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## Complete spans and ovoids on Hermitian varieties

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**Abstract**<sup>1</sup>. This is a survey of the two talks presented at the Dipartimento di Matematica of the Università della Basilicata in November of 2002. One talk dealt with complete spans (or partial spreads) on Hermitian varieties, and the other talk concerned the construction of ovoids on a Hermitian surface over odd characteristic.

### 1. INTRODUCTION TO SPANS

In  $PG(r, q^2)$  a *non-singular Hermitian variety* is defined to be the set of all absolute points of a non-degenerate unitary polarity, and it is denoted by  $H(r, q^2)$ . A *t-span* of  $H(r, q^2)$  is a set of  $t$  disjoint generators (that is, spaces of largest dimension lying on  $H(r, q^2)$ ), and it is said to be *complete* if it is not contained in a  $(t + 1)$ -span.

If a  $t$ -span partitions the points of  $H(r, q^2)$ , it is called a *spread*. It is known that for odd dimensions  $r$ ,  $H(r, q^2)$  does not admit a spread (see Table AVI.2 in [8]). It is also known that  $H(4, 4)$  does not admit a spread. For  $q \neq 2$  the existence of spreads in  $H(r, q^2)$ , for even  $r \geq 4$ , is an open problem. When spreads do not exist, quite naturally the emphasis is on constructing complete  $t$ -spans and obtaining reasonable upper and lower bounds for the size  $t$  of such spans. Here we investigate the cases  $r = 3$  and  $r = 5$  in detail, beginning with  $r = 3$ .

A Hermitian surface  $H \cong H(3, q^2)$  has the following properties, for which [7, Chapter 19] is an excellent source:

- (1) The number of points on the Hermitian surface  $H$  is  $(q^2 + 1)(q^3 + 1)$ .
- (2) Any line of  $\Sigma = PG(3, q^2)$  meets  $H$  in 1 or  $q + 1$  or  $q^2 + 1$  points. The latter lines are the *generators* of  $H$ , and they are  $(q + 1)(q^3 + 1)$  in number. The intersections of size  $q + 1$  are Baer sublines, whereas lines meeting  $H$  in one point are called *tangent lines*.
- (3) Through every point  $P$  of  $H$  there pass exactly  $q + 1$  generators, and these generators are coplanar. The plane containing these generators, say  $\pi_P$ , is the polar plane of  $P$  with respect to the unitary polarity defining  $H$ . The tangent lines through  $P$  are precisely the remaining  $q^2 - q$  lines of  $\pi_P$  incident with  $P$ , and  $\pi_P$  is called the *tangent plane* of  $H$  at  $P$ .

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- (4) Every plane of  $\Sigma$  which is not a tangent plane of  $H$  meets  $H$  in a non-degenerate Hermitian curve.
- (5) The number of null polarities commuting with the unitary polarity associated with  $H$  is  $q^2(q^3+1)$ . The importance of such commuting null polarities is that they map generators to generators.

There are several characterizations of Hermitian surfaces as point sets in  $\Sigma = PG(3, q^2)$ . One useful characterization is given by the following theorem.

**Theorem 1.1.** [7, Theorem 19.5.13] *Suppose  $H$  is a set of points in  $PG(3, q^2)$ , where  $q$  is any prime power, such that every line meets  $H$  in  $1, n$  or  $q^2 + 1$  points for some fixed integer  $n$ , where  $1 \leq n \leq q^2 - 1$  and  $n \neq \frac{1}{2}q^2 + 1$ . Suppose further that every point in  $H$  lies on at least one  $n$ -secant. Then  $n = q + 1$  and  $H \cong H(3, q^2)$ .*

As mentioned previously,  $H(3, q^2)$  has no spread. This first was proved by Segre in [12]. In [5] Ebert and Hirschfeld found a lower bound for the size of complete  $t$ -spans of  $H \cong H(3, q^2)$ , and they also proved the following theorem.

**Theorem 1.2.** [5, Theorem 3.2] *The  $q^2 + 1$  generators meeting each of two skew generators of  $H$  form a complete span.*

In their investigation Ebert and Hirschfeld did not find complete spans of the Hermitian surface  $H$  of size less than  $q^2 + 1$ . In this paper we provide a geometric construction for other complete  $q^2 + 1$ -spans of  $H$ , and also discuss examples of smaller complete spans. Our main tool will be general linear complexes (of lines). It is well known that every general linear complex of  $\Sigma_0 = PG(3, q)$  admits a spread; that is, a collection of  $q^2 + 1$  mutually skew lines in the complex. Such a collection of lines is often called a *symplectic spread* of the underlying projective space  $\Sigma_0$ . We shall soon see that symplectic spreads which are extended over a quadratic extension field yield complete spans in an appropriate Hermitian surface  $H(3, q^2)$ .

The incidence structure formed by the points and lines of  $H$  is the dual of the incidence structure formed by the points and lines of an elliptic quadric  $Q^-(5, q)$  of  $PG(5, q)$  [11]. Thus with a  $t$ -span of  $H$  there corresponds a set of points of  $Q^-(5, q)$ , no two of which are on a line of the quadric. Such a set of points is called a *partial ovoid* of  $Q^-(5, q)$ . Hence, if we find a complete  $t$ -span of  $H$ , we also have found a complete partial ovoid of  $Q^-(5, q)$ . Notice that  $Q^-(5, q)$  has no ovoids.

## 2. EXTENDING A GENERAL LINEAR COMPLEX

It is well known that the points of  $\Sigma_0 = PG(3, q)$  and the  $(q + 1)(q^2 + 1)$  lines of a general linear complex of  $\Sigma_0$  form a symplectic polar space  $\mathcal{W} = W(3, q)$ , and conversely any symplectic polar space determines a general linear complex. The following result, which implicitly was known from the point of view of generalized quadrangles, will now be proved from a purely geometrical point of view in the setting of  $\Sigma = PG(3, q^2)$ .

**Theorem 2.1.** *Let  $\mathcal{L}$  be a general linear complex of  $PG(3, q)$ , where  $q$  is any prime power. If  $H$  is the point set of  $PG(3, q^2)$  covered by the lines of  $\mathcal{L}$  when extended to  $PG(3, q^2)$ , then  $H$  is a Hermitian surface.*

This result is proved by a sequence of lemmas. Clearly  $\Sigma_0$  is contained in  $H$  from the properties of a general linear complex. Also note that the extended lines

of  $\mathcal{L}$ , which are  $(q^2 + 1)(q + 1)$  in number, cannot meet in a point of  $\Sigma \setminus \Sigma_0$  since they are lines of  $\Sigma_0$ . This implies that

$$|H| = (q^2 + 1)(q + 1)(q^2 - q) + (q^2 + 1)(q + 1) = (q^2 + 1)(q^3 + 1),$$

which is the number of points lying on a Hermitian surface  $H(3, q^2)$ . Now let  $\ell$  be an arbitrary line of  $\Sigma$  and consider the three possibilities for  $\ell$  with respect to  $\Sigma_0$ , treated as a Baer subgeometry of  $\Sigma$ .

**Lemma 2.1.** *If  $\ell$  is a line of  $\Sigma_0$ , then either  $|\ell \cap H| = q + 1$  or  $|\ell \cap H| = q^2 + 1$ .*

*Proof:* If  $\ell$  is a line of  $\mathcal{L}$ , then  $\ell$  is contained in  $H$  by definition and  $|\ell \cap H| = q^2 + 1$ . Thus assume that  $\ell$  is a line of  $\Sigma_0$  which is not contained in  $\mathcal{L}$ . Then  $|\ell \cap H| \geq q + 1$  as  $\Sigma_0$  is contained in  $H$ . Suppose that  $P \in \ell \cap H$  and  $P \notin \Sigma_0$ . Thus  $P$  lies on a unique extended line  $m$  of  $\mathcal{L}$  by definition of  $H$ . Now  $\ell$  is not a line of  $\mathcal{L}$  and thus  $\ell \neq m$ . Then  $P = \ell \cap m$ , and we have  $P \in \Sigma_0$  since  $\ell$  and  $m$  are lines of  $\Sigma_0$ . This contradicts the assumption that  $P \notin \Sigma_0$ , and hence  $|\ell \cap H| = q + 1$ .

**Lemma 2.2.** *If  $|\ell \cap \Sigma_0| = 1$ , then  $|\ell \cap H| = 1$  or  $|\ell \cap H| = q + 1$ .*

*Proof:* Let  $P = \ell \cap \Sigma_0$ . Note that  $|\ell \cap H| \geq 1$  as  $\Sigma_0$  is contained in  $H$ . Suppose  $Q \in \ell \cap H$  and  $Q \notin \Sigma_0$ . Then  $Q$  lies on a unique line  $m$  of  $\mathcal{L}$  by definition of  $H$ . Clearly  $\ell \neq m$  since  $\ell$  is not a line of  $\Sigma_0$ . Hence the plane  $\pi = \langle \ell, m \rangle$  meets  $\Sigma_0$  in at least the  $q + 1$  points of  $m \cap \Sigma_0$  plus the point  $P$ . Therefore  $\pi$  is a plane of  $\Sigma_0$ .

Now suppose  $Q'$  is another point of  $\ell \cap H$ , other than  $P$ , and hence  $Q' \notin \Sigma_0$ . As above, this would generate a plane  $\pi'$  of  $\Sigma_0$  containing  $\ell$ . Since  $\ell$  is not a line of  $\Sigma_0$ , necessarily  $\pi = \pi'$ . Thus the points of  $\ell \cap H$ , other than  $P$ , arise from extended lines of  $\mathcal{L}$  meeting  $\ell$ , all of which must lie in  $\pi$ . Since every plane of  $\Sigma_0$  contains exactly  $q + 1$  lines of  $\mathcal{L}$ , and these lines form a planar pencil, it follows that  $|\ell \cap H| = q + 1$ . Note that  $P$  lies on one of the lines in the above planar pencil. The result now follows.

**Lemma 2.3.** *If  $\ell \cap \Sigma_0$  is empty, then either  $|\ell \cap H| = 1$  or  $|\ell \cap H| = q + 1$  or  $|\ell \cap H| = q^2 + 1$*

*Proof:* Since  $\ell \cap \Sigma_0$  is empty, there is a unique line of  $\Sigma_0$  passing through each point of  $\ell$ . Moreover, these  $q^2 + 1$  lines are mutually skew and form a regular spread  $\mathcal{S}_0$  of  $\Sigma_0$  (see Theorem 5.3 in [2]). By considering Plücker coordinates for the lines of  $\Sigma_0$  (see Table 1.5.10 in [7]), we see that every regular spread meets a general linear complex in 1,  $q + 1$  or all of its  $q^2 + 1$  lines. Thus  $|\ell \cap H| = 1, q + 1$  or  $q^2 + 1$  from the definition of  $H$ .

Theorem 2.1 now follows from Lemmas 2.1, 2.2, 2.3 and Theorem 1.1.

### 3. CONSTRUCTION OF COMPLETE $(q^2 + 1)$ -SPANS

The above model for  $H = H(3, q^2)$  is now used to construct complete spans.

**Theorem 3.1.** *Let  $\mathcal{S}$  be a symplectic spread of  $PG(3, q)$  for any prime power  $q$ , and let  $\mathcal{L}$  denote the general linear complex containing the lines of  $\mathcal{S}$ . Let  $H$  be the Hermitian surface constructed as in Theorem 2.1. Then the extended lines of  $\mathcal{S}$  form a complete  $(q^2 + 1)$ -span of  $H$ .*

*Proof:* First we note that the extended lines of the symplectic spread  $\mathcal{S}$  are mutually skew generators of  $H$  and thus form a  $(q^2 + 1)$ -span of  $H$ . Suppose that  $\ell$  is a generator of  $H$  disjoint from each extended line of  $\mathcal{S}$ . Since  $\mathcal{S}$  covers all points of  $\Sigma_0$ , necessarily  $\ell \cap \Sigma_0$  is empty. Let  $\mathcal{S}_0$  be the regular spread of  $\Sigma_0$  obtained as transversals to  $\ell$ . Since the points of  $\ell$  are all contained in  $H$ , each point lies on a unique line of  $\mathcal{L}$ , and this is the only line of  $\Sigma_0$  through a given point of  $\ell$ . Therefore  $\mathcal{S}_0$  is a regular spread contained in  $\mathcal{L}$ , and  $\mathcal{S}_0$  and  $\mathcal{S}$  are two symplectic spreads lying in the same linear complex. From [1]  $\mathcal{S}_0$  and  $\mathcal{S}$  share  $1 \bmod p$  lines, where  $q$  is a power of the prime  $p$ . In particular, this implies that some extended line of  $\mathcal{S}$  must meet  $\ell$ , contradicting our assumption on  $\ell$ . Thus the extended lines of  $\mathcal{S}$  form a complete  $(q^2 + 1)$ -span of the Hermitian surface  $H$ , proving Theorem 3.1.

From Theorem 3.1, using the regular and Lüneburg symplectic spreads, it follows that  $Q^-(5, q)$  contains complete partial ovoids of size  $q^2 + 1$  admitting the groups  $Sz(q)$  and  $P\Omega_4^-(q)$ , respectively. However, there are other possibilities for a partial ovoid of size  $q^2 + 1$  on  $Q^-(5, q)$ , as the following remarks will demonstrate.

**Remark 3.1.** Extensive searching using the software package MAGMA [3] revealed many complete spans of size  $q^2 + 1$  for various “small” values of  $q$ . Although the projective equivalences have not been sorted out, there are many different types. Namely, inequivalent symplectic spreads will typically yield inequivalent complete spans via Theorem 3.1. The stabilizer of the spread inherits as a collineation group of the Hermitian surface leaving the associated complete span invariant. In addition, many complete spans of size  $q^2 + 1$ , for various values of  $q$ , were found which have a trivial stabilizer. Moreover, most of these spans are mutually inequivalent under the stabilizer of the Hermitian surface.

**Remark 3.2.** In [5] a lower bound of  $2q + 2$  is proven for the size of any complete span of  $H(3, q^2)$ ,  $q \geq 4$ , whereas a general upper bound of  $q^3 - q^2 + q + 1$  is shown in [15]. Extensive, but not exhaustive, searching for  $q = 4, 5, 7$  and  $8$  indicates that perhaps both upper and lower bounds for the size of a complete span in  $H(3, q^2)$  should be quadratic in  $q$ . Namely, for  $q = 4$  the complete spans found have sizes between  $17 = q^2 + 1$  and  $25 = (q + 1)^2$ . For  $q = 5$  the sizes of the complete spans found are between  $26 = q^2 + 1$  and  $39$ . For  $q = 7$  the sizes found are between  $46$  and  $60$ , while for  $q = 8$  the sizes found are between  $57$  and  $74$ . Note that, in particular, complete spans of size less than  $q^2 + 1$  were found for  $q = 7$  and  $8$ . However, searching was random, so the probability of finding “special” or “unusual” complete spans was low. Hence it is conceivable that there are much bigger or much smaller complete spans than the ones found for the given values of  $q$ . Nonetheless, it appears that the known bounds are not very good, and perhaps the model for  $H(3, q^2)$  given in Theorem 2.1, together with Lemmas 2.1, 2.2 and 2.3, might be useful in improving these bounds. Current efforts in this direction have so far been unsuccessful.

#### 4. GENERALIZATIONS

The above construction can be generalized to the construction of  $t$ -spans of the Hermitian variety  $H(r, q^2)$ , where  $r \geq 5$  is odd. Recall that it is known that spreads do not exist in this case. It is always possible to construct a symplectic subgeometry  $W_{2m-1}(q)$  as a subset of the point set of  $H(2m - 1, q^2)$ . The symplectic geometry

$W_{2m-1}(q)$  has a  $(m-1)$ -spread, and extending scalars from  $GF(q)$  to  $GF(q^2)$  yields a  $q^m + 1$ -span of  $H(2m-1, q^2)$ .

While it is presently unknown if the above spans are complete for arbitrary  $m$ , this is certainly true for  $m = 2, 3$ . As the case  $m = 2$  was previously discussed, we now consider  $m = 3$ .

**Theorem 4.1.** *Let  $\mathcal{S}$  be a spread of  $\mathcal{W} = W(5, q)$  for any prime power  $q$ , and let  $\mathcal{L}$  denote the collection of generators of  $\mathcal{W}$ . Let  $H = H(5, q^2)$  be the Hermitian variety obtained by extending the planes of  $\mathcal{L}$  over the extension field  $GF(q^2)$ , similarly to the construction in Theorem 2.1. Then the extended planes of  $\mathcal{S}$  form a complete  $(q^3 + 1)$ -span of  $H$ .*

The proof of this result (whose details are omitted in this summary) is a somewhat tricky counting argument that first shows no generator of  $H$  is disjoint from the Baer subgeometry  $\Sigma_0$  containing the points of  $\mathcal{W}$ . This argument breaks down if you try to increase the dimension, say to  $H(7, q^2)$ . Namely, it appears that there are generators (solids) of  $H(7, q^2)$  that are disjoint from the Baer subgeometry covered by the embedded symplectic polar space  $W(7, q)$ . Nonetheless, it is still possible that an extended spread of  $W(7, q)$  is a complete span, but so far this is unproven.

A general upper bound of  $q^2(q^2 + q - 1)$  for the size of a complete span in  $H(5, q^2)$  is proved in [15]. Limited computer searching for complete spans in  $H(5, 9)$  did not yield any examples of cardinality larger than  $q^3 + 1 = 28$ . The smallest example found has size 17. While this is very limited data, and again our searching was random rather than targeted, it seems likely that the above upper bound is far from tight. Perhaps, as in the 3-dimensional setting previously discussed, the above model for  $H(5, q^2)$  and the proof technique alluded to above will shed some light on how to improve this upper bound.

## 5. INTRODUCTION TO OVOIDS

An *ovoid*  $\mathcal{O}$  of a non-singular Hermitian variety  $H(r, q^2)$ ,  $r \geq 3$ , is a set of points in  $H(r, q^2)$  which has exactly one point in common with every generator of  $H(r, q^2)$ . In even dimensions  $r$ , Thas [13] proved that  $H(r, q^2)$  has no ovoid. In odd dimensions  $r$ , the existence problem is still open for  $r > 3$ , apart from some special cases settled with a negative answer by Moorhouse (see [10], [16]).

Here we are interested in ovoids of  $H(3, q^2)$ , where the generators are the lines lying on  $H(3, q^2)$  and the size of an ovoid is  $q^3 + 1$ . As previously discussed, the intersection of the Hermitian surface  $H(3, q^2)$  with any of its non-tangent planes is a Hermitian curve  $H(2, q^2)$ , which is easily seen to be an ovoid (called the *classical ovoid*). The following construction, due to Payne and Thas [11], provides non-classical ovoids of  $H(3, q^2)$  for every  $q$ . Given a classical ovoid  $\mathcal{O}$  of  $H(3, q^2)$ , choose two distinct points  $P_1$  and  $P_2$  on  $\mathcal{O}$ . Then the line  $\ell$  through  $P_1$  and  $P_2$  meets  $\mathcal{O}$  in  $q + 1$  points. Replace these points with those in the intersection of  $H(3, q^2)$  with the polar line of  $\ell$ . The resulting set contains no conjugate pairs of points and has the same size as  $\mathcal{O}$ . Hence it is an ovoid. More generally, by starting from any ovoid  $\mathcal{O}$  of  $H(3, q^2)$ , one can consider the hyperbolic line  $L$  spanned by two points  $P_1$  and  $P_2$  of  $\mathcal{O}$ . Then, if all isotropic points of  $L$  lie on  $\mathcal{O}$ , one can show that  $(\mathcal{O} \cup L^\perp) \setminus L$  is an ovoid of  $H(3, q^2)$ . Such a procedure is called *derivation*.

There are so many mutually inequivalent ovoids of  $H(3, q^2)$  known that their classification seems to be possible only under some extra condition(s). For instance, in [4] all transitive ovoids of  $H(3, q^2)$ ,  $q$  even, are classified. We now concentrate on the construction of a class of ovoids of  $H(3, q^2)$ ,  $q$  odd, which involve the linear group  $PGL_2(q)$  in their automorphism group. From now on,  $q = p^h$  will denote an odd prime power.

## 6. THE OVOID CONSTRUCTION

The following technical lemma is easily proved, using elementary properties of conics in odd characteristic.

**Lemma 6.1.** *Let  $H(2, q^2)$  be a Hermitian curve of  $PG(2, q^2)$ . Let  $B$  be a Baer subplane of  $PG(2, q^2)$  intersecting  $H(2, q^2)$  in a conic  $C$  of  $B$ . Then the stabilizer  $G$  of  $C$  in  $PGU_3(q^2)$  has three orbits on  $H(2, q^2)$ .*

Clearly, one of the above orbits is  $C$ . The other two orbits each have size  $q(q^2 - 1)/2$ . One consists of the points of  $H(2, q^2) \setminus C$  that lie on secants to  $C$ , and the other consists of the points of  $H(2, q^2) \setminus C$  that lie on the lines of the Baer subplane  $B$  that are passants of  $C$ .

Introducing coordinates, we let  $H = H(3, q^2)$  be the Hermitian surface of  $PG(3, q^2)$ ,  $q$  odd, with equation  $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$ , where  $X_0, \dots, X_3$  are homogeneous coordinates in  $PG(3, q^2)$ . Let  $\{Q_a \mid a \in GF(q^2) \setminus \{0\}, a^{q+1} = 1\}$  denote a family of  $q+1$  quadrics of  $PG(3, q^2)$ , where  $Q_a$  has equation  $aX_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$ . Straightforward computations show that each of these quadrics is hyperbolic, and any two of them intersect in the conic  $\bar{C}$  with equation  $X_1^2 + X_2^2 + X_3^2 = 0$ . Note that  $\bar{C}$  lies in the plane  $\bar{\pi}$  with equation  $X_0 = 0$ . Let  $\pi$  denote the Baer subplane of  $\bar{\pi}$  whose normalized point coordinates lie in the subfield  $F = GF(q)$ , and let  $C = \bar{C} \cap \pi$  denote the associated subconic of  $\bar{C}$  in  $\pi$ . Furthermore, let  $U = H \cap \bar{\pi} \cong H(2, q^2)$  be the Hermitian curve, given by the equation  $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$ , that one obtains by intersecting the Hermitian surface  $H$  with the plane  $\bar{\pi}$ .

**Lemma 6.2.** *Using the above notation,  $C = H \cap \pi = U \cap \bar{C} = H \cap \bar{C}$ .*

*Proof:* Since  $x^{q+1} = x^2$  for  $x \in F$ , it suffices to show that  $U \cap \bar{C} \subseteq C$ . Let  $P = (0, x_1, x_2, x_3) \in U \cap \bar{C}$ . Then  $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0 = x_1^2 + x_2^2 + x_3^2$ . If  $x_1 = 0$ , then without loss of generality we may assume  $x_2 = 1$  and  $x_3^{q+1} = -1 = x_3^2$ , implying that  $x_3 \in F$  and  $P \in \pi \cap U = C$ . If  $x_1 \neq 0$ , then we may assume  $x_1 = 1$  and  $x_2^{q+1} + x_3^{q+1} = -1 = x_2^2 + x_3^2$ . In particular,  $x_2^{2q} + x_3^{2q} = (-1)^q = -1$ ,  $x_3^{2q} = -(1 + x_2^{2q})$  and  $x_3^2 = -(1 + x_2^2)$ . Thus  $x_3^{2q+2} = (1 + x_2^2)(1 + x_2^{2q})$  on the one hand, and  $x_3^{2q+2} = (1 + x_2^{q+1})^2$  on the other hand. This implies that  $(x_2 - x_2^q)^2 = 0$  and  $x_2 \in F$ . This further implies that  $x_3 \in F$  from the equation  $x_2^{q+1} + x_3^{q+1} = x_2^2 + x_3^2$ , and hence  $P \in C$  as before.

More straightforward computations show that half of the above  $q+1$  hyperbolic quadrics, say  $Q_{a_1}, \dots, Q_{a_{(q+1)/2}}$ , intersect  $H(3, q^2)$  in an elliptic quadric embedded in some Baer subgeometry of  $PG(3, q^2)$ . Namely, let  $a_i = \beta^{(q-1)(2i-1)}$  for  $i = 1, 2, \dots, (q+1)/2$ , where  $\beta$  is a primitive element of  $GF(q^2)$ , so that  $a_i^{(q+1)/2} = -1$  for all  $i$ . Next let  $\eta_i = \beta^{-(2i-1)}$  for  $i = 1, 2, \dots, (q+1)/2$ , and consider the Baer subgeometry  $B_i = \{(x_0, x_1\eta_i, x_2\eta_i, x_3\eta_i) : x_0, x_1, x_2, x_3 \in F\}$ . Similar computations to

those given in the proof of Lemma 6.2 show that  $Q_{a_i} \cap H = Q_{a_i} \cap B_i = H \cap B_i$  is an elliptic quadric  $\mathcal{O}_i$  defined over  $F$ . In so doing, the fact that  $\lambda = \omega^{2i-1}$  is a nonsquare in  $F$  is used, where  $\omega = \beta^{q+1}$  is a primitive element of the subfield  $F$ . Note that the coordinate description of the Baer subgeometries implies that for  $i \neq j$ ,  $B_i \cap B_j = \pi \cup \{(1, 0, 0)\}$ . It then follows from Lemma 6.2 that  $\mathcal{O}_i \cap \mathcal{O}_j = C$  for  $i \neq j$ . Using these coordinates, one then easily proves that the union  $\mathcal{E} = \bigcup \mathcal{O}_i$ ,  $i = 1, \dots, (q+1)/2$ , is a partial ovoid of  $H$  of size  $(q^3 + q + 2)/2$ .

The next step is to adjoin to the partial ovoid  $\mathcal{E}$  one of the two orbits of  $G$  on the Hermitian curve  $U = H \cap \bar{\pi}$ , where  $G$  is the stabilizer of  $C$  in  $PGU_3(q^2)$  as defined in Lemma 6.1.

**Theorem 6.1.** *The union of  $\mathcal{E}$  and the  $G$ -orbit  $\mathcal{B}$  on  $U \cong H(2, q^2)$  corresponding to the secants of  $C$  is an ovoid  $\mathcal{O}$  of  $H = H(3, q^2)$ .*

*Proof:* It suffices to show that no two points of  $\mathcal{O}$  determine a generator of  $H$ . To that end, suppose  $P \in \mathcal{B}$  and  $R \in \mathcal{E}$  such that  $m = PR$  is a generator of  $H$ . Then  $P \in U$  lies on some secant line  $\ell$  of  $C$ . Let  $\ell \cap C = \{A_1, A_2\}$ . Also  $R \in \mathcal{O}_i \setminus C$  for some  $i$ . Thus  $\Gamma = \langle \ell, m \rangle$  is a plane of  $B_i$ , and we let  $\Gamma_0 = \Gamma \cap B_i$  denote the corresponding Baer subplane. Now  $\Gamma_0$  meets  $\mathcal{O}_i$  in a conic  $D_0$  defined over the field  $F$ . Moreover,  $\Gamma$  meets the hyperbolic quadric  $Q_{a_i}$  in a conic  $D$  which contains  $D_0$  as a subconic. Hence  $D \cap H = \Gamma \cap Q_{a_i} \cap H = \Gamma \cap \mathcal{O}_i = \Gamma \cap B_i \cap Q_{a_i} = D \cap B_i = D_0$ . Recall that  $\Gamma \cap H$  is a tangent plane to  $H$  at some point  $V \in m$ . Since  $A_1, A_2$  and  $P$  are distinct collinear points of  $H \cap \Gamma$  (lying on the line  $\ell$ ) and since  $V \notin \ell$ , the distinct lines  $VA_1, VA_2$  and  $VP = m$  are necessarily generators of  $H$ . But  $\mathcal{O}_i$  is a partial ovoid of  $H$ , and hence none of these generators can meet  $\mathcal{O}_i$  in more than one point. As  $D_0 \subset \mathcal{O}_i$  and  $D_0 = D \cap H$ , each of these three generators must be tangent to the conic  $D$  in the plane  $\Gamma$ . Hence we have three concurrent tangents to a conic in a plane of odd characteristic, a contradiction. This proves the result.

**Proposition 6.1.** *The automorphism group  $J = \text{Aut}(\mathcal{O})$  of  $\mathcal{O}$  is a subgroup of the stabilizer of the point  $P = (1, 0, 0, 0)$  in  $PGU_4(q^2)$ . In particular,  $J/K \cong PGL_2(q)$ , where  $K$  is the cyclic homology group of order  $(q+1)$  with center  $P$  and axis  $\pi$ .*

*Proof:* Clearly,  $J$  is a subgroup of the stabilizer of  $P$  in  $PGU_4(q^2)$ , the stabilizer being isomorphic to  $GU_3(q^2)$ . Let  $K$  be the homology group with center  $P$  and axis  $\pi$ . Then  $GU_3(q^2)/K$  is isomorphic to  $PGU_3(q^2)$ . By construction  $K$  fixes  $\mathcal{O}$ , and thus  $J/K \leq PGU_3(q^2)$ . Again, by construction  $J$  stabilizes the conic  $C$ . From [9], the stabilizer of  $C$  in  $PGU_3(q^2)$ , which is isomorphic to  $PGL_2(q)$ , is maximal in  $PGU_3(q^2)$ . Now, consider the induced action of  $J$  on the plane  $\pi$ . The kernel of this action is exactly  $K$ . Since  $J/K \leq PGU_3(q^2)$  and  $J$  fixes  $C$ , [9] implies that  $J/K \cong PGL_2(q)$ .

**Proposition 6.2.** *The ovoid  $\mathcal{O}$  constructed above can be obtained from multiple derivation from the classical ovoid.*

*Proof:* From the previous Proposition  $\text{Aut}(\mathcal{O})$  contains the cyclic homology group  $K$  of order  $q+1$  with center  $P$  and axis  $\pi$ . The result now follows from [6]

## 7. SPREADS OF $Q^-(5, q)$

As remarked previously, the incidence structure formed by the points and lines of  $H(3, q^2)$  is the dual of the incidence structure formed by the points and lines

of the elliptic quadric  $Q^-(5, q)$  of  $PG(5, q)$ . Under this correspondence, ovoids of  $H(3, q^2)$  and 1-spreads of  $Q^-(5, q)$  are equivalent objects. Recall that a 1-spread of  $Q^-(5, q)$  is a partition of the point set of  $Q^-(5, q)$  into lines. Let  $L$  be a fixed line of some 1-spread  $\mathcal{S}$  of  $Q^-(5, q)$ . For every other line  $M$  of  $\mathcal{S}$ , the subspace  $\langle L, M \rangle$  has dimension three and intersects  $Q^-(5, q)$  in a non-singular hyperbolic quadric  $Q^+(3, q)$ . Let  $\mathcal{R}_{L, M}$  be the regulus of  $Q^+(3, q)$  containing  $L$  and  $M$ . One says that  $\mathcal{S}$  is *locally Hermitian with respect to  $L$*  if  $\mathcal{R}_{L, M}$  is contained in  $\mathcal{S}$  for all lines  $M$  of  $\mathcal{S}$  different from  $L$ . If  $\mathcal{S}$  is locally Hermitian with respect to all the lines of  $\mathcal{S}$ , then  $\mathcal{S}$  is said to be *Hermitian* (see [14]). One can show that  $\mathcal{S}$  is Hermitian if and only if the corresponding ovoid of  $H(3, q^2)$  is classical.

A careful analysis of the ovoid  $\mathcal{O}$  constructed in Theorem 6.1 shows that one can always find a chord (Baer subline) of  $H(3, q^2)$  containing two points of  $\mathcal{O}$  which is not completely contained in  $\mathcal{O}$ . In fact, one can find such a chord passing through any particular point of  $\mathcal{O}$ , thus yielding the final result.

**Theorem 7.1.** *The 1-spread  $\mathcal{S}$  of  $Q^-(5, q)$  corresponding to the ovoid  $\mathcal{O}$  constructed in Theorem 6.1 admits the group  $PGL(2, q)$  and is not locally Hermitian.*

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