Chapter 5

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Gauge-natural structure of spin bundles

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Abstract We shall here review the theory of gauge natural structures. The aspects relevant to differential geometry are emphasized though gauge natural structures are mainly of physical origin. As an example and application, the theory of spin frames on manifolds is recalled in detail. In this case, the gauge natural formulation defines a beautiful and general framework which allows to

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define spin structures on manifolds before specifing a (pseudo)-Riemannian metric. A comparison with a similar framework based on coverings of the general linear group is also given. We shall also stress how these two frameworks enforce each other even if they are shown to be inequivalent. Finally, we shall mention open problems which, although physically motivated, are purely geometric in nature.

5.1 Introduction

The interactions between theoretical physics and differential geometry are very well known and historically documented. Geometry has always provided the theoretical framework for "physical space" since Galileo-Newton physics was formulated using Euclidean geometry as a model. In spite of the fact that the formulation of Euclidean geometry was based on practical issues, the axiomatic framework in which it was developed placed it on a theoretical ground. On the other hand, when non-Euclidean geometries were discovered from a purely theoretical viewpoint they were later used as a framework for the newcoming physics of General Relativity. In the last century the feedback of theoretical physics on differential geometry has been unprecedented. In fact, theoretical physics has produced non-trivial geometrical problems (e.g. spin structures or the classification of group transformations) while advanced tools developed by geometry revealed crucial in physical problems (e.g., (co)-homological and algebraic techniques). Moreover, theoretical physicists have recently begun to develop their own tools (e.g. gauge theories, supermanifolds, etc.) and to apply the corresponding techniques to genuinely geometric problems (e.g. exotic differential structures on $\mathbf{R}^4$). We believe that this interaction between different disciplines is at the origin of the tremendous expansion that both theoretical physics and geometry underwent in recent decades. Also for these reasons we believe that the geometrization of physics, as well as all geometrical problems which are physically motivated, will turn out to be important and useful for both disciplines.

Of course, the exchange of information between geometry and physics is often hindered by different languages which may have been developed by separate communities. A vivid example is given by gauge theories. They have proved, in fact, to be important for geometry, even if in the beginning they have been developed mostly by physicists interested in purely physical aspects. As a result many issues which are instead relevant to geometry have been unclear for a long time. More recently, these issues have been clarified and a geometric language has been developed on a strong basis. The framework is that of *gauge natural field theories* (see [4], [20], [7], [8], [10], [6], [11]), which definitely clarifies the global structure of the relevant objects, thence providing a natural and coherent bridge between geometry and physics. This aim is achieved by selecting a categorial language as the most suited framework to discuss global properties and by assuming bundle theory as a natural setting for global variational calculus. Strangely enough, the category language is also motivated by the very basic physical inputs which are the core of gauge theories. In fact, most of the physical meaning of gauge theories is encoded in the group of gauge transformations and category theory turns out to be the most suited framework to select a preferred class of maps. Just as vector spaces should not be defined on their own but together with linear maps, the category language exhibits the deep relation between gauge natural theories and gauge transformations.

We shall here review the basic gauge natural notation. As an application the theory of spin frames and Frauendiener structures are discussed in detail. The gauge natural framework allows us to define deformations of spin structures more general than the standard metric-preserving deformations which are currently used (expecially in the physical literature) on (pseudo)-Riemannian manifolds. Roughly speaking gauge natural structures allow us to define a spin structure on differential manifold *before* fixing any metric structure. This is very important for physical application as it was discussed in detail in [7]. We believe that it is also of some interest for differential geometry. In fact, since spin frames exhibit all the characteristics of a gauge theory they are certainly important as an example of geometrization of gauge theories. Finally we shall compare our framework with a framework based on double coverings of the linear group (see [3]) and discuss analogies and differences between the two.

5.2 Motivation: the frame bundle

Let M be a paracompact, connected manifold of dimension m (conventionally *spacetime*) and let $(x^\mu_{(\alpha)})$ denote local coordinates on M; here $\mu = 1, \dots, m$ is a spacetime index, while (α) is a label which runs on an atlas. Transition functions on M are given by $x^\mu_{(\beta)} = x^\mu_{(\beta)}(x^\nu_{(\alpha)})$. The frame bundle $L(M)$ over M is the bundle of all basis $\{e_a\}$ of the tangent space T_xM at some point

$x \in M$; here $a = 1, \dots, m$ is a numeric index. A point in $L(M)$ will be denoted by (x, e_a). Any system of local coordinates $(x^\mu_{(\alpha)})$ defines a preferred (local) basis $\partial^{(\alpha)}_\mu)$ in the tangent bundle. Thus, for any other frame (x, e_a), we have

$$e_a = e^{(\alpha)\mu}{}_a \ \partial^{(\alpha)}_\mu \ , \qquad\qquad \|e^{(\alpha)\mu}{}_a\| \in \mathrm{GL}(m)$$

In that way (x^μ, e^μ_a) are *natural fibered coordinates on* $L(M)$ for each chart label (α). The associated trivialization is called *natural trivialization.* If we denote by J^μ_ν the jacobian of the transition function on M, transition functions of $L(M)$ are given by:

$$e^{(\beta)\mu}{}_a = J^\mu_\nu \, e^{(\alpha)\nu}{}_a \, , \qquad\qquad J^\mu_\nu = \frac{\partial x^\mu_{(\beta)}}{\partial x^\nu_{(\alpha)}}$$

The frame bundle $L(M)$ is a *principal bundle*, so that there exists a canonical right action of the standard fiber $\mathrm{GL}(m)$ on the bundle $L(M)$ which is free, vertical and fiber transitive. In this case the action is given by:

$$R_\theta \,(x, \ e_a) = (x, \ e_b \, \theta^b_a) \, , \qquad\qquad \|\theta^b_a\| \in \mathrm{GL}(m)$$

As usual, *principal morphisms* are the fibered morphisms which preserve the canonical right action, i.e. the morphisms $\Phi = (\phi, f)$ such that:

$$\phi(x, \ e^{(\alpha)\mu}{}_a) = (f^\mu(x), \ \varphi^{(\alpha)\mu}{}_\nu(x) \, e^{(\alpha)\nu}{}_a)$$

where $\varphi^{(\alpha)}$ are local maps on M valued in $\mathrm{GL}(m)$.

The morphism Φ is *global* if and only if:

$$\varphi^{(\beta)\mu}{}_\nu = J^\mu_\rho \, \varphi^{(\alpha)\rho}{}_\sigma \, \bar{J}^\sigma_\nu$$

where we set $\bar{J}^\sigma_\nu$ for the inverse Jacobian.

This is all that can be done about the frame bundle when it is merely regarded as a principal bundle. Notice in particular that we defined principal morphisms, but no action of diffeomorphisms of M is defined on $L(M)$; in other words there is still no way of lifting a (local) diffeomorphism $f : M \to M$ to a (local) principal bundle morphism $\hat{f} : L(M) \to L(M)$.

Let us recall that on an arbitrary principal bundle $\mathcal{P} = (P, M, \pi, G)$ we have an exact sequence of groups:

$$0 \quad \longrightarrow \quad \mathrm{Aut}_V(\mathcal{P}) \quad \stackrel{i}{\longrightarrow} \quad \mathrm{Aut}(\mathcal{P}) \quad \stackrel{p}{\longrightarrow} \quad \mathrm{Diff}(M) \quad \longrightarrow \quad 0$$

where $i : \mathrm{Aut}_V(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$ is the standard inclusion of vertical automorphisms into automorphisms and $p : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Diff}(M)$ is the projection onto the base manifold M. Equivalently, from an infinitesimal viewpoint, we can consider the following exact sequence of Lie algebras:

$$0 \longrightarrow \chi_V(\mathcal{P}) \stackrel{i}{\longrightarrow} \chi_R(\mathcal{P}) \stackrel{p}{\longrightarrow} \chi(M) \longrightarrow 0$$

where $\chi(M)$ is the Lie algebra of vector fields over M, while $\chi_V(\mathcal{P})$ and $\chi_R(\mathcal{P})$ denote vertical right-invariant and right-invariant vector fields over $\mathcal{P}$, respectively. Notice that the natural lift of diffeomorphisms corresponds to a canonical splitting of the exact sequences (5.2) and (5.2). In general, however, these two sequences do not split canonically. If we regard (5.2) as an exact sequence of vector spaces, non-canonical splittings $\omega : \chi(M) \to \chi_R(\mathcal{P})$ are induced by fixing any principal connection. However, these splittings preserve the Lie algebra structure only if the connection is flat and even in these cases, the splitting is not canonical since it depends on the connection.

Under this viewpoint, the frame bundle is very peculiar. On the frame bundle, in fact, there exists a canonical splitting $L : \mathrm{Diff}(M) \to \mathrm{Aut}(L(M))$ of the exact sequence (5.2), and consequently of the sequence (5.2), given by:

$$L(f) : L(M) \to L(M) : (x,\, e_a) \mapsto (f(x),\, Tf(e_a))$$

where $Tf(e_a)$ denotes the tangent map of f acting on the tangent vectors e_a. Locally, we have $e_a = e^\mu_a\ \partial_\mu$ and $Tf(e_a)$ is given by:

$$Tf(e_a) = J^\mu_\nu\, e^{(\alpha)}{}^\nu_a\, \partial^{(\alpha)}_\mu$$

The splitting of the sequences above, or equivalently the natural lift here introduced on the frame bundle, corresponds to the so-called *soldering form.*

Because of the splitting, we can define the *"complement"* of vertical automorphisms in the group $\mathrm{Aut}(L(M))$. Namely, any automorphim $\Phi = (\phi, f) \in \mathrm{Aut}(L(M))$ decomposes canonically in the composition of an *horizontal* and a *vertical* automorphism, i.e. $\Phi = \Phi_{(V)} \circ \Phi_{(H)}$, where $\Phi_{(H)} = L(f)$ and $\Phi_{(V)} = \Phi \circ L(f^{-1})$, which is obviously vertical.

In literature the bundles which allow a canonical lift of the base diffeomorphisms are called *natural bundles*. For our purposes it is convenient to define it as it follows. *Natural bundles* are fiber bundles $\mathcal{B} = (B, M, \pi, F)$ such that the exact sequence:

$$0 \longrightarrow \mathrm{Aut}_V(\mathcal{B}) \stackrel{i}{\longrightarrow} \mathrm{Aut}(\mathcal{B}) \stackrel{p}{\longrightarrow} \mathrm{Diff}(M) \longrightarrow 0$$

canonically splits. Morphisms of natural bundles are base diffeomorphisms represented on the bundle by means of the canonical action induced by the splitting, i.e. horizontal morphisms. These together form a category.

Therefore, the bundle $L(M)$ is at the same time a principal bundle and a natural bundle; the different categorial structures arise when all principal automorphisms or just the natural lifts of base diffeomorphisms are allowed as morphisms. In the sequel we shall regard $L(M)$ as a natural bundle unless explicitely stated. This will turn out to be particularly important in physical applications, where the choice of the principal structure in place of the natural one may result in different group transformations leaving the theory invariant.

More generally, one could hope that natural bundles be enough for the application to fundamental physics. But this is clearly false. Even at the simplest level, i.e. in Maxwell electromagnetic theory, the field is a principal connection of a $U(1)$-principal bundle having (x^μ, α) as local fibered coordinates, $\alpha \in U(1)$. Fields are locally represented by $A = A_\mu(x)\, \underset{\cdot}{\mathrm{x}}^\mu \otimes T$, where T is a generator of the $\mathbf{u}(1)$ Lie algebra (e.g., $T = i$). Electromagnetism is invariant with respect to *gauge transformations* of the form:

$$A'_\mu = A_\mu + \partial_\mu \varphi(x)$$

where φ is a family of local real valued functions on M. These are vertical transformations, i.e. they project onto the identity. Thus they cannot be the lift of a base diffeomorphism. These transformations may be interpreted as the action on the principal connection generated by automorphisms of the principal bundle on which the connection is defined. Accordingly, even if a natural structure could be given to the configuration bundle, gauge transformations are not natural morphisms in general. On the other hand the category of general fiber bundles has a structure which is not sufficient to deal in a satisfactory way with important issues such as conserved quantities. Thus we need to extend the *natural category* to some intermediate category which allows to encompass gauge transformations. The *gauge natural category* we are going to introduce has proved to be in a good position to solve all these problems.

5.3 Gauge natural bundles

¿From a mathematical viewpoint the category of gauge natural bundles have been introduced by the Cech school (see for example e.g. [20] and references quoted therein). Its application to fundamental physics has been developed later (see [7], [8], [10], [9]), even if gauge natural bundles have been implicitely used for a long time when dealing with gauge theories (see [1], [22], [16] and references quoted therein).

Let us first recall some standard notation. Let $\mathcal{B} = (B, M, \pi, F)$ be a fiber bundle. A *local section* is a map $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = \mathrm{id}_U$, where $U \subset M$ is an open set in the base manifold. Let $\Gamma_x(\pi)$ be the class of all local sections defined in a neighbourhood of $x \in M$. We can define an equivalence relation $\sim^k_x$ in $\Gamma_x(\pi)$. Two local sections σ and ρ are equivalent if they have *contact of order k at x*, i.e. if their k-order Taylor expansions around x coincide. This relation is intrinsic and we denote by $j^k_x\sigma$ the equivalence class which has σ as representative. The set J^kB is the disjoint union over $x \in M$ of all equivalent classes. If two sections σ and ρ have contact k at x they also have contact $(k-h)$ for any $h \geq 0$. Thence we can define a family of projections:

$$M \overset{\pi}{\longleftarrow} B \overset{\pi^1_0}{\longleftarrow} J^1B \longleftarrow \ \dots \overset{\pi^k_{k-1}}{\longleftarrow} J^kB \longleftarrow \ \dots \ \longleftarrow$$

An obvious meaning is given to the projections $\pi^k_{k-h} : J^kB \to J^{k-h}B$ and to the projections $\pi^k : J^kB \to M$. The corresponding bundles are called *k-order jet bundles*. The bundles induced by each π^k_{k-1} are affine but the affine structure is lost for $h > 1$. If $(x^\mu,\ y^i)$ are fibered coordinates on B, natural fibered coordinates on J^kB are denoted by the obvious standard notation $(x^\mu,\ y^i,\ y^i_\mu,\ \dots,\ y^i_{\mu_1\dots\mu_k})$, where $y^i_{\mu_1\dots\mu_k}$ are symmetric in lower indices since they represent the coefficients of Taylor expansions.

We remark that J^k is a covariant functor on the category of fiber bundles. In fact, if $\Phi = (\phi, f)$ is a bundle morphism $\Phi : \mathcal{B} \to \mathcal{B}'$ (with f a diffeomorphism), we can define $j^k\Phi = (j^k\phi, f)$ as follows:

$$\begin{array}{ccc} B & \xrightarrow{\phi} & B' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} J^kB & \xrightarrow{j^k\phi} & J^kB' \\ {\scriptstyle\pi^k}\downarrow & & \downarrow{\scriptstyle\pi'^k} \\ M & \xrightarrow{f} & M' \end{array} \tag{5.1}$$

$$j^k\phi : J^kB \to J^kB' : j^k_x\sigma \mapsto j^k_{f(x)}[\phi \circ \sigma \circ f^{-1}]$$

The morphism $j^k\Phi : J^k\mathcal{B} \to J^k\mathcal{B}'$ is called the *k-order prolongation of* Φ. Analogously, one can define the *k-order prolongation* $j^k\xi$ of a vector field ξ over M, as well as the *k-order prolongation* $j^k\sigma$ of a section σ of $\mathcal{B}$.

The second ingredient needed to define gauge natural bundles are s-frame bundles $L^s(M)$ (s is any positive integer) which are a direct generalization of the frame bundle $L(M)$ introduced above. The *s-frame bundle* $L^s(M)$ is defined as:

$$L^s(M) = \{j^s_0\epsilon \,:\, \epsilon : \mathbf{R}^m \to M \text{ is locally invertible at } 0 \in \mathbf{R}^m\}$$

It is a bundle with projection $\tau^s_M : L^s(M) \to M : j^s_0\epsilon \mapsto \epsilon(0)$. If we choose local coordinates (x^μ) and (x^a) in M and $\mathbf{R}^m$, respectively, then $(x^\mu, \epsilon^\mu_a, \ldots, \epsilon^\mu_{a_1\ldots a_s})$ are natural fibered coordinates on $L^s(M)$. Notice that for $s = 1$ the ordinary frame bundle is recovered. As for $L(M)$, also $L^s(M)$ is a principal bundle with the group $\mathrm{GL}^s(m)$ defined by:

$$\mathrm{GL}^s(m) = \left\{ j^k_0\alpha \mid \alpha : \mathbf{R}^m \to \mathbf{R}^m \quad \alpha(0) = 0 \quad \text{locally invertible} \right\}$$

where the group moltiplication is induced by the composition, i.e.:

$$j^k_0\alpha \cdot j^k_0\beta = j^k_0(\alpha \circ \beta)$$

The bundle $L^s(M)$ is principal because of the canonical right action which is given by:

$$j^k_0\epsilon \cdot j^k_0\alpha = j^k_0(\epsilon \circ \alpha)$$

The bundle $L^s(M)$ is also a natural bundle, since we can naturally lift a base diffeomorphism f as follows:

$$f : M \to M' \qquad \rightsquigarrow \qquad L^s(f) : L^s(M) \to L^s(M') : j^s_0\epsilon \mapsto j^s_0[f \circ \epsilon]$$

Analogously, one can define the lift $L^s(\xi)$ of a base vector field $\xi \in \chi(M)$.

Now we are ready to define gauge natural bundles of order (r, s) associated to a principal bundle $\mathcal{P} = (P, M, p, G)$.

Definition 4 let (r, s) be two positive integers such that $s \geq r$. The *gauge natural prolongation* $W^{(r,s)}(\mathcal{P})$ *of order* (r, s) *of* $\mathcal{P}$ is, by definition

$$W^{(r,s)}(\mathcal{P}) := J^r\mathcal{P} \times_M L^s(M) \qquad\qquad (s \geq r)$$

where $\times_M$ denotes the fibered product.

A point of $W^{(r,s)}(\mathcal{P})$ is of the form $(j^r_x\sigma, j^s_0\epsilon,)$, where $j^l_p(\cdot)$ denotes the l-jet prolongation of a map evaluated at p, $\epsilon : \mathbf{R}^m \to M$ is locally invertible, $\epsilon(0) = x$ and $\sigma : M \to \Sigma$ is a local section around the point $x \in M$. The bundle $W^{(r,s)}(\mathcal{P})$ is a principal bundle having the following structure group:

$$W^{(r,s)}(G) := J^r(G) \odot \mathrm{GL}^s(m) \qquad (s \geq r)$$

where $J^r(G)$ denotes the group

$$J^r(G) = \{j^r_0 a : a : \mathbf{R}^m \to G\}$$

and group multiplication is induced by the group operation in G. The group multiplication on $W^{(r,s)}(G)$ is defined by the following rule:

$$(j^r_0 a, j^s_0\alpha) \odot (j^r_0 b, j^s_0\beta) = \Big(j^r_0((a \circ \beta) \cdot b), j^s_0(\alpha \circ \beta)\Big)$$

having denoted by $\cdot$ the group multiplication in G.

The right action of $W^{(r,s)}(G)$ on $W^{(r,s)}(\mathcal{P})$ is then defined by:

$$(j^r_x\sigma, j^s_0\epsilon) \odot (j^r_0 a, j^s_0\alpha) = \Big(j^r_0(\sigma \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1})), j^s_0(\epsilon \circ \alpha)\Big) \qquad (5.2)$$

where $\cdot$ denotes the canonical right action of G on $\mathcal{P}$. Notice that if $r > s$ the right action 5.2 would not be well-defined, depending on the representative chosen for $j^s_0\epsilon$ and $j^s_0\alpha$.

Again, $W^{(r,s)}$ turns out to be a functor on the category of G-principal bundles. In fact, let $\Phi : \mathcal{P} \to \mathcal{P}'$ be a principal morphism over a diffeomorfism $f : M \to M'$. We define a morphism $W^{(r,s)}(\Phi)$ by setting:

$$W^{(r,s)}(\Phi) : W^{(r,s)}(\mathcal{P}) \to W^{(r,s)}(\mathcal{P}'), \qquad (5.3)$$

$$(j^r_x\sigma, j^s_0\epsilon) \mapsto (j^r_x(\Phi \circ \sigma \circ f^{-1}), j^s_0(f \circ \epsilon))$$

Definition 5 a *gauge natural bundle of order* (r,s) *associated to* $\mathcal{P}$ is any bundle associated to the gauge natural prolongation $W^{(r,s)}(\mathcal{P})$ $(s \geq r)$.

Gauge natural bundles are characterised (see [4], [20]) by the canonical rapresentation of automorphisms of the structure bundle $\mathcal{P}$ induced by 5.3. Namely, let $\mathcal{B} = W^{(r,s)}(\mathcal{P}) \times_\lambda F$ be the gauge natural bundle associated to $\mathcal{P}$ through an action λ of $W^{(r,s)}(G)$ on F. A point in $\mathcal{B}$ is thence the λ-orbit denoted by $[\rho, v]_\lambda$ with $\rho \in W^{(r,s)}(\mathcal{P})$ and $v \in F$. Let $\Phi : \mathcal{P} \to \mathcal{P}$ be an automorphism over a diffeomorfism $f : M \to M$; then we define the *induced automorphism* Φ_λ by

$$\Phi_\lambda : \mathcal{B} \to \mathcal{B} : [\rho, v]_\lambda \mapsto [W^{(r,s)}\Phi(\rho), v]_\lambda$$

which is well-defined.

We remark that gauge natural bundles may be seen as functors $W^{(r,s)}_\lambda$ from the category of principal bundles with a fixed structure group G into the category of fiber bundles. In fact, once the integer pair (r, s) is fixed (and $s \geq r$) and an action $\lambda : W^{(r,s)}(G) \times F \to F$ over a manifold F is chosen, then to any G-principal bundle $\mathcal{P} = (P, M, \pi, G)$ we can associate a (gauge natural) fiber bundle $W^{(r,s)}_\lambda(\mathcal{P}) = W^{(r,s)}\mathcal{P} \times_\lambda F$. Accordingly, to any principal morphism $\Phi = (\phi, f)$ we can associate the (gauge natural) morphism Φ_λ. These functors $W^{(r,s)}_\lambda$ have the following properties which in fact are sufficient to characterise them:

a) $W^{(r,s)}_\lambda(\mathcal{P})$ is fibered over the same base M of $\mathcal{P} = (P, M, \pi, G)$;

b) any principal morphism $\Phi : \mathcal{P} \to \mathcal{P}'$ projecting onto $f : M \to M'$ induces a fibered morphism $W^{(r,s)}_\lambda(\Phi) : W^{(r,s)}_\lambda(\mathcal{P}) \to W^{(r,s)}_\lambda(\mathcal{P}')$ also projecting over f;

c) if $U \subset M$ is an open subset and $i : \pi^{-1}(U) \to P$ is the inclusion then $W^{(r,s)}_\lambda(i)$ is the inclusion of the subbundle $W^{(r,s)}_\lambda(\pi^{-1}(U))$ into $W^{(r,s)}_\lambda(\mathcal{P})$.

This characterization of gauge natural bundles is equivalent to the one we have chosen above. More precisely one can prove (see [20]) that if W is a functor having the properties mentioned above, then there exists (and it is unique up to bundle isomorphism) a (*finite*) pair (r, s) and a representation λ such that $W = W^{(r,s)}_\lambda$. This is completely analogous to what was earlier known for natural bundles, which are in fact a particular case of our more general setting (when taking $G = \{e\}$, see [20]). Natural bundles turn then out to be always associated to some frame bundle of (*finite*) orders. The functorial characterization is important also for physics because it stresses the canonical nature of gauge natural bundles as well as their analogy to natural bundles; on the contrary, in applications it is often useful to work on some concrete representation of gauge natural bundles. One can say that the two viewpoints are in a suitable sense complementary.

¿From a mathematical viewpoint gauge natural bundles have quite a rich structure, due basically to the analogy with natural bundles. This structure allows in fact one to cope with most constructions one can perform with natural bundles. ¿From a physical viewpoint this framework enables us to treat at the same time and under a unifying formalism both gravity and gauge theories together with Bosonic and Fermionic matter (see [20], [8], [10] and [9]).

5.4 Gauge Natural Field Theories

A gauge natural structure on a given bundle $\mathcal{B}$ selects a preferred subgroup of transformations in $\mathrm{Aut}(\mathcal{B})$, namely $\mathrm{Aut}(\mathcal{P})$ represented on $\mathcal{B}$. A given gauge natural structure, however, is not canonical on the bundle. For example, the tangent bundle can be associated both to the frame bundle $L(M)$ and to the orthonormal frame bundle $\mathrm{SO}(M, g)$ of any Riemannian metric g on M. These two structures of gauge natural bundle, of course, are rather different. This is a desired feature for physical applications, where the group of gauge transformations is known to be a superimposed structure which is usually regarded as containing most of the fundamental physical information of the theory.

The transformations in $\mathrm{Aut}(\mathcal{B})$ selected by a gauge natural structure are called *(generalized) gauge transformations*, while they are called *pure gauge transformations* if they are vertical. To define a field theory one then chooses a Lagrangian of order k, i.e. a bundle morphism $L : J^k\mathcal{B} \to A^m(M)$, where $A^m(M)$ denotes the bundle of m-forms over M. The bundle $\mathcal{B}$ is called the *configuration bundle of the theory.* Fiber local coordinates are (x^μ, y^i) and y^i are called the *dynamical fields.* The local expression of the Lagrangian is thence:

$$L = \mathcal{L}(x^\mu, y^i, y^i_\mu, \dots, y^i_{\mu_1\dots\mu_k})\,\mathbf{ds} \tag{5.4}$$

where $\mathbf{ds}$ is a local basis of m-forms over M.

The action functional acting on a section σ of $\mathcal{B}$ is defined as:

$$A_D(\sigma) = \int_D L \circ j^k\sigma \tag{5.5}$$

for any m-region $D \subset M$. We recall that by *region* we mean a compact submanifold with a boundary ∂D which is also a compact submanifold. Standard variational techniques are then applied to obtain the physical properties of the system (see [8], [11], [1], [17], [12]), but here we are not directly interested in the details of these aspects. We just mention that if we denote by $V^*(J^k\mathcal{B})$ the vector bundle which is dual to the bundle $V(J^k\mathcal{B})$ of vertical vectors on $J^k\mathcal{B}$, we recall that one can define a bundle morphism $\delta L : J^k\mathcal{B} \to V^*(J^k\mathcal{B}) \otimes A^m(M)$, called *the variation of* L, defined by:

$$\delta L = \Big[\frac{\partial \mathcal{L}}{\partial y^i}\,\mathrm{y}^i_{\cdot} + \frac{\partial \mathcal{L}}{\partial y^i_\mu}\,\mathrm{y}^i_{\cdot\mu} + \ldots + \frac{\partial \mathcal{L}}{\partial y^i_{\mu_1\ldots\mu_k}}\,\mathrm{y}^i_{\cdot\mu_1\ldots\mu_k}\Big] \otimes \mathbf{ds} \tag{5.6}$$

(Recall that vertical vectors in the bundle $\pi : Y \to X$ are those tangent vectors which belong to the kernel of the tangent map $T\pi$).

If we consider a deformation of fields, which in this framework is represented by a vertical vector field $X = (\delta y^i)\,\partial_i$ on $\mathcal{B}$, the ordinary variation of the Lagrangian L is given by:

$$\delta_X L = < \delta L \mid j^k X > \tag{5.7}$$

where $j^k X$ is a section of the vertical bundle $J^k V(\mathcal{B}) \approx V(J^k\mathcal{B})$ (the isomorphism is canonical) and $< \ \mid \ >$ denotes the canonical duality.

An infinitesimal generator $\Xi = \xi^\mu\,\partial_\mu + \xi^i\,\partial_i$ of gauge transformations of the configuration bundle $\mathcal{B}$ is a *Lagrangian symmetry* if the Lagrangian is covariant, i.e. if the following identity holds true:

$$< \delta L \mid j^k \mathcal{L}_\Xi > \circ j^k\sigma = \mathcal{L}_\xi (L \circ j^k\sigma) \tag{5.8}$$

for any section σ. Here $\mathcal{L}_\xi = i_\xi \circ \mathrm{d} + \mathrm{d} \circ i_\xi$ is the standard Lie-derivative of forms on M, while $\mathcal{L}_\Xi = (\xi^\mu\, y^i_\mu - \xi^i)\,\partial_i$ is called *formal Lie derivative* because, once it is computed along a section σ, one obtaines the generalized Lie derivative $\mathcal{L}_\Xi\sigma = T\sigma(\xi) - \Xi \circ \sigma$ (see [20], [12], [24]).

Notice that the Lie derivative $\mathcal{L}_\Xi$ is taken with respect to a vector field Ξ on the configuration bundle $\mathcal{B}$. This differs from the standard situation in differential geometry in which Lie derivatives are performed with respect to vector fields on M. That is a direct consequence of the fact that diffeomorphisms of M do not act on fields, while generalized gauge transformation do. This is one of the key features of gauge theories. One has thence to learn how to cope with the fact that relevant transformations are defined at bundle level.

However, what is here essential to us is to stress that generalized gauge transformations are required to be *Lagrangian symmetries*. The gauge natural structure is thence a way of encoding from the very beginning the group of symmetries. In turns, it constrains the Lagrangians which can be used: this is one of the reasons why gauge theories seem to be the most reasonable framework for physical theories.

5.5 Yang Mills Theories

Before addressing our attention to spin frames let us briefly discuss a classical example of gauge theory: Yang-Mills theory.

Let (M, g) be a (pseudo)-Riemannian manifold of dimension m (as usual, paracompact and connected) and G be a Lie group called the *gauge group*, which is assumed to admit an *Ad*-invariant metric κ. Usually, unitary groups (namely, $U(1)$, SU(2), ..., SU(n)) and generators T_A in the Lie algebra such that $\kappa(T_A, T_B) = \delta_{AB}$ are chosen. Dynamical fields are principal connections on some principal bundle $\mathcal{P} = (P, M, \pi, G)$. It can be shown (see [10], [14]) that principal connections on $\mathcal{P}$ correspond to sections of the bundle $J^1\mathcal{P}/G$, which is thence the configuration bundle. Local fibered coordinates on $J^1\mathcal{P}/G$ are (x^μ, A^A_μ).

A principal morphism $\Phi = (\phi, f)$ acts on dynamical fields in a canonical way. If we assume at least for notational convenience that G is a matrix group (and thence the indices A denote a pair ${}^a{}_b$) and Φ is locally defined by:

$$\begin{cases} x'^\mu = f^\mu(x) \\ g'^a_b = \varphi^a_c(x)\, g^c_b \end{cases} \tag{5.9}$$

then its action on dynamical fields $A^a{}_{b\mu}$ is:

$$A'^a{}_{b\mu} = \bar{J}^\nu_\mu \varphi^a_c (A^c{}_{d\nu} - \partial_\nu \varphi^c_d)\bar{\varphi}^d_b \tag{5.10}$$

where we set $\bar{\varphi}$ for the pointwise inverse of φ. Equation (5.10) is the local expression for gauge transformations. Notice that while one can set $f = id_M$ to consider *pure gauge transformations*, the tentative prescription $\varphi = e_G$ has only the local character. In fact, by changing a local chart on the configuration bundle we have:

$$\varphi^{(\beta)}(x) = \gamma(x') \cdot \varphi^{(\alpha)}(x) \cdot \bar{\gamma}(x) \tag{5.11}$$

where γ and $\bar{\gamma}$ denote transition functions in $\mathcal{P}$ and their inverse, respectively. Thus changing chart does not fix the group identity, unless the bundle is trivial (i.e. $P \approx M \times G$).

We can now define the curvature of the principal connection by:

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + c^A_{\cdot\,BC} A^B_\mu A^C_\nu \tag{5.12}$$

where $c^A_{\cdot\,BC}$ are structure constants of the Lie algebra of G. Gauge transformations act on the curvature by the adjoint representation; again for a matrix group, we have:

$$F'^a_{\ b\mu\nu} = \bar{J}^\rho_\mu \, \bar{J}^\sigma_\nu \, \varphi^a_c \, F^c_{d\rho\sigma} \, \bar{\varphi}^d_b \tag{5.13}$$

Using the canonical metric $[\ ,\ ]$ induced on $\mathcal{P}$ by g and κ, we can build the invariant

$$[F, F] = \kappa_{AB} \, F^A_{\mu\nu} \, F^B_{\rho\sigma} \, g^{\mu\rho} \, g^{\nu\sigma} = F^A_{\mu\nu} \, F^{\mu\nu}_A \tag{5.14}$$

Thence the Lagrangian:

$$L_{YM} = -\frac{1}{4}[F, F] \, \sqrt{g} \, \mathbf{ds} \tag{5.15}$$

is covariant with respect to (pure) gauge transformations. But in physics covariance with respect to (pure) gauge transformations is not sufficient. Some group of transformations of spacetime (e.g. the Lorentz group, the Poincaré group, or the general diffeomorphism group) is needed. The first possibility is to fix a Lorentzian structure on M, e.g. by fixing Minkowski spacetime. Being then $g \equiv \eta$ fixed from the beginning in the Lagrangian (5.15) we can consider gauge transformations projecting onto Lorentz (or Poincaré) spacetime transformations. These gauge transformations are symmetries of the Lagrangian (5.15), since they project over isometries of spacetime. A field theory defined in this way satisfies the axioms of Special Relativity. It is not a gauge natural theory since just a small subgroup of generalized gauge transformations (those projecting onto isometries) are symmetries.

A more general possibility is to keep the metric g unfixed on M and to regard it as a dynamical field. In this way generalized gauge transformations are also required to act on the metric field as follows

$$g'_{\mu\nu} = \bar{J}^\rho_\mu \, g_{\rho\sigma} \, \bar{J}^\sigma_\nu \tag{5.16}$$

As a consequence any generalized gauge transformation is now a symmetry of the Lagrangian, which thence defines a gauge natural theory. [Of corse two field equations are now produced, one for the gauge field (called the *Yang-Mills equation*) and one for the metric g. To endow the theory with a physical meaning one has to add a *"kinetic term"* for the metric field to the Lagrangian (5.15), e.g.:

$$L_H = \sqrt{g}R\,\mathbf{ds} \tag{5.17}$$

where R denotes the scalar curvature of the metric g. The Lagrangian L_H is called the *Hilbert-Einstein Lagrangian* and it is also covariant with respect to generalized gauge transformations. The total Lagrangian $L = L_H + L_{YM}$ thence defines a gauge natural theory which is a genuine general relativistic field theory, called *Einstein-Yang-Mills theory.* The Yang-Mills field equations describe the evolution of the Yang-Mills field, including the effect due to gravitation, while the field equation for the metric field (now called the *Einstein-Yang-Mills field equation*) describes the evolution of the gravitational field generated by the Yang-Mills field acting as a source.]

Yang-Mills theories are the prototype of virtually any relevant field theory used in fundamental physics, with the exception of General Relativity. With $G = U(1)$ then Yang-Mills theory becomes commutative and reduces to Maxwell electromagnetism; the electroweak model of Weinberg and Salam is based on a Yang-Mills theory with $G = \mathrm{SU}(2)\times U(1)$ (and a scalar field called Higgs Boson which is responsable of a *"spontaneous symmetry breaking"* and thence of the dramatic differences between electromagnetic and weak interactions); finally, strong nuclear interactions are related to a SU(3)-based Yang-Mills model. We should also mention that at least electroweak model agrees fantastically with experiments. As far as experimentalists can go, there is no discrepancy with the theory. One can say that Yang-Mills theory is the most successful and tested theory in the history of physics.

We remark that Yang-Mills theories, as general gauge natural theories, introduce a new prime actor in the representation of physical world. In General Relativity (as well as in Differential Geometry) the arena is provided by the spacetime manifold M. On M dynamical fields are defined and identified by field equations. The general attitude is that local field equations identify the local expression of a metric g. One can then consider the maximal analytic extension of the metric g which in turn defines M globally, so that anything is determined by field equations *and* boundary conditions. [Actually this viewpoint seems reductive, since there is no purely physical evidence that M has to be analytic. If M is not required to be analytic no (unique) maximal extension can be provided. In our opinion, M has thence to be provided as a physical input of the theory on the same foot of boundary conditions. Nevertheless nothing essential really changes in the sequel and we can get stuck to the generally accepted viewpoint.]

Analogously, in a gauge natural theory the local solution has to provide both M and the principal bundle $\mathcal{P}$. Luckily enough the solution of field equations actually determines the structure bundle $\mathcal{P}$. The reason is very simple and it is once again based on gauge transformations. It is enough to notice that there exists compact supported gauge transformations (remember that M is paracompact by hypothesis). If M has to be interpreted as the spacetime and the evolution of fields has to be uniquely determined by initial conditions (and physicists are quite strict on this point!), then two configurations which are related by a gauge transformation has necessaryly to describe the same physical system. [If not, one can easily use compact supported gauge transformation to produce two different configurations with the same initial data.]

Accordingly, while extending from one chart to another we feel free to perform a local pure gauge transformation before glueing. We want in fact a global description of the physical system and, as we said, two different configurations differing by a gauge transformation actually defines the same physical system. In other words, we are forced to accept as a description of the physical system, a family of local configurations which differ by local pure gauge transformations on the overlaps. These local pure gauge transformations contain the information which encodes the structure bundle $\mathcal{P}$. In fact, if dynamical fields has to be global sections of the configuartion bundle $\mathcal{B}$ the pure gauge transformations performed while glueing have to be transition functions of the bundle $\mathcal{B}$. In the gauge natural framework $\mathcal{B}$ is associated to $\mathcal{P}$ so that transition functions of $\mathcal{P}$, and thence $\mathcal{P}$ itself, are determined.

Let us finally see how the construction of $\mathcal{P}$ out of these pieces of information works out in a simple example. Let us choose $M = S^2$ and consider the Maxwell electromagnetic field theory, i.e. Yang-Mills with $G = U(1)$. One can verify that, choosing spherical coordinates (θ, ϕ) on S^2, a possible local solution is given by:

$$A_{N} = \frac{m}{2}(1 - \cos\theta)\, \mathrm{d}\phi \otimes T \tag{5.18}$$

It is defined anywhere out of $S = (0, 0, -1) \in S^2$, while it cannot be continued as a form on the whole sphere. But if one computes its the curvature:

$$F = \mathrm{d}A_{N} = \frac{m}{2}(\sin\phi\, \mathrm{d}\theta \wedge \mathrm{d}\phi)\, \otimes T \tag{5.19}$$

it can be recognised to be well-defined on the whole sphere. In fact, (5.18) can be globalized as a principal connection, i.e. by performing a suitable gauge transformation before glueing. Let us define $\gamma = \exp(im\phi)$ and set:

$$A_{_S} = A_{_N} - \gamma^{-1}\mathrm{d}\gamma = -\frac{m}{2}(1+\cos\theta)\,\mathrm{d}\phi \otimes T \tag{5.20}$$

which is well-defined anywhere but in $N = (0,0,1) \in S^2$. Thus $A_{_N}$ and $A_{_S}$ are local expressions of a global principal connection on a non-trivial bundle having γ as transition functions.

For $m = 1$, the bundle $\mathcal{P}$ is the Hopf bundle defined by the projection:

$$\pi : S^3 \subset \mathbf{C}^2 \to S^2 \subset \mathbf{C}\times\mathbf{R} : (z^1, z^2) \mapsto (2z^1\bar{z}^2, |z^1|^2 - |z^2|^2) \tag{5.21}$$

Since $(z^1, z^2) \in S^3$, i.e. $|z^1|^2 + |z^2|^2 = 1$, then $\pi(z^1, z^2) \in S^2$.

5.6 Spin Frames

We can now introduce spin frames as an example of gauge natural structures (see [9], [19]). Let then M be an orientable, connected, paracompact, C^∞-manifold of finite dimension m. We assume that M meets the (possible) topological obstructions which ensure the existence on it of a pseudo-Riemannian metric of signature $\eta = (r, s)$ and that M is a spin manifold, i.e. it has vanishing second Stiefel-Whitney class. Under these hypotheses we say that M is a *η-manifold*. We *do not* however choose any metric on M nor a spin structure (in the standard sense, see [21]) on (M, g).

Definition 6 *if $(\sigma, M, \pi, \mathrm{Spin}(\eta))$ is a principal bundle, we call spin frame on σ a vertical principal morphism $\Lambda : \sigma \to L(M)$ related to the composed homomorphism of the structure groups $\hat{\ell} = i\mathbf{C}irc\ell : \mathrm{Spin}(\eta) \to \mathrm{GL}(m)$. The following diagrams are thence commutative:*

$$\begin{array}{ccc} \sigma & \xrightarrow{\Lambda} & L(M) \\ & \searrow & \downarrow \tau \\ & & M \end{array} \qquad \begin{array}{ccc} \sigma & \xrightarrow{\Lambda} & L(M) \\ R_S \downarrow & & \downarrow R_{\hat{\ell}(S)} \\ \sigma & \xrightarrow{\Lambda} & L(M) \end{array} \tag{5.22}$$

In general, nothing can be said about the existence of spin frames on a general Σ. For example, if M is non-parallelizable and we choose the trivial bundle $\Sigma = M \times \mathrm{Spin}(\eta)$, of course no spin frame exists. In fact, if there existed a spin frame Λ on Σ, we could use a global section σ of Σ, which exists by the triviality hypothesis, to produce a global section of $L(M)$, namely $\Lambda \circ \sigma$. This proves M is parallelizable, which contradicts the hypotheses.

However, since M is a η-manifold, the following proposition holds:

Proposition 7 *On any η-manifold M, there exists a* $\mathrm{Spin}(\eta)$*-principal bundle* $(\Sigma, M, \pi, \mathrm{Spin}(\eta))$ *such that at least a spin frame* $\Lambda : \Sigma \to L(M)$ *exists.*

Sketch of the proof: let us fix a metric g of signature η on the η-manifold M and denote by $\mathrm{SO}(M, g)$ the bundle of g-orthonormal frames. This is a principal sub-bundle (a reduction) of $L(M)$ and we denote by $i : \mathrm{SO}(M, g) \to L(M)$ the canonical inclusion. Consider a family $e_a^{(\alpha)}$ of local sections of $\mathrm{SO}(M, g)$. They induce a trivialization on $\mathrm{SO}(M, g)$ and consequently on $L(M)$. Let us denote by γ the cocycle of transition functions which are valued in $\mathrm{SO}(\eta)$. We can try to define a lifted cocycle $\hat{\gamma}$ valued in $\mathrm{Spin}(\eta)$ which projects over γ, i.e. such that $\ell \circ \hat{\gamma} = \gamma$. The topological obstruction to the existence of such a lifted cocycle is the second Stiefel-Whitney class (see [21], [18]), which automatically vanishes being M an η-manifold. Below we shall give another proof based on [3], which better shows the independence on the signature η.

Then the cocycle $\hat{\gamma}$ defines a principal bundle $(\Sigma, M, \pi, \mathrm{Spin}(\eta))$. Of course there exists a trivialization $\sigma^{(\alpha)}$ of Σ having $\hat{\gamma}$ as transition functions. We can define a global morphism $\bar{\Lambda} : \Sigma \to \mathrm{SO}(M, g)$ such that $\bar{\Lambda}(\sigma^{(\alpha)}) = e_a^{(\alpha)}$. Since $\hat{\gamma}$ is the lift of γ, then $\bar{\Lambda}$ is global.

Let now consider $\Lambda = i \circ \bar{\Lambda} : \Sigma \to L(M)$. It can be easily shown that Λ is a spin frame on Σ. Then we explicitely build a bundle Σ such that at least one spin frame actually exists on it. Q.e.d.

Notice that nothing is said about uniqueness. In general we may have more that one possible bundle Σ depending on the choices of the metric g and the lifted cocycle $\hat{\gamma}$.

Definition 7 *a* $\mathrm{Spin}(\eta)$*-principal bundle* $(\Sigma, M, \pi, \mathrm{Spin}(\eta))$ *is called a structure bundle if there exists at least one spin frame on* Σ*. Proposition* (7) *shows that, on any η-manifold, there exists at least one structure bundle.*

¿From now on, M will be a η-manifold and Σ a structure bundle. One can argue if actually there exist η-manifonds having non-equivalent structure bundles. The answer is positive as one can prove with a simple example.

Let us consider the manifold $M = \mathbf{R}^3 - \{0\} \approx \mathbf{R}^+ \times S^2$. We can define two inequivalent $\mathrm{Spin}(\eta)$-principal bundles on M: one, called Σ_1, is the trivial one, the other, called Σ_2, is the extension of the pull-back of the Hopf bundle $\pi : S^3 \to S^2$ on M via the standard projection $M \to S^2$. Some details about the definition of Σ_2 can be found in the appendix A. What is relevant now is that Σ_1 and Σ_2 are inequivalent, i.e. Σ_2 is non-trivial, and that both of them are structure bundles over M, as shown in appendix A.

Since one can have on the same base manifold M more than one structure bundle, problems about naturality of spin frames arise. In fact, if $f : M \to M$ is a base diffeomorphism, it is not clear how to define its lift to spin frames. In particular, it is not clear if such a lift can change the structure bundle. A proposal to overcome this problem can be found in literature (see [3]) and we shall analyze it below in Section 8. We want here to present an alternative viewpoint (see [7], [9], [5]) which gets rid of naturality and produces a gauge natural theory for spin frames. The two approaches are deeply related though not completely equivalent.

We first introduce the metric $g(\Lambda)$ induced on M by a spin frame Λ on a structure bundle Σ.

Property 1
If $\Lambda : \Sigma \to L(M)$ *is a spin frame on* Σ *then* $\mathrm{Im}(\Lambda) \subset L(M)$ *is a principal sub-bundle having* $\mathrm{SO}(\eta)$ *as structure group.*

Proof: let $\sigma^{(\alpha)} : U_\alpha \to \mathrm{Spin}(\eta)$ be a family of local sections which correspond to a trivialization of Σ and let us consider the family of local sections $u^{(\alpha)} = \Lambda(\sigma^{(\alpha)})$ induced on $L(M)$. Every point in Σ is of the form $p = \sigma^{(\alpha)} \cdot S$ for some α and some $S \in \mathrm{Spin}(\eta)$; thence every point in $\mathrm{Im}(\Lambda)$ is of the form $\Lambda(p) = u^{(\alpha)} \cdot \hat{\ell}(S)$.

Changing trivialization domain one has $\sigma^{(\beta)} = \sigma^{(\alpha)} \cdot \hat{\gamma}_{(\alpha\beta)}$, where $\hat{\gamma}_{(\alpha\beta)} : U_{\alpha\beta} \to \mathrm{Spin}(\eta)$ are the transition functions of Σ. The sections induced in $L(M)$ are thence related as follows

$$u^{(\beta)} = u^{(\alpha)} \cdot \ell(\hat{\gamma}_{(\alpha\beta)}) \tag{5.23}$$

Since the cocycle $\ell(\hat{\gamma}_{(\alpha\beta)})$ takes its values in $\mathrm{SO}(\eta)$ and acts on local sections as shown by (5.23), then $\mathrm{Im}(\Lambda)$ is a $\mathrm{SO}(\eta)$-principal subbundle of $L(M)$. Q.e.d.

Since for every $\mathrm{SO}(\eta)$-principal sub-bundle of $L(M)$ there is one and only one metric g on M which has exactly that sub-bundle as its orthonormal frame bundle, we can give the following:

Definition 8 *if* $\Lambda : \Sigma \to L(M)$ *is a spin frame on* Σ *then the metric* $g(\Lambda)$, *which has* $\mathrm{Im}(\Lambda) \subset L(M)$ *as orthonormal frame bundle, is called the metric induced by* Λ.

We stress that, although the notion of spin frame on Σ is deeply related to ordinary spin structures (see [21], [7]), they are not equivalent from a categorial viewpoint. In fact, ordinary spin structures, i.e. the pair $(\Sigma, \bar{\Lambda})$ of a $\mathrm{Spin}(\eta)$-principal bundle Σ and of a vertical principal morphism $\bar{\Lambda} : \Sigma \to \mathrm{SO}(M, g)$, are defined in the category of (pseudo)-Riemannian manifolds (M, g). On the contrary, spin frames on Σ are defined on the category of structure bundles. Usually a lot of metrics on M may be induced by different spin frames on the same structure bundle Σ. We can make this claim precise by means of the following statements.

Definition 9 *a (pseudo)-Riemannian metric g over M is called Σ-admissible if there exists a spin frame Λ on the structure bundle Σ such that g is the metric induced by Λ.*

Theorem 17
Let M be a 4-dimensional, Lorentzian, non-compact, orientable, spin manifold. Then any structure bundle Σ is trivial and all metrics are Σ-admissible.

Proof: this is a simple corollary of a theorem due to Geroch (see [15]), which asserts that for every spin structure $(\Sigma, \bar{\Lambda})$ on a 4-dimensional, Lorentzian, non-compact spin manifold (M, g) any spin bundle Σ is necessarily trivial. Thus, every metric is induced by some spin frame on the trivial structure bundle $\Sigma = M \times \mathrm{Spin}(\eta)$. Q.e.d.

We stress that the above class includes all relevant examples for General Relativity. Dimension and signature requests are in fact physically obvious; spacetimes are required to be spin manifolds because of the hope of describing Fermions on them; moreover they are assumed to be non-compact since it has been shown that on compact Lorentzian manifolds one always has causality problems, which have to be avoided for a correct physical interpretation of the theory ([15] and references quoted therein).

Theorem 18 *Let M be a η-manifold in the Riemannian signature. Then for every structure bundle Σ every metric is Σ-admissible.*

Proof: let us consider a Riemannian structure (M, g) on M. Being M an η-manifold one can build a spin structure $(\Sigma, \bar{\Lambda})$ for $\mathrm{Spin}(m, 0)$. By definition every structure bundle on M can be obtained in this way. Let us now show that any other Riemannian metric g' on M can be induced by a spin frame on a fixed Σ.

To this purpose, let us fix a family of local sections $\sigma^{(\alpha)}$ of Σ inducing a trivialization. Using $\bar{\Lambda} : \Sigma \to \mathrm{SO}(M, g)$ one can produce a trivialization of $\mathrm{SO}(M, g)$, induced by the family of local sections $u^{(\alpha)} = \bar{\Lambda}(\sigma^{(\alpha)})$. If $\hat{\gamma}_{(\alpha\beta)}$ are $\mathrm{Spin}(m)$-valued transition functions for Σ, i.e. $\sigma^{(\beta)} = \sigma^{(\alpha)} \cdot \hat{\gamma}_{(\alpha\beta)}$ holds, then $\ell(\hat{\gamma}_{(\alpha\beta)})$ are transition functions on $\mathrm{SO}(M, g)$. We remark that $i_g : \mathrm{SO}(M, g) \to L(M)$ is a sub-bundle of the frame bundle and that $i_g(u^{(\alpha)})$ induce a trivialization of $L(M)$ whose transition functions are again $\ell(\hat{\gamma}_{(\alpha\beta)})$.

Since the signature is positive definite one can apply Gram-Schmidt procedure to get orthonormal bases. For example, starting from $u^{(\alpha)}$, which is an orthonormal basis for g, we can apply Gram-Schmidt procedure to get an orthonormal basis $u'^{(\alpha)}$ for g'. This provides local isomorphisms on the given trivialization:

$$\phi^{(\alpha)} : \mathrm{SO}(M, g) \to \mathrm{SO}(M, g') \tag{5.24}$$

Let us now remark that the following algebraic lemma holds, the proof of which is given in the Appendix B:

Lemma 2 *If $u^{(\alpha)}$ and $u^{(\beta)} = u^{(\alpha)} \cdot a_{(\alpha\beta)}$ are two orthonormal bases for g (thence $a_{(\alpha\beta)} \in \mathrm{SO}(m)$) and one applies Gram-Schmidt procedure with respect to another metric g', obtaining respectively two g'-orthonormal bases $u'^{(\alpha)}$ and $u'^{(\beta)}$, then these are related by $u'^{(\beta)} = u'^{(\alpha)} \cdot a_{(\alpha\beta)}$ through the same orthonormal matrix $a_{(\alpha\beta)}$.*

This lemma ensures that the local morphisms $\phi^{(\alpha)}$ glue together to give a global, vertical, principal morphism $\phi : \mathrm{SO}(M, g) \to \mathrm{SO}(M, g')$. As a consequence, if $i_g \circ \bar{\Lambda}$ is a spin frame on Σ inducing g, then $i_{g'} \circ \phi \circ \bar{\Lambda}$ is a spin frame on Σ inducing g'. Thus also g' is Σ-admissible, as we claimed. Q.e.d.

We thence identified two classes of η-manifolds on which any metric g of signature η is Σ-admissible on any structure bundle Σ. Consequently, the given examples show that the choice of a structure bundle Σ contains much less information than the fixing of the (pseudo)-Riemannian structure. We believe that this minimality of requirements to be fixed in the beginning is a first strong advantage of spin frames over spin structures. This is particularly true in physical applications where the initial structures should be in principle endowed with a physical meaning. As long as ordinary spin structures are concerned, the (pseudo)-Riemannian structure on spacetime M has to be interpreted as a gravitational background which is physically meaningless since, from a fundamental viewpoint, spinors are expected to interact with the gravitational field as well.

On the contrary, what we will show in Appendix A proves that on $M = \mathbf{R}^3 - \{0\}$ not any metric is Σ_1-admissible. In fact, one can consider the pull-back metric g_1 on M of the Minkowski metric defined on (R^3, η). Obviously g_1 allows a global orthonormal frame (e_a) and it is thence Σ_1-admissible. Let us now consider the metric g_2 such that radial vectors are timelike, while angular vectors are spacelike. By considering coordinates (r, θ, ϕ) on M such a metric would be for example:

$$g_2 = -\mathrm{d}r \otimes \mathrm{d}r + r^2 \left(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi\right) \tag{5.25}$$

Theorem 19
The metric g_2 is Σ_2-admissible but it is not Σ_1-admissible.

Proof: the proof is based on the argument given in appendix A which proves that Σ_2 is not trivial. The ultimate reason is that, if g_2 were Σ_1-admissible, then there would be a global g_2-orthonormal frame (e_a). The spacelike unit vectors would then induce never-vanishing vector fields on S^2 which is impossible because of well known topological arguments. Q.e.d.

It is interesting to notice that the metrics g_1 and g_2 cannot be *"deformed"* one into the other because of a degree argument. In fact, for any metric g of signature $(2, 1)$ on M, an orthonornal frame (e_a) defines a timelike direction. If the metric is also time-orientable it defines a timelike unit eigen-vector e_0. For each metrics g_1 and g_2, we can regard e_0, restricted to the unitary sphere, as a map from S^2 to S^2. For g_1, the timelike vector is constant so that the degree of φ_1 is $\deg(\varphi_1) = 0$. For g_2, φ_2 is the identity and consequently $\deg(\varphi_2) = 1$. Accordingly, as long as deformations of metrics produce homotopies of the corresponding maps $\varphi : S^2 \to S^2$ then the metrics g_1 and g_2 cannot be deformed one into the other.

It would be interesting to address this problem in its more general formulation: which are the conditions on M and Σ which ensure that any metric is Σ-admissible? Examples are known of manifolds M for which no matter which structure bundle Σ is chosen all metrics of a given signature are Σ-admissible (e.g., the case of propositions 17 and 18). In other cases (see proposition 19) this is not true. And, moreover, if the still unknown conditions are not satisfied, a complete characterization of the equivalence classes as well as the quotient space, is still missing. Some step in that direction can be found in [5]. The arguments there developed, as well as the arguments above, show that there actually exist manifolds on which any metric is Σ-admissible (as we recalled above) and manifolds on which the set of metrics disconnects instead into equivalence classes. The degree argument above suggests that there is a relation between Σ-admissibility and accessibility with respect to deformations though that relation is to be clarified in general. The characterization of the accessibility classes would be particularly important for physical applications, where one fixes a structure bundle Σ, regards spin frames as dynamical fields and the metric structure as a derived structure.

5.7 The Bundle of Spin Frames

If spin frames have to be regarded as dynamical fields in a field theory, they should be represented as global sections of a suitable configuration bundle. If the theory have to be a gauge natural theory, such a configuration bundle have to be a gauge natural bundle associated to the structure bundle Σ fixed in the beginning.

Let us then consider the principal bundle $W^{(1,0)}\Sigma = L(M) \times_M \Sigma$ which is a $(\mathrm{Spin}(\eta) \times \mathrm{GL}(m))$-principal bundle and let us choose the following action on $\mathrm{GL}(m)$:

$$\begin{aligned} \rho : \mathrm{Spin}(\eta) \times \mathrm{GL}(m) \times \mathrm{GL}(m) &\to \mathrm{GL}(m) \\ : (S, a^\nu_\mu, e^\mu_a) &\mapsto a^\nu_\mu\, e^\mu_c\, \ell^c_a(S^{-1}) \end{aligned} \tag{5.26}$$

Definition 10 *Let us denote by Σ_ρ the associated bundle to $W^{(0,1)}(\Sigma)$ via the representation ρ. It will be called the spin frame bundle on Σ.*

Theorem 20
There is a one-to-one correspondence between spin frames on Σ and sections of $\Sigma_\rho \to M$.

Proof: let $\Lambda : \Sigma \to L(M)$ be a spin frame on Σ and $\sigma^{(\alpha)}$ be a system of local sections on Σ canonically inducing a trivialization on Σ. We define a system of local sections on $L(M)$ by $u^{(\alpha)} = \Lambda(\sigma^{(\alpha)})$, which induce canonically a trivialization of $L(M)$. By these definitions the morphism Λ may be written locally in the form $\Lambda(\sigma^{(\alpha)} \cdot S) = u^{(\alpha)} \cdot \ell(S)$.

Let us now define the following local sections of Σ_ρ:

$$s^{(\alpha)} : x \mapsto [\sigma^{(\alpha)}, u^{(\alpha)}, \mathbf{1}]_\rho \tag{5.27}$$

where $[\ ]_\rho : W^{(0,1)}(\Sigma) \times \mathrm{GL}(m) \to \Sigma_\rho$ denotes the canonical projection and $\mathbf{1}$ is the identity matrix. These local sections glue together to give a global section s of Σ_ρ; in fact we have:

$$s^{(\beta)}(x) = [\sigma^{(\beta)}, u^{(\beta)}, \mathbf{1}]_\rho = \tag{5.28}$$

$$= [\sigma^{(\alpha)} \cdot \hat{\gamma}_{(\alpha\beta)}, u^{(\alpha)} \cdot \ell(\hat{\gamma}_{(\alpha\beta)}), \mathbf{1}]_\rho =$$

$$= [\sigma^{(\alpha)}, u^{(\alpha)}, \ell(\hat{\gamma}_{(\alpha\beta)}) \cdot \mathbf{1} \cdot \ell(\hat{\gamma}^{-1}_{(\alpha\beta)})]_\rho = s^{(\alpha)}(x)$$

We remark that the section s does not depend on the particular $\sigma^{(\alpha)}$ chosen for the costruction but just on the spin frame Λ.

In this way we have produced a map F which associates to each spin frame Λ a section $F(\Lambda) = s$ of Σ_ρ; to show that this map is one-to-one we shall explicitly produce its inverse application. Let $s : M \to \Sigma_\rho$ be a section, $\sigma^{(\alpha)}$ be a system of local sections on Σ and $\partial^{(\alpha)}$ be a system of local sections on $L(M)$, both systems inducing trivializations.

With these choices we have standard representatives for points in Σ_ρ under the form $[\sigma^{(\alpha)}, \partial^{(\alpha)}, a^{(\alpha)}]_\rho$ and local expressions for the section s in the following form:

$$s^{(\alpha)}(x) = [\sigma^{(\alpha)}, \partial^{(\alpha)}, a^{(\alpha)}(x)]_\rho \tag{5.29}$$

which of course glue together to give s; i.e., the following holds:

$$s^{(\beta)}(x) = [\sigma^{(\beta)}, \partial^{(\beta)}, a^{(\beta)}(x)] = \tag{5.30}$$

$$= [\sigma^{(\alpha)} \cdot \hat{\gamma}_{(\alpha\beta)}, \partial^{(\alpha)} \cdot \gamma_{(\alpha\beta)}, a^{(\beta)}(x)]$$

where $\hat{\gamma}_{(\alpha\beta)}$ and $\gamma_{(\alpha\beta)}$ are the transition functions on Σ and $L(M)$ with respect to the trivializations chosen.

Thence we have the following relations

$$a^{(\alpha)}(x) = \gamma_{(\alpha\beta)} \cdot a^{(\beta)}(x) \cdot \ell(\hat{\gamma}^{-1}_{(\alpha\beta)}) \tag{5.31}$$

Let us then define the following local morphisms:

$$\Lambda^{(\alpha)} : \pi^{-1}(U_\alpha) \to \tau^{-1}(U_\alpha) : \sigma^{(\alpha)} \cdot S \mapsto \partial^{(\alpha)} \cdot a^{(\alpha)} \cdot \ell(S) \tag{5.32}$$

which glue together; in fact, the following holds:

$$\Lambda^{(\beta)}(\sigma^{(\alpha)} \cdot S) = \Lambda^{(\beta)}(\sigma^{(\beta)} \cdot \hat{\gamma}_{(\beta\alpha)} \cdot S) = \tag{5.33}$$

$$= \partial^{(\alpha)} \cdot \gamma_{(\alpha\beta)} \cdot a^{(\beta)} \cdot \ell(\hat{\gamma}_{(\beta\alpha)}) \cdot \ell(S) =$$

$$= \partial^{(\alpha)} \cdot a^{(\alpha)} \cdot \ell(S) = \Lambda^{(\alpha)}(\sigma^{(\alpha)} \cdot S)$$

and this allows us to define a global morphism $\Lambda_s : \Sigma \to L(M)$ induced by the section s. It is a spin frame on Σ and it will be also denoted by $\bar{F}(s)$.

The map $\bar{F}$ so defined is the inverse application of F as defined above. In fact, if we have a spin frame Λ on the bundle Σ given by $\Lambda(\sigma^{(\alpha)}\cdot S) = u^{(\alpha)}\cdot\ell(S)$, then we have the section $F(\Lambda) = s$, where $s : x \mapsto [\sigma^{(\alpha)}, u^{(\alpha)}, \mathbf{1}]_\rho$ and the spin frame associated to this section is $\bar{F}(s)$, given by $\Lambda : \sigma^{(\alpha)} \cdot S \mapsto u^{(\alpha)} \cdot \mathbf{1} \cdot \ell(S)$, which coincides with the spin frame we started from.

Conversely, if we start from a section

$$s : x \mapsto [\sigma^{(\alpha)}, \partial^{(\alpha)}, a^{(\alpha)}]$$

we associate to it a spin frame $\bar{F}(s)$ given by

$$\Lambda : \sigma^{(\alpha)} \cdot S \mapsto \partial^{(\alpha)} \cdot a^{(\alpha)} \cdot \ell(S),$$

to which the section $F(\Lambda)$ given by

$$s : x \mapsto [\sigma^{(\alpha)}, \partial^{(\alpha)} \cdot a^{(\alpha)}, \mathbf{1}]_\rho = [\sigma^{(\alpha)}, \partial^{(\alpha)}, a^{(\alpha)}]_\rho$$

is again associated. Again, this section concides with the section we started from.

The two applications F and $\bar{F}$ are thence inverse one of each other and F is then a bijection, as we claimed. Q.e.d.

Let us now denote by $\mathrm{Met}(M;\eta)$ the bundle of metrics on M of signature η; it is a natural bundle, in fact associated to the frame bundle $L(M)$ by the following representation on $V = \mathbf{R}^{\frac{1}{2}m(m+1)}$

$$\Lambda : \mathrm{GL}(m) \times V \to V : (J^\mu_\rho, g_{\rho\sigma}) \mapsto \bar{J}^\rho_\mu \, g_{\rho\sigma} \, \bar{J}^\sigma_\nu \tag{5.34}$$

where $\bar{J}^\rho_\mu$ is the inverse matrix of J^μ_ρ.

If $\partial_{(\alpha)}$ is a trivialization of $L(M)$, then a point $g = g^{(\alpha)}_{\mu\nu} dx^\mu \otimes dx^\nu$ in $\mathrm{Met}(M;\eta)$ is of the form $[\partial_{(\alpha)}, g^{(\alpha)}_{\mu\nu}]_\Lambda$; fibered local coordinates on $\mathrm{Met}(M;\eta)$ are $(x^\mu, g^{(\alpha)}_{\mu\nu})$ and transition functions are given by

$$g^{(\alpha)}_{\mu\nu} = J^\rho_\mu \, g'^{(\beta)}_{\rho\sigma} \, J^\sigma_\nu \,, \qquad\qquad \partial^{(\beta)}_\mu = \partial^{(\alpha)}_\nu J^\nu_\mu \tag{5.35}$$

Sections of $\mathrm{Met}(M;\eta)$ correspond to metrics of signature η on M. They exist globally whenever M is a η-manifold.

Proposition 8 *For each bundle Σ_ρ of spin frames there exists a canonical strong epimorphism $g_\Sigma : \Sigma_\rho \to \mathrm{Met}(M;\eta)$*

Proof: let us choose the usual trivializations induced by local sections $\sigma^{(\alpha)}$ and $\partial^{(\alpha)}$; a point in Σ_ρ is thence of the form

$$\sigma = [\sigma^{(\alpha)}, \partial^{(\alpha)}, e^\mu_a]_\rho.$$

Let us denotes by e^a_μ the inverse matrix of $e^\mu_a \in \mathrm{GL}(m)$ and define the local morphisms

$$g^{(\alpha)}(\sigma) = [\partial^{(\alpha)}, g^{(\alpha)}_{\mu\nu}]_\Lambda\,, \qquad g^{(\alpha)}_{\mu\nu} = e^a_\mu\,\eta_{ab}\,e^b_\nu \tag{5.36}$$

These local morphisms glue together to give a global morphism, because of the following

$$\begin{aligned} g^{(\beta)}(\sigma) = g^{(\beta)}([\sigma^{(\beta)}\cdot\hat\gamma_{(\beta\alpha)}, \partial^{(\beta)}\cdot\gamma_{(\beta\alpha)}, e^\mu_a]_\rho) = \\ = g^{(\beta)}([\sigma^{(\beta)}, \partial^{(\beta)}, e'^\mu_a]_\rho) = \\ = [\partial^{(\beta)}, e'^a_\mu\,\eta_{ab}\,e'^b_\nu]_\Lambda = [\partial^{(\alpha)}\cdot\gamma_{(\alpha\beta)}, e'^a_\mu\,\eta_{ab}\,e'^b_\nu]_\Lambda = \\ = [\partial^{(\alpha)}, e^a_\mu\,\eta_{ab}\,e^b_\nu]_\Lambda = g^{(\alpha)}(\sigma) \end{aligned} \tag{5.37}$$

where we set $e'^a_\mu = \ell^b_a(G_{(\beta\alpha)})\,e^b_\nu\,\bar J^\nu_\mu$.

This is an epimorphism since every symmetric matrix $g^{(\alpha)}_{\mu\nu}$ of signature η admits orthonormal frames, i.e. there always exists $e^\mu_a \in \mathrm{GL}(m)$ so that $g^{(\alpha)}_{\mu\nu} = e^a_\mu\,\eta_{ab}\,e^b_\nu$. Q.e.d.

This epimorphism $g_\Sigma : \Sigma_\rho \to \mathrm{Met}(M;\eta)$ is called *the inducing metric morphism (associated to the structure bundle Σ)*; in fact, if $\sigma : M \to \Sigma_\rho$ is a section (i.e. a spin frame on Σ) then $g_\Sigma(\sigma)$ is a section of $\mathrm{Met}(M;\eta)$ corresponding to the metric uniquely associated to that spin frame. Accordingly, we shall say that the metric g is Σ-*admissible* if there exists a spin frame Λ on Σ which induces g, i.e. one has $g = g_\Sigma(\Lambda) \equiv g(\Lambda)$.

Of course, spin frames on Σ are not natural objects. On the contrary, automorphisms of the structure bundle Σ canonically act on spin frames. Let

$$\sigma_{ab} = a^c_d \eta_{c[a} \frac{\partial}{\partial a^{b]}_d} \tag{5.38}$$

be a local basis of vertical right invariant vector fields on Σ. Any infinitesimal generator of automorphisms of Σ is of the form $\Xi = \xi^\mu(x)\,\partial_\mu + \xi^{ab}(x)\,\sigma_{ab}$ and it projects over $\xi = \xi^\mu(x)\,\partial_\mu \in \chi(M)$. Thanks to the canonical action of $\mathrm{Aut}(\Sigma)$ on Σ_ρ we can define the Lie derivative

$$\mathcal{L}_\Xi \sigma = T\sigma(\xi) - \Xi \circ \sigma$$

of a section σ of Σ_ρ (i.e. of a spin frame) with respect to Ξ. If σ is locally given by $\sigma : x \mapsto (x, e_a^\mu(x))$ and $\gamma^\mu_{\Lambda\sigma}$ are the Christoffel symbols of the metric $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$ induced by σ, i.e. the components of its Levi-Civita connection, then we obtain:

$$\mathcal{L}_\Xi e_a^\mu = \nabla_\nu \xi^\mu\, e_a^\nu + e_b^\mu\, \eta_{ca}\, (\xi_{(V)})^{bc} \tag{5.39}$$

where we set:

$$\begin{aligned} \nabla_\nu \xi^\mu &= \mathrm{d}_\nu \xi^\mu + \gamma^\mu_{\Lambda\sigma} \xi^\Lambda \\ \xi^{bc}_{(V)} &= \xi^{bc} + \gamma^b_{c\Lambda} \xi^\Lambda \\ \gamma^b_{c\Lambda} &= e^b_\nu (\gamma^\nu_{\rho\Lambda}\, e^\rho_c + \mathrm{d}_\mu e^\nu_c) \end{aligned} \tag{5.40}$$

Here we used the fact that:

$$\nabla_\nu e^\mu_a = \mathrm{d}_\nu e^\mu_a + \gamma^\mu_{\Lambda\nu}\, e^\Lambda_a - \gamma^b_{a\nu}\, e^\mu_b = 0 \tag{5.41}$$

because of the definition of $\gamma^b_{c\Lambda}$. The functions $\gamma^{ab}_\Lambda = \gamma^a_{c\Lambda}\eta^{ca}$ define the *spin connection* induced by the Levi-Civita connection $\gamma^\mu_{\Lambda\sigma}$ (i.e. ultimately by the spin frame itself) on Σ. Thence we have a pair connections $(\gamma^\mu_{\Lambda\sigma}, \gamma^{ab}_\Lambda)$ which can be regarded together as a principal connection on $L(M) \times \Sigma$. This principal connection induces a connection on the associated bundles (in particular Σ_ρ) which allows us to define the covariant derivative of spin frames (5.41).

We stress that spin frames provide also a way to understand from a global and geometrical viewpoint the vielbein formalism used in physics. In fact, a spin frame is essentially a family of local frames $e_a = e^\mu_a\,\partial_\mu$ on M constrained by the requirement to have *transition functions* with values in the orthogonal group $\mathrm{SO}(\eta)$. The above constraint is required so that a spin frame induces a global metric on M. In other words, a spin frame contains the information needed to specify a global metric g *together with* a way to locally select an orthonormal frame for g. [This is exactly the extra ingredient to define the Dirac operator on a manifold.]

In this way one can see that spin frames are non-natural objects. In fact, despite frames are made of vectors which can be dragged along diffeomorphisms of M, spin frame cannot. If one tries to define the action of a diffeomorphism $f : M \to M$ as:

$$f_*(e_a) = Tf(e_a) \tag{5.42}$$

one still obtains a family of local frames e'^{μ}_{a}; however, the new family does not satisfy the aforementioned constraint. In general, the transition functions of the new family are not valued in $\mathrm{SO}(\eta)$, any longer.

One can say that spin frames are the global counterpart of the vielbein formalism. We remark that in general, the globalization of the vielbein formalism is not straightforward or canonical. Extra global informations are in fact needed: they are (equivalent to) the information encoded in the choice of the structure bundle Σ. We finally remark that one can adapt the argument found in [15] to see that this extra information can be in principle observed by physical experiments.

5.8 Double coverings of the Frame Bundle

A different approach can be found in [3]. This approach is based on the choice of a double covering of the general frame bundle, in place of our structure bundle. In fact, on any orientable spin manifold M one can build a bundle $\tilde{\tau} : \tilde{L}^+(M) \to M$ which is a double covering of the (equioriented) frame bundle $\tau : L^+(M) \to M$. [Notice that it is not unique, at least in general.]

One can define in this way the analogous of the Dirac operator on M acting on vector bundles associated to $\tilde{\tau}$ and the double covering group $\tilde{\mathrm{GL}}^+(m)$ is replaced to the spin group $\mathrm{Spin}(\eta)$. This approach has been tried to define spinors with enhanced properties of *naturality*. However, this approach is not generally accepted by physicists because of a simple algebraic drawback: the group $\tilde{\mathrm{GL}}^+(m)$ has no linear finite dimensional representation. Thence its spinors have infinitely many degrees of freedom which have no basis in physical phenomenology.

In [3], a different approach is considered. Let us fix a particular double covering bundle $\tilde{\ell} : \tilde{L}^+(M) \to L^+(M)$ and suppose that M is a η-manifold.

Then a metric g selects a sub-bundle $\mathrm{SO}(M,g)$ in $L^+(M)$, i.e.:

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{i_g} & \tilde{L}^+(M) \\ \tilde{\Lambda}\big\downarrow & & \big\downarrow\tilde{\ell} \\ \mathrm{SO}(M,g) & \xrightarrow{i_g} & L^+(M) \\ & \searrow & \big\downarrow\tau \\ & & M \end{array} \tag{5.43}$$

Then one can define the pull-back covering $\tilde{\Lambda} : \Sigma_g \to \mathrm{SO}(M,g)$ so that $(\Sigma_g, \tilde{\Lambda})$ is an ordinary spin structure on (M,g), or, equivalently, $\Lambda = i_g \circ \tilde{\Lambda} : \Sigma_g \to L^+(M) \subset L(M)$ is a spin frame on Σ_g. Notice that in this way a canonical correpondence is established between metrics and spin structures. Once the bundle $\tilde{\tau} : \tilde{L}^+(M) \to M$ is chosen, any metric g uniquely determines a spin structure $(\Sigma_g, \tilde{\Lambda})$. Thence one can define in a canonical way how deformations of the metric reverberate on deformations of spin structures. This does not contradict our claim about non-naturality of spin frames: it simply explains that the extra information needed to deform spin structures can be specified at once by choosing the bundle $\tilde{\tau} : \tilde{L}^+(M) \to M$.

Anyway, one can now define a canonical action of diffeomorphisms on spinors, which are defined as sections of a vector bundle associated to Σ_g. Furthermore, spinors have finitely many degrees of freedom, since they are still related to representations of the group $\mathrm{Spin}(\eta)$.

We remark that in this formalism one can easily see that on an η-manifold M the vanishing of the second Stiefel-Whitney class is enough to guarantee the existence of spin structures in any signature. In fact, (see [3]) the second Stiefel-Whitney class is the obstruction to existence of $\tilde{\tau} : \tilde{L}^+(M) \to M$, where no signature is involved. Then once a metric of a specific signature exists, a spin structure for it can be explicitely built.

¿From a categorial viewpoint the formalism presented here is completely different from the one of spin frames: if anyone of them is realized in physics is still an open problem.

5.9 Vielbein and Frauendiener Structures

The construction of spin frames can be extended to groups G other than $\mathrm{Spin}(\eta)$. First of all we can choose $G = \mathrm{SO}(\eta)$ to obtain the global counterpart of the so-called *vielbein formalism* (see [2], [17]). Let us in fact define a *structure bundle* Σ to be any principal bundle $(\Sigma, M, \pi, \mathrm{SO}(\eta))$ such that there exists at least one global, vertical, equivariant morphism $\Lambda : \Sigma \to L(M)$ with respect to the canonical immersion $i : \mathrm{SO}(\eta) \to \mathrm{GL}(m)$, i.e.:

$$\begin{array}{ccc} P & \xrightarrow{\Lambda} & L(M) \\ & \searrow & \downarrow \\ & & M \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\Lambda} & L(M) \\ R_a\downarrow & & \downarrow R_{i(a)} \\ P & \xrightarrow{\Lambda} & L(M) \end{array} \tag{5.44}$$

Any such morphism is then called a *vielbein* or a *moving frame* over M. Let $\sigma^{(\alpha)}$ be a local trivialization of Σ; then we have:

$$\Lambda(\sigma^{(\alpha)}) = V_a^{(\alpha)} = e_a^\mu \partial_\mu^{(\alpha)} \tag{5.45}$$

Notice that V_a are a family of local frames with transition functions valued in $\mathrm{SO}(\eta)$ and therefore they define a global metric $g_{\mu\nu} = e^a_\mu\, \eta_{ab}\, e^b_\nu$.

When we change local trivialization $\sigma^{(\beta)} = \sigma^{(\alpha)} \cdot \bar{\gamma}$ on Σ and $\partial_\mu^{(\alpha)} = \partial_\nu^{(\beta)}\, J^\nu_\mu$ on $L(M)$ the vielbein components change accordingly:

$$V_a^{(\beta)} = \Lambda(\sigma^{(\beta)}) = \Lambda(\sigma^{(\alpha)} \cdot \bar{\gamma}) = \Lambda(\sigma^{(\alpha)}) \cdot i(\bar{\gamma}) = V_b^{(\alpha)}\, \bar{\gamma}^b_a \tag{5.46}$$

$$e^{(\beta)}{}^\mu_a = J^\mu_\nu\, e^{(\alpha)}{}^\nu_b\, \bar{\gamma}^b_a$$

We can thence define the action

$$\Lambda : \mathrm{SO}(\eta) \times \mathrm{GL}(m) \times \mathrm{GL}(m) \to \mathrm{GL}(m) \tag{5.47}$$

$$: (\gamma, J, e) \mapsto J \cdot e \cdot \bar{\gamma}$$

and define the *bundle of vielbein* to be the associated bundle $\Sigma_\Lambda = (\Sigma \times_M L(M)) \times_\Lambda \mathrm{GL}(m)$.

On the basis of the previous definitions the theory can be developed following what has been done for spin frames. In particular automorphisms (ϕ, f) of Σ can be represented on the bundle of vielbein by setting:

$$\begin{cases} x'^\mu = f^\mu(x) \\ \gamma'^a_b = \varphi^a_c\, \gamma^c_b \end{cases} \quad \rightsquigarrow \quad \begin{cases} x'^\mu = f^\mu(x) \\ e'^\mu_a = J^\mu_\nu\, e^\nu_b\, \bar{\varphi}^b_a \end{cases} \tag{5.48}$$

We stress that vielbein are thence definitely non-natural objects. As for spin frames spacetime diffeomorphisms do not act on vielbein and transformations (5.48) are gauge natural transformations. Therefore vielbein are not to be identified with local sections of the frame bundle, in particular for what concerns Lie derivations and deformation.

Another, and more subtle, example of the same construction is given by what we shall call Frauendiener structures (see [13], [23]). Here the *structure bundle* is a principal bundle

$$(\Sigma, M, \pi, \mathrm{GL}(m))$$

having $\mathrm{GL}(m)$ as structure group. Of course one can choose $\Sigma = L(M)$ (endowed with the structure of principal bundle and forgetting about its natural structure) but it is just one possibility. Once again a *Frauendiener frame* is an equivariant, vertical morphism $\Lambda : \Sigma \to L(M)$, i.e.:

$$\begin{array}{ccc} P & \xrightarrow{\Lambda} & L(M) \\ & \searrow & \downarrow \\ & & M \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\Lambda} & L(M) \\ R_a\downarrow & & \downarrow R_a \\ P & \xrightarrow{\Lambda} & L(M) \end{array} \tag{5.49}$$

and at least one Frauendiener frame is required to exist. [We stress that in definition 5.49 the bundle $L(M)$ is locally meant with its natural structure].

Notice that transition functions of the local frames V_a are now valued in $\mathrm{GL}(m)$ and consequently no global metric is induced on M but only local metrics are.

Then we can locally choose the action

$$\mu : \mathrm{GL}(m) \times \mathrm{GL}(m) \times \mathrm{GL}(m) \to \mathrm{GL}(m) \tag{5.50}$$

$$: (\gamma, J, e) \mapsto J \cdot e \cdot \bar{\gamma}$$

and define the *Frauendiener bundle* to be the associated bundle $\Sigma_\mu = (\Sigma \times_M L(M)) \times_\mu \mathrm{GL}(m)$.

5.10 Conclusion and Perspectives

We analyzed in detail the gauge natural structure of spin frames and we gave a geometrical and global formulation of spin structures, vielbein and Frauendiener structures.

We already mentioned an open problem about spin frames: a complete and satisfactory characterization of Σ-admissible metrics on M. It would be particularly interesting to know if such classes coincide with accessibility classes with respect to some notion of deformation or with respect to some topological structure of metrics over M.

We recall that one could expect this problem to be related to some quantum effect. In fact, as long as quantization is interpreted as the sum over all *possible histories* (following path integral Feynmann quantization) and provided that one day in the future we shall have a coherent quantum theory of gravitation, one can reasonably expect that some quantum effect will be able to resolve the difference between the two limit situations in which all *possible histories* run over all metrics or over all Σ-admissible metrics.

Another interesting problem which is also purely geometrical in nature is the following. Let us consider a particular gauge natural bundle $\mathcal{B}$ on which $\mathrm{Aut}(\mathcal{P})$ acts by means of gauge tranformations. Is it possible and under which conditions to define the action of spacetime diffeomorphisms on $\mathcal{B}$? Or does it *contain* and under which conditions a natural sub-bundle? Or can at least some quotient of $\mathcal{B}$ be endowed with a natural structure? We have in fact an example of the last situation. Suppose one considers two copies of the spin frame bundle Σ_Λ. The product of these two copies is again gauge natural. But with two spin frames (e^μ_a, f^ν_a) we can build natural objects, e.g. $e^a_\mu f^\nu_a$ or the induced metrics $e^a_\mu\, \eta_{ab}\, e^b_\nu$ and $f^a_\mu\, \eta_{ab}\, f^b_\nu$. All these examples correspond to some suitable quotient of the gauge natural bundle $\Sigma_\Lambda \times \Sigma_\Lambda$ such that the quotient bundle is natural. As an example of the first situation we mantion the following: the tangent bundle (as a gauge natural bundle associated to $\mathrm{SO}(M, g)$) can be obviously endowed with a natural structure, provided one extends the structure group from $\mathrm{SO}(\eta)$ to the whole linear group $\mathrm{GL}(m, \mathbf{R})$.

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5.10.1 Appendix A: Structure bundles on $M = \mathbf{R}^3 - \{0\} \approx S^2 \times \mathbf{R}^+$

We shall here define a structure bundle over $M = \mathbf{R}^3 - \{0\} \approx S^2 \times \mathbf{R}^+$ with signature $(2, 1)$. Of course a possible choice is the trivial bundle $\Sigma_1 = M \times \mathrm{Spin}(2, 1)$. This is a structure bundle because M is parallelizable. Let us in fact consider a global frame (e_a) in M and a global section σ of Σ_1, which exists by triviality. Then we can define a global morphism $\Lambda : \Sigma_1 \to L(M)$ such that $\Lambda(\sigma) = (e_a)$. This condition completely characterizes the principal morphism Λ since $\Lambda(\sigma \cdot S) = (e_a \cdot \hat{\ell}(S))$.

We can also define another inequivalent structure bundle Σ_2. Let us fix a different trivialization of $L(M)$ by noticing that we have a canonical foliation of M into $S^2 \times \mathbf{R}^+$. Using this foliation we define *angular vectors* to be tangent vectors to some leaf S^2, while *radial vectors* are the generators of the foliations. On M there exists a global radial vector field e_0. Using stereographic coordinates $\{x^1_{(N)}, x^2_{(N)}\}$ on the sphere S^2 we can define a local frame $(e_0^{(N)}, e_1^{(N)}, e_2^{(N)})$ on M given by:

$$e_0^{(N)} = e_0, \qquad e_1^{(N)} = \frac{1+|x|^2}{2}\frac{\partial}{\partial x^1_{(N)}}, \qquad e_2^{(N)} = \frac{1+|x|^2}{2}\frac{\partial}{\partial x^2_{(N)}} \tag{5.51}$$

where we set $|x|^2 = (x^1_{(N)})^2 + (x^2_{(N)})^2$. Analogously, by using stereographic coordinates $\{x^1_{(S)}, x^2_{(S)}\}$ from the south pole we get another frame $(e_0^{(S)}, e_1^{(S)}, e_2^{(S)})$ given by:

$$e_0^{(S)} = e_0, \qquad e_1^{(S)} = \frac{1+|x|^2}{2}\frac{\partial}{\partial x^1_{(S)}}, \qquad e_2^{(S)} = -\frac{1+|x|^2}{2}\frac{\partial}{\partial x^2_{(S)}} \tag{5.52}$$

These two frames define a trivialization of $L(M)$ and its transition functions are

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{(x^1_{(S)})^2-(x^2_{(S)})^2}{|x_{(S)}|^2} & -2\frac{x^1_{(S)}x^2_{(S)}}{|x_{(S)}|^2} \\ 0 & 2\frac{x^1_{(S)}x^2_{(S)}}{|x_{(S)}|^2} & -\frac{(x^1_{(S)})^2-(x^2_{(S)})^2}{|x_{(S)}|^2} \end{pmatrix} \in \mathrm{SO}(2,1) \tag{5.53}$$

Let us now define the group $\mathrm{Spin}(2,1)$ together with the covering map $\ell : \mathrm{Spin}(2,1) \to \mathrm{SO}(2,1)$. The even Clifford algebra $\mathbf{C}l^+(2,1)$ is generated by the elements $\{1, e_{01}, e_{12}, e_{02}\}$. The even Clifford algebra can be identified with 2×2 real matrices via the following identification:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e_{01} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_{02} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

An element of the even Clifford algebra is in the spin group if it is in the form:

$$S = a1 + be_{01} + ce_{12} + de_{02} \qquad\qquad a^2 + c^2 - b^2 - d^2 = 1 \tag{5.54}$$

The covering map $\ell : \mathrm{Spin}(2,1) \to \mathrm{SO}(2,1)$ is then in the form:

$$\ell(S) = \begin{pmatrix} a^2+b^2+c^2+d^2 & 2(ab-cd) & 2(ad+bc) \\ 2(ab+cd) & a^2+b^2-c^2-d^2 & 2(ac+bd) \\ 2(ad-bc) & 2(-ac+bd) & a^2-b^2-c^2+d^2 \end{pmatrix} \quad (5.55)$$

The transition functions γ given by equation (5.53) can be lifted to $\mathrm{Spin}(2,1)$ obtaining:

$$\hat{\gamma} = \frac{x^2_{(S)}}{|x_{(S)}|}\mathbf{1} - \frac{x^1_{(S)}}{|x_{(S)}|} e_{12} \quad (5.56)$$

Being the covering of M we are using made by two open sets, it automatically induces a cocycle which can be used to produce a $\mathrm{Spin}(2,1)$-principal bundle Σ_2. It is a structure bundle since one can define a spin frame $\Lambda : \Sigma_2 \to L(M)$ such that $\Lambda(\sigma^{(N)}) = (e_a^{(N)})$ and $\Lambda(\sigma^{(S)}) = (e_a^{(S)})$, once we chose two local sections $\sigma^{(N)}$ and $\sigma^{(S)}$ inducing a trivialization with transition functions $\hat{\gamma}$.

The bundle Σ_2 is inequivalent to Σ_1 since Σ_2 in non-trivial. In fact, if it were trivial it would admit a global section Σ. We can define a global frame $\Lambda(\sigma) = e_a$ on M. Now the versors e_1 and e_2 are spacelike with respect to the induced metric $g(\Lambda)$ and thence non-radial. Accordingly, they project over never-vanishing vectorfields on the leaves S^2 which is clearly impossible because of the Euler characteristic of S^2. This contradicts the hypothesis that Σ_2 is trivial.

The bundle Σ_2 can be identified as the extension of the pull-back of the Hopf bundle. In fact, one can consider the injection $\mathrm{Spin}(2,0)$ into $\mathrm{Spin}(2,1)$ induced by the corresponding injection of the Clifford algebras. The transition functions (5.56) can be considered as transition functions valued in $\mathrm{Spin}(2,0) \approx U(1)$, which define the Hopf bundle (5.21).

5.10.2 Appendix B: The Gram-Schmidt procedure

Let V be a vector space and g, g' two symmetric, non degenerate, bilinear forms on V.

Lemma 3 *if $u^{(\alpha)}$ and $u^{(\beta)} = u^{(\alpha)} \cdot a_{(\alpha\beta)}$ are two orthonormal bases for g (thence $a_{(\alpha\beta)} \in \mathrm{SO}(m)$) and one applies Gram-Schmidt procedure with respect to g', obtaining respectively two g'-orthonormal bases $u'^{(\alpha)}$ and $u'^{(\beta)}$, then these are related by $u'^{(\beta)} = u'^{(\alpha)} \cdot a_{(\alpha\beta)}$ through the same orthonormal matrix $a_{(\alpha\beta)}$.*

Proof: by the Gram-Schmidt procedure, the frame $u'^{(\alpha)}$ is defined so that the following hold:

$u'^{(\alpha)}_1 = \Lambda u^{(\alpha)}_1$ with $\Lambda > 0$ and $|u'^{(\alpha)}_1|^2 = 1$

$u'^{(\alpha)}_2 = \Lambda_1 u^{(\alpha)}_1 + \Lambda_2 u^{(\alpha)}_2$ with $\Lambda_2 > 0$, $|u'^{(\alpha)}_2|^2 = 1$

and $u'^{(\alpha)}_1 \cdot u'^{(\alpha)}_2 = 0$

$u'^{(\alpha)}_3 = \Lambda_1 u^{(\alpha)}_1 + \Lambda_2 u^{(\alpha)}_2 + \Lambda_3 u^{(\alpha)}_3$ with $\Lambda_3 > 0$, $|u'^{(\alpha)}_3|^2 = 1$,

$u'^{(\alpha)}_1 \cdot u'^{(\alpha)}_3 = 0$ and $u'^{(\alpha)}_2 \cdot u'^{(\alpha)}_3 = 0$

and so on.

These conditions identify uniquely the frame $u'^{(\alpha)}$. Let us also remark that the right multiplication by the orthogonal matrix $a_{(\alpha\beta)}$ corresponds to an orthogonal transformation $A : V \to V$.

Let us then prove that $A(u'^{(\alpha)})$ is the frame obtained by applying the Gram-Schmidt procedure to the frame $u^{(\beta)} = A(u^{(\alpha)})$ with respect to the metric g'. In fact we have:

$A(u'^{(\alpha)}_1) = \Lambda A(u^{(\alpha)}_1) = \Lambda u^{(\beta)}_1$ with $\Lambda > 0$; moreover orthonormal transformations preserve the norms so that $|A(u'^{(\alpha)}_1)|^2 = 1$.

$A(u'^{(\alpha)}_2) = \Lambda_1 A(u^{(\alpha)}_1) + \Lambda_2 A(u^{(\alpha)}_2) = \Lambda_1 u^{(\beta)}_1 + \Lambda_2 u^{(\beta)}_2$ with $\Lambda_2 > 0$; moreover orthonormal transformations preserve the internal products so that $|A(u'^{(\alpha)}_2)|^2 = 1$ and $A(u'^{(\alpha)}_1) \cdot A(u'^{(\alpha)}_2) = 0$.

and so on. Q.e.d.

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