

## Coupled system of semilinear wave equations

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**Abstract**<sup>1</sup>. In this note we deal with coupled systems of semilinear wave equations. We shall show that the discrepancy of the propagation speeds of the waves has a major effect on the behavior of the solution to the system. In Section 3 the global existence result for the system with multiple speeds is established for a wider class of nonlinearity, provided the interaction via the nonlinearity is strong in some sense. We shall discuss the case where the last assumption does not fulfilled in Section 4. What is interesting in this case is the fact that the order of the propagation speeds comes into play.

### 1. INTRODUCTION

In this note we consider the following coupled systems of semilinear wave equations with multiple speeds of propagation in three space dimensions:

$$(1) \quad (\partial_t^2 - c_j^2 \Delta)u_j = F_j(u_1, u_2, \dots, u_N) \quad (t, x) \in [0, \infty) \times \mathbb{R}^3,$$

with small initial data

$$(2) \quad u_j(0, x) = \varphi_j(x), \quad \partial_t u_j(0, x) = \psi_j(x), \quad x \in \mathbb{R}^3,$$

where  $j = 1, 2, \dots, N$ ,  $c_j > 0$ ,  $\varphi_j \in C^1(\mathbb{R}^3)$ ,  $\psi_j \in C(\mathbb{R}^3)$ , and  $F_j$  is a Lipschitz continuous function vanishing at the origin  $(u_1, u_2, \dots, u_N) = 0$ . The main question here is formulated as follows.

*Problem:* Find sharp conditions for the *small data global existence* and *blowup* for (1). Here *small data global existence* means that the initial value problem (1)–(2) admits a unique global (mild) solution for any “small” initial data. While, *blowup* means that *small data global existence* does NOT hold. In other words, one can find a pair of initial data  $(\varphi_j, \psi_j)$  such that the lifespan of the corresponding solution is finite.

We are going to answer the above problem based mainly on the joint work with Prof. M. Ohta [16]. As for the same question for the case where the nonlinearity depends also on the derivatives of unknown function, we only refer [3, 11, 12, 19] and the references cited therein, and do not go into this direction.

This note is organized as follows. In the rest of this section, we give notations, and discuss the single wave equations in order to present a general idea for our

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problem. In section 2, we give conditions for *small data global existence* under such an assumption that the vanishing orders of the nonlinearities  $F_j$  are the same, together with *non-resonant* assumption on them (see (12) and (13) below). In section 3, we mention about the case where the above assumptions are not valid. In this case, the characterization of the behavior of solutions to (10) will be much complicated.

### 1.1. Notation.

- By  $f \lesssim g$  we mean there is a positive constant  $C$ , independent of  $f, g$  and their variables, such that  $f \leq Cg$ .
- For any  $x \in \mathbb{R}^n$ , the symbol  $\langle x \rangle$  denotes  $\sqrt{1 + |x|^2}$ .
- We put  $a \vee b := \max\{a, b\}$  and  $a \wedge b := \min\{a, b\}$  for  $a, b \in \mathbb{R}$ .
- For a set  $A$ ,  $\chi_A$  denotes the characteristic function of  $A$ .

**1.2. Single wave equation.** In this subsection we recall some known results for the initial value problem to single semilinear wave equations in general spatial dimensions  $n(\geq 2)$ :

$$(3) \quad (\partial_t^2 - c^2 \Delta)u = |u|^p, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

$$(4) \quad u(0, x) = \varphi(x), \quad \partial_t u(0, x) = \psi(x), \quad x \in \mathbb{R}^n,$$

where  $p > 1$ ,  $c > 0$ ,  $\varphi \in C_0^\infty(\mathbb{R}^3)$ , and  $\psi \in C_0^\infty(\mathbb{R}^3)$ . For the problem W. Strauss [24] introduced a number  $p_0(n)$  which is the positive root of the following quadratic equation:

$$(5) \quad (n-1)p^2 - (n+1)p - 2 = 0.$$

For instance,  $p_0(3) = 1 + \sqrt{2}$ . The importance of this number is the fact that it plays the role as the critical exponent for the problem (3)–(4). Though the number seems to be strange at a first glance, one can understand it based on the *scale invariance* of the semilinear equation. The *scale invariance* means that if  $u(t, x)$  is a solution of (3), then  $D_{\lambda, p}u(t, x)$  also satisfies the same equation for any  $\lambda > 0$ , where we denoted by  $D_{\lambda, p}u(t, x)$  the dilation of  $u(t, x)$  defined by

$$(6) \quad D_{\lambda, p}u(t, x) = \lambda^{m_0} u(\lambda t, \lambda x), \quad m_0 := \frac{2}{p-1} \quad (\lambda > 0).$$

Then the quadratic equation (5) follows from the self-similarity of the function

$$\dot{w}(r, t) = (t+r)^{-\frac{n-1}{2}} |ct-r|^{-\left(\frac{n-1}{2}p - \frac{n+1}{2}\right)}, \quad r, t \in [0, \infty),$$

that is, from the dilation invariance  $D_{\lambda, p}\dot{w}(|x|, t) = \dot{w}(|x|, t)$  for any  $\lambda > 0$ .

To understand the function  $\dot{w}(r, t)$ , let us take the specific spatial dimensions  $n = 3$ . Then the solution  $u_0(t, x)$  of the homogeneous wave equation  $(\partial_t^2 - c^2 \Delta)u = 0$  with (4) satisfies

$$|u_0(t, x)| \lesssim \langle t + |x| \rangle^{-1}, \quad \text{supp } u_0(t, x) \subset \{(t, x) : |ct - |x|| \leq R\},$$

where  $R$  is the diameter of the support of the initial data. Therefore, if we denote by  $L_c[F](t, x)$  the solution operator of the inhomogeneous wave equation  $(\partial_t^2 - c^2 \Delta)u = F$  with zero initial data, then straight forward computation gives

$$|L_c[|u_0|^p](t, x)| \lesssim \langle t + |x| \rangle^{-1} \langle ct - |x| \rangle^{-(p-2)},$$

provided  $p > p_0(3)(> 2)$ . We see that the function appeared in the right hand side is just the inhomogeneous version of  $\dot{w}(|x|, t)$  with  $n = 3$ . This explains that the behavior of the solution to (3)–(4) is well characterized by  $\dot{w}(|x|, t)$ .

Now we briefly mention known results. It was shown that *blowup* occurs for either  $1 < p < p_0(n)$  or  $p = p_0(n)$  and  $n = 2, 3$  (see T. Sideris [23], J. Schaeffer [22]). Notice that due to the “bad” sign of the nonlinearity, the solution likely blows up for small values of  $p$ .

On the other hand, the existence part was firstly solved by F. John [10] for  $n = 3$ . After that, there are so many contributions on this issue. (See e.g., [8, 9, 2, 20, 28] and the references cited therein). Then V. Georgiev, H. Lindblad and C. Sogge [7] showed that *small data global existence* holds for general  $n \geq 2$  by proving the weighted version of Strichartz estimate, when  $p_0(n) < p < (n+3)/(n-1)$  and the initial data is compactly supported. The proof of the weighted Strichartz estimate is simplified by D. Tataru [26] by using the Fourier transform on the hyperboloid. Based on the estimate, P. D’Ancona, V. Georgiev and the author [4] was able to show *small data global existence* without assuming the compactness of the support of the initial data. Of course, the initial data should be decay in some sense. In fact, if we take the initial data in such a way that

$$(7) \quad \varphi(x) \equiv 0, \quad \psi(x) \geq \varepsilon \langle x \rangle^{-m-1}$$

with  $m < m_0 = 2/(p-1)$ , then *blowup* occurs even for  $p > p_0(n)$  and arbitrary small  $\varepsilon$ . (see H. Takamura [25]). Here  $m_0$  is the number related to the exponent of the *scale invariance* for (3). On the contrary, if we consider the space  $Y(m)$  defined by

$$(8) \quad Y(m) = \{(\varphi, \psi) \in C^1(\mathbb{R}^n) \times C(\mathbb{R}^n) : \|(\varphi, \psi)\|_{Y(m)} < \infty\},$$

$$\|(\varphi, \psi)\|_{Y(m)} = \sum_{|\alpha| \leq 1} \sup_{x \in \mathbb{R}^n} \langle x \rangle^{m+|\alpha|} |\partial_x^\alpha \varphi(x)| + \sup_{x \in \mathbb{R}^n} \langle x \rangle^{m+1} |\psi(x)|,$$

then *small data global existence* holds for such initial data  $(\varphi, \psi) \in Y(m)$  with  $m \geq m_0$ , provided  $p > p_0(n)$ . This result is obtained by F. Asakura [1], H. Pecher [21] for  $n = 3$  and by K. Kubota [18], K. Tutaya [27] for  $n = 2$  (see also [13] for the case where  $n \geq 4$  and the initial data is radially symmetric).

## 2. COUPLED SYSTEMS

We turn our attention to the coupled systems (1). The following system was studied by D. Del Santo, V. Georgiev and E. Mitidieri [5] and found the critical curve in  $p$ - $q$  plane when  $c_1 = c_2$ :

$$(9) \quad \begin{cases} (\partial_t^2 - c_1^2 \Delta) u_1 = |u_2|^p, & (t, x) \in [0, \infty) \times \mathbb{R}^n, \\ (\partial_t^2 - c_2^2 \Delta) u_2 = |u_1|^q, & (t, x) \in [0, \infty) \times \mathbb{R}^n \end{cases}$$

where  $p, q > 1$  and  $n \geq 2$ . (See also [6, 14] and the references cited therein). In [15], it was shown that even if  $c_1 \neq c_2$ , the critical curve does not change. This means that the unequal propagation speeds does not help us to show *small data global existence* for wider values  $p$  and  $q$ , hence one may suppose that the interaction in the right hand side is too weak.

Here we consider the case where the interaction in the nonlinearity is stronger than the above system. More precisely, we specify the nonlinear function  $F_j$  ( $j = 1, \dots, N$ ;  $N \geq 2$ ) as

$$F_j(u_1, u_2, \dots, u_N) = \sum_{k,l=1}^N A_j^{kl} |u_k|^{p_{jkl}} |u_l|^{q_{jkl}},$$

where  $p_{jkl}, q_{jkl} \geq 1$  and  $A_j^{kl}$  are constants. Then our problem takes the form of

$$(10) \quad (\partial_t^2 - c_j^2 \Delta) u_j = \sum_{k,l=1}^N A_j^{kl} |u_k|^{p_{jkl}} |u_l|^{q_{jkl}}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^3,$$

$$(11) \quad u_j(0, x) = \varphi_j(x), \quad \partial_t u_j(0, x) = \psi_j(x), \quad x \in \mathbb{R}^3.$$

To simplify the situation, we assume that there is  $\alpha \geq 1$  such that

$$(12) \quad p_{jkl} + q_{jkl} \equiv \alpha + 1 \quad \text{for any } j, k, l = 1, \dots, N.$$

This means that the degree of each term of the nonlinearity coincides each other. While, as we shall see in section 3, if they are different from each other, then the characterization of the problem becomes more complicated.

When the problem (10)–(11) with (12) has a common propagation speed, it follows from the result of the single wave equation that *small data global existence* holds if  $\alpha > \sqrt{2}$  and the initial data decays suitably, and that *blowup* occurs if  $1 < \alpha \leq \sqrt{2}$  (recall that  $p_0(3) = 1 + \sqrt{2}$ ). Our purpose here is to exploit the effect of the discrepancy between the propagation speeds. In other words, the question is if it is possible to show *small data global existence* even if  $\alpha$  is less than  $\sqrt{2}$ . The crucial point is to make use of the interaction between  $u_k$  and  $u_l$  with  $k \neq l$ . Thus we need to exclude the self-interaction terms from  $F_j$ . To do so, we shall pose the *non-resonant* assumption on  $F_j$ , i.e.

$$(13) \quad A_j^{kk} = 0 \quad \text{for any } j, k = 1, \dots, N.$$

In what follows, we shall treat the problem (10) in the integral form:

$$(14) \quad u_j = K_{c_j}[\varphi_j, \psi_j] + L_{c_j}[F_j(u_1, \dots, u_N)] \quad \text{in } [0, \infty) \times \mathbb{R}^3,$$

where we have set

$$(15) \quad K_c[\varphi, \psi](t, x) = \frac{t}{4\pi} \int_{|\omega|=1} \psi(x + ct\omega) d\omega + \partial_t \left( \frac{t}{4\pi} \int_{|\omega|=1} \varphi(x + ct\omega) d\omega \right),$$

$$(16) \quad L_c[F](t, x) = \int_0^t \frac{t-s}{4\pi} \int_{|\omega|=1} F(s, x + c(t-s)\omega) d\omega ds.$$

It follows from (8), Lemmas 2.1 and 2.2 with  $\nu = 0$  in [16] that

$$(17) \quad |K_c[\varphi, \psi](t, x)| \lesssim \|(\varphi, \psi)\|_{Y(m)} \langle t + |x| \rangle^{-1} \langle ct - |x| \rangle^{-(m-1)}.$$

Therefore it is natural to consider the weighted  $L^\infty$ -space  $X$  defined below, following [10]:

$$(18) \quad X = \{u = (u_1, u_2, \dots, u_N) \in C([0, \infty) \times \mathbb{R}^3)^N : \\ \|u\|_X \equiv \max_{1 \leq j \leq N} \|u_j\|_{X(c_j, 1, m-1)} < \infty\},$$

$$(19) \quad \|v\|_{X(c, \mu, \kappa)} = \sup_{(t, x) \in [0, \infty) \times \mathbb{R}^3} \langle t + |x| \rangle^\mu \langle ct - |x| \rangle^\kappa |v(t, x)|,$$

where  $c > 0$ ,  $\mu, \kappa \geq 0$ . Our main result of this section is stated as follows.

**Theorem 2.1.** *Let  $c_1, \dots, c_N$  be different from each other and let  $1 \leq \alpha < 2$ . Assume that  $(\varphi_j, \psi_j) \in Y(m)$  ( $i = 1, \dots, N$ ) with  $\varepsilon := \max_{1 \leq j \leq N} \|(\varphi_j, \psi_j)\|_{Y(m)}$  and that (12) and (13) hold.*

- (i) Let  $1 < \alpha < 2$ . If  $m \geq 2/\alpha$ , then small data global existence holds. More precisely, if  $\varepsilon$  is sufficiently small, then there exists a unique global solution  $(u_1, \dots, u_N)$  of (14) in  $X$ .
- (ii) Let  $\alpha = 1$ . If  $m > 2$ , then small data global existence holds. While, if  $m = 2$ , then blowup occurs. Moreover, if  $\varepsilon$  is sufficiently small, then there is a constant  $A > 0$ , independent of  $\varepsilon$ , such that

$$(20) \quad T(\varepsilon) \leq \exp(A\varepsilon^{-1}),$$

where  $T(\varepsilon)$  stands for the least upper bound of all  $T > 0$  such that there exists a unique solution  $u(t, x)$  of (14) in  $[0, T) \times \mathbb{R}^3$  satisfying  $\chi_{[0, T)}(t)u(t, x) \in X$ . Besides, if  $\varepsilon$  is sufficiently small, then there is a constant  $B > 0$ , independent of  $\varepsilon$ , such that

$$(21) \quad T(\varepsilon) \geq \exp(B\varepsilon^{-1}).$$

- Remark 2.1.** (i) Since the theorem says that *small data global existence* holds for any  $1 \leq \alpha < 2$  if the initial data is suitably chosen, we see that the discrepancy between the propagation speeds has a major effect on the behavior of the solution under the *non-resonant* assumption (13).
- (ii) The statements of the theorem remains true, even if we replace the nonlinear terms  $|u_k|^{p_{jkl}}|u_l|^{q_{jkl}}$  in (10) by  $|u_k|^{p_{jkl}-1}|u_l|^{q_{jkl}-1}u_1u_2$ .

*Outline of the Proof:* Thanks to the assumption (12), the existence part of the theorem essentially follows from the corresponding results of the author and K.Tsugawa [17] for  $1 < \alpha < 2$ , and the author and M.Ohta [15] for  $\alpha = 1$ . We shall look for a solution of (14) in  $X$  defined by (18). Employing the explicit representation formula (16), we are able to establish the following bilinear estimate:

**Proposition 2.1.** Let  $a_0, a_1, a_2 > 0$  and  $\mu_1, \mu_2, \kappa_1, \kappa_2 \geq 0$ . If  $a_1 \neq a_2$ , then we have

$$(22) \quad \begin{aligned} \langle t + |x| \rangle \langle a_0 t - |x| \rangle^{\kappa_0} |L_{a_0}[uv](t, x)| &\lesssim \\ &\lesssim [1 + M_0 \log(1 + \langle a_0 t - |x| \rangle)] \|u\|_{X(a_1, \mu_1, \kappa_1)} \|v\|_{X(a_2, \mu_2, \kappa_2)}, \end{aligned}$$

where  $\kappa_0$  and  $M_0$  are defined by

$$\kappa_0 = \begin{cases} \mu_1 + \mu_2 - 2 + (\kappa_1 \wedge \kappa_2) & \text{if } \kappa_1 \vee \kappa_2 \geq 1, \mu_1 + \mu_2 + (\kappa_1 \wedge \kappa_2) > 2 \\ \mu_1 + \mu_2 - 3 + \kappa_1 + \kappa_2 & \text{if } \kappa_1 \vee \kappa_2 < 1, \mu_1 + \mu_2 + \kappa_1 + \kappa_2 > 3 \end{cases},$$

$$M_0 = \begin{cases} 1 & \text{if } \kappa_1 \vee \kappa_2 = 1 \\ 0 & \text{if } \kappa_1 \vee \kappa_2 \neq 1 \end{cases}.$$

If  $\kappa_1 \vee \kappa_2 \neq 1$ , then the above proposition follows from Proposition 2.1 in [16]. Besides, the other case can be treated in a similar fashion by using Lemma 2.2 in [16]. In the course of the proof, the main assumption  $a_1 \neq a_2$  is used in the following way: If we take  $(t, x)$  near the light cone  $a_1 t = |x|$ , then the weight  $\langle a_2 t - |x| \rangle$  behaves like  $\langle t + |x| \rangle$ . On the contrary, if  $(t, x)$  is close to the other light cone  $a_2 t = |x|$ , then the weight  $\langle a_1 t - |x| \rangle$  is equivalent to  $\langle t + |x| \rangle$ . In conclusion, one can extract some additional decay in  $\langle t + |x| \rangle$ , provided  $a_1 \neq a_2$ .

Applying the estimate as  $a_0 = c_j$ ,  $a_1 = c_k$ ,  $a_2 = c_l$ ,  $\mu_1 = p_{jkl}$ ,  $\mu_2 = q_{jkl}$ ,  $\kappa_1 = p_{jkl}(m-1)$ , and  $\kappa_2 = q_{jkl}(m-1)$ , we have

$$\begin{aligned} \langle t + |x| \rangle \langle c_j t - |x| \rangle^{\kappa_0} |L_{c_j}[|u_k|^{p_{jkl}}|u_l|^{q_{jkl}}](t, x)| &\lesssim \\ &\lesssim [1 + M_0 \log(1 + \langle c_j t - |x| \rangle)] \| |u_k|^{p_{jkl}} \|_{X(c_k, p_{jkl}, \kappa_1)} \| |u_l|^{q_{jkl}} \|_{X(c_l, q_{jkl}, \kappa_2)}, \end{aligned}$$

from which one can derive

$$(23) \quad \|L_{c_j} [|u_k|^{p_{jkl}} |u_l|^{q_{jkl}}]\|_{X(c_j,1,m-1)} \lesssim \|u_k\|_{X(c_k,1,m-1)}^{p_{jkl}} \|u_l\|_{X(c_l,1,m-1)}^{q_{jkl}},$$

provided either  $m \geq 2/\alpha$  for  $1 < \alpha < 2$  or  $m > 2$  for  $\alpha = 1$ . Indeed, if  $1 < \alpha < 2$  and  $m \geq 2/\alpha$ , then  $\kappa_0 > m-1$  for  $\kappa_1 \vee \kappa_2 \geq 1$ , while  $\kappa_0 \geq m-1$  for  $\kappa_1 \vee \kappa_2 < 1$ . On the other hand, if  $\alpha = 1$  and  $m > 2$ , then we have  $p_{jkl} = q_{jkl} = 1$ ,  $\kappa_1 \vee \kappa_2 = m-1 > 1$ , hence  $\kappa_0 = m-1$ . Thus we get (23). Once such an estimate is obtained, we are able to prove the existence of a solution to (14) in  $X$  by standard argument, provided the assumption of (i) holds.

Next we consider the case where  $\alpha = 1$  and  $m = 2$ , so that  $\mu_1 = \mu_2 = \kappa_1 = \kappa_2 = 1$ . If  $2c_*t \geq |x|$  with  $c_* = \max\{c_1, \dots, c_N\}$ , then Proposition 2.1 gives

$$(24) \quad \langle t + |x| \rangle \langle c_j t - |x| \rangle |L_{c_j} [|u_k| |u_l]|(t, x)| \lesssim \log(2+t) \|u_k\|_{X(c_k,1,1)} \|u_l\|_{X(c_l,1,1)}.$$

While, if  $2c_*t \leq |x|$  and  $|x - y| = c_i(t - s)$  ( $1 \leq i \leq N$ ), then we get

$$|x| - |y| \leq 2|x - y| \leq 2c_*(t - s),$$

which yields  $2c_*s \leq |y|$ . Thus  $\langle c_i s - |y| \rangle$  is equivalent to  $\langle s + |y| \rangle$  in this case. Therefore, following the computation made in the proof of Proposition 2.1 in [16], we obtain

$$\langle t + |x| \rangle \langle c_j t - |x| \rangle |L_{c_j} [|u_k| |u_l]|(t, x)| \lesssim \|u_k\|_{X(c_k,1,1)} \|u_l\|_{X(c_l,1,1)}.$$

Due to the presence of the logarithmic tail in (24), we have the estimate of the lifespan as in (21). We omit further details.

Now we turn our attention to the blowup part of the theorem. Clearly, we have  $(0, \varepsilon \langle x \rangle^{-3}) \in Y(2)$ , hence we can choose  $\varphi_j(x) \equiv 0$  and  $\psi_j(x) = \varepsilon \langle x \rangle^{-3}$  as the initial data. Then it follows from Lemmas 2.1 and 2.4 in [16] that

$$(25) \quad \begin{aligned} K_{c_j}[0, \psi_j](t, x) &= \frac{\varepsilon}{2c_j|x|} \int_{|c_j t - |x||}^{c_j t + |x|} \rho \langle \rho \rangle^{-3} d\rho \\ &\geq C\varepsilon(t + |x|)^{-1} (c_j t - |x|)^{-1} \end{aligned}$$

for  $c_j t > |x|$  and  $j = 1, \dots, N$ . To proceed further, we introduce the following quantity:

$$(26) \quad \begin{aligned} \langle u \rangle_{c, \mu, \kappa}(y) &= \inf_{(t, x) \in \Sigma(c, y)} (t + |x|)^\mu (ct - |x|)^\kappa |u(t, x)|, \\ \Sigma(c, y) &= \{(t, x) \in [0, \infty) \times \mathbb{R}^3 : ct - |x| \geq cy\} \end{aligned}$$

for  $c, y > 0$  and  $\mu, \kappa \geq 0$ . Then we see from (25) that

$$(27) \quad \langle K_{c_j}[0, \psi_j] \rangle_{c_j, 1, 1}(y) \geq C\varepsilon.$$

The main step of the proof is to reduce (14) to a system of integral equations for

$$(28) \quad U_j(y) \equiv \langle u_j \rangle_{c_j, 1, 1}(y).$$

To realize this, we employ the following (for the proof, see Lemma 2.5 in [16]):

**Proposition 2.2.** *Let  $a_0, a_1, a_2, \alpha, \kappa^* > 0$ ,  $\mu \in \mathbb{R}$  and  $0 \leq \kappa \leq \kappa^*$ . If  $a_1 \leq a_2$ , then we have*

$$(29) \quad \langle L_{a_0}[R(f)] \rangle_{a_0, 1, \kappa_0}(y) \geq C \int_\alpha^y \left(1 - \frac{\eta}{y}\right)^2 f(\eta) d\eta$$

for  $y \geq \alpha$ , where  $\kappa_0 = \mu + \kappa - 2$  and  $R(f)(t, x)$  is defined by

$$R(f)(t, x) = \frac{1}{(t + |x|)^\mu (a_2 t - |x|)^\kappa} f\left(\frac{a_1 t - |x|}{a_1}\right)$$

for  $(t, x) \in \Sigma(a_1, \alpha)$ , and  $R(f) \equiv 0$  otherwise. Besides,  $C$  is independent of  $\kappa^*$  and  $\alpha$ .

We may assume  $c_k \leq c_l$  without loss of generality. Then for  $(t, x) \in \Sigma(c_k, 1)$  we have  $(t, x) \in \Sigma(c_k, (c_k t - |x|)/c_k) \cup \Sigma(c_l, (c_k t - |x|)/c_k)$ , hence

$$|u_k(t, x)| |u_l(t, x)| \geq \frac{U_k((c_k t - |x|)/c_k) U_l((c_k t - |x|)/c_k)}{(t + |x|)^2 (c_k t - |x|) (c_l t - |x|)},$$

in view of (28). Therefore, we get from Proposition 2.2 with  $\mu = 2$ ,  $\kappa = 1$ ,  $\alpha = 1$ ,  $a_0 = c_j$ ,  $a_1 = c_k$ ,  $a_2 = c_l$  and  $f(\eta) = U_k(\eta) U_l(\eta)/\eta$ ,

$$\langle |L_{c_j} [|u_k(t, x)| |u_l(t, x)|] \rangle_{c_j, 1, 1}(y) \geq C \int_1^y \left(1 - \frac{\eta}{y}\right)^2 U_k(\eta) U_l(\eta) \frac{d\eta}{\eta}$$

for  $y \geq 1$ . Setting  $U(y) = \min_{1 \leq j \leq N} U_j(y)$  and recalling (27), we arrive at

$$U(y) \geq C_1 \varepsilon, \quad U(y) \geq C_2 \int_1^y \left(1 - \frac{\eta}{y}\right)^2 U(\eta)^2 \frac{d\eta}{\eta}$$

for  $y \geq 1$ . Hence we see from Lemma 6.3 with  $\alpha = 1$ ,  $\beta = 0$ ,  $\kappa = 1$  and  $p = 2$  in [16] that the lifespan of  $U(y)$  is bounded by  $\exp(C^* \varepsilon^{-1})$ , which gives the thesis.

### 3. AN EXAMPLE FOR MORE GENERAL CASE

In this section, we consider the case where (12) and (13) do not hold. As a typical example, we take the following simple system:

$$(30) \quad \begin{cases} (\partial_t^2 - c_1^2 \Delta) u_1 = |u_1| |u_2|, & (t, x) \in [0, \infty) \times \mathbb{R}^3, \\ (\partial_t^2 - c_2^2 \Delta) u_2 = |u_1|^q, & (t, x) \in [0, \infty) \times \mathbb{R}^3 \end{cases}$$

with the initial data

$$(31) \quad u_j(0, x) = \varphi_j(x), \quad \partial_t u_j(0, x) = \psi_j(x), \quad x \in \mathbb{R}^3 \quad (j = 1, 2).$$

We assume  $q > 2$  and  $\varphi_j, \psi_j \in C_0^\infty(\mathbb{R}^3)$ . Then Theorems 1.4 and 1.5 in [16] show the following.

**Theorem 3.1.** (i) *If  $2 < q < 3$ , then blowup occurs.*

(ii) *If  $q > 3$ , then small data global existence holds.*

(iii) *Let  $q = 3$ . If  $c_1 > c_2 > 0$ , then blowup occurs. While, when  $0 < c_1 < c_2$ , small data global existence holds.*

*Outline of the Proof:* Here we concentrate on the case of  $q = 3$ , since it is the most delicate one in sense that the result depends on the propagation speeds  $c_1$  and  $c_2$ . To prove the global existence part, we need to modify the space  $X$  given by (18) as follows:

$$\begin{aligned} W_j &= \{f \in C([0, \infty) \times \mathbb{R}^3) : \|f\|_{W_j} < \infty\}, \\ \|f\|_{W_j} &= \sup_{(t, x) \in [0, \infty) \times \mathbb{R}^3} w_j(|x|, t) |f(t, x)|. \end{aligned}$$

for  $j = 1, 2$ , where

$$\begin{aligned} w_1(r, t) &= \langle t+r \rangle \langle c_1 t - r \rangle^\kappa z_1(r, t), \\ z_1(r, t) &= \begin{cases} 1, & (r, t) \in \Omega_1 \\ \{1 + \frac{c_2 t - r}{r} \log(2 + c_2 t - r)\}^{-1}, & (r, t) \in \Omega_2 \\ \log^{-2}(2 + |c_1 t - r|), & (r, t) \in \Omega_3 \end{cases}, \\ w_2(r, t) &= \begin{cases} \langle t+r \rangle \langle c_2 t - r \rangle^{3\kappa}, & (r, t) \in \Omega_1 \\ \langle t+r \rangle \langle c_2 t - r \rangle, & (r, t) \in \Omega_2 \\ \langle t+r \rangle \langle c_2 t - r \rangle, & (r, t) \in \Omega_3 \end{cases}, \end{aligned}$$

with  $1/3 < \kappa < 1$ , and

$$\begin{aligned} \Omega_1 &= \{(r, t) \in [0, \infty)^2 : c_2 t \leq r\}, \\ \Omega_2 &= \{(r, t) \in [0, \infty)^2 : (c_1 + c_2)t/2 \leq r \leq c_2 t\}, \\ \Omega_3 &= \{(r, t) \in [0, \infty)^2 : r \leq (c_1 + c_2)t/2\}. \end{aligned}$$

If we could take  $z_1(r, t) \equiv 1$ , then  $w_2(r, t)$  will become like  $\langle t+r \rangle \langle c_2 t - r \rangle^{3\kappa}$ . But we are not able to realize this, so that the order of  $w_2(r, t)$  is different according to the region. More precisely,  $w_2(r, t)$  takes the standard form only in the outside of the light cone  $c_2 t = r$ , i.e.  $\Omega_1$  (notice that  $3\kappa > 1$ ). Nevertheless, if  $c_1 < c_2$ , then we have the following estimates (for the proof, see Propositions 5.1 and 5.2 in [16]):

**Proposition 3.1.** *Let  $0 < c_1 < c_2$  and  $1/3 < \kappa < 1$ . Then we have*

$$\|L_{c_1}[fg]\|_{W_1} \lesssim \|f\|_{W_1} \|g\|_{W_2}, \quad \|L_{c_2}[f^3]\|_{W_2} \lesssim \|f\|_{W_1}^3.$$

Once we obtain above estimates, we can show the existence of the solution  $(u_1, u_2)$  in  $W_1 \times W_2$  for the small initial data.

Finally we consider the blowup part of the theorem. We take the initial data in such a way that  $u_j(0, x) = 0$  and  $\partial_t u_j(0, x) = \varepsilon g_j(x)$ , where  $\varepsilon > 0$  and  $g_j \in C(\mathbb{R}^3)$  satisfies

$$g_j(x) \geq 0 \text{ for all } x \in \mathbb{R}^3, \quad g_1(0) > 0.$$

Then it follows from Lemma 8.1 in [16], with  $p_1 = p_2 = 1$  and  $q_1 = 3$ , that

$$(32) \quad \langle u_1 \rangle_{c_1, 1, 2}(y) \geq C_1 \varepsilon^4, \quad \langle u_2 \rangle_{c_2, 1, 1}(y) \geq C_2 \varepsilon^3, \quad y \geq 1.$$

For  $0 \leq \kappa \leq 2$  we set

$$U_{1, \kappa}(y) = \langle u_1 \rangle_{c_1, 1, \kappa}(y), \quad U_2(y) = \langle u_2 \rangle_{c_2, 1, 1}(y).$$

Then (32) implies

$$(33) \quad U_{1, 2}(y) \geq C_1 \varepsilon^4, \quad U_2(y) \geq C_2 \varepsilon^3, \quad y \geq 1.$$

Since  $c_2 < c_1$ , for  $(t, x) \in \Sigma(c_2, 1)$  we have  $(t, x) \in \Sigma(c_2, (c_2 t - |x|)/c_2) \cup \Sigma(c_1, (c_2 t - |x|)/c_2)$ . Therefore, we get

$$|u_1(t, x) u_2(t, x)| \leq \frac{U_{1, \kappa}((c_2 t - |x|)/c_2) U_2((c_2 t - |x|)/c_2)}{(t + |x|)^2 (c_1 t - |x|)^\kappa (c_2 t - |x|)},$$

hence by Proposition 2.2, there exists positive constant  $C_3 = C_3(c_1, c_2)$  such that for any  $\kappa \in [0, 2]$ ,

$$(34) \quad U_{1, \kappa}(y) \geq C_3 \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1, \kappa}(\eta) U_2(\eta)}{\eta} d\eta, \quad y \geq 1,$$

While, for  $(t, x) \in \Sigma(c_1, 1)$  we have  $(t, x) \in \Sigma(c_1, (c_1 t - |x|)/c_1)$ , hence

$$|u_1(t, x)|^3 \leq \frac{U_{1,\kappa}((c_1 t - |x|)/c_1)^3}{(t + |x|)^3 (c_1 t - |x|)^{3\kappa}}.$$

Applying Proposition 2.2 once again, we see that there exist positive constant  $C_4$  such that

$$(35) \quad U_2(y) \geq C_4 \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,\kappa}(\eta)^3}{\eta^{3\kappa}} d\eta, \quad y \geq 1.$$

Note that  $C_3$  and  $C_4$  do not depend on  $\kappa \in [0, 2]$ .

By (33) and (34), we have

$$(36) \quad U_{1,\kappa}(y) \geq 16b \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,\kappa}(\eta)}{\eta} d\eta, \quad y \geq 1,$$

where  $b = C_2 C_3 \varepsilon^3 / 16$ . Especially, (33) and (36) give

$$(37) \quad U_{1,2}(y) \geq a, \quad U_{1,2}(y) \geq 16b \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U_{1,2}(\eta)}{\eta} d\eta, \quad y \geq 1$$

with  $a = C_1 \varepsilon^4$ . Then we get from Lemma 6.2 in [16]

$$(38) \quad U_{1,2}(y) \geq \frac{a}{4} y^b, \quad y \geq 1.$$

For fixed  $y \geq 1$ , let  $(t, x) \in \Sigma(c_1, y)$ , so that  $(c_1 t - |x|)/c_1 \geq 1$ . Then (38) yields

$$|u_1(t, x)|(t + |x|)(c_1 t - |x|)^2 \geq \frac{a}{4} \left(\frac{c_1 t - |x|}{c_1}\right)^b, \quad i.e. \quad U_{1,2-b}(y) \geq \frac{a}{4c_1^b}$$

for  $y \geq 1$ . Repeating this procedure  $n$  times, we obtain

$$(39) \quad U_{1,2-nb}(y) \geq \frac{a}{4^n c_1^{nb}}, \quad y \geq 1.$$

Moreover, we have

$$(40) \quad U_{1,2-nb}(y) \geq \frac{a}{4^{2n} c_1^{nb}} y^{nb}, \quad y \geq 1.$$

In fact, for  $(t, x) \in \Sigma(c_1, y)$ , (39) with  $n$  replaced by  $2n$  implies

$$|u_1(t, x)|(t + |x|)(c_1 t - |x|)^{2-2nb} \geq \frac{a}{4^{2n} c_1^{2nb}}, \quad y \geq 1.$$

Combining this with  $c_1 t - |x| \geq c_1 y$ , we get (40).

Let  $m$  be the smallest natural number satisfying  $3(2 - mb) \leq 1$ . Being  $b = C_2 C_3 \varepsilon^3 / 16$ , we see that  $C_5 \varepsilon^{-3} \leq m \leq C_5 \varepsilon^{-3}$  with a positive constant  $C_5$ , independent of  $\varepsilon$ . Therefore,

$$\frac{a}{4^{2m} c_1^{mb}} y^{mb} \geq C y^{C_*} \exp(4 \log \varepsilon - 2C_5 \varepsilon^{-3} \log 4)$$

with  $C_* = C_2 C_3 C_5 / 16$ , since  $a = C_1 \varepsilon^4$ . Now, we take a positive constant  $C_6$  and suppose that  $y \geq \alpha^* := \exp(C_6 \varepsilon^{-3})$  so that  $U_{1,\kappa_m}(y) \geq 1$ . This can be realized, because of (40) and the fact that  $\varepsilon^3 \log \varepsilon$  has a minimum for  $\varepsilon > 0$ . Then (35) with  $\kappa = \kappa_m$  yields  $U_2(y) \geq 1$  for  $y \geq \alpha$ , taking  $\varepsilon$  is sufficiently small, if it is necessary (recall that  $3\kappa_m \leq 1$ ).

Finally, if we set  $U(z) = \min\{U_{1,\kappa_m}(\alpha^*z), U_2(\alpha^*z)\}$ , then we see from (34) and (35) that

$$U(z) \geq 1, \quad U(z) \geq C_7 \int_1^z \left(1 - \frac{\zeta}{z}\right)^2 \frac{U(\zeta)^2}{\zeta} d\zeta, \quad z \geq 1,$$

where  $C_7 = \min\{C_3, C_4\}$ . Since it is easy to show that  $U(z)$  blows up in a finite time, we find that the solution of (30) blows up in a finite time  $T^*(\varepsilon)$ . Moreover, there exists a constant  $C^*$  such that  $T(\varepsilon) \leq \exp(C^*\varepsilon^{-3})$ . We finish the outline of the proof.

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