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Extensions of Gronwall's inequality with logarithmic terms

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Abstract[1]. This article is a self-contained account of some new results on some nonlinear Gronwall type inequalities which also include some logarithmic terms. These extend many known results including some results used by [4, 5] and are generalizations of the main result of [9].

An example of the type of inequality we study is

$$u^2(t) \le c_0^2 + \int_0^t 2c_1(s)u(s) + 2c_2(s)u^2(s) + 2c_3u^2(s)\log(u(s))\,ds, \tag{1}$$

where $c_i, i = 1, 2, 3$ are non-negative L^1 functions. We establish an explicit L^∞ bound for u in terms of the coefficients $c_i, i = 0, \dots, 3$. An inequality such as (1) may also be written in terms of fractional powers, for example in the form

$$v(t) \le c_0 + \int_0^t c_1(s)v^{1/2}(s) + c_2(s)v(s) + c_3v(s)\log(v(s))\,ds, \tag{2}$$

but we prefer the version with integer powers for which we can give an elementary inductive proof.

We also give some new results when the coefficients c_i depend on the t variable too. In one case we generalize a result used in [7], in another we generalize a result of [3].

1. Introduction

The Gronwall inequality is a well-known tool in the study of differential equations and Volterra integral equations which is used for proving *inter alia* uniqueness and stability results. It is often useful in establishing *a priori* bounds which can help establish the existence of global solutions.

For a review of the many types of inequality that have been proved we refer to the book [6] which contains several hundred references. There are a number of versions of Gronwall type results and determining priority is not an easy task. We have relied as a source book on [6].

We only hint at a few of the applications and mention some works which use inequalities similar to ones we develop below. For some applications to global existence for evolution equations and nonlinear Schrödinger equations involving logarithmic terms see [5], for applications in elasticity see [4], for applications (without logarithmic terms) to bounding the errors arising in a scheme for linear Volterra integral equations see [7].

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One classical version of Gronwall's inequality reads as follows. It is essentially due to Bellman, see for example [6], though Bellman's original result had c_0 constant. [For notation used see section 2.]

Theorem 1.1. *Suppose that* $u \in L^\infty_+[0,T]$ *satisfies the inequality*

$$u(t) \le c_0(t) + \int_0^t c_1(s)u(s)\,ds \quad \text{for a.e. } t \in [0,T],$$

where $c_0 \in L^\infty_+$ *is nondecreasing,* $c_1 \in L^1_+$. *Then*

$$u(t) \le c_0(t)\exp\left(\int_0^t c_1(s)\,ds\right) \text{ a.e.} \tag{1}$$

Another somewhat different inequality is the following, apparently first given (for continuous functions) by Liang Ou-Iang [8].

Theorem 1.2. *Suppose that* $u^2(t) \le c_0^2 + 2\int_0^t h(s)u(s)\,ds$ *for a.e.* $t \in [0,T]$, *where* c_0 *is a constant,* $h \in L^1_+$, *and* $u \in L^\infty_+$. *Then, for a.e.* $t \in [0,T]$,

$$u(t) \le c_0 + \int_0^t h(s)\,ds.$$

If $h \in L^2$, by using the inequality $2hu \le h^2 + u^2$ this case could be reduced to the classical case but would give a different type of conclusion.

A few years ago [9] I discovered a nonlinear version of Gronwall's inequality that simultaneously contains both of these results as well as many other inequalities of Gronwall type given in the literature, for example results of Perov and Gamidov, see [6], pp. 360–362.

One advantage of my approach is that explicit bounds are found; other methods, such as use of the comparison principle, will apply in theory but may be hard, or impossible, to use.

In the present article I shall give a self-contained account of some new results on Gronwall type inequalities which allow faster growth by including some logarithmic terms. These extend some results used by [4, 5] and are generalizations of the main result of [9].

The following illustrates the type of inequality we study in our main result, Theorem 3.2. Suppose that a non-negative L^∞ function $u \ge 1$ satisfies the inequality

$$u^2(t) \le c_0^2 + \int_0^t 2c_1(s)u(s) + 2c_2(s)u^2(s) + 2c_3u^2(s)\log(u(s))\,ds, \tag{2}$$

where $c_i, i = 1,2,3$ are non-negative L^1 functions. Then we establish an explicit L^∞ bound for u in terms of the coefficients $c_i, i = 0,\dots,3$. An inequality such as (2) may also be written in terms of fractional powers, for example in the form

$$v(t) \le c_0 + \int_0^t c_1(s)v^{1/2}(s) + c_2(s)v(s) + c_3v(s)\log(v(s))\,ds, \tag{3}$$

but we prefer the version with integer powers for which we can give elementary proofs.

The proof is by reduction to a basic result which gives an explicit L^∞ bound in terms of the given function c_i for a function $u \ge 1$ which satisfies an inequality

$$u(t) \le c_0(t) + \int_0^t c_1(s)u(s) + c_2(s)u(s)\log(u(s))\,ds. \tag{4}$$

We obtain the L^∞ bound that can be deduced from this in Theorem 3.1 and use it to prove our main result.

In section 4 we also give some new results when the coefficients c_i also depend on the t variable. We give two types of result, in one case we generalize a result used in [7], in another we generalize a result of [3].

2. Basic results

We first give proofs of some essentially known results. These will be used to prove a number of extensions.

Notation. We write $AC[0,T], L^\infty[0,T], L^1[0,T]$ to stand for the space of absolutely continuous functions, essentially bounded and Lebesgue integrable functions respectively on an interval $[0,T]$. We usually omit reference to the fixed interval $[0,T]$. We write a subscript $+$ on such spaces to mean functions in the space that are non-negative a.e. For a function $c \in L^\infty$, we use the notation $c^*(t) = \text{ess sup}_{0\le s\le t} c(s)$.

We only give results for functions defined on an interval $[0,T]$, simple modifications can be made to treat intervals of the form $[\alpha,\beta]$.

We first prove a result that proves the classical Gronwall inequality quoted in Theorem 1.1. This is a well-known result but we give the proof for completeness because we need this result in our work below. It is essentially a special case of a result of Beesack [2] and Willett; see Theorem 1, p.356 of [6] and the remark following for some further information.

Theorem 2.1. *Suppose $a, b \in L^1_+$ and $v \in AC_+$ satisfies*

$$v'(t) \le a(t) + b(t)v(t), \text{ for a.e. } t \in [0,T]. \tag{5}$$

Then

$$v(t) \le \exp(B(t))\big[v(0) + \int_0^t a(s)\exp(-B(s))\,ds\big], \tag{6}$$

for $t \in [0,T]$, where $B(t) = \int_0^t b(s)\,ds$.

Proof. From (5) it follows that

$$\big(v(t)\exp(-B(t))\big)' \le a(t)\exp(-B(t)), \text{ a.e.}$$

Integrating this inequality proves (6). Note that the function $v\exp(-B)$ is AC because B is AC, exp is Lipschitz on bounded intervals, so the composition $\exp(-B)$ is AC, and a product of AC functions is also AC.

Proof of Theorem 1.1. Fix τ in $(0,T]$ and let $w(t) := c_0(\tau) + \int_0^t c_1(s)u(s)\,ds$. Then $w \in AC$, $u(t) \le w(t)$ for a.e. $t \in [0,\tau]$ and

$$w'(t) = c_1(t)u(t) \le c_1(t)w(t) \text{ for a.e. } t \in [0,\tau].$$

This is now a special case of Theorem 2.1 so we obtain

$$w(t) \le \exp(C_1(t))w(0),\ t \in [0,\tau],$$

and since $w(0) = c_0(\tau)$ this gives $w(t) \leq c_0(\tau)\exp(C_1(t))$ for all $t \in [0,\tau]$. In particular, $w(\tau) \leq c_0(\tau)\exp(C_1(\tau))$ and, as τ is arbitrary, (1) holds.

Remark 2.1. Note that if c_0 is not assumed to be nondecreasing then this proof applies if c_0 is replaced by c_0^*.

3. Logarithmic Gronwall inequalities

We now have our first generalization of the results of the previous section which is the base result for our general inequality involving logarithmic terms.

Theorem 3.1. *Suppose that $c_0 \in L_+^\infty$, $c_1, c_2 \in L_+^1$ and that $u \in L_+^\infty$. Suppose that $u \geq 1, c_0 \geq 1$ and, for a.e. $t \in [0,T]$, u satisfies the inequality*

$$u(t) \leq c_0(t) + \int_0^t c_1(s)u(s) + c_2(s)u(s)\log(u(s))\,ds. \tag{7}$$

Then, for a.e. $t \in [0,T]$,

$$u(t) \leq c_0^*(t)^{\exp(C_2(t))} \exp\big[\exp(C_2(t)) \int_0^t c_1(s)\exp(-C_2(s))\,ds\big], \tag{8}$$

where $C_2(t) = \int_0^t c_2(s)\,ds$.

Proof. Fix $\tau \in (0,T]$ and let

$$w(t) := c_0^*(\tau) + \int_0^t c_1(s)u(s) + c_2(s)u(s)\log(u(s))\,ds.$$

Then $u(t) \leq w(t)$ for a.e $t \in [0,\tau]$, $w \in AC$ so w' exists a.e. and

$$w'(t) = c_1(t)u(t) + c_2(t)u(t)\log(u(t)) \leq c_1(t)w(t) + c_2(t)w(t)\log(w(t)),$$

or

$$\frac{w'}{w} \leq c_1(t) + c_2(t)\log(w(t)), \text{ a.e.}$$

Thus $v = \log w$ satisfies

$$v'(t) \leq c_1(t) + c_2(t)v(t), \text{ a.e.}$$

Note that $v \in AC$ since $w(t) \geq 1$. Theorem 2.1 now gives

$$v(t) \leq \exp(C_2(t))\big[v(0) + \int_0^t c_1(s)\exp(-C_2(s))\,ds\big],\ t \in [0,\tau].$$

Taking the exponential of both sides and noting that $\tau \in (0,T]$ is arbitrary gives (8).

Corollary 3.1. *Suppose that $c_0 \in L_+^\infty$, $b \in L_+^1$, $a \geq 1$ and $u \in L_+^\infty$ and that, for a.e. $t \in [0,T]$, u satisfies the inequality*

$$u(t) \leq c_0(t) + \int_0^t b(s)(a + u(s))\log(a + u(s))\,ds. \tag{9}$$

Then

$$u(t) \leq (a + c_0^*(t))^{\exp(\int_0^t b(s)\,ds)} - a, \text{ a.e.}$$

Proof. Let $U(t) := a + u(t)$, then

$$U(t) \leq (a + c_0(t)) + \int_0^t b(s)U(s)\log(U(s))\,ds.$$

Theorem 3.1 applies and this gives the result.

Corollary 3.1 is an extension of Corollary 16 (p.139) of Haraux [5], who only considered constant coefficients. His proof uses a comparison principle.

Corollary 3.2. *Suppose that $a \in L^\infty_+, a \geq 1$, $b \in L^1_+$ and $u \in L^\infty_+, u \geq 1$ and that, for a.e. $t \in [0,T]$,*

$$u(t) \leq a(t) + \int_0^t b(s)u(s)\log u(s)\,ds. \tag{10}$$

Then

$$u(t) \leq (a^*(t))^{\exp(\int_0^t b(s)\,ds)}, \text{ a.e.}$$

Proof. This is a special case of Theorem 3.1

Corollary 3.2 is a slight extension of the lemma in Engler [4] who has a constant.

We now give a new nonlinear generalization of these results which also includes a logarithmic term and shares features of the previous result of the author [9] where the coefficients W_i were first defined.

Theorem 3.2. *Let $m \geq 1$ be an integer. Suppose that $c_0 \in L^\infty_+$, $c_i \in L^1_+$, $i = 1, \ldots m+1$. Suppose that $c_0 \geq 1$, $u \geq 1$ and for a.e. $t \in [0,T]$,*

$$\begin{aligned} u^m(t) \leq c_0^m(t) + m\int_0^t c_1(s)u(s) + c_2(s)u^2(s) + \cdots \\ \cdots + c_m(s)u^m(s) + c_{m+1}(s)u^m(s)\log(u(s))\,ds. \end{aligned} \tag{11}$$

Then, it follows that

$$\begin{aligned} u(t) \leq (W_m(c_0^*(t), c_1(t), \ldots, c_{m-1}(t)))^{\exp(C_{m+1}(t))} \\ \times \exp\big[\exp(C_{m+1}(t))\int_0^t c_m(s)\exp(-C_{m+1}(s))\,ds\big], \text{ a.e.} \end{aligned} \tag{12}$$

where the coefficients W_k are defined recursively by

$$W_1(c_0) = c_0$$

$$W_{k+1}(c_0, c_1, \ldots, c_k) = W_k\big(\big[c_0^k + k\textstyle\int_0^t c_1\big]^{1/k}, c_2, \ldots, c_k\big), \quad k \geq 1.$$

Proof. We first suppose c_0 is a constant. The proof is then inductive by showing that if (11) implies (12) for an integer $m-1$ then it does also for m. For $m = 1$ the hypothesis reads

$$u(t) \leq c_0 + \int_0^t c_1 u + c_2 u\log(u), \quad \text{a.e.}$$

and the conclusion is a consequence of Theorem 3.1. Now suppose that the result holds for $m-1$ and consider the inequality

$$u^m(t) \leq c_0^m + m\int_0^t c_1 u + \cdots + c_m u^m + c_{m+1}u^m\log(u), \text{ a.e.}$$

Let

$$w^m := c_0^m + m\int_0^t c_1 u + \cdots + c_m u^m + c_{m+1}u^m \log(u).$$

Differentiating gives

$$\begin{aligned} w^{m-1}w' =& c_1 u + \cdots + c_m u^m + c_{m+1}u^m\log(u) \\ \le& c_1 w + \cdots + c_m u^{m-1} w + c_{m+1}u^{m-1} w \log(u) \end{aligned}$$

so that

$$w^{m-2}w' \le c_1 + c_2 u + \cdots + c_m u^{m-1} + c_{m+1}u^{m-1}\log(u).$$

Integrating this inequality gives

$$w^{m-1} \le c_0^{m-1} + (m-1)\int_0^t c_1 + c_2 u + \cdots + c_m u^{m-1} + c_{m+1}u^{m-1}\log(u)$$

so that

$$u^{m-1} \le \big[c_0^{m-1} + (m-1)\int_0^t c_1\big] + \int_0^t c_2 u + \cdots + c_m u^{m-1} + c_{m+1}u^{m-1}\log(u).$$

By the inductive hypothesis this gives

$$\text{(13)} \qquad u \le W_{m-1}\big([c_0^{m-1} + (m-1)\textstyle\int_0^t c_1]^{1/(m-1)}, c_2, \ldots, c_{m-1}\big)^{\exp(C_{m+1})} \times \exp\big[\exp(\displaystyle\int_0^t c_m(s)\exp(-C_{m+1}(s))\,ds)\big].$$

Using the definition of W_m this concludes the inductive step. When c_0 is not a constant we proceed as in the proof of Theorem 3.1 using $c_0^*(\tau)$.

Corollary 3.3. *Let $m \ge 1$ be an integer. Suppose that*

$$\text{(14)} \quad u^m(t) \le c_0^m(t) + m\int_0^t c_1 u(s) + c_2 u^2 + \cdots + c_m u^m\,ds, \ \textit{for a.e. } t \in [0,T],$$

where $c_0 \in L_+^\infty$, $u \in L_+^\infty$, and $c_i \in L_+^1$ for $i \ge 1$. Then,

$$\text{(15)} \qquad u(t) \le W_m(c_0^*(t), c_1(t), \ldots, c_{m-1}(t))\exp\left(\int_0^t c_m(s)\,ds\right) \ \textit{a.e.}$$

where the function W_m is as in Theorem 3.2.

Proof. Follow the proof of Theorem 3.2 with $c_{m+1} = 0$. Note that now it suffices to have $u \ge 0$ and $c_0 \ge 0$.

Remark 3.1. Corollary 3.3 is a result of the author, save that it was assumed in [9] that c_0 was nondecreasing. As shown in [9] this result contains many known results which had previously been proved one-by-one, [6].

We give an example to show how Theorem 3.2 may be applied to yield explicit bounds.

Example 3.1. Suppose that $1 \le u \in L^\infty$ and satisfies

$$u(t) \le c_0^3 + \int_0^t 3c_1(s)u^{1/3}(s) + 3c_2(s)u^{2/3}(s) + c_4(s)u(s)\log(u(s))\,ds$$

where $c_0 \geq 1$ is a constant and $c_i \in L^1$ for $i = 1, 2, 4$. Then

$$u(t) \leq \left([c_0 + 3C_1]^{1/3} + 2C_2\right)^{(3/2)\exp(C_4)}.$$

where $C_i = \int_0^t c_i(s)\, ds$.

To see this let $v(t) := u^{1/3}(t)$. Then v satisfies

$$v^3(t) \leq c_0^3 + \int_0^t 3c_1(s)v(s) + 3c_2(s)v^2(s) + 3c_4 v^3(s)\log(v(s))\, ds.$$

By Theorem 3.2 we can conclude that

$$v(t) \leq W_4(c_0, c_1, c_2, 0)^{\exp(C_4)}. \tag{16}$$

Now

$$\begin{aligned} W_4(c_0, c_1, c_2, 0) =& W_3([c_0 + 3C_1]^{1/3}, c_2, 0) \\ =& W_2(\{[c_0 + 3C_1]^{1/3} + 2C_2\}^{1/2}, 0) \\ =& ([c_0 + 3C_1]^{1/3} + 2C_2)^{1/2}. \end{aligned}$$

Substituting into (16) and using $u = v^3$ gives the result claimed.

There is another extension of Corollary 3.1 involving logarithmic terms in every place. This one can be reduced more quickly to a standard one.

Theorem 3.3. *Let $m \geq 1$ be an integer. Suppose that, $c_0 \geq 1$, $u \geq 1$ and that u satisfies*

$$u^m(t) \leq c_0^m + m\int_0^t c_1 u\log(u) + c_2 u^2\log(u) + \cdots + c_m u^m \log(u), \tag{17}$$

where $c_i \in L^1_+$ for $i \geq 1$. Then, it follows that

$$u(t) \leq c_0^{\exp\int_0^t (c_0 + \cdots + c_m)}. \tag{18}$$

Proof. Clearly we have

$$u^m(t) \leq c_0^m + m\int_0^t (c_1 + c_2 + \cdots + c_m)u^m \log(u).$$

Setting $v = u^m$ and noting that $m\log(u) = \log(v)$ we have

$$v \leq c_0^m + \int_0^t Cv\log(v)$$

where $C := c_1 + c_2 + \cdots + c_m$. By Corollary 3.1 we obtain

$$v \leq c_0^{m\exp\int_0^t C}.$$

Taking the m-th root proves the result.

Remark 3.2. In Theorems 3.2, 3.3 we assume that $u \geq 1$. Often we do not know this and we may have an inequality of the following type for $u \geq 0$:

$$\begin{aligned} u^m(t) \leq c_0^m(t) + m\int_0^t & c_1(s)u(s) + c_2(s)u^2(s) + \cdots \\ & \cdots + c_m(s)u^m(s) + c_{m+1}(s)u^m(s)|\log(u(s))|\, ds. \end{aligned} \tag{19}$$

In this case we can use the inequality

$$u|\log(u)| \le 1/e + u\log(1+u), \quad (e = \exp(1)),$$

to return to a case we have discussed.

4. More general coefficients

In this section we show that there are some results of the same type when the coefficients are allowed to depend on the variable t as well as on s. These results extend a result proved and used by [7] which was a special case of one of the more general results of [1]. We first give some extensions of this result and then, using a different idea, give another result which extends a result of Dafermos [3].

Theorem 4.1. *Let $m \ge 1$ be an integer. Suppose that for $i = 1, 2, \dots, m$, and for $s, t \in [0, T]$, we have $0 \le g_i(s,t) \le h(s)$ where h is an L^1 function, Suppose that $c_0, u \in L^\infty_+$, and u satisfies the inequality*

$$u^m(t) \le c_0^m(t) + m\int_0^t g_1(s,t)u(s) + \cdots + g_m(s,t)u^m(s)\,ds, \tag{20}$$

for a.e. $t \in [0,T]$. Then,

$$u(t) \le W_m(c_0^*(t), g_1^*(t,t), \dots, g_{m-1}^*(t,t))\exp\Big(\int_0^t g_m^*(s,t)\,ds\Big), \quad a.e. \tag{21}$$

where, for each s, $g_i^*(s,t) := \operatorname{ess\,sup}_{0\le\tau\le t} g(s,\tau)$.

Proof. Fix $\tau \in (0,T]$. Let

$$w^m(t) := (c_0^*(\tau))^m + m\Big[\int_0^t g_1^*(s,\tau)u(s) + \cdots + g_m^*(s,\tau)u^m(s)\,ds\Big]$$

Then $u(t) \le w(t)$ for a.e. $t \in [0,\tau]$ and

$$w^m(t) \le (c_0^*(\tau))^m + m\int_0^t g_1^*(s,\tau)w(s) + \cdots + g_m^*(s,\tau)w^m(s)\,ds,$$

for $t \in [0,\tau]$. Since $g_i(s,\tau) \le h(s)$ for $h \in L^1$, we have $g_i^*(\cdot,\tau) \in L^1$, and we deduce from Corollary 3.3 that

$$w(t) \le W_m(c_0^*(\tau), g_1^*(t,\tau), \dots, g_{m-1}^*(t,\tau))\exp\Big(\int_0^t g_m^*(s,\tau)\,ds\Big).$$

Taking $t = \tau$ and noting that τ is arbitrary completes the proof.

Remark 4.1. This result includes the Gronwall inequality proved in [7] (the case $m = 1$) which has a much longer proof. In fact they also assume more, one of their assumptions is that $\partial g_i/\partial t$ exists and satisfies $0 \le \partial g_i(s,t)/\partial t$ in which case $g_i^* = g_i$. It also includes a result of Movljankulov and Filatov (1972) stated as Theorem 2, p.359 of [6] who also seem to use the g^* idea (for continuous functions). Their result also contains that of [7].

We can prove a similar result when we have a logarithmic term.

Theorem 4.2. *Let $m \geq 1$ be an integer. Suppose that for $i = 1, 2, \dots, m$, and for $s, t \in [0, T]$, we have $0 \leq g_i(s,t) \leq h(s)$ where h is an L^1 function. Suppose that $c_0, u \in L^\infty_+$, $c_0 \geq 1, u \geq 1$ and u satisfies the inequality*

$$u^m(t) \leq c_0^m(t) + m\Big[\int_0^t g_1(s,t)u(s) + \cdots + g_m(s,t)u^m(s) + g_{m+1}(s,t)u^m(s)\log(u(s))\,ds\Big], \tag{22}$$

for $t \in [0, T]$. Then,

$$u(t) \leq (W_m(c_0^*(t), g_1^*(t,t), \dots, g_{m-1}^*(t,t)))^{\exp(G^*_{m+1}(t))} \times \exp\Big[\exp(G^*_{m+1}(t)) \int_0^t g_m^*(s,t)\exp(-G^*_{m+1}(t))\,ds\Big], \tag{23}$$

where $g_i^*(s,t)$ is as in Theorem 4.1 and $G^*_{m+1}(t) = \int_0^t g^*_{m+1}(s,t)\,ds$.

Proof. The proof is similar to that of Theorem 4.1 using Theorem 3.2

Now we give a result which generalizes the result of Dafermos [3] by using a slight modification of the idea from [3]. We impose stronger assumptions than we did in Theorem 4.2 but in some cases we get a stronger conclusion. This is illustrated by the example following the Theorem.

Theorem 4.3. *Let $m \geq 1$ be an integer. Suppose that for $i = 1, 2, \dots, m$, and for $s, t \in [0, T]$, we have $0 \leq g_i(s,t) \leq h(s)$ where h is an L^1 function, and that $\partial g_i/\partial t$ exists and satisfies $0 \leq \partial g_i(s,t)/\partial t \leq a(t)g_i(s,t)$ for some $a \in L^\infty_+$. Suppose that $c_0, u \in L^\infty_+$, $c_0 \geq 1, u \geq 1$ and u satisfies the inequality*

$$u^m(t) \leq c_0^m(t) + m\Big[\int_0^t g_1(s,t)u(s) + \cdots + g_m(s,t)u^m(s) + g_{m+1}(s,t)u^m(s)\log(u(s))\,ds\Big], \tag{24}$$

for $t \in [0, T]$. Then,

$$u(t) \leq (W_m(c_0^*(t), g_1(t,t), \dots, g_{m-1}(t,t)))^{\exp(G_{m+1}(t))} \times \exp\Big[\exp(G_{m+1}(t)) \int_0^t [a(s)/m + g_m(s,s)]\exp(-G_{m+1}(s))\,ds\Big], \tag{25}$$

where $G_{m+1}(t) = \int_0^t g_{m+1}(s,s)\,ds$.

Proof. As previously we suppose that c_0 is a positive constant. Let

$$w^m(t) := c_0^m + m\int_0^t \sum_{i=1}^m g_i(s,t)u^i(s) + g_{m+1}(s,t)u^m(s)\log u(s)\,ds.$$

Note that

$$m\int_0^t \sum_{i=1}^m g_i(s,t)u^i(s) + g_{m+1}(s,t)u^m(s)\log u(s)\,ds \leq w^m(t).$$

The hypotheses made allow us to differentiate under the integral sign to obtain

$$\begin{aligned}
w^{m-1}w' &= \sum_{i=1}^{m} g_i(t,t)u^i(t) + \int_0^t \sum_{i=1}^{m} \frac{\partial g_i(s,t)}{\partial t} u^i(s)\,ds \\
&\quad + g_{m+1}(t,t)u^m(t)\log u(t) + \int_0^t \frac{\partial g_{m+1}(s,t)}{\partial t} u^m(s)\log u(s)\,ds \\
&\le \sum_{i=1}^{m} g_i(t,t)u^i(t) + g_{m+1}(t,t)u^m(t)\log u(t) \\
&\quad + \int_0^t \sum_{i=1}^{m} a(t)[g_i(s,t)u^i(s) + g_{m+1}(s,t)u^m(s)\log u(s)]\,ds \\
&\le \sum_{i=1}^{m-1} g_i(t,t)u^i(t) + [a(t)/m + g_m(t,t)]w^m(t) + g_{m+1}(t,t)u^m(t)\log u(t).
\end{aligned}$$

This yields the inequality

$$w^m(t) \le w^m(0) + m\int_0^t \Big[\sum_{i=1}^{m-1} g_i(s,s)w^i(s) + [a(s)/m + g_m(s,s)]w^m(s) + g_{m+1}(s,s)w^m(s)\log(w(s))\Big]\,ds. \tag{26}$$

Theorem 3.2 now applies to give the conclusion.

Remark 4.2. When $g_{m+1} = 0$ this result is valid with $u, c_0 \ge 0$ and is a slight generalization of a result of [9], which had $a =$ const., and which contained the result of [3].

Example 4.1. We consider the example from Dafermos [3]. Suppose $g \in L^1_+$ and that $u \in L^\infty_+$ satisfies the inequality

$$u^2(t) \le M^2 + \int_0^t 2Ng(s)u(s) + (2\gamma + 4\beta t)u^2(s)\,ds$$

where N, M, γ, β are non-negative constants. Then in Theorems 4.2, 4.3 we have $g_1(t,s) = Ng(s)$ and $g_2(t,s) = \gamma + 2\beta t$ so that $g_i^*(t,s) = g_i(t,s)$ but note that $G_2^*(t) \ne G_2(t)$. Observe that

$$\partial g_2(s,t)/\partial t = 2\beta \le a g_2(s,t) = a(\gamma + 2\beta t)$$

for $a = 2\beta/\gamma$. The conclusion of Theorem 4.2 is

$$u(t) \le W_2(M, g(t,t))\exp\Big(\int_0^t g_2(s,t)\,ds\Big) \tag{27}$$

while the conclusion of Theorem 4.3 is

$$u(t) \le W_2(M, g(t,t))\exp\Big(\int_0^t [a/2 + g_2(s,s)]\,ds\Big) \tag{28}$$

with $a = 2\beta/\gamma$. Carrying out the calculations shows that (27) gives a better estimate than (28) if and only if

$$\gamma t + 2\beta t^2 \le \beta t/\gamma + \gamma t + \beta t^2$$

that is, if and only if $t \le 1/\gamma$.

Example 4.1 shows that Theorem 4.3 gives a worse result than Theorem 4.2 for $t \leq 1/\gamma$ but a better result for $t > 1/\gamma$. Thus the two results are not directly comparable.

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