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Pseudo-differential equations and hyperbolic systems

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Abstract[1]. This paper summarizes recent works by Vladimir Georgiev, Guido Ziliotti and the author. The analysis of pseudo-differential equations leads to pointwise decay estimates for first order hyperbolic systems. Some applications to physical models are discussed.

1. Introduction

Starting from Jörgens' pioneer paper [7], a great deal of work has been devoted to study the nonlinear wave equations

$$u_{tt} - \Delta u = f(x, t, u, \partial_t u, \nabla u). \tag{1}$$

Here $u = u(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ represents a wave, $\Delta_x u$ is the Laplacian of u with respect to the space variable and $\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_1} u)$ is the spatial gradient of u.

In several works one can appreciate how a decay estimate (for $t \to +\infty$) for the solution of the linear wave equation

$$u_{tt} - \Delta u = 0$$

gives information on the lifespan and on the behaviour of the solutions of (1). We refer the reader for example to [3], [5], [8], [17] or to the monographs [6], [15].

On the other hand, the wave equations can be easily reduced in the form of first order system

$$\partial_t U - \sum_{j=1}^{n} A_j \partial_j U = F,$$

where $U = U(x, t) = (\partial_t u, \partial_{x_1} u, \ldots, \partial_{x_n} u)$, $x \in \mathbb{R}^n$, $F = F(x, t) = F(x, t; U, \partial_t U, \nabla U) \in \mathbb{R}^{n+1}$ and A_j is a $(n + 1) \times (n + 1)$ matrix with constant entries $a_{1,j+1} = a_{j+1,1} = 1$ and $a_{ij} = 0$ otherwise. For other aspects on the linear wave equation one can see [1].

For this reason the generalization of decay estimates to first order systems seems natural. In order to obtain such inequalities for a generic symmetric first order system we reduce it to pseudo-differential equations, that is

$$w_t - i\lambda(D)w = G,$$

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where $w = w(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and λ is a pseudo-differential operator of convolution type. Section 2 will be devoted to clarify this reduction. Then we shall establish several decay estimates for pseudo-differential equations and finally we shall apply them to nonlinear first order systems.

The advantage of this approach is that we pass from a system to a single equation. On the other hand, we deal with pseudo-differential operators so that a difficulty appears: we loose the information on finite speed of propagation.

As byproduct one can also reobtain the well known estimates for the wave equations. These estimates can be also applied to study the nonlinear Cauchy Problem

$$\begin{cases} u_t - i\lambda(D)u = \phi(u), \\ u(0,x) = g(x). \end{cases}$$

In [11] we prove global existence theorems provided $\phi(u)$ is a smooth function which satisfies $\phi(u) \simeq |u|^p$ near $u = 0$, p is greater than a critical exponent, and g is small in a suitably Sobolev norm. We do not present these results here, in fact our main goal is the application to systems which describe physical models. In particular we deal with nonlinear crystal optics systems that is Maxwell system in anisotropic media.

First, we obtain a von Wahl type estimate for pseudo-differential equations and for first order systems. Then we establish a Klainerman-type estimate: we need to associate a generalized Sobolev space to a first order system. This means that we shall look for the group of symmetries of the system. For the proof of both estimates we shall assume that the eigenvalues $\lambda_k(\xi)$ of the symbol $\sum_i A_i\xi_i$ have constant multiplicity and that the hypersurface $\Sigma_{\lambda_k} = \{\xi \in \mathbb{R}^n | \lambda_k(\xi) = 1\}$ is strictly convex. The relevance of the convexity of Σ_{λ_k} is discussed by Liess in [9] and by Sugimoto, in [16]. If Σ_{λ_k} is convex, but not strictly convex, a loss of decay rate can appear.

Here we do not present any proof. These can be found in the joint works with Vladimir Georgiev and Guido Ziliotti [4], [11], [12], [13] in which these ideas are developed.

1.1. **Notations.**

- As usual, for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ we put $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
- The standard inner product in $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$.
- The symbol A^* means the complex conjugate of the matrix A.
- Given a space B of real functions, considering a vector function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, by $\|\Phi\|_B$ we mean the norm of $|\Phi|$ in B.
- Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. The support of the function $x \to F(x,t)$, for fixed $t \in \mathbb{R}$, is denoted by $\text{supp}_x F(x,t)$.
- To any given function $\lambda : \mathbb{R}^n \to \mathbb{R}$ we associate the level surface $\Sigma_\lambda := \{\xi \in \mathbb{R}^n \mid \lambda(\xi) = 1\}$

In this paper the choice for the coefficients of Fourier transform is

$$\hat{f}(x) = \mathcal{F}(f) = (2\pi)^{-n/2} \int e^{-i\langle x,\xi\rangle} f(\xi)\, d\xi.$$

If f depends on (x,t) then we use Fourier transform with respect to the space variable only.

We write $p(x,D)$ for a pseudo-differential operator with real symbol $p(x,\xi)$:

$$P(x,D)(f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{f}(\xi)\, d\xi.$$

In particular the symbol of a differential operator $p(x,D) = \sum_{|\alpha|\leq k} c_\alpha(x)\partial^\alpha$ is given by

$$p(x,\xi) = \sum_{|\alpha|\leq k} (-i)^{|\alpha|} c_\alpha(x)\xi^\alpha.$$

Moreover, if the symbol is independent of x we say that the correspondent operator $p(D)$ is a convolution type operator. One of the main feature of these operators is that two convolution operators commute, that is $[P(D), Q(D)] = 0$ where $[P,Q] = PQ - QP$.

2. From a first order system to pseudo-differential equations

We consider first order hyperbolic systems

$$\partial_t U - \sum_{j=1}^{n} A_j \partial_j U = F, \tag{2}$$

where $U = U(x,t) = (U_1, \ldots, U_N)$, $x \in \mathbb{R}^n$, $F = F(x,t) = F(x,t;U,\nabla U) \in \mathbb{R}^N$ and A_j are $N \times N$ symmetric matrices. We can diagonalize the symbol $-iA(\xi)$, where

$$A(\xi) = \sum_{j=1}^{n} A_j \xi_j. \tag{3}$$

Hence, there exists an orthogonal matrix $M(\xi)$ such that $M^{-1}(\xi)A(\xi)M(\xi) = D(\xi)$, where $D(\xi) = diag(\lambda_1(\xi), \ldots, \lambda_N(\xi))$ is the diagonal matrix of the characteristic roots.

Fix $\xi \in \mathbb{R}^n$; we denote by $\pi_m(\xi)$ the projection on the eigenspace related to $\lambda_m(\xi)$ whenever $\lambda_m(\xi) \neq 0$. We recall that

$$\begin{aligned}
&\pi_j(\xi)\pi_k(\xi) = \delta_{j,k}\pi_k(\xi),\\
&\pi_j(\xi)\pi_{\ker A(\xi)} = 0,\ \pi_{\ker A(\xi)}\pi_{\ker A(\xi)} = \pi_{\ker A(\xi)},\\
&I = \pi_{\ker A(\xi)} + \sum_j \pi_j(\xi) \quad A(\xi) = \sum_j \lambda_j(\xi)\pi_j(\xi).
\end{aligned}$$

Applying the inverse of Fourier transform, we get the operators $\pi_m(D), \pi_0(D)$ having symbols $\pi_m(\xi), \pi_{\ker A(\xi)}$. In particular

$$\pi_j(D)\pi_k(D) = \delta_{j,k}\pi_k(D), \tag{4}$$

$$\pi_0(D)\pi_j(D) = 0,\ \pi_0(D)\pi_0(D) = \pi_0(D), \tag{5}$$

$$I = \pi_0(D) + \sum_j \pi_j(D), \tag{6}$$

$$\sum_{j=1}^{n} A_j\partial_j = i\sum_j \lambda_j(D)\pi_j(D). \tag{7}$$

These relations give

$$U(x,t) = \pi_0(D)U(x,t) + \sum_j \pi_j(D)U(x,t),$$
$$F(x,t) = \pi_0(D)F(x,t) + \sum_j \pi_j(D)F(x,t).$$

We can write (2) in the form

$$\partial_t \pi_0(D)U + \sum_j (\partial_t - i\lambda_j(D))(\pi_j(D)U) = \pi_0(D)F + \sum_j \pi_j(D)F.$$

Applying $\pi_h(D)$ and $\pi_0(D)$ we find the pseudo-differential equations

$$(\partial_t - i\lambda_h(D))(\pi_h(D)U) = \pi_h(D)F, \quad \forall\, h$$
$$\partial_t \pi_0(D)U = \pi_0(D)F(x,t).$$

Moreover, if $\pi_0(D)F = 0$, then we have the conservation law

$$\pi_0(D)U(x,t) = \pi_0(D)U(x,0). \tag{8}$$

3. Decay estimates for pseudo-differential equations

3.1. $L^1 - L^\infty$ **inequality.** In [11] we obtain a generalization of the classical Von Wahl's estimate (cf. [17]) for the wave equation

$$\begin{cases} \Box u = 0, \\ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases}$$

In that case one has

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{-\frac{n-1}{2}} \left(\|u_0\|_{W^{[\frac{n}{2}]+1,1}} + \|u_1\|_{W^{[\frac{n}{2}],1}} \right). \tag{9}$$

Theorem 3.1. *Let $\lambda(D)$ be a pseudo-differential operator with a real symbol $\lambda(\xi)$ homogeneous of degree 1 which satisfies either*

(i) *for any $\xi \in \mathbb{R}^n$ $\lambda(\xi) \ge 0$ and Σ_λ is a strictly convex set of $\mathbb{R}^n$*

or

(ii) *for any $\xi \in \mathbb{R}^n$ $\lambda(\xi) \le 0$ and $\Sigma_{-\lambda}$ is a strictly convex set of $\mathbb{R}^n$.*

Let $u(t,x)$ be the solution to the Cauchy Problem

$$\begin{cases} u_t - i\lambda(D)u = 0 \quad x \in \mathbb{R}^n \\ u(0,x) = g(x) \end{cases}$$

with $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Then:

$$\|u(t)\|_\infty \le Ct^{-\frac{n-1}{2}} \|g\|_{W^{[n/2]+h,1}} \quad \forall t \ge 1 \tag{10}$$

with $h = 1$ for even n and $h = 2$ for odd n.

A detailed proof of this theorem can be found in [10].

We remark that the hypothesis $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ can be relaxed by the aid of a density argument.

On the contrary, the geometric assumption on Σ_λ can not be avoided without loosing information on the decay rate. We are requiring that the Gaussian curvature of Σ_λ is strictly positive. So that $\{\lambda(\xi) \le 1\}$ is a convex set on one side of the tangent space to Σ_λ.

The proof of this theorem is based on stationary phase method applied to the solution

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle + it\lambda(\xi)} g(y)\, d\xi\, dy.$$

After integration by parts with respect to y, using polar coordinates

$$\begin{cases} \rho = \lambda(\xi), \\ \omega = \xi/\lambda(\xi), \end{cases}$$

we reduce to the estimate of the kernel

$$(11) \qquad \int_0^{+\infty} \rho^{n-1-k} \int_{\Sigma_\lambda} e^{i\rho t\left(\frac{\langle x-y,\omega\rangle}{t}+1\right)} |\nabla\lambda(\omega)|^{-1}\, d\sigma_{n-1}\, d\rho.$$

Hence we describe the surface Σ_λ by means of charts and we arrive at a sum of integrals over $\mathbb{R}^{n-1}$. To these integrals, we can apply stationary phase method with parameter ρt. This in turn gives the decay rate $t^{-(n-1)/2}$.

Further, we see that this decay rate coincides with the wave equation one given in (9) This is not surprising since one can write

$$\partial_{tt} - \Delta = (\partial_t - i\sqrt{-\Delta})(\partial_t - i\sqrt{-\Delta}).$$

3.2. $L^2 - L^\infty$ **inequality.** We now present a non-homogeneous estimate for pseudo-differential equations, that is we fix our attention on $u_t - i\lambda(D)u = F$.

A first result is obtained when the support of F does not contain the hyperplane $y = 0$.

Lemma 3.1. *Let $t \geq 1$ and λ be homogeneous of degree 1. Assume that*

$$\mathrm{supp}_y F(y,s) \subseteq \{|y| \geq 3C_0 t\}, \quad 0 \leq s \leq t$$

where $C_0 = \max_{|\omega|=1} |\nabla\lambda(\omega)|$. Then, for $|x| \leq C_0 t$, the solution of $u_t - i\lambda(D)u = F$ having zero initial data satisfies

$$|u(x,t)| \quad \leq \quad C(t+|x|)^{-\frac{n}{2}} \int_0^t \|F(\cdot,s)\|_2\, ds.$$

This lemma can be proved by using integration by parts in the kernel of the representation

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(t,s,x,y) F(s,y)\, dy\, ds,$$

where the kernel K is given by the following oscillatory integral

$$K(t,s,x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(\langle x-y,\xi\rangle + (t-s)\lambda(\xi))}\, d\xi.$$

In order to deal with a generic data F and to estimate $u(x,t)$ when $|x| \geq C_0 t$, we use the generalized Sobolev spaces on Σ_λ. For the wave equation, this approach was introduced in [8].

Our first step is to find a family of operators Y that satisfy the commutator relations

$$(12) \qquad [Y, \partial_t - i\lambda(D)] = c(\partial_t - i\lambda(D)), \quad c \in \mathbb{R}.$$

It is clear that $\partial_t, \partial_{x_j}, j = 1, \cdots, n$ are $n+1$ vector fields that commute with $\partial_t - i\lambda(D)$.

For the case of $\lambda(\xi) = |\xi|$ one can use $Y = x_j\partial_{x_k} - x_k\partial_{x_j}$ the generators of $so(n)$. In order to treat with a generic $\lambda(\xi)$ homogeneous of degree 1, following [2], we consider the pseudo-differential operators $\Omega_{j,k}(\lambda)$ with symbol $-i/2(x_k\partial_j\lambda^2(\xi) - x_j\partial_k\lambda^2(\xi))$, that is

$$\Omega_{j,k}(\lambda) = -i/2\left(x_k\partial_j\lambda^2(D) - x_j\partial_k\lambda^2(D)\right).$$

We see that these operators commute with $\lambda(D)$:

$$[\lambda(D), \Omega_{j,k}(\lambda)] = 0.$$

hence they satisfy (12) with $c = 0$.

Let us denote by S the scaling operator

$$S = t\partial_t + \langle x, \nabla_x\rangle.$$

Note that $[\lambda(D), S] = -i\langle\nabla\lambda(D), \nabla_x\rangle = \lambda(D)$; then $[\partial_t - i\lambda(D), S] = \partial_t - i\lambda(D)$. This relation is a natural analogue of the well-known property of the wave equations: $[\partial_{tt} - \Delta, S] = 2(\partial_{tt} - \Delta)$. This operator verifies (12) with $c = 1$.

At this point we can introduce the Sobolev spaces associated with the generators

$$\{\partial_{x_j}, \partial_t, S, \Omega_{j,k}(\lambda)\}_{j,k=1,\dots,n,\, j<k}. \tag{13}$$

For brevity these generators shall be denoted by $Y_1, ..., Y_N$. Here $N = (n^2+n+4)/2$. The next lemma assures that the operators (13) generate a Lie algebra.

Lemma 3.2. *Let Y_l, Y_m be two elements of* (13)*. Then there exist $c^r_{l,m}(D)$ pseudo-differential operators of convolution type with symbol homogeneous of degree* 0*, such that*

$$[Y_l, Y_m] = \sum_{r=1}^{N} c^r_{l,m}(D)Y_r.$$

Let $\mathcal{A}_\lambda$ be the Lie algebra generated by $Y_1, ..., Y_N$. To $\mathcal{A}_\lambda$ we associated the generalized Sobolev norm

$$\|f\|_{k,\mathcal{A}_\lambda} := \sum_{|\alpha|\le k} \|Y^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

By using these spaces, in [4] we established the following $L^2 - L^\infty$ estimate.

Theorem 3.2. *Let $n \ge 3$. Consider a smooth function $\lambda : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ such that λ is homogeneous of degree 1; $\lambda(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$; Σ_λ is strictly convex. Let $t \ge 1$. There exists a suitable constant $\sigma > 0$ such that for the solution of*

$$\begin{cases} u_t - i\lambda(D)u = F & x \in \mathbb{R}^n \\ u(x, 0) = 0 \end{cases} \tag{14}$$

the following estimates hold:

$$|u(x,t)| \leq (1+t)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq[n/2]+1} \int_0^t \|Y^\alpha F(\cdot,s)\|_2\, ds + $$
$$+ \sum_{1\leq|\alpha|\leq[n/2]} \|Y^\alpha F(\cdot,t)\|_2 \qquad \text{if } |x|\leq \sigma t;$$

$$|u(x,t)| \leq (1+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq[n/2]+k} \int_0^t \|Y^\alpha F(\cdot,s)\|_2\, ds \qquad \text{if } |x|\geq \sigma t;$$

$$|u(x,t)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq[n/2]+k} \int_0^t \|Y^\alpha(\lambda) F(\cdot,s)\|_2\, ds + $$
$$+ \sum_{1\leq|\alpha|\leq[n/2]} \|Y^\alpha(\lambda) F(\cdot,t)\|_2.$$

Here $k=1$ for even n, $k=2$ for odd n.

The same result holds when λ is negative and $\Sigma_{-\lambda}$ is strictly convex.

For the case $|x|\leq\sigma t$, to overcome the lack of finite speed of propagation we use Lemma (3.1). Moreover we combine stationary phase method and extended Hardy inequality (see [3]):

$$\||\xi|^{-b}\hat{f}(\xi)\|_2 \leq \||x|^b f\|_2 \quad b\in[0,n/2).$$

While considering $|x|\geq\sigma t$, usual Sobolev spaces on Σ_λ come into play. These spaces are defined by means of the norm

(15) $$\|u\|^2_{H^s(\Sigma_\lambda)} := \sum_{|\alpha|\leq s} \|\tilde{\Omega}^\alpha(\lambda)u\|^2_{L^2(\Sigma_\lambda)} \quad s\in\mathbb{N}.$$

Here $\tilde{\Omega}_{jk}$ are the operators

(16) $$\tilde{\Omega}_{jk} = (\partial_j\lambda^2)(D)\partial_{\xi_k} - (\partial_k\lambda^2)(D)\partial_{\xi_j}.$$

Such operators span the tangent space to Σ_λ. In term of symbols we see that

(17) $$(\Omega_{j,k}(\lambda)f)^\wedge(\xi) = \frac{1}{2}\tilde{\Omega}_{j,k}(\lambda)\hat{f}(\xi).$$

A crucial result for the proof of Theorem (3.2) is the following.

Lemma 3.3. *Consider a smooth function $\lambda:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ homogeneous of degree 1. Suppose Σ_λ is strictly convex. Let $s\in\mathbb{N}, s>(n-1)/2$, then*

$$\left|\int_{\Sigma_\lambda} e^{i\langle x,\omega\rangle} g(\omega)\, d\sigma_{n-1}(\omega)\right| \leq C|x|^{-\frac{n-1}{2}}\|g\|_{H^s(\Sigma_\lambda)} \quad x\neq 0.$$

The definition of Sobolev spaces on Σ_λ having fractional order is given in [13]. In the same paper one can find a proof of the last result without the assumption $s\in\mathbb{N}$. Again this proof is based on stationary phase method.

4. Decay estimates for first order systems

Throughout this section we consider the symbol $A(\xi)$ defined in (3) and assume that

(18) $A_j^* = A_j$ for $j = 1, \cdots, n$;

(19) $A(\xi)$ has real eigenvalues of constant multiplicity for $\xi \in \mathbb{R}^n \setminus \{0\}$;

(20) the surface Σ_{λ_k} is strictly convex for each non-zero eigenvalue λ_k.

One can see that (18) and (19) assure that the number of distinct non-zero eigenvalues is even and these eigenvalues are homogeneous functions of degree 1 in ξ. For this reason the projections $\pi_k(\xi)$ are zero order matrix symbols.

4.1. $L^1 - L^\infty$ **inequality.** In order to apply Theorem 3.1 to first order systems, we need a suitable modification of that result. Let us consider the Cauchy problem

$$\begin{cases} u_t - i\lambda(D)u = 0, & x \in \mathbb{R}^n \\ u(x,0) = Qg(x), \end{cases}$$

where $Qg = \mathcal{F}^{-1}(M(\xi)\hat{g}(\xi))$. If $M(\xi)$ is smooth and homogeneous of degree zero, then it is possible to avoid Q in the norm on the right side of (10).

As seen in Section 2, the homogeneous Cauchy Problem

$$\begin{cases} \partial_t U(x,t) - \sum_{j=1}^n A_j \partial_j U(x,t) = 0 & x \in \mathbb{R}^n, t \geq 0, \\ U(x,0) = G(x) \end{cases} \tag{21}$$

reduces to (8) and to

$$(\partial_t - i\lambda(D))\pi_h(D)U = 0 \quad x \in \mathbb{R}^n$$
$$\pi_h(D)U(0,x) = \pi_h(D)UG(x).$$

Applying Theorem 3.1 and the above remarks we arrive at the following result.

Theorem 4.1. *Let* $n \geq 3$. *Consider* (21) *with initial data* $G \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$. *Assume that* $A(\xi)$ *defined in* (3) *satisfy the assumption* (18), (19), (20). *Suppose in addition that for any* $\xi \in \mathbb{R}^n$, $\hat{G}(\xi) \perp \ker A(\xi)$. *Then for any* $t \geq 1$ *one has* $\hat{U}(t,\xi) \perp \ker A(\xi)$ *and*

$$\|U(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-(n-1)/2}\|G\|_{W^{3,1}(\mathbb{R}^n)}.$$

4.2. $L^2 - L^\infty$ **inequality.** Coming back to (2), in correspondence with non-zero eigenvalues we can take the Lie algebra $\mathcal{A}_{\lambda_m}$ generated by $\{\nabla_x, \partial_t, S, \Omega_{j,k}(m) = \partial_j\lambda_m^2(D)x_k - \partial_k\lambda_m^2(D)x_j\}$. On the other hand, we can consider the operator

$$\Omega_{j,k} = \sum_m \Omega_{j,k}(m)\pi_m(D). \tag{22}$$

One can establish the invariance properties of $\Omega_{j,k}$ with respect to the operator $\partial_t - \sum_j A_j\partial_j$:

$$[A, \Omega_{j,k}] = 0.$$

In order to consider the Lie algebra $\mathcal{A}$ generated by $\{\nabla_x, \partial_t, S, \Omega_{j,k}\}$, it is necessary to see that for $\Gamma_1, \Gamma_2 \in \mathcal{A}$ the operator $[\Gamma_1, \Gamma_2]$ is a linear combination of the operators belonging to $\mathcal{A}$ modulo zero-order coefficients . In [4] we proved the following.

Lemma 4.1. *If $A(\xi)$ satisfies* (18), (19) *then for any $\Gamma_l, \Gamma_m \in \mathcal{A}$ there exists $c^r_{l,m}$ a matrix-pseudo-differential operator of convolution type with symbol homogeneous of order* 0 *such that*

$$[\Gamma_l, \Gamma_m] = \sum_{r=1}^{N} c^r_{l,m}(D)\Gamma_r, \quad N = (n^2 + n + 4)/2.$$

This lemma gives the possibility to define Sobolev spaces associated with $\mathcal{A}$ by means of the norm

$$\|F\|_{k,\mathcal{A}} := \sum_{|\alpha|\le k} \|\Gamma^\alpha F\|_{L^2(\mathbb{R}^n)}.$$

In [4] we prove that there exist $C_1, C_2 > 0$ such that

$$C_1 \sum_l \sum_{|\alpha|\le s} \|\Gamma^\alpha(l)\pi_l u\|_2 \le \sum_{|\alpha|\le s} \|\Gamma^\alpha u\|_2 \le C_2 \sum_l \sum_{|\alpha|\le s} \|\Gamma^\alpha(l)\pi_l u\|_2 \tag{23}$$

being $\Gamma \in \mathcal{A}$ and $\Gamma(l) \in \mathcal{A}_{\lambda_l}$.

Combining this equivalence with Theorem 3.2 and the results contained in Section 2 we have the following.

Theorem 4.2. *Let $n \ge 3$ and $t \ge 1$. Let $A(\xi) = \sum_j A_j\xi_j$ satisfy the assumptions* (18), (19), (20) *and* $\ker A(\xi) = \{0\}$ *for all ξ. There exists a suitable constant $\sigma > 0$ such that the solution of* (2) *having zero initial data satisfies the following estimates:*

$$|U(x,t)| \le C(1+t)^{-\frac{n-1}{2}} \sum_{|\alpha|\le[n/2]+1} \int_0^t \|\Gamma^\alpha F(\cdot,s)\|_2\, ds + C \sum_{1\le|\alpha|\le[n/2]} \|\Gamma^\alpha F(\cdot,t)\|_2 \qquad \text{if } |x| \le \sigma t \tag{24}$$

$$|U(x,t)| \le C(1+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\le[n/2]+k} \int_0^t \|\Gamma^\alpha F(\cdot,s)\|_2\, ds \qquad \text{if } |x| \ge \sigma t \tag{25}$$

$$|U(x,t)| \le C(1+t+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\le[n/2]+k} \int_0^t \|\Gamma^\alpha F(\cdot,s)\|_2\, ds + C \sum_{1\le|\alpha|\le[n/2]} \|\Gamma^\alpha F(\cdot,t)\|_2 \tag{26}$$

for any $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$ such that the right side of the estimate is finite. Here $k = 1$ for even $n \ge 3$ and $k = 2$ for odd $n \ge 3$ Moreover Γ belongs to $\mathcal{A}$ the Lie algebra generated by $\{\partial_t, \nabla, S, \Omega_{j,k}\}$.

A first consequence of this result is the following.

Corollary 4.1. *Under the same assumption of the previous theorem one has*

$$|U(x,t)| \le C(1+t+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\le[n/2]+k} \sup_{0\le s\le t} (1+s)^a \|\Gamma^\alpha F(\cdot,s)\|_2 \tag{27}$$

Here $k = 1$ for even $n \ge 3$ and $k = 2$ for odd $n \ge 3$. Moreover $a = (n-1)/2$ if $n \ge 4$ and $a = 1 + \varepsilon, \varepsilon > 0$ if $n = 3$.

In order to guarantee that all the components of u decay, when $\ker A(\xi) \neq \{0\}$ we need some additional hypotheses which can be related to the elliptic complex assumptions quoted in [14]. Having in mind (8) we find the following.

Theorem 4.3. *Let $n \geq 3$. Consider a Cauchy Problem associated to* (2). *Assume that $A(\xi) = \sum_j A_j \xi_j$ satisfies the assumptions* (18), (19), (20). *Suppose in addition that*

$$\hat{F}(t) \perp \ker A(\xi), \quad \hat{u}(0) \perp \ker A(\xi). \tag{28}$$

Then (27) *holds.*

5. Quasilinear Maxwell systems

Many equations of Mathematical Physics can be written in the form of a first order system (2) where A_j are self-adjoint matrices (see [1]). For example

- Dirac-Pauli system;
- Wave equations with different speeds of propagation;
- Equations of elasticity in some anisotropic media;
- Maxwell system for uniaxial crystals.

These applications are discussed in [4]. Here, for brevity we present only the last model since one can compare the case of uniaxial crystals with the one of biaxial crystals in which the assumption (20) fails.

Let us consider Maxwell equations in anisotropic media:

$$\begin{cases} \epsilon_0 \partial_t E = \mathrm{curl} H + F_1, \\ \partial_t H = -\mathrm{curl} E + F_2, \end{cases} \tag{29}$$

where $E(x,t), H(x,t), F_1(x,t), F_2(x,t) \in \mathbb{R}^3$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. Moreover ϵ_0 is a diagonal 3 by 3 matrix: $\epsilon_0 = diag(a^2, b^2, c^2)$. The fields E, H represent respectively the electric and the magnetic vector fields.

This system describes the propagation of the light in crystals. In particular, in uniaxial crystal, there is a single axis along which light can propagate without exhibiting double refraction; along other axis a light beam splits into two different components which travel at different velocities. This corresponds to the choice $\epsilon_0 = diag(a^2, b^2, b^2)$. On the contrary, in biaxial crystals conical refraction takes place: a ray incident on a surface of the crystal in a certain direction splits into a family of rays which lie along a cone. To examine this case one takes $\epsilon_0 = diag(a^2, b^2, c^2)$ with three different entries. We shall see that the decay rate is different in uniaxial and biaxial cases.

Taking $U = (\epsilon_0^{1/2} E, H)$, we can rewrite this system as

$$\partial_t U - BU = F,$$

with $F = (\epsilon_0^{-1/2} F_1, F_2)$ and

$$B := \begin{bmatrix} 0 & \epsilon_0^{-1/2} \mathrm{curl} \\ -\mathrm{curl} \epsilon_0^{-1/2} & 0 \end{bmatrix}.$$

In Fourier transform coordinates we get

$$\begin{cases} \hat{U}' - iB(\xi)\hat{U} = \hat{F}, \\ \hat{U}(0) = (\epsilon_0^{1/2}\hat{E}_0, \hat{H}_0), \end{cases}$$

where

$$B(\xi) := \begin{bmatrix} 0 & -\epsilon_0^{-1/2}\xi\wedge \\ \xi\wedge\epsilon_0^{-1/2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix}, \tag{30}$$

$$\tilde{B} = \begin{bmatrix} 0 & -|a|^{-1}\xi_3 & |a|^{-1}\xi_2 \\ |b|^{-1}\xi_3 & 0 & -|b|^{-1}\xi_1 \\ -|c|^{-1}\xi_2 & |c|^{-1}\xi_1 & 0 \end{bmatrix}.$$

The characteristic equation $\det(B(\xi) - \lambda\mathbf{I}) = 0$ is given by

$$\lambda^2(\lambda^4 - \psi(\xi)\lambda^2 + \phi(\xi)|\xi|^2) = 0, \tag{31}$$

where

$$\begin{aligned} \psi(\xi) &= (|b|^{-1} + |c|^{-1})\xi_1^2 + (|a|^{-1} + |c|^{-1})\xi_2^2 + (|a|^{-1} + |b|^{-1})\xi_3^2, \\ \phi(\xi) &= |b|^{-1}|c|^{-1}\xi_1^2 + |a|^{-1}|c|^{-1}\xi_2^2 + |a|^{-1}|b|^{-1}\xi_3^2. \end{aligned}$$

In the case $b = c$, $B(\xi)$ has eigenvalues

$$\begin{aligned} &\lambda_{1,2} = 0, \\ &\lambda_{3,4}(\xi) = \pm\sqrt{b^{-2}\xi_1^2 + a^{-2}\xi_2^2 + a^{-2}\xi_3^2}, \\ &\lambda_{5,6}(\xi) = \pm b^{-1}|\xi|. \end{aligned}$$

The most relevant point is that Σ_{λ_j} is strictly convex for $j \geq 3$ and our theorems apply.

On the contrary, if ϵ_0 has three different entries, Σ_λ is not strictly convex, in fact taking $\lambda = 1$ in (31), one finds Fresnel surface with four singular points (see [15]). This is a consequence of the variable multiplicity of the characteristic roots. Our technique is not available.

Before stating decay estimates for Maxwell systems in anisotropic media we have to consider the conditions due to the presence of null eigenvalues. We notice that

$$(\hat{G}_1, \hat{G}_2) \perp \operatorname{Ker} B(\xi) \Leftrightarrow \operatorname{div}\epsilon_0^{1/2}G_1 + \operatorname{div} G_2 = 0.$$

Hence we have to assume that

$$\operatorname{div}\epsilon_0 E_0 + \operatorname{div} H_0 = 0, \quad \operatorname{div} F_1 + \operatorname{div} F_2 = 0. \tag{32}$$

These elliptic conditions have a clear physical meaning. If we take $\operatorname{div}\epsilon_0 E_0 = 0 = \operatorname{div} H_0$ we get the other two laws of Maxwell system in the case of null charge density:

$$\operatorname{div}\epsilon_0 E(t) = 0 = \operatorname{div} H(t) \tag{33}$$

Finally, for the homogeneous Maxwell system in uniaxial crystals, Theorem 4.1 can be rewritten in the following form.

Theorem 5.1. *Let us denote by* $e^{tB}(E_0, H_0)$ *the solution of* (29) *with data* $F_1 = F_2 = 0$ $(E_0, H_0) \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}^6)$. *Suppose* $\epsilon_0 = diag(a^2, b^2, b^2)$ *and* $\operatorname{div}\epsilon_0 E_0 = 0 = \operatorname{div} H_0$ *holds. Then for all* $t \geq 0$ *one has* (33) *and the following decay estimate*

$$\|e^{tB}(E_0, H_0)\|_{L^\infty(\mathbb{R}^3)} \leq C(1+t)^{-1}\|(E_0, H_0)\|_{W^{3,1}(\mathbb{R}^3)}. \tag{34}$$

On the contrary, if ϵ_0 has three different entries, in [9] Liess established the following estimate:

Theorem 5.2. *Considering* (29) *with* $\epsilon_0 = diag(a^2, b^2, c^2)$, $a \neq b \neq c \neq 0$. *Let* $e^{tB}(E_0, H_0)$ *the solution of this system with initial data* $(E_0, H_0) \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}^6)$ *which satisfies* $\operatorname{div} \epsilon_0 E_0 = 0 = \operatorname{div} H_0$. *For great enough* $m \in \mathbb{N}$, *one can find a constant* $C > 0$ *so that*

$$\|e^{tB}(E_0, H_0)\|_{L^\infty(\mathbb{R}^3)} \leq C(1+t)^{-1/2}\|(E_0, H_0)\|_{W^{m,1}(\mathbb{R}^3)}.$$

By using the results of Section 4.2 we also have information on the non-homogeneous case.

Theorem 5.3. *Let* $t \geq 1$. *Let* $U = (\epsilon_0^{1/2}E, H)$ *such that* (E, H) *solves* (29) *with* $\epsilon_0 = diag(a^2, b^2, b^2)$ *and initial data which satisfy* (32). *For any* $\varepsilon > 0$, *we have*

$$|U(x,t)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq 2} \sum_{l=3}^{6} \sup_{0\leq s\leq t} (1+s)^{(1+\varepsilon)} \|\Gamma^\alpha(l) F(\cdot, s)\|_2$$

where $\Gamma(3), \Gamma(4)$ *belong to the Lie algebra generated by*

$$\{\partial_t, \nabla_x, S, (a^{-2}x_1\partial_2 - b^{-2}x_2\partial_1), (a^{-2}x_1\partial_3 - b^{-2}x_3\partial_1), a^{-2}(x_2\partial_3 - x_3\partial_2)\}$$

and $\Gamma(5), \Gamma(6)$ *belong to the Lie algebra generated by* $\{\partial_t, \nabla_x, S, x_j\partial_k - x_k\partial_j\}_{j,k=1,2,3}$.

One of the main applications of the previous theorems are global existence results for the nonlinear systems. Here we present the quasi-linear perturbation of Maxwell system in anisotropic media

$$\begin{cases} \partial_t(\epsilon_0 E + \Phi(E)) = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E, \end{cases} \tag{35}$$

with small initial data (E_0, H_0). Here the nonlinear perturbation Φ is a smooth function of E with polynomial growth near $E = 0$: ϕ grows like $|E|^p$. Since we treat with small amplitude solutions, then a global existence result will require p sufficiently large.

Theorem 5.4. *Let* $f :\to \mathbb{R}^3 \to \mathbb{R}$ *be a smooth real function such that* $\Phi(E) = \nabla f(E)$ *and*

$$|\Phi(E)| = O(|E|^p) \quad \text{near } E = 0 \quad p \in \mathbb{N}, p \geq 4.$$

Consider the quasi-linear system (35) *with* $\epsilon_0 = diag(a^2, b^2, b^2)$, $a \neq b \neq 0$ *and initial data* (E_0, H_0) *verifying*

$$\begin{cases} \operatorname{div}(\epsilon_0 E_0 + \Phi(E_0)) = 0, \\ \operatorname{div} H_0 = 0. \end{cases} \tag{36}$$

There exists a small $0 < \varepsilon < 1$ *such that, if*

$$\|E_0\|_{W^{6,1}} + \|H_0\|_{W^{6,1}} + \|E_0\|_{H^7} + \|H_0\|_{H^7} < \varepsilon, \tag{37}$$

then (35) *has a unique global solution* $(E, H) : \mathbb{R}^3 \to \mathbb{R}^6$. *Moreover* $u \in \mathcal{C}([0, +\infty); W^{6,1} \cap H^7)$.

The assumption (37) can be relaxed by taking small weighted Sobolev norms like in [13].

In [12] we prove previous theorem. We deal with

$$\begin{cases} \partial_t M = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl}\epsilon_0^{-1} M + \operatorname{curl}\Psi(M) \end{cases} \tag{38}$$

with initial data $(\epsilon_0 E_0 + \Phi(E_0), H_0)$. In fact by means of implicit function theorem from

$$M(E) = \epsilon_0 E + \Phi(E),$$

in a neighborhood of $M = 0$, we get the following representations:

$$E = \epsilon_0^{-1} M + \Psi(M), \quad |\Psi(M)| = O(|M|^p).$$

In turn (38) can be rewritten in the form of (2) where $A(\xi) = B(\xi)$ is described in (30), $U = (\epsilon_0^{-1/2} M, H) = (U_1, U_2)$ and $F(t) = (0, -\mathrm{curl}(\Psi(\epsilon_0^{-1/2} U_1(t)))$. The existence of U can be obtained by means of contraction mapping principle with norm

$$X(u)(t) = \sup_{0 \leq \tau \leq t} \Big\{ \sup_{|\alpha| \leq 3} (1+\tau) \|D^\alpha u(\tau)\|_\infty + \sum_{|\alpha| \leq 7} \frac{1}{2} \int_{\mathbb{R}^3} \langle \epsilon_0^{-1} D^\alpha M, D^\alpha M \rangle + \\ + |D^\alpha H|^2 \, dx \Big\}.$$

The L^∞ term of this norm is estimated by using Theorem 5.1. Instead, the second term instead involves energy estimates. In both cases one needs nonlinear inequalities in Sobolev spaces.

For biaxial crystals, the result is weaker, since Theorem 5.2 gives a lower decay.

Theorem 5.5. *Let $f :\to \mathbb{R}^3 \to \mathbb{R}$ be a smooth real function such that $\Phi(E) = \nabla f(E)$ and*

$$|\Phi(E)| = O(|E|^p) \quad \text{near } E = 0 \quad p \in \mathbb{N},\, p \geq 5.$$

Consider the quasi-linear system (35) *with $\epsilon_0 = diag(a^2, b^2, c^2)$, $a \neq b \neq c \neq 0$ and initial data (E_0, H_0) verifying* (36). *There exists a small $0 < \varepsilon < 1$ and a sufficiently large k such that if*

$$\|E_0\|_{W^{k,1}} + \|H_0\|_{W^{k,1}} + \|E_0\|_{H^k} + \|H_0\|_{H^k} < \varepsilon,$$

then (35) *admits a unique global solution $(E, H) : \mathbb{R}^3 \to \mathbb{R}^6$.*

6. Conclusion

The results presented here suggest some open problems.

We observe that in Theorem 5.5 the number of derivatives of data that we require small is not fixed. In fact we use Liess' result that does not point out this aspect. One can try to find the optimal k in that theorem.

Moreover, we underline that the exponent $p \geq 4$ in Theorem 5.4 is too large, since the decay rate gives a critical exponent equal to $2 = 1 + 2/(n-1)$. One can expect to cover at least the case $p = 3$ by using L^∞-L^2 estimates. Another interesting problem is related to the case $p = 2$ in which null-type conditions would play an important role.

Object of our recent study is to gain a Moser type inequality in the generalized Sobolev spaces introduced in Section 4.2:

$$\| |U|^p \|_{k,\mathcal{A}} \leq C \|U\|_{L^\infty}^{p-1} \| |U|^p \|_{k,\mathcal{A}} \quad p > 1.$$

Finally one can hope to relax some assumptions on the symbol $A(\xi)$. For example one can investigate if it is possible to take $A(\xi)$ uniformly symmetrizable matrix. Another direction consists in the generalization of Liess' result to other systems in which several principal curvatures of characteristic surfaces vanish.

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