Chapter 1

Nash implicit functions, by G.D'AMBRA

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Nash-Gromov implicit function theorems and induced geometric structures[1]

Abstract The main purpose of this paper[2] is to review, without any ambition of being exaustive, some of the basic aspects of Gromov's general theory of induced geometric structures with focus on his *implicit function theorem for infinitesimally invertible operators* and some of its applications. The essential meaning of this powerful result, which applies to a broad class of differential operators, is that

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[2] This survey is based on the text of a lecture centered on isometric immersions and induced geometric structures given by the Author in Potenza in February 1997.

when a certain linear algebraic system of equations connected naturally with a specific differential operator is solvable, and when an appropriate topology is introduced on the image and inverse image, then the operator in question is an *open* mapping, that is, *locally* invertible near any point of its image. In [7], which is the main reference for the subject, different implicit function theorems are (abstractly) deduced for differential operators with a sufficiently simple linearization. The analytical results combined with other (mainly topological) methods, provide a general framework for the solution of many problems related to the possibility of inducing geometric structures. Among others, this is one of the main reasons which today make this topic worth of a systematic study. In this survey we mainly discuss isometric immersions of Riemannian manifolds and connection inducing maps. All the material we have included here is by now mostly well known. However, it is not gathered together in one easily accessible place.

1.1 Introduction and motivation

Historically, the theory of isometric immersions may be considered as a precursor to the general theory of induced geometric structures developed by M. Gromov in [7]. In this section, we briefly recall the terminology and the basic facts on Riemannian isometric immersions (a more precise account of the matter may be found in [21], together with a detailed bibliography). From now on, all manifolds will be considered C^∞ smooth, unless otherwise specified.

1.1.1 Immersions, embeddings

Let (M, g) and (N, h) be two Riemannian manifolds. A smooth map $f : M \to N$ is called a *isometric immersion* (a Riemannian immersion) if it satisfies $f^*h = g$. Recall that in differential topology a C^∞ smooth map f between smooth manifolds M, N is said to be an *embedding* if it is a diffeomorphism onto a submanifold of N and to be an *immersion* if it is locally an embedding, that is, it it has, for the smoothness required, a nonzero differential. Of course, a smooth Riemannian immersion is always an immersion. In the sequel (unless otherwise specified) the term immersion is meant to include embeddings, too.

1.1.2 General isometric immersions and induced geometric structures

Here we are interested in general 'isometric' immersions. That is, we shall discuss isometric C^∞ maps $f : (M, g) \to (N, h)$ which induce given geometric structures on M (for given g on M and h on N). According to the general approach indicated in [7], a convenient set up for this notion is as follows. Let N be a smooth manifold with a fixed geometric structure h, such as a Riemannian metric, a connection (in some bundle over N) a Pfaffian system on N, etc.. Then a smooth map $f : M \to N$ *induces* a structure on M of the same type as h, say $g = f^*h$, which can also be written as

$$g = \mathcal{D}(f) \tag{1.1}$$

where $\mathcal{D} = \mathcal{D}_h$ is a (nonlinear differential) operator from the space $\{\mathcal{F}\}$ of maps $M \to N$ to the space $\{\mathcal{G}\}$ of pertinent structures on M, defined by $\mathcal{D}_h(f) = f^*h$. The global analytic and geometric study of $\mathcal{D}_h$ was started by J. Nash in his two seminal papers [17] and [18] for the case where h is the standard (Euclidean) Riemannian metric

$$h = \sum_{i=1}^{q} dx_i^2$$

on $N = \mathbf{R}^q$. Nash's work was motivated by the *isometric immersion problem for Riemannian manifolds* asking for a solution to the inducing relation (1.1) for a given Riemannian metric $g \in \{\mathcal{G}\}$. This is the well known problem about the possibility to realize abstract Riemannian manifolds as submanifolds of Euclidean spaces endowed with the induced Riemannian metric. The question here can be local or global. Given a Riemannian manifold M with the metric g, does there exist an immersion of M in some Euclidean space $\mathbf{R}^q$ such that the metric on M induced by this immersion coincides with g ? A local version of this question is as follows: is there an isometric immersion in $\mathbf{R}^q$ of a sufficiently small neighborhood of a given point $p_0 \in M$? Although these questions were, for almost a century, intensively studied by many mathematicians, before J. Nash only very special results had been obtained (cf. [21], [6]). In 1956, J. Nash was able to solve the isometric immersion problem in global form and in an almost completely general situation. The main result in [18] is the following

Theorem 1 *Every compact Riemannian manifold* M *with* C^r*-metric* ($r = 3, 4, \cdots \infty$) *can be isometrically embedded into an arbitrary small domain in the Euclidean space* $\mathbf{R}^q$ *for* $q = 3n(n+9)/2$.

Remark 1 It is actually convenient to introduce the critical number $s_n = n(n+1)/2$ (this will be motivated by the discussion of the local problem, cf. section 1.1.3) and write $q = 3s_n + 12n$ in the above statement. Notice that (as can be seen in Gromov's theorem stated below) the compactness assumption turned out to be non-essential, but the original result by J. Nash was significantly weaker for *non-compact* manifolds as it required a much larger q, namely $q = 3(n+9)s_n$. This value of the dimension of an Euclidean space $\mathbf{R}^q$ isometrically containing all, possibly non-compact, n-dimensional manifolds was improved for $n \geq 4$ by R. Greene, [8], to $q = 24s_n + 22n + 14$. The dimension q of the ambient space was later reduced to $s_n + 3n + 5$ by a rather direct application of methods in general isometric immersion theory (i.e. the h-principle discovered by M. Gromov, [7], cf. Definition 1 of this survey). This theory allowed all (compact or not) $M = M^n$ and an arbitrary, not necessarily Euclidean, target Riemannian manifold N of dimension $q \geq s_n + 3n + 5$. The detailed proof of the existence theorem for isometric immersions $M^n \to \mathbf{R}^{s_n+3n+5}$ (following from this general theory) was presented in the survey article by M. Gromov & V. Roklin, [6], and the complete result (i.e. the h-principle) appeared much later in [7] with some lowering of the estimate of q to $q \geq s_n + 2n + 3$. Finally, we mention a recent result of M. Günter, [9], who indicated an isometric embedding construction working for even smaller q, i.e. applicable in the presence of a free map and for $q \geq s_n + n + 5$. Moreover, he obtains an isometric embedding for $q \geq \max\{s_n + 2n\, ,\, s_n + n + 5\}$.

1.1.3 The local problem

Let M be a n-dimensional manifold with metric $g = \sum_{i,j=1}^n g_{ij} dx_i dx_j$ and let $f : M \to \mathbf{R}^q$ be a C^∞-smooth map $f = (f^1, \cdots, f^q)$. The problem of isometrically embedding a neighborhood of a point $p_0 \in M$, in $\mathbf{R}^q$ is equivalent to that of finding a local solution to the following system of nonlinear PDEs of the first order

$$\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \rangle_h = g_{ij}\,, \quad 1 \leq i, j \leq n, \tag{1.2}$$

where $\langle v, w\rangle_h = \sum_{\alpha=1}^{q} v^\alpha w^\alpha$. Analitically speaking, we have to solve, for a given positive definite, symmetric matrix of functions $g_{ij} = g_{ij}(p_0)$ on M, the system (1.2) which consists of $s_n = \frac{n(n+1)}{2}$ equations in the q unknowns f^α.

1.1.3.A For the general case, when instead of $\mathbf{R}^q$ we have an arbitrary Riemannian manifold (N, h), the system (1.2) is $\langle \nabla_i f, \nabla_j f\rangle_h = g_{ij}$, $1 \leq i, j \leq n$, where $\langle\ ,\ \rangle_h$ is the scalar product on $T_w(N)$, $w = f(p)$, induced by the Riemannian metric h of N, and $\nabla_i f$ is the first (covariant) derivative of f, with respect to the local coordinates x_i on M, i.e. $\nabla_i f = Df\left(\frac{\partial}{\partial x_i}\right)$.

1.1.3.B Comments and a few other results (cf. [4]). (i) If $q = s_n = \frac{1}{2}n(n+1)$ then the system (1.2) is determined, and so there are grounds for thinking that we can always find a solution, for any functions $g_{ij}(p)$. This idea goes back to L. Schlaefly who was the first to conjecture it (cf. [20], the earliest publication (1873) on isometric immersions). There he claimed (without exhibiting a proof) that any n-dimensional Riemannian manifold can be isometrically immersed in an Euclidean space of dimension $s_n = \frac{n(n+1)}{2}$. In 1926, M. Janet, [13], published a proof of the local form of this conjecture for analytic Riemannian manifolds and in 1931 C. Burstin, [1], completed the proof, by filling in some gaps of M . Janet's proof for the case $n = 2$. Independently, E. Cartan, [2], gave another (completely different) proof based on his theory of differential forms, so the result is known as the Burstin-Janet-Cartan theorem. This (referred hereafter as the *B-J-C theorem*) can be considered to be the first general immersion theorem in Riemannian geometry. It claims that *for any analytic Riemannian manifold M and any point $p \in M$ there exists a neighborhood of p in M that can be isometrically analitically immersed in $\mathbf{R}^{s_n}$.* In this theorem $\mathbf{R}^{s_n}$ cannot be replaced by $\mathbf{R}^{s_n - 1}$, i.e. the number s_n is the smallest possible. The C^∞-version of the B-J-C theorem has not been proved yet. However, we mention here the result of R. Greene, [8], who showed that in the class C^∞, a local immersion is possible in some Euclidean space $\mathbf{R}^q$ ($q > s_n + n$). Also, two years later, M. Gromov improved q to $q = s_n + n + \left[\frac{n}{2}\right]$ ($[a]$ is the integer part of a).

1.1.3.B.1 Let us give here a brief account (following [13]) of the proof of the B-J-C theorem, which is fundamental for the local theory. First, a definition. For a C^2-map f of a C^2-manifold M of dimension n into $\mathbf{R}^q$, we denote by $T_p^2(f)$ the *osculating space* spanned by the first and second derivatives

$$\frac{\partial f}{\partial x_i}(p),\ \ \frac{\partial^2 f}{\partial x_i \partial x_j}(p),\ \ 1 \leq i, j \leq n, \tag{1.3}$$

of the vector $f = (f^1, \cdots, f^q)$, with respect to the local coordinates at $p \in M$. It is clear that $T_p^2(f)$ does not depend upon the choice of local coordinates. If $\dim T_p^2(f) = n + s_n$ (i.e. it is maximal, so that the vectors (1.3) are linearly independent) at every point $p \in M$, then the map is said to be *free*. It is obvious that a free map is a differentiable immersion and that an n-dimensional manifold cannot be freely mapped into $\mathbf{R}^{s_n+n-1}$. The map

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n; x_1^2, \{x_i x_j\}_{1 \leq i < j \leq n}, x_n^2)$$

defines a free analytic immersion of $\mathbf{R}^n$ in $\mathbf{R}^{n+s_n}$ (for another example of free map, see 1.3.2.C.1). The main part of the proof of the B-J-C theorem relies on a Lemma saying that *any point of a n-dimensional analytic Riemannian manifold admits a neighborhood which can be isometrically analitically immersed in* $\mathbf{R}^{n+s_n}$. Then the derivation of the B-J-C statement from the above lemma can be sketched as follows (the reader can consult [21] for a detailed proof). Both the arguments start by assuming that the Lemma is true for manifolds of dimension $n-1$. Let M be a n-dimensional analytic Riemannian manifold and $p_0 \in M$. Construct locally a $(n-1)$-dimensional submanifold M_0 passing through p_0 and geodesic at p_0 (i.e. M consists of geodesics through p_0) and then (by applying the Lemma to M_0) we get a *free isometric analytic* immersion in $\mathbf{R}^q$ of some neighborhood (in M_0) of p_0, where $q = (n-1) + s_{n-1} = s_n - 1$. To derive the B-C-J theorem from the Lemma one extends this immersion - in such a way to preserve its analyticity and isometry - to some neighborhood of p_0 in M, having previously extended $\mathbf{R}^{s_n-1}$ into $\mathbf{R}^{s_n}$ and having shrinked the original neighborhood (of p_0 in M_0) if necessary. Such an extension can be obtained by the Cauchy-Kowalewsy theorem, as a solution of the Cauchy problem for a certain system of PDEs. The proof of the Lemma goes along the same lines than the proof of the theorem, with the only difference that $\mathbf{R}^{s_n-1}$ is extended in $\mathbf{R}^{s_n+n}$, rather than $\mathbf{R}^{s_n}$, and the extension is built so that to be free. Since the Lemma is obvious for $n = 0$, the proof follows by induction. In both the proofs (of the lemma and of the Theorem) the freedom of the local embedding of M_0 in $\mathbf{R}^{s_n-1}$ is needed to reduce the system of PDEs to a form satisfying the hypothesis of the Cauchy-Kowalewsky theorem.

1.1.4 Negative results

If $q < s_n$, then the system (1.2) is overdetermined and therefore it seems unlikely that we can always find a solution f. Apparently, this fact is nowadays looked at as trivial. Yet, although statements such as *not every n-dimensional Riemannian manifold can be locally isometrically immersed in $\mathbf{R}^q$ with $q < s_n$* are currently met in the mathematical literature, proofs or exact statements about non local isometric immersibility of specific n-dimensional manifolds in some $\mathbf{R}^q$, $q < s_n$, remain rather scarce. For $q = s_n - 1$, $n > 2$, there are non known results of this kind (the only existing results concern, instead of $\mathbf{R}^{s_n-1}$, Euclidean spaces of a comparatively low dimension). The following statement is quoted from Appendix 1 of [6]: *for $r \geq n(s_n - 1)$ and $r - 1 \leq k \leq \infty$ the set of Riemannian metrics that are locally induced on a smooth n-dimensional manifold with a distinguished point by local C^r-immersions in $\mathbf{R}^q$ for $q < s_n$ is nowhere dense in the set of all C^k-Riemannian metrics on M endowed with the usual C^k-topology.* The proof of this result, given in section 1.1.4.A below, does not involve any geometry and only depends on the fact that, for $q < s_n$, the system (1.2) is overdetermined (cf. [21]). Indeed, it can be regarded as a special case of a general theorem on systems of differential equations where the number of equations exceeds the number of unknown functions.

1.1.4.A Let $x_1, \cdots, x_n$ be local coordinates in M with the origin at the distinguished point and let $x_1, \cdots, x_{s_n-1}$ be the standard coordinates in $\mathbf{R}^{s_n-1}$. We are given g_{ij} (the components of a metric g induced by a C^r embedding of a neighborhood $\mathcal{U} = \mathcal{U}(0) \subset M$), and we seek for $f : \mathcal{U} \to \mathbf{R}^{s_n-1}$ such that

$$\sum_{\alpha=1}^{s_n-1} \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\alpha}{\partial x_j} = g_{ij}.$$

Now,the g_{ij} must also satisfy all the equations obtained from these by partial differentiation of orders up to $r - 1$. If we evaluate all these equations at 0, we get polynomial formulae expressing the $s_n \frac{(n+r-1)!}{n!(r-1)!}$ order derivatives

$$\frac{\partial^{r_1+\cdots+r_n} g_{ij}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}(0), \quad 0 \leq r_1 + \cdots + r_n \leq r - 1, \tag{1.4}$$

in terms of the $(s_n - 1)[\frac{(n+r)!}{n!r!} - 1]$ order derivatives

$$\frac{\partial^{r_1+\cdots+r_n} f^\alpha}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}, \quad 1 \leq r_1 + \cdots + r_n \leq r, \tag{1.5}$$

We can view the derivatives (1.5) as the coordinates of a point in $\mathbf{R}^a$ with $a = (s_n - 1)[\frac{(n+1)!}{n!r!} - 1]$, the derivatives (1.4) as the coordinates of a point in $\mathbf{R}^b$ with $b = s_n \frac{n+r-1)!}{n!(r-1)!}$, and the polynomial formulae as coordinate relations defining a map $F : \mathbf{R}^a \to \mathbf{R}^b$. Since $a < b$, this means that the set of all possible derivatives (1.4) (thought of as a point in $\mathbf{R}^a$) is the image of a polynomial map defined on a lower dimensional space $\mathbf{R}^a$ so that the image $f(\mathbf{R}^a)$ is nowhere dense in $\mathbf{R}^b$. This image contains the $(r-1)$-jets (at the distinguished point) of all the metrics induced by local C^r-embeddings of M in $\mathbf{R}^{s_n - 1}$, and since the (natural) map of the space of C^k-metrics on M into $\mathbf{R}^b$ (mapping each metric into its $(r-1)$-jet at the distinguished point) is open, the metrics locally induced by local C^r-embeddings of M into $\mathbf{R}^{s_n - 1}$ form a nowhere dense set in the space of all C^k-metrics.

1.2 Final comments

We end this section with some more comments related to the previous discussion and by recalling a few more results relevant to the isometric immersion theory. In doing this, we shall merely indicate what these results are and our discussion will be brief since there are nowadays several good references (e.g. [21]) where all this material is well presented.

i) There are purely topological questions which must be considered in the isometric immersion theory. First of all, one has Whitney's theorem saying that every n-dimensional manifold can be smoothly immersed into $\mathbf{R}^{2n-1}$ and embedded into $\mathbf{R}^{2n}$. But the essential analytic aspects of the isometric immersion theory, e.g. the *local* isometric immersions, have no analogue in differential topology where all local problems are trivial by definition.

ii) All of Nash's results are valid for metrics and immersions of class C^r, $r \geq 3$. Nothing is known about the case $r = 2$.

iii) The situation changes drastically if we consider immersions of class C^1. The first results along this line were those by J. Nash ([17]) supplemented by N.H. Kuiper, [15]. For a compact n-dimensional Riemannian manifold M, their results show that if M admits any immersion in $\mathbf{R}^q$, with $q \geq n+1$, then it also admits an isometric C^1-immersion.

iv) (Compare to Definition 1). The *h-principle for smooth immersions* $M^n \to \mathbf{R}^q$, $q > n$ (cf. Hirsch, [11]) asserts that *every smooth map* $M^n \to \mathbf{R}^q$ *can be homotoped into a smooth immersion iff it can be covered by a continuous bundle map* $\varphi : T(M^n) \to T(\mathbf{R}^q)$ *which is injective on each fiber of* $T(M^n)$. In particular, the following statements a) and b) are true for *stably* parallelizable[3] manifolds M.

a) *Every smoth parallelizable* M^n *admits an immersion into* $\mathbf{R}^{n+1}$. (This is an immediate corollary of Hirsch's theorem: see (B_2') at p. 8 in [7]).

b) *Every parallelizable* M^n *admits a* C^1*-isometric immersion into* $\mathbf{R}^{n+1}$. (This is achieved by combining the C^1-Nash-Kuiper result with a)).

v) The notion of h-principle in the sense of [7] applies to the solution of an arbitrary differential equation and/or equality (also of a non equality such as the one describing immersions $M^n \to \mathbf{R}^q$ by the non-degeneracy of the differential). Yet, as demonstrated to the Author by M. Gromov, no single case of this h-principle has been proved or disproved for C^∞-smooth isometric immersions for $n \geq 2$ and $q \geq 4$. In fact, the h-principle established by M. Gromov applies to *free* isometric immersions ([7], p. 9). The freedom condition automatically limits the local dimension by $s_n + n$ (while the *global* construction in [7] requires $q \geq s_n + 2n + 3$). On the other hand, the B-J-C theorem (in section 1.1.3.B above) and the general philosophy behind the h-principle suggest, according to M . Gromov[4], that every C^∞-smooth n-dimensional Riemannian manifold admits an isometric (possibly non-free) immersion in $\mathbf{R}^{s_n+n}$ where $s_n = \frac{n(n+1)}{2}$ is the "local dimension" of the B-J-C theorem. This is confirmed by M. Gromov's result allowing isometric C^∞-embeddings of compact surfaces $M^2 \to \mathbf{R}^5$ ([7], p. 298)[5]. On the other hand, there are many geometric obstructions for isometric C^∞-immersions $M^2 \to \mathbf{R}^3$ but none was found[6] for isometric C^∞-immersions $M^n \to \mathbf{R}^{s_n}$ for $n \geq 3$. Also notice that every compact 3-manifold is known to admit an isometric embedding in $\mathbf{R}^{13}$ (cf. [7], p. 305), a result which is the best available nowadays.

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[3] Recall that M is *parallelizable* if its tangent bundle is trivial, $T(M) = M \times \mathbf{R}^n$. Also M is *stably* parallelizable if $M \times \mathbf{R}$ is parallelizable.

[4] Private communication.

[5] Actually, it is an old conjecture by S.S. Chern that every surface can be isometrically immersed in $\mathbf{R}^4$.

[6] Again, demonstrated to the Author by M. Gromov.

1.3 Nash-Gromov implicit functions

As we mentioned in the introduction, the first theorem of the implicit function type was given by J. Nash, who discovered it in the course of his solution to the isometric immersion problem. From the conceptual point of view, this theorem of J. Nash is a function space analogue of the classical theorem saying that a differentiable map of one Euclidean space into another is surjective in the neighborhood of any point at which its derivative is surjective. The original proof of J. Nash is hard to follow and requires rather involved techniques, some of which were created precisely for the isometric immersion problem. One of the main steps in Nash's proof (cf. section 1.3.2.B) amounts to inverting algebraically the linearized equations corresponding to the system (1.2) in the previous section. It turns out that (cf. [7], sect. 2.3.1, where Nash's theory is developed in the context of general differential operators) this idea applies to many other instances of non-linear systems of partial differential equations. In all these cases one needs an appropriate version of the implicit function theorem to pass from (solutions of) the linearized system to (solutions of) the nonlinear system. A short discussion of the analytical results entering the proof of Nash's isometric embedding theorem will be given in section 1.3.2. After that we shall state the generalized version of the implicit function theorem proved by M . Gromov for inifinitesimally invertible operators (see Theorem 2). This is a result which, besides its importance in itself, represents the starting point for the general theory of induced geometric structures. An application of it to the problem of inducing connections between principal bundles will be shown in section 1.4.

1.3.1 Jet language and the h-principle

This section consists in a short (recollectional) discussion meant to introduce the basic language in [7]. However, we shall avoid the tecnicalities in [7] (dealing with more general situations) and the understanding of basic facts will only require some jet formalism. This is needed in order to set up the (natural) context in which interpretations of PDEs become geometrically meaningful. Let $\pi : X \to M$ be a smooth fibration and let $X^{(r)}$ be the space of germs of r-jets of smooth sections $M \to X$. Thus each $X^{(r)}$ is a bundle over X whose fibre at each $x \in X$ consists of linear maps ψ from $T^r_{\pi(x)}M$ (the tangent space of M at $\pi(x)$, of order r) to $T_x^r X$, such that for all $1 \leq s \leq r$,

$$\psi\left(T^s_{\pi(x)}M\right) \subset T^s_x M.$$

By taking the r^{th} order jet of a smooth section $f : M \to X$ we get a smooth section $j^r f : M \to X^{(r)}$. Clearly $j^r f$ is, locally, nothing than the string of all partial derivatives of f up to order r.

Definition 1 A *differential relation* imposed on the sections of $\pi : X \to M$ is a subset $\mathcal{R} \subset X^{(r)}$. A *solution* to $\mathcal{R}$ is a section $f : M \to X$ such that $(j^r f)(M) \subset \mathcal{R}$. Thus we may identify the solutions to a differential relation $\mathcal{R}$ with the *holonomic* sections $M \to X^{(r)}$ which map into $\mathcal{R}$.

Usually, it is easy to construct a continuous section $f : M \to \mathcal{R}$, or else one such section is given. Then the obvious way to obtain a solution to $\mathcal{R}$ is to *deform by homotopy* the section f into a holonomic one, if this is possible. All this is not expressed formally (a formal discussion of the global aspects of M. Gromov's general theory is beyond the limits of the present (expository) paper), yet it will be enough for our purposes to accept the following statements (a) to (c):

a) we say that $\mathcal{R}$ *satisfies the h-principle* (h stays for *homotopy*) if every continuous section $M \to \mathcal{R}$ can be deformed homotopically into a holonomic section;

b) the h-principle is a very strong claim which allows to solve a differential relation $\mathcal{R}$. It does not hold in general, but (cf. [7]) it holds under a variety of assumptions on $\mathcal{R}$ (compare to the discussion in section 1.4.8.D).

c) As it was pointed out at the beginning of these notes, our exposition is intended to give an account of the *local* aspects of the general theory of induced geometric structures. So priority is, of course, given to points where the generalized implicit function theorem plays the main role. Yet, an attempt will be made to persuade the reader of the practical value of the whole of M. Gromov's method based on the h-principle, and this has the implicit function theorem as a starting point.

Now, there is a long way from the implicit function theorem (which claims that some map $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$ is open, see Theorem 2) to actually solving the relation $\mathcal{D}(f) = g$ (compare to section 1.2.(i)), i.e. to showing that $\mathcal{D}$ is onto. As we mentioned previously, J. Nash invented a very special approach to isometric immersions $M \to \mathbf{R}^q$ which has *a priori* a rather limitted applicability. The general method due to M. Gromov consists of an assemblying process of local solutions of the given relation, one solution $f_p(p')$ for each $p \in M$, and p' ranging in some small neighborhood $\mathcal{U}_p \subset M$. This needs a rather elaborate use of topological sheaves ([7], [21]). Eventually one obtains in this way more than just solvability of the equation $\mathcal{D}(f) = g$, but also a description of the homotopy type of the space of solutions in terms of some space of sections of some auxiliary jet fibration over M (cf. section 1.4.8.D). Here are some examples (see also sections Theorem 8 and Remark 5 (iv)).

1.3.1.A.1. Immersions Let M, N be differentiable manifolds of dimensions n and q respectively, $n \leq q$. Let $X = M \times N$ and let $\pi : X \to M$ be the natural projection. A section of $X^{(1)} \to M$ is a map $p \mapsto ((p, w), \psi)$ where $p \in M,\ w \in N$ and ψ is a linear map $T_pM \to T_wN$. A *holonomic* section is then a map

$$p \mapsto ((p, f(p)), df : T_p(M) \to T_{f(p)}(N)),$$

where f is a map from M to N and hence the holonomic section may be simply identified with the pair (f, df). We define the Immersion Relation $\mathcal{R} = Im$ on this $X^{(1)}$ to consist only of those $((p, w), \psi)$ where ψ is injective. It follows that the holonomic sections of Im in this case consist of immersions $M \to N$. The immersion theory of Hirsch and Smale ensures the validity of the h-principle quoted in section 1.2 (iv).

1.3.1.A.2. Free isometric immersions (cf. 1.1.3.B.1). Let M be a Riemannian manifold, $\dim(M) = n$, and let $X = M \times \mathbf{R}^q$, where $q \geq \frac{1}{2}(n+2)(n+3)$. As before, $\pi : X \to M$ is the natural projection. Now define $\mathcal{R} \subset X^{(2)}$ to be the set of all $((p, w), \psi)$ where $p \in M,\ w \in \mathbf{R}^q$, and ψ is a non-singular linear map $T_p^2(M) \to T_w^2\mathbf{R}^q$ su ch that the restriction of ψ to $T_p(M)$ is an isometric linear map into $T_w(\mathbf{R}^q)$. By the same arguments as in section 1.3.1.A.1, one sees immediately that a holonomic section of this $\mathcal{R}$ is just an isometric immersion $M \to \mathbf{R}^q$ whose second order differential is everywhere non-singular. Such immersions, as we saw above, are called *free isometric immersions.* A theorem of M. Gromov shows that $\mathcal{R}$ satifies the h-principle ([7], p. 12; cf. also section 1.3.2.C.2 of this survey).

1.3.2 Linearization and inversion of differential operators

We come now to the main topic of this survey. Consider a nonlinear differential operator $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$, where $\{\mathcal{F}\}$ and $\{\mathcal{G}\}$ are some function spaces. We shall explain how a series of statements - of the implicit function theorem type - can be proved (i.e. propositions which reduce properties of $\mathcal{D}$ such as being open, invertible, etc., to the corresponding properties of its linearization (differential)). In a series of cases such theorems allow one to solve for $f \in \{\mathcal{F}\}$ the equation $\mathcal{D}(f_0 + f) = g_0 + g$ if a solution f_0 of $\mathcal{D}(f_0) = g_0$ is known, and if g is small in some sense (cf. section 1.3.2.E below).

We shall discuss first the fundamental (and most well known) examples, related to the problem of isometric immersions (namely that of the metric inducing operator). In section 1.4 we shall describe in some detail another example related to the problem of the construction and classification of connection inducing maps. As we shall see, both these problems (and the same thing happens in all other cases we consider) need to be reformulated in the light of the general program developed in [7] and (clearly) depending on the geometric structure under scrutiny. Once this is done, one immediately realizes the efficiency of the method as it will be often possible to derive specific statements as direct corollaries of the general implicit function theorem just by specializing to the differential operator at hand the basic properties of infinitesimally invertible operators and the consequent analytical results (see [7], p. 114-119).

1.3.2.A. Example (The metric inducing operator) The operator $\mathcal{D}$ relating to C^1 maps $M \to \mathbf{R}^q$ the induced quadratic form $f^*(h)$ for the standard metric $h = \sum_{i=1}^{q} dx_i^2$ on $\mathbf{R}^q$ is a differential operator of the first order. In the case when g is the given Riemannian metric on M, we write

$$\mathcal{D}(f) = g = \{g_{ij}\} = \left\{ \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle \right\}, \quad 1 \leq i, j \leq n. \tag{1.6}$$

That is, locally (with respect to some coordinate chart $(x_1, \cdots, x_n)$ at $p \in M$) we have a quadratic differential operator of the first order sending maps $f = (f^1(p), \cdots, f^n(p))$ to matrix valued functions $\{g_{ij}(p)\}_{1 \leq i,j \leq n}$ where

$$\langle \partial_i f, \partial_j f \rangle_h = \sum_{\alpha=1}^{q} \frac{\partial f^\alpha}{\partial x_i}(p) \frac{\partial f^\alpha}{\partial x_j}(p).$$

Observe that one cannot write such a formula for $g = f^*(h)$ globally. The only thing we have is a (nonlinear) differential operator $f \mapsto g = f^*(h)$ defined on all of M, which in every local coordinate system on M is expressible by (1.6). In fact, this helps a clearer understanding of the definition of the notion of a (nonlinear) differential operator $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$ between spaces of sections of certain vector bundles over M. Each function space is given locally by systems of finitely many functions on M and an operator $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$ is called a *differential* operator of order r if it is *local*, i.e. if the value $\mathcal{D}(f)(p)$ is expressible in terms of partial derivatives of (the components of) f at p (for formal definitions, see [7], p. 114).

We proceed by briefly describing the analytic part of Nash's proof of Theorem 1 (we follow [4]).

1.3.2.B. Linearized equations We begin with a map $f : M \to \mathbf{R}^q$ inducing some $g = f^*(h)$ written, as before, $g = \langle \partial_i f, \partial_j f \rangle$. Let us slightly modify f by replacing it with $f + \varphi$ for a "small" $\varphi : M \to \mathbf{R}^q$. The new induced metric $g_\varphi = (f + \varphi)^*(h)$ decomposes into a sum

$$g_\varphi = g + L(\varphi) + Q(\varphi) \tag{1.7}$$

where

$$L(\varphi) = \langle \partial_i f, \partial_j \varphi \rangle + \langle \partial_j f, \partial_i \varphi \rangle \tag{1.8}$$

and

$$Q(\varphi) = \langle \partial_i \varphi, \partial_j \varphi \rangle \tag{1.9}$$

(for notational simplicity we drop the i, j indices in the left hand member of the above equations). Note that $L(\varphi)$ is a *linear* differential operator with respect to φ whose coefficients are some combinations of the derivatives of f. The operator $Q(\varphi)$ is *quadratric* in φ and if φ is "small" one may think of $Q(\varphi)$ as being something like $(\varphi')^2$ (an accent denotes derivatives) which is much smaller than φ. It is worth noticing that the decomposition (1.7) is typical for all differential operators. Namely

$$\mathcal{D}(f + \varphi) = \mathcal{D}(f) + L(\varphi) + Q(\varphi)$$

where $L = L(\varphi)$ is a linear differential operator and $Q = Q(\varphi)$ is quadratic in the derivatives of φ. Now we want to solve the equation $L(\varphi) = \psi$ for some given ("small") metric $\psi = \{\psi_{ij}\}$. We can neglect the "small" term $Q(\varphi) = \langle \partial_i \varphi, \partial_j \varphi \rangle$. We use the same "trick" as in [18] and add to (1.8) an extra equation expressing the condition for the unknown $\varphi = \varphi(p) : M \to \mathbf{R}^q$ to be orthogonal at each $p \in M$ to the image of the differential of f at p, i.e. to $D_p f(T_p(M)) \subset \mathbf{R}^q = T_{f(p)}\mathbf{R}^q$. The idea is to study simultaneously the two groups of equations

$$\langle \varphi, \partial_i f \rangle = 0, \ \ i = 1, \cdots n, \tag{1.10}$$

$$\langle \partial_i f, \partial_j \varphi \rangle + \langle \partial_j f, \partial_i \varphi \rangle = \psi_{ij}, \ \ i, j = 1, \cdots, n, \tag{1.11}$$

To some surprize, it turns out that this is easier than just solving (1.11). We differentiate (1.10) and obtain

$$\partial_j \langle \varphi, \partial_i f \rangle = \langle \partial_j \varphi, \partial_i f \rangle + \langle \varphi, \partial_{ij} f \rangle = 0. \tag{1.12}$$

Interchange i and j and combine (1.12)-(1.11) so that to transform the system (1.10)-(1.11) into the *equivalent* system

$$\langle \varphi, \partial_i f \rangle = 0, \ \ i = 1, \cdots, n,$$

$$\langle \varphi, \partial_{ij} f \rangle = -\frac{1}{2}\psi_{ij}, \ \ i, j = 1, \cdots, n. \tag{1.13}$$

The new system (1.10) and (1.13) is not differential anymore, but linear algebraic with respect to φ and so it is solvable provided the matrix of its coefficients is non-singular, i.e. the vectors $\partial_i f$, $\partial_{ij} f : M \to \mathbf{R}^q$ are linearly independent at each $p \in M$. Note that there are $n + s_n = \frac{1}{2}n(n+3)$ of these functions for $s_n = \frac{n(n+1)}{2}$ is the number of the second derivatives and so, in order to be independent they need "lots of room". Namely, one needs $q \geq s_n + n$.

Remark 2 i) The maps $\partial_i f, \partial_{ij} f$ are defined with respect to a choice of local coordinates on M. Yet, it is easy to see that their linear independence does not depend upon the choice of local coordinates. Now, if the vectors

$$\frac{\partial f}{\partial x_i}(p), \ \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \in \mathbf{R}^q$$

are linearly independent for any $p \in M$ then the map is free and we conclude that, on the space of free maps $M \to \mathbf{R}^q$, the system (1.10) and (1.13) is solvable at every p.

ii) To get a solution over the whole manifold M one has to specify a canonical procedure for the choice, at every $p \in M$, of a solution $\varphi(p)$ which is also a global one. (This problem appears only for $q > n + s_n$, while for $q = n + s_n$ the solution is unique). The choice is made by picking up at each $p \in M$ the solution φ minimizing the norm $\|\varphi\| = \langle \varphi, \varphi \rangle^{1/2}$. This is equivalent to choosing the closest point to the origin in the plane of solutions of the system (1.10) and (1.13). With this minimal $\varphi = \varphi(f, \psi)$ we get the infinitesimal inverse I of the operator $\mathcal{D}$ by setting $I(f, \psi) = \varphi(f, \psi)$.

iii) It is worth mentioning that when (N, h) is an arbitrary (pseudo) Riemannian manifold the above calculation for the solution of the linearized equations goes through with the covariant derivatives ∇_i^h and ∇_{ij}^h in place of $\partial/\partial x_i$ and $\partial^2/\partial x_i \partial x_j$, and the minimal φ may be taken with any auxiliary Riemannian metric.

1.3.2.C.1. Examples of free maps There are relatively few examples of free maps. An interesting example is provided by the *Veronese map* $f : S^n \to \mathbf{R}^{q+1}$, $q = n + s_n$, defined by the monomials $x_i x_j$ on $\mathbf{R}^{n+1}$, $1 \leq i, j \leq n+1$. The image $f(S^n)$ is diffeomorphic to $\mathbf{R}P^n$ and lying in a hyperplane $H \approx \mathbf{R}^q \subset \mathbf{R}^{q+1}$. Thus one gets a free (Veronese) embedding of $\mathbf{R}P^n$ into $\mathbf{R}^q$ for $q = \frac{1}{2}n(n+3)$.

Remark 3 A theorem of J. Nash shows that a generic map is free for $q \geq s_n + 2n$. This means that free maps are open and dense among all C^2-maps for $q \geq s_n + 2n$ (the proof relies on Thom's transversality theorem, cf. e.g. [7], p. 33).

1.3.2.D. A closing note The essential meaning of all the preceeding discussion can be now summarized as follows. We saw, in section 1.3.2.B, that the linearized system corresponding to the isometric immersion relation $L(\varphi) = \psi$ is solvable for all ψ, provided that the inducing map f satisfies a certain *regularity* condition, called *freedom*. In particular, the system is solvable for generic f if $q \geq s_n + 2n$. Unexpectedly enough, the original PDE system has been solved by a purely algebraic procedure. The solution φ can be easily expressed as $\varphi = I(\psi)$ where $I = I_f$ is a linear differential operator with respect to ψ whose coefficients are rational functions in the derivatives of f, so that these functions *have no poles at free maps.* This naturally leads to the following definition: a (nonlinear) differential operator $\mathcal{D}$ is called *infinitesimally invertible* on some space $\{\mathcal{F}\}$ of functions (sections of the fibration $X \to M$ where $\mathcal{D}$ applies), if for all $f \in \{\mathcal{F}\}$, the linearization $L = L_f : T_f(\{\mathcal{F}\}) \to \{\mathcal{G}\}$ of $\mathcal{D}$ has a right inverse $I = I_f$ and where the coefficients of $I_f(\psi)$ (itself a differential operator) are smooth functions depending on the derivatives of some order (cf. [7] for the formal definition).

1.3.2.D.1. As a result of all the above we may conclude that *the metric inducing operator is infinitesimally invertible on* (the space of) *free maps.*

It might seem that very few differential operators satisfy the property of being infinitesimally invertible, but the opposite is true. A theorem of M . Gromov asserts ([7], p. 156) that a *generic* differential operator $\mathcal{D}$ from s-tuples of functions to q-tuples of functions, $q > s$, is infinitesimally invertible on an open dense set $\Omega = \Omega(\mathcal{D})$ of q-tuples of functions. Furthermore, in many cases of geometric interest the operator $I = L^{-1}$ can be constructed following the pattern seen for the case of the metric inducing operator.

Remark 4 (On the solution of the equation $\mathcal{D}(f_0 + \varphi) = \mathcal{D}(f_0) + \psi$ for small ψ). Observe that, once we can solve $L_{f_0}(f_0) = \psi$ we get $\mathcal{D}(f_0 + \varphi_0) = \mathcal{D}(f_0) + \psi + Q(\psi)$, with a quadratic error $Q(\psi_1)$. Then we can repeat this with $f_1 = f_0 + \varphi_0$ and $\psi_1 = Q(\psi - \psi_0)$, and again with $f_2 = f_1 + \varphi_1$ and $\psi_2 = Q(\psi_1)$, etc.. At each step, the error for the next step ψ_i will be quadratic in ψ_{i-1}. By taking this into account, there are grounds for hoping for convergence if ψ is "small". This is justified in the real analytic case but not in the smooth case. Nevertheless, by applying J. Nash's technique ([18]) one can combine the above with some smoothing operators and enforce the convergence. (For some applications of this tecnique see [10] and [16]).

A direct consequence is the general statement below (basic for applications)

Theorem 2 ([*7*], p. 118) *If* $\mathcal{D} : \{\mathcal{F}\} \to \{\mathcal{G}\}$ *is an infinitesimally invertible operator then it is open with respect to the* C^∞*-topology in the given space of functions.*

As a special case of Theorem 2, one has that if $\mathcal{D}$ is a structure inducing operator then the inducible structures contain a nonempty C^∞-open subset.

1.3.3 Nash's theorem

It is now appropriate to mention that the most well known instance of a result of the kind of Theorem 2 is the (now classical)

Theorem 3 (Nash, [*18*]) *If* $q \geq s_n + 2n$ *then the space* $\overline{\mathcal{G}} \subset \{\mathcal{G}\}$ *of* C^∞*-metrics on* M *inducible from* h *on* $\mathbf{R}^q$ *contains a nonempty open subset* $\mathcal{U}$*, i.e.* $\mathcal{U} \subset \overline{\mathcal{G}} \subset \{\mathcal{G}\}$.

In terms of the metric inducing operator $\mathcal{D} = \mathcal{D}_h$ (i.e. for $\mathcal{D}_h(f) =: f^*(h)$) Nash's statement may be rephrased as follows: *There exists a nonempty open subset* $\mathcal{U} \subset \{\mathcal{G}\}$ *such that* $\mathcal{U} \subset \mathcal{D}_h(\{\mathcal{F}\}) \subset \{\mathcal{G}\}$. To end this paragraph, let us mention that a stronger version of Nash's statement was established by M. Gromov ([7], sect. 3.1). M. Gromov's result (stated for the general case where the range of f is a Riemannian manifold (N, h)) ensures that if $\dim N \geq s_n + 3n + 5$ then all Riemannian metrics on M are inducible from h on N. In other words the set of inducible metrics coincides with $\{\mathcal{G}\}$ $(\overline{\mathcal{G}} = \{\mathcal{G}\})$.

The above result (cf. section 1.3.2.D.1) is, in a number of reformulations and refinements (as for instance those in the previous section) fundamental to the whole Nash-Gromov theory. We mentioned in the introduction that Nash's machinery as described in [7] for the metric inducing operator applies in a similar way to many other geometric cases where the pertinent operator has a sufficiently simple linearization. Related to this, the discussion in the previous two sections demostrates that the first objective to be reached each time one investigates the realizability of any geometric structure h is to prove, for the corresponding differential operator $\mathcal{D} = \mathcal{D}_h$, a statement of the kind in section 1.3.2.D.1. This will, roughly speaking, ensure that if the dimension of the receiving space N is "big enough" then there exists a nonempty open subset, say $\mathcal{R} \subset \{\mathcal{F}\}$, such that the restriction of $\mathcal{D}_h$ to this $\mathcal{R}$ is an open map in the considered function space topologies.

1.4 An application: Inducing connections

In this section, we describe another example where a statement of the implicit function theorem type can be proved along the same lines it was done above for the metric inducing operator. This example is related to the construction of a universal principal bundle with connection (cf. [3], [19]). The corresponding structure inducing problem can be formulated as follows.

1.4.1 The problem

Given two C^∞-smooth principal bundles $X \to M$ and $Y \to N$ with the same structure group G and with C^∞-connections Γ on X and Δ on Y we ask when there exists a C^∞-smooth map $f : M \to N$ such that the induced bundle $f^*(Y)$ over M with the induced connection $f^*(\Delta)$ is isomorphic to

(X, Γ). As it is shown in [3], under certain regularity conditions imposed to the inducing maps $M \to N$, the (nonlinear differential) operator $\mathcal{D} = \mathcal{D}_\Delta$ which assigns to each smooth map $f : M \to N$ the connection $f^*(\Delta)$ on X (for a fixed Δ on Y) is an open map. Here is a specific statement (compare to Theorem 2)

Theorem 4 ([*3*], p. 77) *If* $\dim N \geq n \dim G$ *and* $\dim N \geq n = \dim M$*, then the space of* C^∞*-connections on* Y *contains a nonempty open subset.*

The proof of this result is of a similar nature as that of Theorem 3 in the previous section. The main point is to work out an effective criterion of infinitesimal invertibility of the differential operator $\mathcal{D} = \mathcal{D}_\Delta$.

1.4.2 Connections

In this section we briefly recall the essential terminology and some basic definition from connection theory. In addition, we state a few classical facts needed in the subsequent discussion.

1.4.2.A. Connections in principal bundles Let $\pi : X \to M$ be a C^∞ principal of structure group G, with the corresponding Lie algebra $\mathbf{g}$. A connection Γ on X consists of the assignment of a G-invariant subbundle H of $T(X)$ such that $T(X) = H \oplus V$. Therefore, for every $x \in X$, a subspace $H_x \subset T_x(X)$ is assigned in such a way that i) $T_x(X) = H_x \oplus V_x,\ x \in X$, ii) for every $g \in G,\ H_{x \cdot g} = (DR_g)H_x$, and iii) $x \mapsto H_x$ is a C^∞ distribution on X. H_x is the *horizontal subspace* of the connection at $x \in X$. If a connecion Γ is given, let $h_\Gamma : T(X) \to H$ be the natural projection. Note that $D\pi[x] : H_x \approx T_{\pi(x)}$ (an isomorphism). Each $\tau \in T_x(X)$ decomposes into a horizontal and a vertical component, viz. $\tau = \tau^{(h)} + \tau^{(vert)}$, with $\tau^{(h)} = h_\Gamma[x] \in H_x$ and $\tau^{(vert)} \in V_x$.

1.4.2.A.1. Connections in vector bundles On a C^∞ vector bundle $E \to M$ a connection is a rule which assigns to a C^∞ vector field τ on M a first order linear differential operator in the space $C^\infty(E)$ of smooth sections $M \to E$. This operator, the *covariant derivative* in the direction τ, is denoted by $\nabla_\tau : C^\infty(E) \to C^\infty(E)$ and must satisfy the formal properties of covariant derivatives (cf. e.g. [14]).

If $x_1, \cdots, x_n$ is a system of local coordinates on M, each connection in E is determined by the n covariant derivative operators $\nabla_i = \nabla_{\partial/\partial x_i} : C^\infty(E) \to C^\infty(E),\ i = 1, \cdots, n$.

Example 1 a) Each trivial bundle $E = M \times \mathbf{R}^m \to M$ carries the *standard flat connection* whose covariant derivatives ∇_i of a section $M \to M \times \mathbf{R}^m$ are just the usual derivatives of the map $f : M \to \mathbf{R}^m$ corresponding to the section.

b) Let E be a vector bundle with a connection ∇ and let $E' \subset E$ be a C^∞ subbundle. Each linear projection $P : E \to E'$, $P^2 = P$, gives a connection ∇' in E', viz. $\nabla' = P \circ \nabla$.

1.4.2.B.1 Assume now that the bundle $E \to M$ is given a *Euclidean structure*, i.e. a field of Euclidean metrics in the fibres $E_p \subset E$, for $p \in M$. A connection ∇ in E is *Euclidean* if it satisfies the Leibnitz rule

$$\tau\langle\varphi, \xi\rangle = \langle\nabla_\tau\varphi, \xi\rangle + \langle\varphi, \nabla_\tau\xi\rangle.$$

Let $E' \subset E$ be a subbundle with the induced Euclidean structure and $P : E \to E'$ the orthogonal projection. If ∇ is Euclidean, then $P \circ \nabla$ is an Euclidean connection in E'.

Let us apply this construction to the canonical rank p bundle over the Grassmann manifold $Gr = Gr_p\mathbf{R}^q$. This bundle, say $H \to Gr$, is a subbundle of the trivial bundle $B = Gr \times \mathbf{R}^q \to Gr$ and has a *canonical connection* $\nabla^H = P \circ \nabla^B$, where ∇^B is the canonical flat connection in B and $P : B \to H$ is the orthogonal projection.

Let $x_1, \cdots, x_{p(q-p)}$ be local coordinates on $Gr = Gr_p(\mathbf{R}^q)$. We regard the points $P \in Gr_p\mathbf{R}^q$ as orthogonal projections $P : \mathbf{R}^q \to \mathbf{R}^p$, of rank p, and view the vectors $Z \in H_P \subset H$ (here H_p is the fibre over $P \in Gr$) as vectors $\tau \in \mathbf{R}^q$ for which $P(\tau) = \tau$. If $Z = Z(x_1, \cdots, x_{p(q-p)})$ is a section of H then

$$\nabla_i^H Z = P(\partial_i Z), \;\; \partial_i = \partial/\partial x_i.$$

1.4.2.C. The above construction (of ∇^H) is, as we shall see, naturally related to the problem of inducing Euclidean connections. Let M' be a C^∞ Riemannian manifold. The tangent bundle $T(M') \to M$ has a distinguished Euclidean connection, the *Levi-Civita connection* $\nabla^{M'}$. if M' is isometrically C^∞ immersed into $\mathbf{R}^q$, then the tangential Gauss map $\gamma : M' \to Gr_p\mathbf{R}^q$, $p = \dim M'$, induces $\nabla^{M'}$ from the canonical connection in the bundle $H \to Gr_p\mathbf{R}^q$.

1.4.2.C.1. Universal bundles (Cf. [23]) We recall that for any topological group G and for any natural number k, one may build a k-classifying space B^k_G, defined up to a homotopy equivalence and endowed with a k-*universal principal* G-bundle $E^k_G \to B^k_G$ such that for any principal G-bundle $X \to M$ over a CW-complex M with $\dim M < k$ one has $X = \varphi^* E^k_G$ where $\varphi : M \to B_G$ is the *classifying map* (which is determined up to a homotopy). In particular for $G = O(p)$, one can take for B^k_G the Grassmann manifold $Gr_p\mathbf{R}^q$, with $q \geq k+p$, and for E^k_G the corresponding *Stiefel manifold* $St(p,q)$ of p-orthonormal frames in $\mathbf{R}^q$ (which is a natural fibration over $Gr_p(\mathbf{R}^q)$). For $G = U(p)$ one has a similar situation entering complex Grassmann and Stiefel manifolds.

Now, the objects of interest are differentiable principal G-bundles $\pi : X \to M$ endowed with connection Γ. If $F : X' \to X$ is a principal bundle map and Δ is a connection on X then $\Delta' = F^*\Delta$ is a connection on X'. Therefore, the objects $(X \to M,\ \Gamma)$ form a category whose morphisms are *connection preserving bundle mappings.* We recall

Theorem 5 ([*19*]) *Given a compact Lie group G and a positive integer k there exists a differentiable principal G-bundle $E^k_G \to B^k_G$ with connection $\nabla^{(k)}_G$ so that any connection on a principal G-bundle $X \to M,\ \dim M < k$, can be induced from the connection $\nabla^{(k)}_G$ by a bundle map $X \to E^k_G$.*

In view of this result the connection $\nabla^{(k)}_G$ is called a k-universal G-connection. In particular for $G = O(p)$, the result in Theorem 5 holds for

$$B^k_G = Gr_p(\mathbf{R}^q),\ E^k_G = St(p,q),$$

and the natural connection on $St(p,q) \to Gr_p(\mathbf{R}^q)$.

1.4.2.C.3. Recall that the Stiefel bundle $St(p,q) \to Gr_p(\mathbf{R}^q)$ comes equipped with a canonical connection (which is $O(q)$-invariant). This is precisely the connection ∇^H in $H \to Gr_p(\mathbf{R}^q)$, built before (as $H \to Gr_p(\mathbf{R}^q)$ is isomorphic to $H' = St(p,q) \times_{O(p)} \mathbf{R}^q$ (the associated bundle with fibre $\mathbf{R}^q$).

1.4.3 Gauss map

(Gauss map, Gauss induced connections, and Nash's isometric immersion theorem)
Our task is to prove the assertion in 1.4.2.C. Recall that for a p-dimensional smooth submanifold Y of the Euclidean space $\mathbf{R}^q$ ($q = p + k$, for some k) one defines the Gauss map

$$\gamma : Y \to Gr_p(\mathbf{R}^q)$$

by $\gamma(y) = T_y(Y) - y$, i.e. $\gamma(y)$ is the p-dimensional subspace of $\mathbf{R}^q$ through the origin which equals $T_y(Y)$ translated by y. For a Riemannian manifold (M, g) consider a submanifold (M', g') where g' is induced by g. Let ∇, ∇' be the Levi-Civita connections of (M, g) and (M', g'), respectively. Consider the restriction $\nabla_{|M'}$ of ∇ to $T(M)_{|M'}$ (note that $T(M')$ is a subbundle of the restricted bundle $T(M)_{|M'}$). One can define on $T(M')$ a connection ∇^*, the *Gauss-induced connection*, by setting

$$\nabla^*_A B = P \circ \nabla_A B \tag{1.14}$$

for all sections $A, B : M \to T(M') \subset T(M)_{|M'}$, where $P : T(M)_{|M'} \to T(M')$ is the orthogonal projection. The Levi-Civita connection ∇' of M' and ∇^* coincide.

Assume that M is isometrically immersed into $\mathbf{R}^q$. By Nash's isometric imbedding theorem there exists an integer q so that g' be induced from the natural flat metric on $\mathbf{R}^q$. The restriction of $T\mathbf{R}^q$ to any submanifold is trivial hence we can view $T\mathbf{R}^q_{|M'}$ as $M' \times \mathbf{R}^q$ so that the orthogonal projection $P : T\mathbf{R}^q_{|M'} \to T(M')$ can be written $P : M' \times \mathbf{R}^q \to T(M')$. Thus we see that $T(M')$ embeds into $M' \times \mathbf{R}^q = T\mathbf{R}^q_{|M'}$ and the connection $P \circ \nabla$ on $T(M')$, where $\nabla = \nabla_{|M'}$ is the Levi-Civita connection on $T\mathbf{R}^q_{|M'}$, is called *Gauss-induced* from ∇ by the natural map $T(M') \hookrightarrow M' \times \mathbf{R}^q$. Equivalently, it may be said that the Levi-Civita connection on $T(M')$ is induced (by the Gauss map $\gamma : M' \to Gr_p\mathbf{R}^q$) from the connection ∇^H on the canonical bundle $H \to Gr_p\mathbf{R}^q$. Observe that the two equivalent formulations above come from two equivalent constructions of the bundle $T(M')$ ($T(M')$ may be viewed both as the image of $T(M') \hookrightarrow T(\mathbf{R}^q)_{|M'}$ and as the pullback bundle $\gamma^*(H)$).

1.4.4 Setting the general problem

Let us go back to the inducing problem for connections, as addressed in section 1.4.1. Let $(X, \Gamma) \to M$ and $(Y, \nabla) \to N$ be two C^∞ principal G-bundles; we look for C^∞ maps $f : M \to N$ so that the induced bundle $f^*(Y) \to M$, with the induced connection $f^*(\nabla)$, be isomorphic to (X, Γ). Equivalently, we want to find maps $F : X \to Y$ so that a) $F : X \to Y$ is a principal bundle morphism covering f, and b) F is a connection preserving map ($F^*(\nabla) = \Gamma$). Every $F : X \to Y$ is uniquely determined by the underlying map f modulo the action of the structure group G ($G = Aut(X, \Gamma)$).

1.4.4.A. The operator $\mathcal{D}_\Delta$. Following the general approach in [7] we think of the "operation" $\Delta \to \Gamma = F^*(\Delta)$ expressing the inducing connection relation as a map (operator) relating to F the induced connection $F^*(\Delta)$. We denote this by

$$F \to \Gamma = \mathcal{D}_\Delta(F) = F^*(\Delta),$$

and interpret it as an operator from $\{\mathcal{F}\}$ to $\{\mathcal{G}\}$, where $\{\mathcal{F}\}$ is the space of C^∞-bundle morphisms $F : X \to Y$ and $\{\mathcal{G}\}$ is the space of C^∞ connections on X. Let us explain now why $\mathcal{D}_\Delta$ is an actual (first order) differential operator. We interpret morphisms $F : X \to Y$ as sections of the bundle $\mathcal{Z} \to M$ associated to the principal bundle $X \to M$ with the standard fibre Y, for the action of G on Y. This $\mathcal{Z}$ naturally fibres over $M \times N$ with fibre $(X_p \times Y_w)/G$ canonically isomorphic to the space of G-equivariant maps $X_p \to Y_w,\ w = f(p)$. On the other hand, every C^{r+1} smooth bundle morphism $F : X \to Y$ is by definition given by a C^{r+1} map $f : M \to N$ and a family of G-equivariant maps $F_p : X_p \to Y_{f(p)}$ which are C^{r+1} smooth in p. This F becomes a C^{r+1} section $M \to \mathcal{Z}$ covering the graph $M \to N$ of f. The range of the operator $F \mapsto F^*(\Delta)$ consists of the space of C^r connections in X. These are C^r sections of the fibration $Q \to M$ whose fibre $Q_p \subset H$, for $p \in M$, can be described as follows. Denote by $X_p^{(1)}$ the space of 1-jets (or differentials) of germs of sections $M \to X$ at p. Namely, $X_p^{(1)}$ consists of linear maps $\ell : T_p(M) \to T - x(X)$ which project on the identity $Id : T_p(M) \to T_p(M)$ by the differential of the projection map $(D\pi \circ \ell = Id_{|T_p(M)})$. The group G acts naturally on $X_p^{(1)}$ and the fibre $Q_p \subset Q$ is $X_p^{(1)}/G$.

1.4.4.B. Assume the dimensions of the base manifolds M and N (of the fibrations X and Y respectively) are fixed ($\dim M = n,\ \dim N = m$) and assume G to be a k-dimensional Lie group. By the preceeding discussion $\dim Q_p = nk$ and $\dim \mathcal{Z}_p = m + k$. Therefore $\mathcal{D}_\Delta(F) = \Gamma$ consists of $\alpha = nk$ equations in $\beta = m + k$ unknown functions.

Remark 5 If the inducing map F (or f) is required to be C^∞ or C^ω (real analytic) then necessarily we must have $m \geq (n-1)k$. This follows from the fact that, for $m < (n-1)k$, the PDE system corresponding to $F^*(\Delta) = \Gamma$ is *overdetermined.* (The same phenomenon was already discussed for the metric inducing operator). It is important to mention that, if $m < (n-1)k$ and Δ is any fixed connection on Y, then the connections Γ locally inducible by C^∞

or C^ω maps $f.X \to Y$ form a *meager* (or *first category*[7]) subset (depending on Δ) of the space of all (C^∞ or C^ω) connections on M.

1.4.4.C.1. In view of the above remark the estimate $m \geq (n-1)k$ should be, at least in the C^ω category the best possible. This is ensured by a local result to be quoted shortly. Finally, we point out that the case $m < n$ is interesting only in very simple cases as $k = 0$ and $k = 1$, as shown by the subsequent lemma.

The following local result is due to E. Cartan[8]:

Theorem 6 (E. Cartan) *Let Δ be a generic C^ω connection on Y. If $m \geq (n-1)k$ and $m \geq n$ then there exists a connection Γ_0 on M, a point $p_0 \in M$ and a $\epsilon_0 > 0$ so that for any real analytic connection Γ satisfying*[9]

$$||\Gamma - \Gamma_0||_{C^1}(p_0) \leq \epsilon_0, \tag{1.15}$$

there is a neighborhood $\mathcal{U}$ in M so that $\Gamma_{|\mathcal{U}}$ can be induced from Δ.

This result is the analogue, for the case where the structure under scrutiny is a connection in a principal G-bundle, of the J-B-C theorem on local isometric immersions. The proof parallels closely that of the J-B-C theorem. Once again, the key idea is to apply the Cauchy-Kowalwsky theorem to solve locally (in a neighborhood of $p_0 \in M$) the PDE system corresponding to the inducing relation $F^*(\Delta) = \Gamma$. Inbdeed, the C^ω assumptions in the statement, combined with the inequality $m \geq \max\{(n-1)k, n\}$ permit to reduce the PDE system in question to the form needed for the applicability of the Cauchy-Kowalewsky theorem[10]. Now, what was really crucial for the proof of the local isometric immersion theorem was the role played by the property of *freedom* which was satisfied there by the local isometric map f. In fact, freedom was the "right" nondegeneracy condition needed in order to bring the PDE system (1.2) to the form required for the applicability of the Cauchy-Kowalewsky theorem.

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[7] A *first category* subset of a topological space is a union of a countable family of nowhere dense subsets.

[8] Pointed out to the Author by M. Gromov, who also suggested the (not entirely rigorous) statement of Theorem 6.

[9] The inequality (1.15) should be read "Γ is C^1 close to Γ_0".

[10] See [6], Theor. 2.4.9, p. 27, and also [7], for an *ad hoc* formal version of the Cauchy-Kowalewsky theorem suitable for our purposes.

Remark 6
As we shall see later the limitation $m \geq (n-1)k$ translates to the condition for the existence of certain linear subspaces (called Ω-regular as they are associated to a linear system naturally related to the curvature Ω of a connection) in an arbitrary linear space T endowed with some bilinear vector-valued form Ω. It will be demonstrated that this regularity condition plays - in the problem of inducing connections - the same role played by the freedom property in the isometric immersion problem.

1.4.5 Inducing connections

(A brief review of the known results) The problem of inducing connections was first studied by Narasimhan and Ramanan [19]. Besides their classification result which we quoted before, they also showed a more general theorem (for the case where the structure group G is any Lie group, not necessarily compact). Furthermore, they give in [19] a precise description of the universal connection Δ for the unitary and orthogonal groups. Namely, if $G = U(p)$ they take the Grassmann manifold $N = Gr_p(\mathbf{C}^q)$ and use the standard connection on the canonical bundle $Y \to Gr_p(\mathbf{C}^q)$ (here Y is the Stiefel manifold of orthonormal p-frames in $\mathbf{C}^q$). The dimension q for which they prove the existence of F is $q = (n+1)(2n+1)p^3$, where $n = \dim M$. Similarly, for $G = O(p)$ their method provides a connection inducing map into the real Grassmann manifold $Gr_p(\mathbf{R}^q)$, with the same q as before.

The result of Narasimhan and Ramanan for $G = O(p)$ was improved by Gromov (cf. 2.2.6 in [7]) who showed the existence of a connection inducing map $f : M \to Gr_p\mathbf{R}^q$ for $q = \max\{p(n+2),\ p(n+1)+n\}$. Furthermore, if the manifold M is parallelizable, then $q = p(n+2)$ suffices to prove the existence of a map $M \to Gr_p\mathbf{R}^q$ inducing a given connection on X.

Remark 7 In both papers [19] and [7] an explicit solution of the local PDE system expressing the condition $F^*(\Delta) = \Gamma$ is found. This is possible because the PDE system in question is such that a solution can be obtained directly by a purely algebraic procedure provided the dimensions of the spaces involved in the problem are not "very small". In the next section we give a short illustration of the local construction for the case where the problem of inducing Euclidean connections is considered. We follow [7], p. 95-96.

1.4.5.A.1. Inducing Euclidean connections Consider two vector bundles E, B over M equipped with arbitrary Euclidean connections ∇ in B and ∇'

in E. We look for connection preserving morphisms $F : E \to B$. That is, F must be an isometric morphism of E onto a subbundle $E' \subset B$ so that $\nabla' = P \circ \nabla$ for the orthogonal projection $P : B \to E' = F(E) \approx E$.

Fix a local chart $(U, x_1, \cdots, x_n)$ on M and a local orthonormal frame $e_k : U \to E,\ k = 1, \cdots, \ell$, where $\ell = rankE$. Consider the Christoffel symbols

$$\Gamma_i^{kj} = \langle e_k, \nabla_i e_j \rangle,\ \ 1 \le i \le n,\ 1 \le k, j \le \ell.$$

Then $\langle e_k, e_j \rangle = \delta_{kj}$ yields $\Gamma_i^{k\ell} = -\Gamma_i^{\ell k}$. The isometric morphism $F : E \to B$ is looked at (locally) as a system of orthonormal sections $F_k : U \to B,\ k = 1, \cdots, \ell$,

$$\langle F_k, F_j \rangle = \delta_{kj}, \tag{1.16}$$

and the condition $\nabla' = P \circ \nabla$ is expressed by the following system of PDEs in the unknowns F_j

$$\langle F_k, \nabla_i F_j \rangle = \Gamma_i^{kj},\ \ 1 \le i \le n,\ \ 1 \le k, j \le \ell. \tag{1.17}$$

If the bundle B has rank q, then each section F_k is given by q real functions and the conditions (1.16)-(1.17) are $\left[\frac{\ell(\ell+1)}{2}\right] + \left[\frac{n\ell(\ell-1)}{2}\right]$ equations in ℓq unknown functions on U. Thus we see that in order to get a local solution for this structure inducing problem we must solve the above equations with (arbitrarily given) $\frac{n\ell(\ell-1)}{2}$ functions $\Gamma_i^{k\ell} = -\Gamma_i^{\ell k}$ on U. The system of these equations is undetermined for $q > \left[\frac{\ell+1}{2} + \frac{n(\ell-1)}{2}\right]$ and so one can predict solvability for $q \approx \frac{\ell n}{2}$. This restriction originates from the requirement of a nondegeneracy condition to be satisfied by the homomorphisms $F : E \to B$ ([7], p. 95) which amounts to the linear independence of the sections F_k and $\nabla_i F_k,\ i = 1, \cdots, n,\ \ k = 1, \cdots, \ell$, in each fibre $B_p \subset B,\ p \in M$. Note that this definition of regularity of the homomorphisms $E \to B$ doesn't depend upon the choice of frame $\{e_k\}$ and coordinates $\{x_i\}$. Also, the connection ∇' plays no role in the definition. Finally, it is clear that regular homomorphisms may exist only for $q \ge \ell(n+1)$. The global solution (i.e. global existence of connection inducing maps) is obtained by a direct application of methods in the topology of sheaves (cf. [7], [22]) and is expressed as a h-principle

Theorem 7 ([7])
The regular connection C^∞ homomorphisms $F : E \to B$ satisfy the h-principle for $q \ge \ell(n+2)$, where $n = \dim M,\ \ell = rankE$ and $q = rankB$.

Remark 8 The condition $q \geq \ell(n+1)$ (found in [19]) for the existence of the connection inducing maps is the same as Gromov's condition. As far as global results are concerned the estimate $q \geq \ell(n+2)$ in [7] is better. This depends only on the different methods (adopted in [19] and [7] respectively) used to build the global solution out of the local one (Gromov employs sheaves while Narasimhan and Ramanan make use of a partition of unity argument borrowed from Nash ([18])).

We end this section by observing that [18] shouold be added to the list of early contributors to the inducing connection problem. Indeed, let us combine Nash's isometric immersion Theorem 1 with the classical results recalled in 1.4.2.C. To this end, start with an integer[11] $q = q(n)$, referred to as *Nash's dimension*, a terminology whose meaning is that each n-dimensional Riemannian manifold admits an isometric immersion into $\mathbf{R}^{q(n)}$. The following is easy

Proposition 1 *Let M be a Riemannian manifold with $\dim M = n$. Any connection in a principal $O(n)$-bundle over M is inducible by a map $M \to Gr_p(\mathbf{R}^q)$.*

Proof. We begin with the (Euclidean) bundle $\pi : E \to M$, $rankE = p$, carrying a Euclidean connection, and fix a Riemannian metric g on M. Notice that E may be viewed as a subbundle $E' \subset T(E)_{|M}$. Let $\Sigma \subset T(E)$ be the horizontal distribution defined by the connection ∇ originally given on $T(E)$. We want to construct a Riemannian metric h on E so that the connection on E ($\equiv E' \subset T(E)_{|M}$), Gauss-induced from the Levi-Civita connection on $T(E)_{|M}$, be the original connection on E. We proceed as follows. The bundle $T(E)$ splits into a direct sum of orthogonal Euclidean subbundles

$$T(E) = \Sigma \oplus T^{vert}(E),$$

so that we may consider the metric

$$\tilde{g} = \pi^*(g)_{|\Sigma}$$

on Σ. Next, on $T^{vert}(E)$ we may take the metric which each vertical space inherits from the fibre E_p, $p \in M$. The direct sum of these two metrics

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[11] Recall that $q(n) = n(3n+11)/2$ for M compact, and $q(n) = n(3n+11)(n+1)/2$ in the noncompact case.

gives a metric h on the total space E so that 1) for any $x \in E$, $\Sigma_x^\perp$ is a vertical subspace, and 2) the projection $\pi : E \to M$ is an isometry on Σ and moreover all maps $E_p \to T(E)$ are isometric. If we endow E with the above metric h, the Levi-Civita connection on $T(E)_{|M}$ Gauss-induces the primitive connection on $E' \subset T(E)_{|M}$. This is proved by first looking at the case $\dim M = 1$ followed by an argument reducing the higher dimensional case to the 1-dimensional case. Then the connection on E is necessarily flat. Next, one observes that on each bundle over M two connections are equivalent if and only if they coincide on each curve immersed in M. Finally, one invokes Nash's Theorem 1 which, when E is compact, ensures the existence of an isometric immersion $(E, h) \to \mathbf{R}^q$ for $q = (p+n)/2$, as $\dim E = p + n$. Now one applies the proposition in section 1.4.2.C to the Riemannian manifold (E, h) and this concludes the proof.

1.4.6 The infinitesimal invertibility of $\mathcal{D}_\Delta$.

Our immediate task here is to work out an effective criterion for the infinitesimal invertibility of the differential operator $\mathcal{D}_\Delta$ which assignes to each smooth bundle morphism $F : X \to Y$ the connection $F^*(\Delta)$ on X for a fixed Δ on Y. According to the general theory in [7], we wish to produce an open set A in the space of 1-jets of germs of sections $M \to \mathcal{Z}$ so that the connection inducing operator $\mathcal{D}_\Delta : F \mapsto F^*(\Delta)$ be *infinitesimally invertible* on A. The operator $\mathcal{D} = \mathcal{D}_\Delta$ is a nonlinear differential operator on the space $\{\mathcal{F}\}$ of sections in $\mathcal{Z}$ with values in the space $\{\mathcal{G}\}$ of sections in Q. We say $\mathcal{D}$ is *infinitesimally invertible* on $A \subset \mathcal{Z}^1$, where $\mathcal{Z}^1$ is the space of 1-jets of sections $M \to \mathcal{Z}$, if for every section $F : M \to \mathcal{Z}$ whose 1-jet sends M to A, the linearization of $\mathcal{D}$ at F, called L_F, admits a right inverse I_F which is a linear differential operator. Here we are interested in the case where I_F is a zero order operator which is a (nonlinear) differential operator of order one in F. The resulting operator $I_F(\cdot)$ (in two variables, F and $\cdot$) is called an infinitesimal inversion of order zero and defect (the order of I in F) one (cf. [3]).

1.4.6.A. To describe the pertinent set A for $\mathcal{D}(F) = F^*(\Delta)$ we invoke the bundle $\tilde{Y} \to N$ associated to Y with standard fibre the Lie algebra $\mathbf{g}$ of G, with the adjoint action of G. The curvature form of the connection Δ on Y is a $\mathbf{g}$-valued 2-form on N and thus a section in $\Lambda^2 T^*(Y) \otimes \mathbf{g}$. Therefore, it is a map $\Omega : T(N) \otimes T(N) \to \tilde{Y}$.

1.4.6.A.1. Ω-regular subspaces Consider a linear subspace $T' \subset T_w(N)$, for $w \in N$. We say T' is Ω-*regular* in $T_w(N)$ if one of the equivalent conditions is satisfied

i) for some (and hence for any) basis $\tau_1, \cdots, \tau_n$ in T' the linear system

$$\Omega_w(\tau_i, \partial) = e_i, \quad i = 1, \cdots, n$$

is solvable in $\partial \in T_w(N)$ for every n-tuple of vectors e_i in the Lie algebra $\mathbf{g}$ of G.

ii) The homogeneous system

$$\Omega_w(\tau_i, \partial) = 0, \quad i = 1, \cdots, n,$$

is nonsingular. Namely, the dimension of the space of solutions is $\dim N - n \dim \mathbf{g}$.

iii) The linear map $T_w(N) \to Hom(T', \mathbf{g})$ given by $\tau \mapsto h_\tau(\tau') = \Omega_w(\tau, \tau')$ is surjective.

1.4.6.A.2. For a clearer understanding of the notion of Ω-regularity it is helpful to analize it for subspaces $T' \subset T$, where T is an arbitrary linear space endowed with a vector-valued 2-form Ω. This is done in section 1.4.8.B below.

Example 2 If Ω is a scalar form, then $T' \subset T$ is Ω-regular if and only if $T' \cap Ker\Omega = 0$, where $Ker\Omega = \{t \in T : \Omega(t, t') = 0, \forall t' \in T\}$. In particular, if Ω is a symplectic form then every subspace in T is Ω-regular ([7], section 3.4). Observe that the curvature Ω of the canonical $O(2)$-bundle over the Grassmann manifold $Gr_2\mathbf{R}^q$ is symplectic (Ω can be regarded as a scalar form as the Lie algebra of $O(2)$ is $\approx \mathbf{R}$).

1.4.6.C. Ω-regular maps. Consider a linear map $\varphi : T_p(M) \to T_w(N)$ and a 1-jet $\phi \in \mathcal{Z}^1$ over φ. (If φ is the 1-jet $j^1_F(p)$ for a morphism $F : X \to Y$ then φ is the differential Df of the underlying map $f : M \to N$ at $p \in M$). We say φ is Ω-regular if it is injective and if the image $\varphi(T_p(M)) \subset T_w(N)$ is Ω-regular for all $p \in M$. We say ϕ is Ω-regular if the underlying map φ is Ω-regular. Then we define the subset $A \subset \mathcal{Z}^1$ as the set of all Ω-regular 1-jets ϕ. According to this terminology, we say that $F : X \to Y$ (as well as the underlying map $f : M \to N$) is Ω-regular if the 1-jet $j^1_F : M \to \mathcal{Z}^1$ sends M to A. This is equivalent to the Ω-regularity of the differential $Df : T(M) \to T(N)$ at every point $p \in M$.

1.4.7 An implicit function theorem for the connection inducing operator

We are now in the position to state the following variant of Nash's implicit function theorem.

Proposition 2 ([*3*])
The connection inducing operator $\mathcal{D} : F \mapsto F^*(\Delta)$ *is infinitesimally invertible on* Ω*-regular morphisms.*

Proof. We first linearize the operator $\mathcal{D}$ at some morphism F. To do this we take a smooth 1-parameter family of morphisms $F_t : X \to Y$ for $t \in [0, 1]$ such that $F_0 = F$ and study the corresponding family of induced connections $\Gamma_t = \mathcal{D}(F_t)$. The linearization of $\mathcal{D}_\Delta$ acts on vector fields $\tilde{\partial}$ which are fields in Y along the map F. Namely, they are sections of the induced bundle $F^*(T(Y)) \to X$ where $F : X \to Y$ and where the fibre is (the linear space) $F^*_x(T(Y)) = T_{F(x)}(Y)$. By definition L_Δ is

$$L_\Delta(\tilde{\partial}) = \frac{d}{dt}\left(\mathcal{D}_\Delta(F_t)\right) = \frac{d}{dt}(\Gamma_t)_{t=0}.$$

Observe that the derivative Γ'_t is a 1-form on M with values in the vector bundle $\tilde{X}$ which is induced from $\tilde{Y}$ by the map $f_0 = f : M \to N$ underlying $F_0 = F : X \to Y$. Let us express this form in terms of the curvature Ω. Take $M' = M \times [0, 1]$ and $X' = X \times [0, 1]$. Consider on X' the connection Γ^* induced by the morphism $X' \to Y$ defined by $(x, t) \mapsto F_t(x)$. To compute Γ'_t we consider on M' the (tautological) vector field $\partial = \partial/\partial t$ and denote by $\tilde{\partial}$ the vector field on X' corresponding to ∂. It is easy to see that the ordinary derivative Γ'_t equals the Lie derivative (with respect to ∂) of the connection Γ^* (which we interpret as a homomorphism $T(X\times t) \to T(X\times t)$). Alternatively, we may say that , denoting by $\partial\Gamma^*$ the Lie derivative of Γ^* with respect to ∂, then $\Gamma'_t = \partial\Gamma^*_{|X\times t}$. Thus to compute $L_\Delta(\tilde{\partial})$ is the same as to compute the Lie derivative of a $(1, 1)$-tensor. We have

$$L(\tilde{\partial})(\tau) = \Omega^*(\tau, \tilde{\partial}^{hor}) + d\tilde{\partial}^{vert}(\tau) \tag{1.18}$$

where τ is a tangent vector on M $(= M \times 0)$, Ω^* is the curvature of Γ^*, $\tilde{\partial} = \tilde{\partial}^{hor} + \tilde{\partial}^{vert}$ is the decomposition of $\tilde{\partial}$ as the sum of its horizontal and vertical components with respect to Γ^*, and d is the *horizontal differential* (cf. [5]) associated to Γ^*.

Now let L_F be the linearization of $\mathcal{D}$ at F and assume that f is Ω-regular. Let us solve the linearized equation

$$L_F(\tilde{\partial}) = \ell \tag{1.19}$$

where ℓ is a given section $M \to \tilde{X}$ and $\tilde{\partial}$ is the unknown infintesimal deformation (a vector field) of F. We use once again Nash's trick, namely add another condition

$$\tilde{\partial}^{vert} = 0 \tag{1.20}$$

Next we introduce another linear algebraic system of equations in the unknown ∂, the projection of $\tilde{\partial}$ to $T(M \times [0,1])$

$$\Omega_0^*(\tau_i, \partial) = \ell, \quad i = 1, \cdots, n, \tag{1.21}$$

where $\tau \in T(M_0)$ and $\Omega_0^* = \Omega^*_{|M\times 0}$. ¿From (1.18) it follows that if $\tilde{\partial}$ is a solution to (1.20) and (1.21) then it is also a solution to (1.19). On the other hand, since F (and also f) is assumed to be Ω-regular we can express, at each $p \in M$, (1.21) as the following nonlinear system of algebraic equations

$$\Omega_0^*(\tau_i, \partial) = \ell(\tau_i), \quad i = 1, \cdots, n, \tag{1.22}$$

for a fixed basis $\tau_1, \cdots, \tau_n$ in $T_p(M_0)$. Hence, the solutions to (1.20)-(1.21) form an affine subspace of dimension $d = \dim N - n \dim \mathbf{g}$. Such a bundle always admits a section ([12]) and this solves completely the local problem. Moreover, one may easily choose a specific section, say ∂_0, with an appropriate partition of unity or with an auxiliary Riemannian metric in the ambient vector bundle. Finally, we define the infinitesimal inversion $I = I_F$ of $\mathcal{D}$ by $I_F(\ell) = \partial_0$ which, according to our construction, satisfies the desired (infinitesimal invertibility) relation $L_F(I_F(\ell)) = \ell$.

1.4.8 A global existence theorem for connection inducing maps

The linearization formula and the consequent infinitesimal invertibility of the differential operator $\mathcal{D}_\Delta$ established in the previous Proposition 2 allow us to apply the generalized version of Nash's implicit function Theorem 2 to our $\mathcal{D}_\Delta$ so that we are led to the following

Corollary 1 ([*3*]) *If the morphism $F : X \to Y$ is Ω-regular then the operator $\mathcal{D}_\Delta$ is an open operator from $\{\mathcal{F}\}$ to $\{\mathcal{G}\}$ at $F \in \{\mathcal{F}\}$ and therefore all the connections Γ in a small neighborhood of the induced structure $\mathcal{D}_\Delta(F) = F^*(\Delta)$ are inducible from Δ.*

(Here the function spaces $\{\mathcal{F}\}$ and $\{\mathcal{G}\}$ are endowed with the C^∞ topology). In view of Corollary 1 our next task is to show the existence of a morphism $F : X \to Y$ satisfying the nondegeneracy condition required by Proposition 2 (the Ω-regularity there). This is needed in order to make sure that the general results we got (by applying the general Gromov method) are nonempty. That is, we need examples of connections Δ on N so that the tangent bundle $T(N)$ contains "sufficiently many" Ω-regular subspaces. This is achieved on the basis of an analysis of the notion of Ω-regularity in a generic situation, namely by considering general bilinear antisymmetric forms $\Omega : T \otimes T \to \mathbf{g}$ where T and $\mathbf{g}$ are arbitrary linear spaces. The following lemma summarizes the results of this analysis (for full details see section 2 in [3]).

Lemma 1 ([*3*], p. 76) *Let T, T' and $\mathbf{g}$ be linear spaces and $\Omega : T \otimes T \to \mathbf{g}$ an antisymmetric bilinear form. Let $T' \subset T$ be a subspace. Set $m = \dim T$, $n = \dim T'$ and $k = \dim \mathbf{g}$. Then, in the following three cases, and only in those, there exists a $\mathbf{g}$-valued 2-form Ω on T for which $T' \subset T$ is Ω-regular*

i) $n = 1$, $m > k$,

ii) $k = 1$ and m is even,

iii) $m \geq nk$ and $m > n$.

1.4.8.C. Let us go back to the canonical $O(p)$-bundle Y over the Grassmann manifold $Gr_p(\mathbf{R}^q)$ with the standard ($O(q)$-invariant) connection Δ. At every point $w \in Gr_p(\mathbf{R}^q)$ the tangent space $T_w(Gr_p(\mathbf{R}^q))$ is identified with the space $T = Hom(\mathbf{R}^p, \mathbf{R}^{q-p})$, the Lie algebra $\mathbf{g}$ of $O(p)$ is represented by antisymmetric $p \times p$ matrices, and the curvature form Ω of the connection Δ is given by the formula

$$\Omega(A_1, A_2) = A_1' A_2 - A_2' A_1$$

for $A_1, A_2 \in \Gamma$, where an accent denotes transposes. Note that $\dim T = p(q-p)$ and $\dim \mathbf{g} = p(p-1)/2$.

1.4.8.D. In this section we briefly indicate how the above version of the implicit function theorem (Corollary 1) combined with Lemma 1 can be used to establish a global existence theorem for connection inducing maps, where "global" refers to the space $\{\mathcal{F}\}$ of maps $F : X \to Y$. (The implicit function theorem deals with a small neighborhood of a given $F \in \{\mathcal{F}\}$). Our global result (indicated below) does in fact solve completely the inducing connection problem. Its proof makes essential use of Gromov's h-principle which reduces, under favourable conditions, the solvability of the differential relation $\mathcal{D}_\Delta = \Gamma$ to the existence of a section of a certain subbundle of the i-th jet bundle of maps $X \to Y$ denoted by $\mathcal{R}^i = \mathcal{R}^i(\mathcal{D}, \Gamma) \subset J^i(X, Y)$, where i must be sufficiently large. Note that Ω-regularity is the "favourable condition" for the applicability of the h-principle. The specific existence Theorem 8 below was established in [3] under a simplifying topological assumption on M and on the bundle X. We also impose a local geometric condition on the curvature form Ω of $T_w \subset T(N)$ at a single point $w \in N$, for which the notion of Ω-isotropy of a subspace $T' \subset T_w$ is needed ($\Omega(t', t'') = 0,\ t', t'' \in T'$).

Theorem 8 ([*3*]) *Let* Γ *be an arbitrary connection on a trivial* $O(p)$*-bundle over a stably parallelizable manifold* M*. If* $q \geq p(n+3)/2$ *then there exists a connection inducing map* $M \to Gr_p(\mathbf{R}^q)$*.*

For a proof see sect. 2 in [3], p. 72-77.

Remark 9
i) One may, in principle, apply Theorem 8 to a nontrivial bundle X over a non-stably parallelizable manifold. Namely, take the trivial rank $2p$ bundle $X' \to M' \supset M$, where M' is a $(2n-1)$-dimensional parallelizable manifold which is the total space of the normal bundle of M and where X' contains X as a subbundle. Then one easily obtains the existence of the inducing connection map to $Gr_p(\mathbf{R}^q)$ for $q = 2p(n+1)$. A comparison between this bound on q and the estimates in [19] and [7] shows that it is too crude.

ii) The construction of a connection inducing map F between principal bundles X and Y amounts to solving a system consisting of $\alpha = (\dim M)(\dim G)$ partial differential equations imposed on $\beta = \dim N + \dim G$ unknown functions. Therefore, for a fixed Δ and for $\alpha > \beta$, a *generic* connection Γ cannot be induced (even locally) from Δ. This means that inducible connections form a meager subset (depending on Δ) in the space of C^∞ connections on M. In particular if Y is the canonical $O(p)$-bundle over $Gr_p(\mathbf{R}^q)$ then

$$\alpha = \frac{1}{2}np(p-1),\ \beta = p(q-p) + \frac{1}{2}(p-1).$$

Hence, a generic connection on M cannot be induced from this Y unless $q \geq (p+1)/2 + n(p-1)/2$. This bound on q agrees asymptotically (for $p, n \to \infty$) with the inequality $q \geq p(n+3)/2$ in Theorem 8.

iii) As mentioned before, for $q \geq p(n+1)$ the PDE system for connection inducing maps $f : M \to Gr_p(\mathbf{R}^q)$ can be reduced to an algebraic system. Yet, as shown by the discussion in section , such a result cannot be proved for $q \approx pn/2$. So Theorem 8 is optimal.

iv) An approach similar to that in [3] for studying connection inducing maps applies to the *problem of inducing subbundles of given codimension* (cf. [5]). Given two manifolds X and Y and smooth bundles $S \subset T(X)$ and $T \subset T(Y)$ (both of fixed codimension k) the problem is to study the existence of maps $f : X \to Y$ which induce S from T. By a construction parallel to that used for the inducing connection operator $\mathcal{D}_\Delta$, one can show that the operator $\mathcal{D}_T : f \mapsto S = f^*(T)$ is an open map provided that certain nondegeneracy conditions are satisfied. The Gromov h-principle machinery allows one to obtain, under certain topological assumptions on X, a global existence theorem for bundle inducing maps $X \to Y$ (cf. Theorem 5.A in [5], p. 102). In particular, the following effective criterion for the existence of integral submanifolds $L \subset Y$ of given dimension ℓ is known.

Theorem 9 ([5]) *A generic smooth C^∞ subbundle of codimension k of the tangent bundle $T(Y)$ admits an integral submanifold $L \subset Y$ of dimension ℓ, provided that $2\ell(k+1) \leq \dim Y$.*

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