a^q, b, \rho b^q, a) : a, b \in GF(q^2), \rho^{q+1} = 1\}.$$

Now, consider the family of q^2+1 lines of $\mathcal{H}$ given by the equations $r_t := X_1 = tX_0; X_2 = -\omega^{-q}t^qX_3$, $t \in GF(q^2) \cup \{\infty\}$.

Exercise 2.3. Prove that the lines r_t form a spread of Σ_0 and intersect $\overline{\mathcal{H}}$ in set $\{(1,t,t^{q+1}) : t \in GF(q^2)\} \cup \{(0,0,1)\}$.

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